Jensen's diamond principle and its relatives

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Abstract. We survey some recent results on the validity of Jensen's diamond principle at successor cardinals. We also discuss weakening of this principle such as club guessing, and anti-diamond principles such as uniformization.

A collection of open problems is included.

Introduction. Cantor's continuum hypothesis has many equivalent formulations in the context of ZFC. One of the standard formulations asserts the existence of an enumeration $\{A_\alpha \mid \alpha < \omega_1\}$ of the set $\mathcal{P}(\omega)$. A non-standard, twisted, formulation of CH is as follows:

$(\exists)_2$ there exists a sequence, $\langle A_\alpha \mid \alpha < \omega_1\rangle$, such that for every subset $Z \subseteq \omega_1$, there exist two infinite ordinals $\alpha, \beta < \omega_1$ such that $Z \cap \beta = A_\alpha$.

By omitting one of the two closing quantifiers in the above statement, we arrive to the following enumeration principle:

$(\exists)_1$ there exists a sequence, $\langle A_\alpha \mid \alpha < \omega_1\rangle$, such that for every subset $Z \subseteq \omega_1$, there exists an infinite ordinal $\alpha < \omega_1$ such that $Z \cap \alpha = A_\alpha$.

Jensen discovered this last principle during his analysis of Gödel's constructible universe, and gave it the name of *diamond*, $\diamondsuit$. In [**28**], Jensen proved that $\diamondsuit$ holds in the constructible universe, and introduced the very first $\diamondsuit$-based construction of a complicated combinatorial object — a Souslin tree. Since then, this principle and generalizations of it became very popular among set theorists who utilized it to settle open problems in fields including topology, measure theory and group theory.

In this paper, we shall be discussing a variety of diamond-like principles for successor cardinals, including *weak diamond*, *middle diamond*, *club guessing*, *stationary hitting*, and λ^+*-guessing*, as well as, anti-diamond principles including the *uniformization property* and the *saturation of the nonstationary ideal.*

An effort has been put toward including a lot of material, while maintaining an healthy reading flow. In particular, this survey cannot cover all known results on this topic. Let us now briefly describe the content of this survey's sections, and comment on the chosen focus of each section.

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Organization of this paper. In Section 1, Jensen's diamond principles, $\diamondsuit_S$, $\diamondsuit_S^*$, $\diamondsuit_S^+$, are discussed. We address the question to which stationary sets $S \subseteq \lambda^+$, does $2^\lambda = \lambda^+$ imply $\diamondsuit_S$ and $\diamondsuit_S^*$, and describe the effect of square principles and reflection principles on diamond. We discuss a GCH-free version of diamond, which is called *stationary hitting*, and a reflection-free version of $\diamondsuit_S^*$, denoted by $\diamondsuit_S^{\lambda^+}$. In this section, we only deal with the most fundamental variations of diamond, and hence we can outline the whole history.

Section 2 is dedicated to describing part of the set theory generated by Whitehead problem. We deal with the weak diamond, Φ_S, and the uniformization property. Here, rather than including all known results in this direction, we decided to focus on presenting the illuminating proofs of the characterization of weak diamond in cardinal-arithmetic terms, and the failure of instances of the uniformization property at successor of singular cardinals.

In Section 3, we go back to the driving force to the study of diamond — the Souslin hypothesis. Here, we focus on aggregating old, as well as, new open problems around the existence of higher souslin trees, and the existence of particular club guessing sequences.

Section 4 deals with non-saturation of particular ideals — ideals of the form $\mathrm{NS}_{\lambda^+} \restriction S$. Here, we describe the interplay between non-saturation, diamond and weak-diamond, and we focus on presenting the recent results in this line of research.

Notation and conventions. For ordinals $\alpha < \beta$, we denote by $(\alpha, \beta) := \{\gamma \mid \alpha < \gamma < \beta\}$, the open interval induced by α and β. For a set of ordinals C, we denote by $\mathrm{acc}(C) := \{\alpha < \sup(C) \mid \sup(C \cap \alpha) = \alpha\}$, the set of all accumulation points of C. For a regular uncountable cardinal, κ, and a subset $S \subseteq \kappa$, let

$$\mathrm{Tr}(S) := \min\{\gamma < \kappa \mid \mathrm{cf}(\gamma) > \omega, S \cap \gamma \text{ is stationary in } \gamma\}.$$

We say that S *reflects* iff $\mathrm{Tr}(S) \neq \emptyset$, is *non-reflecting* iff $\mathrm{Tr}(S) = \emptyset$, and *reflects stationarily often* iff $\mathrm{Tr}(S)$ is stationary.

For cardinals $\kappa < \lambda$, denote $E^\lambda_\kappa := \{\alpha < \lambda \mid \mathrm{cf}(\alpha) = \kappa\}$, and $[\lambda]^\kappa := \{X \subseteq \lambda \mid |X| = \kappa\}$. $E^\lambda_{>\kappa}$ and $[\lambda]^{<\kappa}$ are defined analogously. Cohen's notion of forcing for adding κ many λ-Cohen sets is denoted by $\mathrm{Add}(\lambda, \kappa)$. To exemplify, the forcing notion for adding a single Cohen real is denoted by $\mathrm{Add}(\omega, 1)$.

Contents

1. Diamond

Recall Jensen's notion of diamond in the context of successor cardinals.

Definition 1.1 (Jensen, [28]). For an infinite cardinal λ and stationary subset $S \subseteq \lambda^+$:

- ▶ $\diamondsuit_S$ asserts that there exists a sequence $\langle A_\alpha \mid \alpha \in S\rangle$ such that:
  - for all $\alpha \in S$, $A_\alpha \subseteq \alpha$;
  - if Z is a subset of λ^+, then the following set is stationary:
$$\{\alpha \in S \mid Z \cap \alpha = A_\alpha\}.$$

Jensen isolated the notion of diamond from his original construction of an $\aleph_1$-Souslin tree from $V = L$; in [28], he proved that $\diamondsuit_{\omega_1}$ witnesses the existence of such a tree, and that:

Theorem 1.2 (Jensen, [28]). *If $V = L$, then $\diamondsuit_S$ holds for every stationary $S \subseteq \lambda^+$ and every infinite cardinal λ.*

In fact, Jensen established that $V = L$ entails stronger versions of diamond, two of which are the following.

Definition 1.3 (Jensen, [28]). For an infinite cardinal λ and stationary subset $S \subseteq \lambda^+$:

- ▶ $\diamondsuit^*_S$ asserts that there exists a sequence $\langle \mathcal{A}_\alpha \mid \alpha \in S\rangle$ such that:
  - for all $\alpha \in S$, $\mathcal{A}_\alpha \subseteq \mathcal{P}(\alpha)$ and $|\mathcal{A}_\alpha| \leq \lambda$;
  - if Z is a subset of λ^+, then the there exists a club $C \subseteq \lambda^+$ such that:
$$C \cap S \subseteq \{\alpha \in S \mid Z \cap \alpha \in \mathcal{A}_\alpha\}.$$
- ▶ $\diamondsuit^+_S$ asserts that there exists a sequence $\langle \mathcal{A}_\alpha \mid \alpha \in S\rangle$ such that:
  - for all $\alpha \in S$, $\mathcal{A}_\alpha \subseteq \mathcal{P}(\alpha)$ and $|\mathcal{A}_\alpha| \leq \lambda$;
  - if Z is a subset of λ^+, then the there exists a club $C \subseteq \lambda^+$ such that:
$$C \cap S \subseteq \{\alpha \in S \mid Z \cap \alpha \in \mathcal{A}_\alpha \ \&\ C \cap \alpha \in \mathcal{A}_\alpha\}.$$

Kunen [34] proved that $\diamondsuit^*_S \Rightarrow \diamondsuit_T$ for every stationary $T \subseteq S \subseteq \lambda^+$, and that $\diamondsuit_{\lambda^+}$ cannot be introduced by a λ^+-c.c. notion of forcing.

Since, for a stationary subset $S \subseteq \lambda^+$, $\diamondsuit^+_S \Rightarrow \diamondsuit^*_S \Rightarrow \diamondsuit_S \Rightarrow \diamondsuit_{\lambda^+} \Rightarrow (2^\lambda = \lambda^+)$, it is natural to study which of these implications may be reversed.

Jensen (see [7]) established the consistency of $\diamondsuit^*_{\omega_1} + \neg\diamondsuit^+_{\omega_1}$, from the existence of an inaccessible cardinal. In [46], it is observed that if $\lambda^{\aleph_0} = \lambda$, then for every stationary $S \subseteq \lambda^+$, $\diamondsuit^*_S$ is equivalent to $\diamondsuit^+_S$. Devlin [8], starting with a model of $V \models \mathrm{GCH}$, showed that $V^{\mathrm{Add}(\lambda^+,\lambda^{++})} \models \neg\diamondsuit^*_{\lambda^+} + \diamondsuit_{\lambda^+}$.[1] Jensen proved that, in general, the implication $\diamondsuit_{\lambda^+} \Rightarrow (2^\lambda = \lambda^+)$, may not be reversed:

Theorem 1.4 (Jensen, see [10]). CH *is consistent together with* $\neg\diamondsuit_{\omega_1}$.

On the other hand, Gregory, in a paper that deals with higher Souslin trees, established the following surprising result.

[1] For this, he argued that if G is $\mathrm{Add}(\lambda^+, 1)$-generic over V, then
(1) $V[G] \models \diamondsuit_S$ for every stationary $S \subseteq \lambda^+$ from V, and
(2) every sequence $\langle \mathcal{A}_\alpha \mid \alpha < \lambda^+\rangle$ that witnesses $\diamondsuit^*_{\lambda^+}$ in V, will cease to witness $\diamondsuit^*_{\lambda^+}$ in $V[G]$.

Theorem 1.5 (Gregory, [25]). *Suppose λ is an uncountable cardinal, $2^\lambda = \lambda^+$. If $\sigma < \lambda$ is an infinite cardinal such that $\lambda^\sigma = \lambda$, then $\diamondsuit^*_{E^{\lambda^+}_\sigma}$ holds.*
In particular, GCH *entails $\diamondsuit^*_{E^{\lambda^+}_{<\mathrm{cf}(\lambda)}}$ for any cardinal λ of uncountable cofinality.*

Unfortunately, it is impossible to infer $\diamondsuit_{\lambda^+}$ from GCH using Gregory's theorem, in the case that $\lambda > \mathrm{cf}(\lambda) = \omega$. However, shortly afterwards, this missing case has been settled by Shelah.

Theorem 1.6 (Shelah, [52]). *Suppose λ is a singular cardinal, $2^\lambda = \lambda^+$. If $\sigma < \lambda$ is an infinite cardinal such that $\sup\{\mu^\sigma \mid \mu < \lambda\} = \lambda$, and $\sigma \neq \mathrm{cf}(\lambda)$, then $\diamondsuit^*_{E^{\lambda^+}_\sigma}$ holds.*
In particular, GCH *entails $\diamondsuit^*_{E^{\lambda^+}_{\neq\mathrm{cf}(\lambda)}}$ for every uncountable cardinal, λ.*

A closer look at the proof of Theorems 1.5, 1.6 reveals that moreover $\diamondsuit^+_{E^{\lambda^+}_\sigma}$ may be inferred from the same assumptions, and, more importantly, that the hypothesis involving σ may be weakened to: "$\sup\{\mathrm{cf}([\mu]^\sigma, \supseteq) \mid \mu < \lambda\} = \lambda$". However, it was not clear to what extent this weakening indeed witnesses more instances of diamonds.

Then, twenty years after proving Theorem 1.6, Shelah established that the above weakening is quite prevalent. In [63], he proved that the following consequence of GCH follows outright from ZFC.

Theorem 1.7 (Shelah, [63]). *If θ is an uncountable strong limit cardinal, then for every cardinal $\lambda \geq \theta$, the set $\{\sigma < \theta \mid \mathrm{cf}([\lambda]^\sigma, \supseteq) > \lambda\}$ is bounded below θ.*
In particular, for every cardinal $\lambda \geq \beth_\omega$, the following are equivalent:
1. *$2^\lambda = \lambda^+$;*
2. *$\diamondsuit_{\lambda^+}$;*
3. *$\diamondsuit^*_{E^{\lambda^+}_\sigma}$ for co-boundedly many $\sigma < \beth_\omega$.*

Let CH_λ denote the assertion that $2^\lambda = \lambda^+$. By Theorems 1.4 and 1.7, CH_λ does not imply $\diamondsuit_{\lambda^+}$ for $\lambda = \omega$, but does imply $\diamondsuit_{\lambda^+}$ for every cardinal $\lambda \geq \beth_\omega$. This left a mysterious gap between ω and $\beth_\omega$, which was only known to be closed in the presence of the stronger cardinal arithmetic hypotheses, as in Theorem 1.5.

It then took ten additional years until this mysterious gap has been completely closed, where recently Shelah proved the following striking theorem.

Theorem 1.8 (Shelah, [68]). *For an uncountable cardinal λ, and a stationary subset $S \subseteq E^{\lambda^+}_{\neq\mathrm{cf}(\lambda)}$, the following are equivalent:*
1. *CH_λ;*
2. *$\diamondsuit_S$.*

Remark. In [31], Komjáth provides a simplified presentation of Shelah's proof. Also, in [44] the author presents a considerably shorter proof.[2]

Having Theorem 1.8 in hand, we now turn to studying the validity of $\diamondsuit_S$ for sets of the form $S \subseteq E^{\lambda^+}_{\mathrm{cf}(\lambda)}$. Relativizing Theorem 1.5 to the first interesting case, the case $\lambda = \aleph_1$, we infer that GCH entails $\diamondsuit^*_{E^{\omega_2}_\omega}$. By Devlin's theorem [8], GCH $\not\Rightarrow \diamondsuit^*_{\aleph_2}$, and consequently, GCH does not imply $\diamondsuit^*_{E^{\aleph_2}_{\omega_1}}$. Now, what about the

[2]See the discussion after Theorem 1.19 below.

unstarred version of diamond? It turns out that the behavior here is analogous to the one of Theorem 1.4.

THEOREM 1.9 (Shelah, see [29]). GCH *is consistent with* $\neg\diamondsuit_S$, *for* $S = E^{\omega_2}_{\omega_1}$.

The proof of Theorem 1.9 generalizes to successor of higher regular cardinals, suggesting that we should focus our attention on successors of singulars. And indeed, a longstanding, still open, problem is the following question.

QUESTION 1 (Shelah). Is it consistent that for some *singular* cardinal λ, CH_λ holds, while $\diamondsuit_{E^{\lambda^+}_{\mathrm{cf}(\lambda)}}$ fails?

In [55, §3], Shelah established that a positive answer to the above question — in the case that λ is a strong limit — would entail the failure of weak square,[3] and hence requires large cardinals. More specifically:

THEOREM 1.10 (Shelah, [55]). *Suppose* λ *is a strong limit singular cardinal, and* $\square^*_\lambda$ *holds. If* $S \subseteq E^{\lambda^+}_{\mathrm{cf}(\lambda)}$ *reflects stationarily often, then* $\mathrm{CH}_\lambda \Rightarrow \diamondsuit_S$.

Applying ideas of the proof of Theorem 1.8 to the proof Theorem 1.10, Zeman established a "strong limit"-free version of the preceding.

THEOREM 1.11 (Zeman, [75]). *Suppose* λ *is a singular cardinal, and* $\square^*_\lambda$ *holds. If* $S \subseteq E^{\lambda^+}_{\mathrm{cf}(\lambda)}$ *reflects stationarily often, then* $\mathrm{CH}_\lambda \Rightarrow \diamondsuit_S$.

The curious reader may wonder on the role of the reflection hypothesis in the preceding two theorems; in [55, §2], Shelah established the following counterpart:

THEOREM 1.12 (Shelah, [55]). *Suppose* CH_λ *holds for a strong limit singular cardinal,* λ. *If* $S \subseteq E^{\lambda^+}_{\mathrm{cf}(\lambda)}$ *is a non-reflecting stationary set, then there exists a notion of forcing* $\mathbb{P}_S$ *such that:*

(1) $\mathbb{P}_S$ *is* λ*-distributive;*
(2) $\mathbb{P}_S$ *satisfies the* λ^{++}*-c.c.;*
(3) S *remains stationary in* $V^{\mathbb{P}_S}$*;*
(4) $V^{\mathbb{P}_S} \models \neg\diamondsuit_S$.

In particular, it is consistent that $\mathrm{GCH} + \square^*_\lambda$ *holds, while* $\diamondsuit_S$ *fails for some non-reflecting stationary set* $S \subseteq E^{\lambda^+}_{\mathrm{cf}(\lambda)}$.

The next definition suggests a way of filtering out the behavior of diamond on non-reflecting sets.

DEFINITION 1.13 ([44]). For an infinite cardinal λ and stationary subsets $T, S \subseteq \lambda^+$:

- ▶ $\diamondsuit^T_S$ asserts that there exists a sequence $\langle \mathcal{A}_\alpha \mid \alpha \in S\rangle$ such that:
  - for all $\alpha \in S$, $\mathcal{A}_\alpha \subseteq \mathcal{P}(\alpha)$ and $|\mathcal{A}_\alpha| \leq \lambda$;
  - if Z is a subset of λ^+, then the following set is non-stationary:

$$T \cap \mathrm{Tr}\{\alpha \in S \mid Z \cap \alpha \notin \mathcal{A}_\alpha\}.$$

[3] The weak square property at λ, denoted $\square^*_\lambda$, is the principle $\square_{\lambda,\lambda}$ as in Definition 3.8.

Notice that by Theorem 1.6, GCH entails $\diamondsuit_{\lambda^+}^{\lambda^+}$ for every regular cardinal λ. Now, if λ is singular, then GCH does not necessarily imply $\diamondsuit_{\lambda^+}^{\lambda^+}$,[4] however, if in addition $\square_\lambda^*$ holds, then GCH does entail $\diamondsuit_{\lambda^+}^{\lambda^+}$, as the following improvement of theorem 1.10 shows.

Theorem 1.14 ([44]). *For a strong limit singular cardinal, λ:*

(1) *if $\square_\lambda^*$ holds, then $\mathrm{CH}_\lambda \Leftrightarrow \diamondsuit_{\lambda^+}^{\lambda^+}$;*
(2) *if every stationary subset of $E_{\mathrm{cf}(\lambda)}^{\lambda^+}$ reflects, then $\diamondsuit_{\lambda^+}^{\lambda^+} \Leftrightarrow \diamondsuit_{\lambda^+}^*$.*

Remark. An interesting consequence of the preceding theorem is that assuming GCH, for every singular cardinal, λ, $\square_\lambda^*$ implies that in the generic extension by $\mathrm{Add}(\lambda^+,1)$, there exists a non-reflecting stationary subset of λ^+. This is a reminiscent of the fact that $\square_\lambda$ entails the existence non-reflecting stationary subset of λ^+.

Back to Question 1, it is natural to study to what extent can the weak square hypothesis in Theorem 1.11 be weakened. We now turn to defining the axiom SAP_λ and describing its relation to weak square and diamond.

Definition 1.15 ([44]). For a singular cardinal λ and $S \subset \lambda^+$, consider the ideal $I[S;\lambda]$: a set T is in $I[S;\lambda]$ iff $T \subseteq \mathrm{Tr}(S)$ and there exists a function $d : [\lambda^+]^2 \to \mathrm{cf}(\lambda)$ such that:

- d is *subadditive*: $\alpha < \beta < \gamma < \lambda^+$ implies $d(\alpha,\gamma) \leq \max\{d(\alpha,\beta), d(\beta,\gamma)\}$;
- d is *normal*: for all $i < \mathrm{cf}(\lambda)$ and $\beta < \lambda^+$, $|\{\alpha < \beta \mid d(\alpha,\beta) \leq i\}| < \lambda$;
- *key property*: for some club $C \subseteq \lambda^+$, for every $\gamma \in T \cap C \cap E_{>\mathrm{cf}(\lambda)}^{\lambda^+}$, there exists a stationary $S_\gamma \subseteq S \cap \gamma$ with $\sup(d``[S_\gamma]^2) < \mathrm{cf}(\lambda)$.

Evidently, if $I[S;\lambda]$ contains a stationary set, then S reflects stationarily often. The purpose of the next definition is to impose the converse implication.

Definition 1.16 ([44]). For a singular cardinal λ, the *stationary approachability property* at λ, abbreviated SAP_λ, asserts that $I[S;\lambda]$ contains a stationary set for every stationary $S \subseteq E_{\mathrm{cf}(\lambda)}^{\lambda^+}$ that reflects stationarily often.

Our ideal $I[S;\lambda]$ is a variation of Shelah's approachability ideal $I[\lambda^+]$, and the axiom SAP_λ is a variation of the *approachability property*, AP_λ.[5] We shall be comparing these two principles later, but let us first compare SAP_λ with $\square_\lambda^*$.

In [44], it is observed that for every singular cardinal λ, $\square_\lambda^* \Rightarrow \mathrm{SAP}_\lambda$, and moreover, $\square_\lambda^*$ entails the existence of a function, $d : [\lambda^+]^2 \to \mathrm{cf}(\lambda)$, that serves as a *unified* witness to the fact *for all* $S \subseteq \lambda^+$, $\mathrm{Tr}(S) \in I[S;\lambda]$. Then, starting with a supercompact cardinal, a model is constructed in which (1) $\mathrm{GCH} + \mathrm{SAP}_{\aleph_\omega}$ holds, (2) every stationary subset of $E_\omega^{\aleph_{\omega+1}}$ reflects stationarily often, and (3) for

[4]Start with a model of GCH and a supercompact cardinal κ. Use backward Easton support iteration of length $\kappa+1$, forcing with $\mathrm{Add}(\alpha^{+\omega+1}, \alpha^{+\omega+2})$ for every inaccessible $\alpha \leq \kappa$. Now, work in the extension and let $\lambda := \kappa^{+\omega}$. Then the GCH holds, κ remains supercompact, and by Devlin's argument [8], $\diamondsuit_{\lambda^+}^*$ fails. Since $\mathrm{cf}(\lambda) < \kappa < \lambda$, and κ is supercompact, we get that every stationary subset of $E_{\mathrm{cf}(\lambda)}^{\lambda^+}$ reflects, and so it follows from Theorem 1.14(2), that $\diamondsuit_{\lambda^+}^{\lambda^+}$ fails in this model of GCH.

[5]For instance, if $\lambda > \mathrm{cf}(\lambda) > \omega$ is a strong limit, then $I[\lambda^+] = \mathcal{P}(E_\omega^{\lambda^+}) \cup I[E_\omega^{\lambda^+};\lambda]$. For the definition of $I[\lambda^+]$ and AP_λ, see [15].

every stationary $S \subseteq E^{\aleph_{\omega+1}}_{\omega}$ and any function d witnessing that $I[S;\aleph_\omega]$ contains a stationary set, there exists another stationary $S' \subseteq E^{\aleph_{\omega+1}}_{\omega}$ such that this particular d does not witness the fact that $I[S';\aleph_\omega]$ contains a stationary set. Thus, establishing:

Theorem 1.17 ([44]). *It is relatively consistent with the existence of a supercompact cardinal, that* $\mathrm{SAP}_{\aleph_\omega}$ *holds, while* $\square^*_{\aleph_\omega}$ *fails.*

Once it is established that SAP_λ is strictly weaker than $\square^*_\lambda$, the next task would be proving that it is possible to replace $\square^*_\lambda$ in Theorem 1.11 with SAP_λ, while obtaining the same conclusion. The proof of this fact goes through a certain cardinal-arithmetic-free version of diamond, which we now turn to define.

Definition 1.18 ([44]). For an infinite cardinal λ and stationary subsets $T, S \subseteq \lambda^+$, consider the following two principles:

- ► $\clubsuit^-_S$ asserts that there exists a sequence $\langle \mathcal{A}_\alpha \mid \alpha \in S\rangle$ such that:
  - for all $\alpha \in S$, $\mathcal{A}_\alpha \subseteq [\alpha]^{<\lambda}$ and $|\mathcal{A}_\alpha| \leq \lambda$;
  - if Z is a *cofinal* subset of λ^+, then the following set is stationary:

$$\{\alpha \in S \mid \exists A \in \mathcal{A}_\alpha(\sup(Z \cap A) = \alpha)\}.$$

- ► $\clubsuit^-_S \restriction T$ asserts that there exists a sequence $\langle \mathcal{A}_\alpha \mid \alpha \in S\rangle$ such that:
  - for all $\alpha \in S$, $\mathcal{A}_\alpha \subseteq [\alpha]^{<\lambda}$ and $|\mathcal{A}_\alpha| \leq \lambda$;
  - if Z is a *stationary* subset of T, then the following set is non-empty:

$$\{\alpha \in S \mid \exists A \in \mathcal{A}_\alpha(\sup(Z \cap A) = \alpha)\}.$$

Notice that $\clubsuit^-_S$ makes sense only in the case that $S \subseteq E^{\lambda^+}_{<\lambda}$. In [44], it is established that the *stationary hitting* principle, $\clubsuit^-_S \restriction \lambda^+$, is equivalent to $\clubsuit^-_S$, and that these equivalent principles are the cardinal-arithmetic-free version of diamond:

Theorem 1.19 ([44]). *For an uncountable cardinal* λ*, and a stationary subset* $S \subseteq E^{\lambda^+}_{<\lambda}$*, the following are equivalent:*

(1) $\clubsuit^-_S + \mathrm{CH}_\lambda$;
(2) $\diamondsuit_S$.

It is worth mentioning that the proof of Theorem 1.19 is surprisingly short, and when combined with the easy argument that $\mathrm{ZFC} \vdash \clubsuit^-_S$ for every stationary subset $S \subseteq E^{\lambda^+}_{\neq \mathrm{cf}(\lambda)}$, one obtains a single-page proof of Theorem 1.8.

It is also worth mentioning the functional versions of these principles.

Fact 1.20. *Let* λ *denote an infinite cardinal, and* S *denote a stationary subset of* λ^+*; then:*

- ► $\diamondsuit_S$ *is equivalent to the existence of a sequence* $\langle g_\alpha \mid \alpha \in S\rangle$ *such that:*
  - *for all* $\alpha \in S$, $g_\alpha : \alpha \to \alpha$ *is some function;*
  - *for every function* $f : \lambda^+ \to \lambda^+$*, the following set is stationary:*

$$\{\alpha \in S \mid f \restriction \alpha = g_\alpha\}.$$

- ► $\clubsuit^-_S$ *is equivalent to the existence of a sequence* $\langle \mathcal{G}_\alpha \mid \alpha \in S\rangle$ *such that:*
  - *for all* $\alpha \in S$, $\mathcal{G}_\alpha \subseteq [\alpha \times \alpha]^{<\lambda}$ *and* $|\mathcal{G}_\alpha| \leq \lambda$*;*
  - *for every function* $f : \lambda^+ \to \lambda^+$*, the following set is stationary:*

$$\{\alpha \in S \mid \exists G \in \mathcal{G}_\alpha \ \sup\{\beta < \alpha \mid (\beta, f(\beta)) \in G\} = \alpha\}.$$

Finally, we are now in a position to formulate a theorem of local nature, from which we derive a global corollary.

THEOREM 1.21 ([44]). *Suppose λ is a singular cardinal, and $S \subseteq \lambda^+$ is a stationary set. If $I[S;\lambda]$ contains a stationary set, then $\clubsuit^-_S$ holds.*

COROLLARY 1.22 ([44]). *Suppose* SAP_λ *holds, for a given singular cardinal, λ. Then the following are equivalent:*

(1) CH_λ*;*
(2) $\diamondsuit_S$ *holds for every $S \subseteq \lambda^+$ that reflects stationarily often.*

Thus, the hypothesis $\square^*_\lambda$ from Theorem 1.11 may indeed be weakened to SAP_λ. Having this positive result in mind, one may hope to improve the preceding, proving that $\mathrm{CH}_\lambda \Rightarrow \diamondsuit_S$ for every $S \subseteq \lambda^+$ that reflects stationarily often, without any additional assumptions. Clearly, this would have settle Question 1 (in the negative!). However, a recent result by Gitik and the author shows that diamond *may fail* on a set that reflects stationarily often, and even on an (ω_1+1)-fat subset of $\aleph_{\omega+1}$:

THEOREM 1.23 (Gitik-Rinot, [22]). *It is relatively consistent with the existence of a supercompact cardinal that the* GCH *holds above ω, while $\diamondsuit_S$ fails for a stationary set $S \subseteq E^{\aleph_{\omega+1}}_\omega$ such that:*

$$\{\gamma < \aleph_{\omega+1} \mid \mathrm{cf}(\gamma) = \omega_1, S \cap \gamma \textit{ contains a club}\} \textit{ is stationary.}$$

In fact, the above theorem is just one application of the following general, ZFC result.

THEOREM 1.24 (Gitik-Rinot, [22]). *Suppose* CH_λ *holds for a strong limit singular cardinal, λ. Then there exists a notion of forcing $\mathbb{P}$, satisfying:*

(1) $\mathbb{P}$ *is λ^+-directed closed;*
(2) $\mathbb{P}$ *has the λ^{++}-c.c.;*
(3) $|\mathbb{P}| = \lambda^{++}$*;*
(4) *in $V^{\mathbb{P}}$, $\diamondsuit_S$ fails for some stationary $S \subseteq E^{\lambda^+}_{\mathrm{cf}(\lambda)}$.*

Note that unlike Theorem 1.14, here the stationary set on which diamond fails, is a generic one.

Utilizing the forcing notion from Theorem 1.24, Gitik and the author were able to show that Corollary 1.22 is optimal: in [22], it is proved that replacing the SAP_λ hypothesis in Corollary 1.22 with AP_λ, or with the existence of a *better scale* for λ, or even with the existence of a *very good scale* for λ, is impossible, in the sense that these alternative hypotheses do not entail diamond on all reflecting stationary sets.[6] In particular:

THEOREM 1.25 (Gitik-Rinot, [22]). *It is relatively consistent with the existence of a supercompact cardinal that* $\mathrm{AP}_{\aleph_\omega}$ *holds, while* $\mathrm{SAP}_{\aleph_\omega}$ *fails.*

Moreover, in the model from Theorem 1.25, every stationary subset of $E^{\aleph_{\omega+1}}_\omega$ reflects. Recalling that $\mathrm{AP}_{\aleph_\omega}$ holds whenever every stationary subset of $\aleph_{\omega+1}$ reflects, we now arrive to the following nice question.

[6] The existence of a better scale at λ, as well as the approachability property at λ, are well-known consequences of $\square^*_\lambda$. For definitions and proofs, see [15].

QUESTION 2. Is it consistent that every stationary subset of $\aleph_{\omega+1}$ reflects, while $\mathrm{SAP}_{\aleph_\omega}$ fails to hold?

To summarize the effect of square-like principles on diamond, we now state a corollary. Let Refl_λ denote the assertion that every stationary subset of $E^{\lambda^+}_{\mathrm{cf}(\lambda)}$ reflects stationarily often. Then:

COROLLARY 1.26. *For a singular cardinal, λ:*

(1) $\mathrm{GCH}+\square^*_\lambda \not\Rightarrow \diamondsuit^*_{\lambda^+}$;
(2) $\mathrm{GCH}+\mathrm{Refl}_\lambda+\square^*_\lambda \Rightarrow \diamondsuit^*_{\lambda^+}$;
(3) $\mathrm{GCH}+\mathrm{Refl}_\lambda+\mathrm{SAP}_\lambda \not\Rightarrow \diamondsuit^*_{\lambda^+}$;
(4) $\mathrm{GCH}+\mathrm{Refl}_\lambda+\mathrm{SAP}_\lambda \Rightarrow \diamondsuit_S$ *for every stationary* $S\subseteq\lambda^+$;
(5) $\mathrm{GCH}+\mathrm{Refl}_\lambda+\mathrm{AP}_\lambda \not\Rightarrow \diamondsuit_S$ *for every stationary* $S\subseteq\lambda^+$.

PROOF. (1) By Theorem 1.12. (2) By Theorem 1.14. (3) By the proof of Theorem 1.17 in [**44**]. (4) By Corollary 1.25. (5) By the proof of Theorem 1.25 in [**22**]. $\square$

The combination of Theorems 1.19 and 1.21 motivates the study of the ideal $I[S;\lambda]$. For instance, a positive answer to the next question would supply an answer to Question 1.

QUESTION 3. Must $I[E^{\lambda^+}_{\mathrm{cf}(\lambda)};\lambda]$ contain a stationary set for every singular cardinal λ?

One of the ways of attacking the above question involves the following reflection principles.

DEFINITION 1.27 ([**44**]). Assume $\theta>\kappa$ are regular uncountable cardinals.

$R_1(\theta,\kappa)$ asserts that for every function $f: E^\theta_{<\kappa}\rightarrow\kappa$, there exists some $j<\kappa$ such that $\{\delta\in E^\theta_\kappa \mid f^{-1}[j]\cap\delta \text{ is stationary}\}$ is stationary in θ.

$R_2(\theta,\kappa)$ asserts that for every function $f: E^\theta_{<\kappa}\rightarrow\kappa$, there exists some $j<\kappa$ such that $\{\delta\in E^\theta_\kappa \mid f^{-1}[j]\cap\delta \text{ is non-stationary}\}$ is non-stationary.

It is not hard to see that $R_2(\theta,\kappa)\Rightarrow R_1(\theta,\kappa)$, and that MM implies $R_1(\aleph_2,\aleph_1)+\neg R_2(\aleph_2,\aleph_1)$. In [**44**], a fact from *pcf* theory is utilized to prove:

THEOREM 1.28 ([**44**]). *Suppose $\lambda>\mathrm{cf}(\lambda)=\kappa>\omega$ are given cardinals.*

The ideal $I[E^{\lambda^+}_{\mathrm{cf}(\lambda)};\lambda]$ contains a stationary set whenever the following set is non-empty:

$$\{\theta<\lambda \mid R_1(\theta,\kappa) \text{ holds}\}.$$

As a corollary, one gets a surprising result stating that a local instance of reflection affects the validity of diamond on a proper class of cardinals.

COROLLARY 1.29 (implicit in [**68**]). *Suppose κ is the successor of a cardinal κ^-, and that every stationary subset of $E^{\kappa^+}_{\kappa^-}$ reflects.*

Then, $\mathrm{CH}_\lambda\Leftrightarrow\diamondsuit_{E^{\lambda^+}_{\mathrm{cf}(\lambda)}}$ for every singular cardinal λ of cofinality κ.

As the reader may expect, the principle R_2 yields a stronger consequence.

THEOREM 1.30 ([**44**]). *Suppose $\theta>\kappa$ are cardinals such that $R_2(\theta,\kappa)$ holds. Then:*

(1) *For every singular cardinal* λ *of cofinality* κ*, and every* $S \subseteq \lambda^+$*, we have*

$$\mathrm{Tr}(S) \cap E^{\lambda^+}_\theta \in I[S;\lambda].$$

(2) *if* λ *is a strong limit singular cardinal of cofinality* κ*, then* $\mathrm{CH}_\lambda \Leftrightarrow \diamondsuit^{E^{\lambda^+}_\theta}_{\lambda^+}$.

Unfortunately, there is no hope to settle Question 3 using these reflection principles, as they are independent of ZFC: by a theorem of Harrington and Shelah [**26**], $R_1(\aleph_2, \aleph_1)$ is equiconsistent with the existence of a Mahlo cardinal, whereas, by a theorem of Magidor [**38**], $R_2(\aleph_2, \aleph_1)$ is consistent modulo the existence of a weakly-compact cardinal. An alternative sufficient condition for $I[S;\lambda]$ to contain a stationary set will be described in Section 4 (See Fact 4.14 below).

2. Weak Diamond and the Uniformization Property

Suppose that G and H are abelian groups and $\pi : H \to G$ is a given epimorphism. We say that π *splits* iff there exists an homomorphism $\phi : G \to H$ such that $\pi \circ \phi$ is the identity function on G. An abelian group G is *free* iff every epimorphism onto G, splits.

Whitehead problem reads as follows.

Question. Suppose that G is an abelian group such that every epimorphism π onto G with the property that $\ker(\pi) \simeq \mathbb{Z}$ — splits;[7]
Must G be a free abelian group?

Thus, the question is whether to decide the freeness of an abelian group, it suffices to verify that only a particular, narrow, class of epimorphism splits. Stein [**71**] solved Whitehead problem in the affirmative in the case that G is a countable abelian group. Then, in a result that was completely unexpected, Shelah [**49**] proved that Whitehead problem, restricted to groups of size ω_1, is independent of ZFC. Roughly speaking, by generalizing Stein's proof, substituting a counting-based diagonalization argument with a guessing-based diagonalization argument, Shelah proved that if $\diamondsuit_S$ holds for every stationary $S \subseteq \omega_1$, then every abelian group of size ω_1 with the above property is indeed free. On the other hand, he proved that if MA_{ω_1} holds, then there exists a counterexample of size ω_1.

Since CH holds in the first model, and fails in the other, it was natural to ask whether the existence of a counterexample to Whitehead problem is consistent together with CH. This led Shelah to introducing the *uniformization property*.

Definition 2.1 (Shelah, [**50**]). Suppose that S is a stationary subset of a successor cardinal, λ^+.

- A *ladder system* on S is a sequence of sets of ordinals, $\langle L_\alpha \mid \alpha \in S\rangle$, such that $\sup(L_\alpha) = \alpha$ and $\mathrm{otp}(L_\alpha) = \mathrm{cf}(\alpha)$ for all $\alpha \in S$;
- A ladder system $\langle L_\alpha \mid \alpha \in S\rangle$ is said to have the *uniformization property* iff whenever $\langle f_\alpha : L_\alpha \to 2 \mid \alpha \in S\rangle$ is a given sequence of local functions, then there exists a global function $f : \lambda^+ \to 2$ such that $f_\alpha \subseteq^* f$ for all limit $\alpha \in S$. That is, $\sup\{\beta \in L_\alpha \mid f_\alpha(\beta) \neq f(\beta)\} < \alpha$ for all limit $\alpha \in S$.

Theorem 2.2 (Shelah, [**53**]; see also [**17**]). *The following are equivalent:*

- *there exists a counterexample of size* ω_1 *to Whitehead problem;*

[7] Here, $\mathbb{Z}$ stands for the usual additive group of integers.

- *there exists a stationary $S \subseteq \omega_1$, and a ladder system on S that has the uniformization property.*

Devlin and Shelah proved [11] that if MA_{ω_1} holds, then every stationary $S \subseteq \omega_1$ and every ladder system on S, has the uniformization property. On the other hand, it is not hard to see that if $\diamondsuit_S$ holds, then no ladder system on S has the uniformization property (See Fact 2.5, below). Note that altogether, this gives an alternative proof to the independence result from [49].

Recalling that $\neg\diamondsuit_{\omega_1}$ is consistent with CH (See Theorem 1.4), it seemed reasonable to suspect that CH is moreover consistent with the existence of a ladder system on ω_1 that has the uniformization property. Such a model would also show that the existence of a counterexample to Whitehead problem is indeed consistent together with CH, settling Shelah's question.

However, a surprising theorem of Devlin states that CH implies that no ladder system on ω_1 has the uniformization property. Then, in a joint paper with Shelah, the essence of Devlin's proof has been isolated, and a weakening of diamond which is strong enough to rule out uniformization has been introduced.

Definition 2.3 (Devlin-Shelah, [11]). For an infinite cardinal λ and a stationary set $S \subseteq \lambda^+$, consider the principle of *weak diamond*.

- ▶ Φ_S asserts that for every function $F : {}^{<\lambda^+}2 \to 2$, there exists a function $g : \lambda^+ \to 2$, such that for all $f : \lambda^+ \to 2$, the following set is stationary:
$$\{\alpha \in S \mid F(f \restriction \alpha) = g(\alpha)\}.$$

Note that by Fact 1.20, $\diamondsuit_S \Rightarrow \Phi_S$. The difference between these principles is as follows. In diamond, for each function f, we would like to guess $f \restriction \alpha$, while in weak diamond, we only aim at guessing the value of $F(f \restriction \alpha)$, i.e., whether $f \restriction \alpha$ satisfies a certain property — is it black or white. A reader who is still dissatisfied with the definition of weak diamond, may prefer one of its alternative formulations.

Fact 2.4 (folklore). *For an infinite cardinal λ and a stationary set $S \subseteq \lambda^+$, the following principles are equivalent:*

- ▶ Φ_S*;*
- ▶ *for every function $F : {}^{<\lambda^+}\lambda^+ \to 2$, there exists a function $g : S \to 2$, such that for all $f : \lambda^+ \to \lambda^+$, the following set is stationary:*
$$\{\alpha \in S \mid F(f \restriction \alpha) = g(\alpha)\}.$$
- ▶ *for every sequence of functions $\langle F_\alpha : \mathcal{P}(\alpha) \to 2 \mid \alpha \in S\rangle$, there exists a function $g : S \to 2$, such that for every subset $X \subseteq \lambda^+$, the following set is stationary:*
$$\{\alpha \in S \mid F_\alpha(X \cap \alpha) = g(\alpha)\}.$$

Back to uniformization, we have:

Fact 2.5 (Devlin-Shelah, [11]). *For every stationary set S, Φ_S (and hence $\diamondsuit_S$) entails that no ladder system $\langle L_\alpha \mid \alpha \in S\rangle$ has the uniformization property.*

Proof (sketch). For all $\alpha \in S$ and $i < 2$, let $c^i_\alpha : L_\alpha \to \{i\}$ denote the constant function. Pick a function $F : {}^{<\lambda^+}2 \to 2$ such that for all $\alpha \in S$ and $i < 2$, if $f : \alpha \to 2$ and $c^i_\alpha \subseteq^* f$, then $F(f) = i$. Now, let $g : \lambda^+ \to 2$ be given by applying Φ_S to F. Then, letting $f_\alpha := c_\alpha^{1-g(\alpha)}$ for all $\alpha \in S$, the sequence $\langle f_\alpha \mid \alpha \in S\rangle$ cannot be uniformized. □

Before we turn to showing that CH $\Rightarrow \Phi_{\omega_1}$, let us mention that since Φ_{λ^+} deals with two-valued functions, its negation is an interesting statement of its own right:

Fact 2.6. *Suppose that Φ_{λ^+} fails for a given infinite cardinal, λ.*

Then there exists a function $F : {}^{<\lambda^+}({}^{\lambda}2) \rightarrow {}^{\lambda}2$ such that for every $g : \lambda^+ \rightarrow {}^{\lambda}2$, there exists a function $f : \lambda^+ \rightarrow {}^{\lambda}2$, for which the following set contains a club:

$$\{\alpha < \lambda^+ \mid F(f \restriction \alpha) = g(\alpha)\}.$$

Roughly speaking, the above states that there exists a decipher, F, such that for every function g, there exists a function f that F-ciphers the value of $g(\alpha)$ as $f \restriction \alpha$.

Since the (easy) proof of the preceding utilizes the fact that weak diamond deals with two-valued functions, it is worth mentioning that Shelah also studied generalization involving more colors. For instance, in [61], Shelah gets weak diamond for more colors provided that NS_{ω_1} is saturated (and Φ_{ω_1} holds).[8]

We now turn to showing that CH $\Rightarrow \Phi_{\omega_1}$. In fact, the next theorem shows that weak diamond is a cardinal arithmetic statement in disguise. The proof given here is somewhat lengthier than other available proofs, but, the value of this proof is that its structure allows the reader to first neglect the technical details (by skipping the proofs of Claims 2.7.1, 2.7.2), while still obtaining a good understanding of the key ideas.

Theorem 2.7 (Devlin-Shelah, [11]). *For every cardinal λ, $\Phi_{\lambda^+} \Leftrightarrow 2^{\lambda} < 2^{\lambda^+}$.*

Proof. $\Rightarrow$ Assume Φ_{λ^+}. Given an arbitrary function $\psi : {}^{\lambda^+}2 \rightarrow {}^{\lambda}2$, we now define a function $F : {}^{<\lambda^+}2 \rightarrow 2$ such that by appealing to Φ_{λ^+} with F, we can show that ψ is not injective.

Given $f \in {}^{<\lambda^+}2$, we let $F(f) := 0$ iff there exists a function $h \in {}^{\lambda^+}2$ such that $h(\mathrm{dom}(f)) = 0$ and $f \subseteq \psi(h) \cup (h \restriction [\lambda, \lambda^+))$.

Let $g : \lambda^+ \rightarrow 2$ be the oracle given by Φ_{λ^+} when applied to F, and let $h : \lambda^+ \rightarrow 2$ be the function satisfying $h(\alpha) = 1 - g(\alpha)$ for all $\alpha < \lambda^+$.

Put $f := \psi(h) \cup (h \restriction [\lambda, \lambda^+))$. Since $f \in {}^{\lambda^+}2$, let us pick some $\alpha < \lambda^+$ with $\alpha > \lambda$ such that $F(f \restriction \alpha) = g(\alpha)$. Since $f \restriction \alpha \subseteq \psi(h) \cup (h \restriction [\lambda, \lambda^+))$, the definition of F implies that $F(f \restriction \alpha) = 0$ whenever $h(\alpha) = 0$. However, $F(f \restriction \alpha) = g(\alpha) \neq h(\alpha)$, and hence $h(\alpha) = 1$. Since, $F(f \restriction \alpha) = g(\alpha) = 0$, let us pick a function h' such that $h'(\alpha) = 0$ and $f \restriction \alpha \subseteq \psi(h') \cup (h' \restriction [\lambda, \lambda^+))$. By definition of f, we get that $\psi(h) = f \restriction \lambda = \psi(h')$. By $g(\alpha) = 0$, we also know that $h(\alpha) = 1 \neq h'(\alpha)$, and hence $h \neq h'$, while $\psi(h) = \psi(h')$.

$\Leftarrow$ Given a function $H : {}^{<\lambda^+}({}^{\lambda}2) \rightarrow {}^{<\lambda^+}({}^{\lambda}2)$, let us say that a sequence $\langle (f_n, D_n) \mid n < \omega \rangle$ is an *H-prospective sequence* iff:

1. $\{D_n \mid n < \omega\}$ is a decreasing chain of club subsets of λ^+;
2. for all $n < \omega$, f_n is a function from λ^+ to ${}^{\lambda}2$;
3. for all $n < \omega$ and $\alpha \in D_{n+1}$, the following holds:

$$H(f_{n+1} \restriction \alpha) = f_n \restriction \min(D_n \setminus \alpha + 1).$$

Note that the intuitive meaning of the third item is that there exists $\beta > \alpha$ such that the content of $f_n \restriction \beta$ is coded by $f_{n+1} \restriction \alpha$.

[8] For the definition of "NS_{ω_1} is saturated" see Definition 4.1 below.

Claim 2.7.1. *Assume* $\neg\Phi_{\lambda^+}$.

Then there exists a function $H : {}^{<\lambda^+}({}^{\lambda}2) \to {}^{<\lambda^+}({}^{\lambda}2)$ *such that for every function* $f : \lambda^+ \to {}^{\lambda}2$, *there exists an* H*-prospective sequence* $\langle (f_n, D_n) \mid n < \omega\rangle$ *with* $f_0 = f$.

Proof. Fix F as in Fact 2.6, and fix a bijection $\varphi : {}^{\lambda}2 \to {}^{<\lambda^+}({}^{\lambda}2)$. Put $H := \varphi \circ F$. Now, given $f : \lambda^+ \to {}^{\lambda}2$, we define the H-prospective sequence by recursion on $n < \omega$. Start with $f_0 := f$ and $D_0 := \lambda^+$. Suppose $n < \omega$ and f_n and D_n are defined. Define a function $g : \lambda^+ \to {}^{\lambda}2$ by letting for all $\alpha < \lambda^+$:

$$g(\alpha) := \varphi^{-1}(f_n \restriction \min(D_n \setminus \alpha + 1)).$$

By properties of F, there exists a function f_{n+1} and a club $D_{n+1} \subseteq D_n$ such that for all $\alpha \in D_{n+1}$, we have $F(f_{n+1} \restriction \alpha) = g(\alpha)$. In particular,

$$H(f_{n+1} \restriction \alpha) = (\varphi \circ F)(f_{n+1} \restriction \alpha) = (\varphi \circ g)(\alpha) = f_n \restriction \min(D_n \setminus \alpha + 1). \quad \square$$

Claim 2.7.2. *Given a function* $H : {}^{<\lambda^+}({}^{\lambda}2) \to {}^{<\lambda^+}({}^{\lambda}2)$, *there exists a function* $H^* : {}^{\omega}({}^{<\lambda^+}({}^{\lambda}2)) \to {}^{\omega}({}^{<\lambda^+}({}^{\lambda}2))$ *with the following stepping-up property.*

For every H*-prospective sequence,* $\langle (f_n, D_n) \mid n < \omega\rangle$, *and every* $\alpha \in \bigcap_{n<\omega} D_n$, *there exists some* $\alpha^* < \lambda^+$, *such that:*

1. $\alpha^* > \alpha$;
2. $\alpha^* \in \bigcap_{n<\omega} D_n$;
3. $H^*(\langle f_n \restriction \alpha \mid n < \omega\rangle) = \langle f_n \restriction \alpha^* \mid n < \omega\rangle$.

Proof. Given H, we define functions $H^m : {}^{\omega}({}^{<\lambda^+}({}^{\lambda}2)) \to {}^{\omega}({}^{<\lambda^+}({}^{\lambda}2))$ by recursion on $m < \omega$. For all $\sigma : \omega \to {}^{<\lambda^+}({}^{\lambda}2)$, let:

$$H^0(\sigma) := \sigma,$$

and whenever $m < \omega$ is such that H^m is defined, let:

$$H^{m+1}(\sigma) := \langle H(H^m(\sigma)(n+1)) \mid n < \omega\rangle.$$

Finally, define H^* by letting for all $\sigma : \omega \to {}^{<\lambda^+}({}^{\lambda}2)$:

$$H^*(\sigma) := \langle \bigcup_{m<\omega} H^m(\sigma)(n) \mid n < \omega\rangle.$$

To see that H^* works, fix an H-prospective sequence, $\langle (f_n, D_n) \mid n < \omega\rangle$, and some $\alpha \in \bigcap_{n<\omega} D_n$. Define $\langle\langle \alpha_n^m \mid n < \omega\rangle \mid m < \omega\rangle$ by letting $\alpha_n^0 := \alpha$ for all $n < \omega$. Then, given $m < \omega$, for all $n < \omega$, let:

$$\alpha_n^{m+1} := \min(D_n \setminus \alpha_{n+1}^m + 1).$$

(1) Put $\alpha^* := \sup_{m<\omega} \alpha_0^m$. Then $\alpha^* \geq \alpha_0^1 > \alpha_1^0 = \alpha$.

(2) If $n < \omega$, then $D_n \supseteq D_{n+1}$, and hence $\alpha_{n+1}^{m+1} \geq \alpha_n^{m+1} > \alpha_{n+1}^m$ for all $m < \omega$. This shows that $\sup_{m<\omega} \alpha_n^m = \sup_{m<\omega} \alpha_{n+1}^m$ for all $n < \omega$.

For $n < \omega$, since $\langle \alpha_n^m \mid m < \omega\rangle$ is a strictly increasing sequence of ordinals from D_n that converges to α^*, we get that $\alpha^* \in D_n$.

(3) Let us prove by induction that for all $m < \omega$:

$$H^m(\langle f_n \restriction \alpha \mid n < \omega\rangle) = \langle f_n \restriction \alpha_n^m \mid n < \omega\rangle.$$

<u>Induction Base</u>: Trivial.

<u>Induction Step:</u> Suppose $m < \omega$ is such that:

$$(\star) \qquad H^m(\langle f_n \restriction \alpha \mid n < \omega\rangle) = \langle f_n \restriction \alpha_n^m \mid n < \omega\rangle,$$

and let us show that:

$$H^{m+1}(\langle f_n \restriction \alpha \mid n<\omega\rangle) = \langle f_n \restriction \alpha_n^{m+1} \mid n<\omega\rangle.$$

By definition of H^{m+1} and equation $(\star)$, this amounts to showing that:

$$\langle H(f_{n+1} \restriction \alpha_{n+1}^m) \mid n<\omega\rangle = \langle f_n \restriction \alpha_n^{m+1} \mid n<\omega\rangle.$$

Fix $n<\omega$. Recalling the definition of α_n^{m+1}, we see that we need to prove that $H(f_{n+1} \restriction \alpha_{n+1}^m) = f_n \restriction \min(D_n \setminus \alpha_{n+1}^m + 1)$. But this follows immediately from the facts that $\alpha_{n+1}^m \in D_{n+1}$, and that $\langle (f_n, D_n) \mid n<\omega\rangle$ is an H-prospective sequence.

Thus, it has been established that:

$$H^*(\langle f_n \restriction \alpha \mid n<\omega\rangle) = \langle f_n \restriction \bigcup_{m<\omega} \alpha_n^m \mid n<\omega\rangle = \langle f_n \restriction \alpha^* \mid n<\omega\rangle. \quad \square$$

Now, assume $\neg\Phi_{\lambda^+}$, and let us prove that $2^{\lambda^+} = 2^\lambda$ by introducing an injection of the form $\psi : {}^{\lambda^+}({}^\lambda 2) \to {}^\omega({}^{<\lambda^+}2)$. Fix H as in Claim 2.7.1, and let H^* be given by Claim 2.7.2 when applied to this fixed function, H.

▶ Given a function $f : \lambda^+ \to {}^\lambda 2$, we pick an H-prospective sequence $\langle (f_n, D_n) \mid n<\omega\rangle$ with $f_0 = f$ and let $\psi(f) := \langle f_n \restriction \alpha \mid n<\omega\rangle$ for $\alpha := \min(\bigcap_{n<\omega} D_n)$.

To see that ψ is injective, we now define a function $\varphi : {}^\omega({}^{<\lambda^+}2) \to {}^{\leq\lambda^+}({}^\lambda 2)$ such that $\varphi \circ \psi$ is the identity function.

▶ Given a sequence $\sigma : \omega \to {}^{<\lambda^+}2$, we first define an auxiliary sequence $\langle \sigma_\tau \mid \tau \leq \lambda^+\rangle$ by recursion on τ. Let $\sigma_0 := \sigma$, $\sigma_{\tau+1} := H^*(\sigma_\tau)$, and $\sigma_\tau(n) := \bigcup_{\eta<\tau} \sigma_\eta(n)$ for limit $\tau \leq \lambda^+$ and $n<\omega$. Finally, let $\varphi(\sigma) := \sigma_{\lambda^+}(0)$.

Claim 2.7.3. *$\varphi(\psi(f)) = f$ for every $f : \lambda^+ \to {}^\lambda 2$.*

Proof. Fix $f : \lambda^+ \to {}^\lambda 2$ and let $\sigma := \psi(f)$. By definition of ψ, $\sigma = \langle f_n \restriction \alpha \mid n<\omega\rangle$ for some H-prospective sequence $\langle (f_n, D_n) \mid n<\omega\rangle$ and $\alpha \in \bigcap_{n<\omega} D_n$. It then follows from the choice of H^*, that there exists a strictly increasing sequence, $\langle \alpha_\tau \mid \tau < \lambda^+\rangle$, of ordinals from $\bigcap_{n<\omega} D_n$, such that $\sigma_\tau := \langle f_n \restriction \alpha_\tau \mid n<\omega\rangle$ for all $\tau<\lambda^+$, and then $\varphi(\psi(f)) = \varphi(\sigma) = \sigma_{\lambda^+}(0) = f_0 \restriction \lambda^+ = f$. $\square$

This completes the proof. $\square$

Evidently, Devlin's pioneering theorem that CH excludes the existence of a ladder system on ω_1 with the uniformization property now follows from Fact 2.5 and Theorem 2.7. It is interesting to note that if one considers the notion of *weak uniformization*, in which the conclusion of Definition 2.1 is weakened from $\sup\{\beta \in L_\alpha \mid f_\alpha(\beta) \neq f(\beta)\} < \alpha$ to $\sup\{\beta \in L_\alpha \mid f_\alpha(\beta) = f(\beta)\} = \alpha$, then we end up with an example of an anti-$\diamondsuit_S$ principle, which is not an anti-Φ_S principle:

Theorem 2.8 (Devlin, see [3]). *It is consistent with* GCH *(and hence with Φ_{ω_1}) that every ladder system on every stationary subset of ω_1 has the weak uniformization property.*

Back to Whitehead problem, Shelah eventually established the consistency of CH together with the existence of a counterexample:

Theorem 2.9 (Shelah, [50]). *It is consistent with* $\mathrm{GCH}+\diamondsuit_{\omega_1}$ *that there exists a stationary, co-stationary, set $S \subseteq \omega_1$ such that any ladder system on S has the uniformization property.*

It is worth mentioning that Shelah's model was also the first example of a model in which $\diamondsuit_{\omega_1}$ holds, while for some stationary subset $S \subseteq \omega_1$, $\diamondsuit_S$ fails .

We now turn to dealing with the uniformization property for successor of uncountable cardinals. By Theorem 1.8 and Fact 2.5, there is no hope for getting a model of GCH in which a subset of $E^{\lambda^+}_{\neq \mathrm{cf}(\lambda)}$ carries a ladder system that has the uniformization property, so let us focus on sets of the critical cofinality. The first case that needs to be considered is $E^{\omega_2}_{\omega_1}$, and the full content of Theorem 1.9 is now revealed.

Theorem 2.10 (Shelah, [29],[51]). *It is consistent with* GCH *that there exists a ladder system on $E^{\omega_2}_{\omega_1}$ with the uniformization property.*

Knowing that $2^{\aleph_1} = \aleph_2$ implies $\diamondsuit_{E^{\omega_2}_{\omega}}$ but not $\diamondsuit_{E^{\omega_2}_{\omega_1}}$, and that $2^{\aleph_1} < 2^{\aleph_2}$ implies Φ_{ω_2} but not $\Phi_{E^{\omega_2}_{\omega_1}}$, one may hope to prove that $2^{\aleph_1} < 2^{\aleph_2}$ moreover implies $\Phi_{E^{\omega_2}_{\omega}}$. However, a consistent counterexample to this conjecture is provided in [56].

Note that Theorem 2.10 states that there exists a particular ladder system on $E^{\omega_2}_{\omega_1}$ with the uniformization property, rather than stating that all ladder systems on $E^{\omega_2}_{\omega_1}$ have this property.[9] To see that Theorem 2.10 is indeed optimal, consider the following theorem.

Theorem 2.11 (Shelah, [62]). *Suppose that λ is a regular cardinal of the form 2^θ for some cardinal θ, and that $\langle L_\alpha \mid \alpha \in E^{\lambda^+}_{\lambda} \rangle$ is a given ladder system.*

If, moreover, L_α is a club subset of α for all $\alpha \in E^{\lambda^+}_{\lambda}$, and $2^{<\lambda} = \lambda$, then there exists a coloring $\langle f_\alpha : L_\alpha \to 2 \mid \alpha \in E^{\lambda^+}_{\lambda} \rangle$ such that for every function $f : \lambda^+ \to 2$, the following set is stationary:

$$\{\alpha \in E^{\lambda^+}_{\lambda} \mid \{\beta \in L_\alpha \mid f_\alpha(\beta) \neq f(\beta)\} \textit{ is stationary in } \alpha\}.$$

In particular, CH *entails the existence of a ladder system on $E^{\omega_2}_{\omega_1}$ that does not enjoy the uniformization property.*

The proof of Theorem 2.10 generalizes to successor of higher regular cardinals, showing that there may exist a ladder system on $E^{\lambda^+}_{\lambda}$ that enjoys the uniformization property. Hence, we now turn to discuss the uniformization property at successor of singulars. We commence with revealing the richer content of Theorem 1.12.

Theorem 2.12 (Shelah, [55]). *Suppose* CH_λ *holds for a strong limit singular cardinal, λ. If $S \subseteq E^{\lambda^+}_{\mathrm{cf}(\lambda)}$ is a non-reflecting stationary set, then there exists a notion of forcing $\mathbb{P}_S$ such that:*

1. *$\mathbb{P}_S$ is λ-distributive;*
2. *$\mathbb{P}_S$ satisfies the λ^{++}-c.c.;*
3. *S remains stationary in $V^{\mathbb{P}_S}$;*
4. *in $V^{\mathbb{P}_S}$, there exists a ladder system on S that has the uniformization property.*

By Theorem 1.23, it is consistent that diamond fails on a set that reflects stationarily often. Now, what about the following strengthening:

[9] Compare with the fact that MA_{ω_1} entails that *every* ladder system on ω_1 has the uniformization property.

Question 4. Is it consistent with GCH that for some singular cardinal, λ, there exists a stationary set $S \subseteq E^{\lambda^+}_{\mathrm{cf}(\lambda)}$ *that reflects stationarily often*, and a ladder system on S that has the uniformization property?

Remark. By Corollary 1.22, SAP_λ necessarily fails in such an hypothetical model.

Now, what about the existence of ladder systems that do *not* enjoy the uniformization property? Clearly, if λ is a strong limit singular cardinal, then Theorem 2.11 does not apply. For this, consider the following.

Fact 2.13 (Shelah, [**65**]). *Suppose* CH_λ *holds for a strong limit singular cardinal,* λ*. Then, for every stationary* $S \subseteq \lambda^+$*, there exists a ladder system on* S *that does not enjoy the uniformization property.*

Proof. Fix a stationary $S \subseteq \lambda^+$. If $S \cap E^{\lambda^+}_{\neq \mathrm{cf}(\lambda)}$ is stationary, then by Theorem 1.8, $\diamondsuit_S$ holds, and then by Fact 2.5, moreover, no ladder system on S has the uniformization property. Next, suppose $S \subseteq E^{\lambda^+}_{\mathrm{cf}(\lambda)}$ is a given stationary set. By the upcoming Theorem 2.14, in this case, we may pick a ladder system $\langle L_\alpha \mid \alpha \in S\rangle$ such that for every function $f : \lambda^+ \to 2$, there exists some $\alpha \in S$ such that if $\{\alpha_i \mid i < \mathrm{cf}(\lambda)\}$ denotes the increasing enumeration of L_α, then $f(\alpha_{2i}) = f(\alpha_{2i+1})$ for all $i < \mathrm{cf}(\lambda)$.

It follows that if for each $\alpha \in S$, we pick $f_\alpha : L_\alpha \to 2$ satisfying for all $\beta \in L_\alpha$:

$$f_\alpha(\beta) = \begin{cases} 0, & \exists i < \mathrm{cf}(\lambda)(\beta = \alpha_{2i}) \\ 1, & \text{otherwise} \end{cases},$$

then the sequence $\langle f_\alpha \mid \alpha \in S\rangle$ cannot be uniformized. $\square$

Remark. Note that the sequence $\langle f_\alpha \mid \alpha \in S\rangle$ that was derived in the preceding proof from the guessing principle of Theorem 2.14, is a sequence of non-constant functions that cannot be uniformized. To compare, the sequence that was derived from weak diamond in the proof of Fact 2.5 is a sequence of constant functions. In other words, weak diamond is stronger in the sense that it entails the existence of a *monochromatic coloring* that cannot be uniformized.

Theorem 2.14 (Shelah, [**65**]). *Suppose* CH_λ *holds for a strong limit singular cardinal,* λ*,* $S \subseteq E^{\lambda^+}_{\mathrm{cf}(\lambda)}$ *is stationary and* $\mu < \lambda$ *is a given cardinal.*

Then there exists a ladder system $\langle L_\alpha \mid \alpha \in S\rangle$ *so that if* $\{\alpha_i \mid i < \mathrm{cf}(\lambda)\}$ *denotes the increasing enumeration of* L_α*, then for every function* $f : \lambda^+ \to \mu$*, the following set is stationary:*

$$\{\alpha \in S \mid f(\alpha_{2i}) = f(\alpha_{2i+1}) \textit{ for all } i < \mathrm{cf}(\lambda)\}.$$

Proof. Without loss of generality, λ divides the order-type of α, for all $\alpha \in S$. Put $\kappa := \mathrm{cf}(\lambda)$ and $\theta := 2^\kappa$. By $2^\lambda = \lambda^+$, let $\{d_\gamma \mid \gamma < \lambda^+\}$ be some enumeration of $\{d : \theta \times \tau \to \mu \mid \tau < \lambda^+\}$.

Fix $\alpha \in S$. Let $\langle c^\alpha_i \mid i < \kappa\rangle$ be the increasing enumeration of some club subset of α, such that $(c^\alpha_i, c^\alpha_{i+1})$ has cardinality λ for all $i < \kappa$. Also, let $\{b^\alpha_i \mid i < \kappa\} \subseteq [\alpha]^{<\lambda}$ be a continuous chain converging to α with $b^\alpha_i \subseteq c^\alpha_i$ for all $i < \kappa$. Recall that we have fixed $\alpha \in S$; now, in addition, we also fix $i < \kappa$.

For all $j < \kappa$, define a function $\psi_j = \psi_{\alpha,i,j} : (c_i^\alpha, c_{i+1}^\alpha) \to {}^{\theta \times b_j^\alpha}(\mu + 1)$ such that for all $\varepsilon \in (c_i^\alpha, c_{i+1}^\alpha)$ and $(\beta, \gamma) \in \theta \times b_j^\alpha$:

$$\psi_j(\varepsilon)(\beta,\gamma) = \begin{cases} d_\gamma(\beta,\varepsilon), & (\beta,\varepsilon) \in \mathrm{dom}(d_\gamma) \\ \mu, & \text{otherwise} \end{cases}.$$

For all $j < \kappa$, since $|{}^{\theta \times b_j^\alpha}(\mu+1)| < \lambda = |(c_i^\alpha, c_{i+1}^\alpha)|$, let us pick two ordinals $\alpha_{i,0}^j, \alpha_{i,1}^j$ with $c_i^\alpha < \alpha_{i,0}^j < \alpha_{i,1}^j < c_{i+1}^\alpha$ such that $\psi_j(\alpha_{i,0}^j) = \psi_j(\alpha_{i,1}^j)$.

For every function $g \in {}^\kappa\kappa$, consider the ladder system $\langle L_\alpha^g \mid \alpha \in S\rangle$, where $L_\alpha^g := \{\alpha_{i,0}^{g(i)}, \alpha_{i,1}^{g(i)} \mid i < \kappa\}$.

Claim 2.14.1. *There exists some $g \in {}^\kappa\kappa$ such that $\langle L_\alpha^g \mid \alpha \in S\rangle$ works.*

Proof. Suppose not. Let $\{g_\beta \mid \beta < \theta\}$ be some enumeration of ${}^\kappa\kappa$. Then, for all $\beta < \theta$, we may pick a function $f_\beta : \lambda^+ \to \mu$ and a club E_β such that for all $\alpha \in S \cap E_\beta$, there exists some $i < \kappa$ such that

$$f_\beta(\alpha_{i,0}^{g_\beta(i)}) \neq f_\beta(\alpha_{i,1}^{g_\beta(i)}).$$

Now, let $h : \lambda^+ \to \lambda^+$ be the function such that for all $\epsilon < \lambda^+$:

$$h(\epsilon) = \min\{\gamma < \lambda^+ \mid \forall(\beta,\varepsilon) \in \theta \times \epsilon\, (d_\gamma(\beta,\varepsilon) \text{ is defined and equals } f_\beta(\varepsilon))\}.$$

Pick $\alpha \in S \cap \bigcap_{\beta<\theta} E_\beta$ such that $h[\alpha] \subseteq \alpha$.

Then we may define a function $g : \kappa \to \kappa$ by letting:

$$g(i) := \min\{j < \kappa \mid h(c_{i+1}^\alpha) \in b_j^\alpha\}.$$

Let $\beta < \theta$ be such that $g = g_\beta$ and fix $i < \kappa$ such that

$$f_\beta(\alpha_{i,0}^{g_\beta(i)}) \neq f_\beta(\alpha_{i,1}^{g_\beta(i)}).$$

Put $j := g(i)$. By definition of $\alpha_{i,0}^j$ and $\alpha_{i,1}^j$, we know that $\psi_{\alpha,i,j}(\alpha_{i,0}^j) = \psi_{\alpha,i,j}(\alpha_{i,1}^j)$ is a function from $\theta \times b_j^\alpha$ to $\mu + 1$.

Put $\gamma := h(c_{i+1}^\alpha)$; then $(\beta,\gamma) \in \theta \times b_j^\alpha$, and hence:

$$\psi_{\alpha,i,j}(\alpha_{i,0}^j)(\beta,\gamma) = \psi_{\alpha,i,j}(\alpha_{i,1}^j)(\beta,\gamma).$$

It now follows from $\alpha_{i,0}^j < \alpha_{i,1}^j < c_{i+1}^\alpha$ and $\gamma = h(c_{i+1}^\alpha)$, that:

$$f_\beta(\alpha_{i,0}^j) = d_\gamma(\beta,\alpha_{i,0}^j) = \psi_{\alpha,i,j}(\alpha_{i,0}^j)(\beta,\gamma) = \psi_{\alpha,i,j}(\alpha_{i,1}^j)(\beta,\gamma) = d_\gamma(\beta,\alpha_{i,1}^j) = f_\beta(\alpha_{i,1}^j)$$

Unrolling the notation, we must conclude that

$$f_\beta(\alpha_{i,0}^{g_\beta(i)}) = f_\beta(\alpha_{i,0}^j) = f_\beta(\alpha_{i,1}^j) = f_\beta(\alpha_{i,1}^{g_\beta(i)}),$$

thus, yielding a contradiction to $\alpha \in E_\beta$. □

Thus, it has been established that there exists a ladder system with the desired properties. □

In light of Theorem 1.24, the moral of Theorem 2.14 is that GCH entails some of the consequences of diamond, even in the case that diamond fails. Two natural questions concerning this theorem are as follows.

Question 5. Is it possible to eliminate the "strong limit" hypothesis from Theorem 2.14, while maintaining the same conclusion?

Question 6. Is Theorem 2.14 true also for the case that $\mu = \lambda$?

Note that an affirmative answer to the last question follows from $\diamondsuit_S$. In fact, even if $2^\lambda > \lambda^+$, but Ostaszewski's principle, $\clubsuit_S$, holds, then a ladder system as in Theorem 2.14 for the case $\mu = \lambda$, exists.

Definition 2.15 (Ostaszewski, [42]). Let λ denote an infinite cardinal, and S denote a stationary subset of λ^+. Consider the following principle.

- ► $\clubsuit_S$ asserts that there exists a sequence $\langle A_\alpha \mid \alpha \in S\rangle$ such that:
  - for all $\alpha \in S$, A_α is a cofinal subset of α;
  - if Z is a *cofinal* subset of λ^+, then the following set is stationary:

$$\{\alpha \in S \mid A_\alpha \subseteq Z\}.$$

It is worth mentioning that unlike $\clubsuit_S^-$, the principle $\clubsuit_S$ makes sense also in the case that $S \subseteq E^{\lambda^+}_\lambda$. In particular, the missing case of Theorem 1.19 may be compensated by the observation that $\diamondsuit_S$ is equivalent to $\clubsuit_S + \mathrm{CH}_\lambda$. It is also worth mentioning that $\clubsuit_{\lambda^+} + \neg\,\mathrm{CH}_\lambda$ is indeed consistent; for instance, in [53], Shelah introduces a model of $\clubsuit_{\omega_1} + \neg\Phi_{\omega_1}$.

Next, consider Theorem 2.14 for the case that $\mu = \mathrm{cf}(\lambda)$. In this case, the theorem yields a collection $\mathcal{L} \subseteq [\lambda^+]^{\mathrm{cf}(\lambda)}$ of size λ^+, such that for every function $f : \lambda^+ \rightarrow \mathrm{cf}(\lambda)$, there exists some $L \in \mathcal{L}$ such that $f \restriction L$ is not injective (in some strong sense). Apparently, this fact led Shelah and Džamonja to consider the following dual question.

Question. Suppose λ is a strong limit singular cardinal.

Must there exist a collection $\mathcal{P} \subseteq [\lambda^+]^{\mathrm{cf}(\lambda)}$ of size λ^+ such that for every function $f : \lambda^+ \rightarrow \mathrm{cf}(\lambda)$ which is non-trivial in the sense that $\bigwedge_{\beta<\mathrm{cf}(\lambda)} |f^{-1}\{\beta\}| = \lambda^+$, there exists some $a \in \mathcal{P}$ such that $f \restriction a$ is injective?

We shall be concluding this section by describing the resolution of the above question. To refine the question, consider the following two definitions.

Definition 2.16. For a function $f : \lambda^+ \rightarrow \mathrm{cf}(\lambda)$, let κ_f denote the minimal cardinality of a family $\mathcal{P} \subseteq [\lambda^+]^{\mathrm{cf}(\lambda)}$ with the property that whenever $Z \subseteq \lambda^+$ satisfies $\bigwedge_{\beta<\mathrm{cf}(\lambda)} |Z \cap f^{-1}\{\beta\}| = \lambda^+$, then there exist some $a \in \mathcal{P}$ with $\sup(f[a \cap Z]) = \mathrm{cf}(\lambda)$.[10]

Definition 2.17. For a singular cardinal λ, we say that λ^+-*guessing* holds iff $\kappa_f \leq \lambda^+$ for all $f \in {}^{\lambda^+}\mathrm{cf}(\lambda)$.

Answering the above-mentioned question in the negative, Shelah and Džamonja established the consistency of the failure of λ^+-guessing.

Theorem 2.18 (Džamonja-Shelah, [13]). *It is relatively consistent with the existence of a supercompact cardinal that there exist a strong limit singular cardinal λ and a function $f : \lambda^+ \rightarrow \mathrm{cf}(\lambda)$ such that $\kappa_f = 2^\lambda > \lambda^+$.*

Recently, we realized that the above-mentioned question is simply equivalent to the question of whether every strong limit singular cardinal λ satisfies CH_λ.

[10] Note that if λ is a strong limit, then we may assume that $\mathcal{P}$ is closed under taking subsets. Thus, we may moreover demand the existence of $a \in \mathcal{P}$ such that $a \subseteq Z$ and $f \restriction a$ is injective.

THEOREM 2.19 ([22]). *Suppose λ is a strong limit singular cardinal. Then:*

$$\{\kappa_f \mid f \in {}^{\lambda^+}\mathrm{cf}(\lambda)\} = \{0, 2^\lambda\}.$$

In particular, if λ is a strong limit singular cardinal, then λ^+-guessing happens to be equivalent to the, seemingly, much stronger principle, $\diamondsuit^+_{E^{\lambda^+}_{\neq \mathrm{cf}(\lambda)}}$.

3. The Souslin Hypothesis and Club Guessing

Recall that a λ^+-Aronszajn tree is a tree of height λ^+, of width λ, and without chains of size λ^+. A λ^+-Souslin tree is a λ^+-Aronszajn tree that has no antichains of size λ^+.

Jensen introduced the diamond principle and studied its relation to Souslin trees.

THEOREM 3.1 (Jensen, [28]). *If $\lambda^{<\lambda} = \lambda$ is a regular cardinal such that $\diamondsuit_{E^{\lambda^+}_\lambda}$ holds, then there exists a λ^+-Souslin tree.*

In particular, $\diamondsuit_{\omega_1}$ entails the existence of an ω_1-Souslin tree.

THEOREM 3.2 (Jensen, see [10]). GCH *is consistent together with the non-existence of an ω_1-Souslin tree.*

REMARK. This is how Jensen proves Theorem 1.4. For a more modern proof of Theorem 3.2, see [2] or [3].

Let V denote the model from Theorem 1.4/3.2, and let $\mathbb{P} := \mathrm{Add}(\omega, 1)$ denote Cohen's notion of forcing for introducing a single Cohen real. Since $V \models \neg\diamondsuit_{\omega_1}$ and since $\mathbb{P}$ is c.c.c., the discussion after Definition 1.3 shows that $V^{\mathbb{P}} \models \neg\diamondsuit_{\omega_1}$. By a theorem of Shelah from [54], adding a Cohen real introduces an ω_1-Souslin tree, and hence $V^{\mathbb{P}}$ is a model of CH witnessing the fact that the existence of an ω_1-Souslin tree does not entail $\diamondsuit_{\omega_1}$.

Now, one may wonder what is the role of the cardinal arithmetic assumption in Theorem 3.1? the answer is that this hypothesis is necessary. To exemplify the case $\lambda = \aleph_1$, we mention that PFA implies $\diamondsuit^+_{E^{\lambda^+}_\lambda}$, but it also implies that $\lambda^{<\lambda} \neq \lambda$ and the *non-existence* of λ^+-Aronszajn trees.[11]

So, $\diamondsuit_{E^{\omega_2}_{\omega_1}}$ *per se* does not impose the existence of an ω_2-Souslin tree. Also, starting with a weakly compact cardinal, Laver and Shelah [37] established that CH is consistent together with the non-existence of an $\aleph_2$-Souslin tree. This leads us to the following tenacious question.

QUESTION 7 (folklore). Does GCH imply the existence of an ω_2-Souslin tree?

An even harder question is suggested by Shelah in [64].

QUESTION 8 (Shelah). Is it consistent that the GCH holds while for some regular uncountable λ, there exists neither λ^+-Souslin trees nor λ^{++}-Souslin trees?

Gregory's proof of Theorem 1.5 appears in the paper [25] that deals with Question 7, and in which this theorem is utilized to supply the following partial answer.

THEOREM 3.3 (Gregory, [25]). *Assume* GCH *(or just* $\mathrm{CH}_\omega + \mathrm{CH}_{\omega_1}$*).*

If there exists a non-reflecting stationary subset of $E^{\omega_2}_\omega$, then there exists an ω_2-Souslin tree.

[11] For an introduction to the Proper Forcing Axiom (PFA), see [9].

It follows that the consistency strength of a negative answer to Question 7 is at least that of the existence of a Mahlo cardinal. Recently, B. Koenig suggested an approach to show that the strength is at least that of the existence of a weakly compact cardinal. Let $\square(\omega_2)$ denote the assertion that there exists a sequence $\langle C_\alpha \mid \alpha < \omega_2\rangle$ such that for all limit $\alpha < \omega_2$: (1) C_α is a club subset of α, (2) if β is a limit point of α, then $C_\alpha \cap \beta = C_\beta$, (3) there exists no "trivializing" club $C \subseteq \omega_2$ such that $C \cap \beta = C_\beta$ for all limit points β of C.

The principle $\square(\omega_2)$ is a consequence of $\square_{\omega_1}$,[12] but its consistency strength is higher — it is that of the existence of a weakly compact cardinal. Thus, Koenig's question is as follows.

Question 9 (B. Koenig). Does GCH $+\square(\omega_2)$ imply the existence of an ω_2-Souslin tree?

In light of Theorem 3.1, to answer Question 7 in the affirmative, one probably needs to find a certain consequence of $\diamondsuit_{E^{\omega_2}_{\omega_1}}$ that, from one hand, follows outright from GCH and which is, on the other hand, strong enough to allow the construction of an $\aleph_2$-Souslin tree. An example of ZFC-provable consequences of diamond is Shelah's family of *club guessing* principles. The next theorem exemplifies only a few out of many results in this direction.

Theorem 3.4 (several authors). *For infinite cardinals $\mu \leq \lambda$, and a stationary set $S \subseteq E^{\lambda^+}_\mu$, there exists a sequence $\overrightarrow{C} = \langle C_\alpha \mid \alpha \in S\rangle$ such that for all $\alpha \in S$, C_α is a club in α of order-type μ, and:*

(1) *if $\mu < \lambda$, then $\overrightarrow{C}$ may be chosen such that for every club $D \subseteq \lambda^+$, the following set is stationary:*
$$\{\alpha \in S \mid C_\alpha \subseteq D\}.$$
(2) *if $\omega < \mu = \mathrm{cf}(\lambda) < \lambda$, then $\overrightarrow{C}$ may be chosen such that for almost all $\alpha \in S$, $\langle \mathrm{cf}(\beta) \mid \beta \in \mathrm{nacc}(C_\alpha)\rangle$ is a strictly increasing sequence cofinal in λ, and for every club $D \subseteq \lambda^+$, the following set is stationary:*
$$\{\alpha \in S \mid C_\alpha \subseteq D\}.$$
(3) *if $V = L$, then $\overrightarrow{C}$ may be chosen such that for every club $D \subseteq \lambda^+$, the following set contains a club subset of S:*
$$\{\alpha \in S \mid \exists\beta < \alpha(C_\alpha \setminus \beta \subseteq D)\}.$$
(4) *if $\omega < \mathrm{cf}(\mu) = \lambda$, then $\overrightarrow{C}$ may be chosen such that for every club $D \subseteq \lambda^+$, the following set is stationary:*
$$\{\alpha \in S \mid \{\beta \in C_\alpha \mid \min(C_\alpha \setminus \beta + 1) \in D\} \text{ is stationary in } \alpha\}.$$

▶ Theorem 3.4(1) is due to Shelah [**59**], and the principle appearing there reflects the most naive form of club guessing. Personally, we are curious whether the guessing may concentrate on a prescribed stationary set T:

Question 10. Suppose that S, T are given stationary subsets of a successor cardinal λ^+. Must there exist a sequence $\langle C_\alpha \mid \alpha \in S\rangle$ with $\sup(C_\alpha) = \alpha$ for all $\alpha \in S$, such that for every club $D \subseteq \lambda^+$, $\{\alpha \in S \mid C_\alpha \subseteq D \cap T\}$ is stationary?

[12] The square property at λ, denoted $\square_\lambda$, is the principle $\square_{\lambda,1}$ as in Definition 3.8.

A positive answer follows from $\clubsuit^-_S$, and a negative answer is consistent for various cardinals λ and non-reflecting sets $S \subseteq \lambda^+$, hence one should focus on sets $S \subseteq E^{\lambda^+}_{\mathrm{cf}(\lambda)}$ that reflect stationarily often.

▶ Theorem 3.4(2) is due to Shelah [**59**], but see also Eisworth and Shelah [**16**]. Roughly speaking, the principle appearing there requires that, in addition to the naive club guessing, the non-accumulation points of the local clubs to be of high cofinality. An hard open problem is whether their assertion is valid also in the case of countable cofinality.

Question 11 (Eisworth-Shelah). Suppose that λ is a singular cardinal of countable cofinality. Must there exist a ladder system $\langle L_\alpha \mid \alpha \in E^{\lambda^+}_{\mathrm{cf}(\lambda)}\rangle$ such that for almost all α, $\langle \mathrm{cf}(\beta) \mid \beta \in L_\alpha\rangle$ is a strictly increasing ω-sequence cofinal in λ, and for every club $D \subseteq \lambda^+$, the set $\{\alpha \in E^{\lambda^+}_{\mathrm{cf}(\lambda)} \mid L_\alpha \subseteq D\}$ is stationary?

While the above question remains open, Eisworth recently established the validity of a principle named *off-center club guessing* [**14**], and demonstrated that the new principle can serve as a useful substitute to the principle of Question 11.

▶ Theorem 3.4(3) is due to Ishiu [**27**], and the principle appearing there is named *strong club guessing*. The "strong" stands for the requirement that the guessing is done on almost all points rather that on just stationary many. Historically, Foreman and Komjáth first proved in [**20**] that strong club guessing may be introduced by forcing (See Theorem 4.17 below), and later on, Ishiu proved that this follows from $V = L$. In his paper, Ishiu asks whether $V = L$ may be reduced to a diamond-type hypothesis. Here is a variant of his question.

Question 12. Suppose that $\diamondsuit^+_{\lambda^+}$ holds for a given infinite cardinal λ. Must there exist a regular cardinal $\mu < \lambda$, a stationary set $S \subseteq E^{\lambda^+}_\mu$, and a ladder system $\langle L_\alpha \mid \alpha \in S\rangle$ such that for every club $D \subseteq \lambda^+$, for club many $\alpha \in S$, there exists $\beta < \alpha$ with $L_\alpha \setminus \beta \subseteq D$?

We mention that $\diamondsuit^+_{\omega_1}$ is consistent together with the failure of strong club guessing over ω_1 (see [**36**]), while, for an uncountable regular cardinal λ, and a stationary $S \subseteq E^{\lambda^+}_\lambda$, $\diamondsuit^*_S$ suffices to yield strong club guessing over S.

▶ Theorem 3.4(4) is due to Shelah [**60**], and a nice presentation of the proof may be found in [**69**]. The prototype of this principle is the existence of a sequence of local clubs, $\langle C_\alpha \mid \alpha \in E^{\lambda^+}_\lambda\rangle$, such that for every club $D \subseteq \lambda^+$, there exists some $\alpha \in E^{\lambda^+}_\lambda$ with $\sup(\mathrm{nacc}(C_\alpha) \cap D) = \alpha$. Now, if $\{\alpha_i \mid i < \lambda\}$ denotes the increasing enumeration of C_α, then Theorem 3.4(3) states that for every club $D \subseteq \lambda^+$, there exists stationarily many $\alpha \in S$, for which not only that $\sup(\mathrm{nacc}(C_\alpha) \cap D) = \alpha$, but moreover, $\{i < \lambda \mid \alpha_{i+1} \in D\}$ is stationary in λ. According to Shelah [**64**], to answer Question 7 in the affirmative, it suffices to find a proof of the following natural improvement.

Question 13 (Shelah). For a regular uncountable cardinal, λ, must there exist a sequence $\langle C_\alpha \mid \alpha \in E^{\lambda^+}_\lambda\rangle$ with each C_α a club in α whose increasing enumeration is $\{\alpha_i \mid i < \lambda\}$, such that for every club $D \subseteq \lambda^+$, there exists stationarily many α, for which $\{i < \lambda \mid \alpha_{i+1} \in D \text{ and } \alpha_{i+2} \in D\}$ is stationary in λ?

To exemplify the tight relation between higher Souslin trees and the preceding type of club guessing, we mention the next principle.

Definition 3.5 ([46]). Suppose λ is a regular uncountable cardinal, T is a stationary subset of λ, and S is a stationary subset of $E^{\lambda^+}_\lambda$.

$\langle T\rangle_S$ asserts the existence of sequences $\langle C_\alpha \mid \alpha\in S\rangle$ and $\langle A^\alpha_i \mid \alpha\in S, i<\lambda\rangle$ such that:

(1) for all $\alpha\in S$, C_α is a club subset of α of order-type λ;
(2) if for all $\alpha\in S$, $\{\alpha_i \mid i<\lambda\}$ denotes the increasing enumeration of C_α, then for every club $D\subseteq\lambda^+$ and every subset $A\subseteq\lambda^+$, there exist stationarily many $\alpha\in S$ for which:

$$\{i\in T \mid \alpha_{i+1}\in D \ \&\ A\cap\alpha_{i+1}=A^\alpha_{i+1}\} \text{ is stationary in } \lambda.$$

It is obvious that $\diamondsuit_S \Rightarrow \langle T\rangle_S$. It is also not hard to see that $\langle T\rangle_S \Rightarrow \diamondsuit_S$ whenever $\mathrm{NS}_\lambda \restriction T$ is saturated.[13] A strengthening of Theorem 3.1 is the following.

Theorem 3.6 (implicit in [30]). *If $\lambda^{<\lambda}=\lambda$ is a regular uncountable cardinal and $\langle\lambda\rangle_{E^{\lambda^+}_\lambda}$ holds, then there exists a λ^+-Souslin tree.*

We now turn to discuss Souslin trees at the of successor of singulars. By Magidor and Shelah [39], if λ is a singular cardinal which is a limit of strongly compact cardinals, then there are no λ^+-Aronszajn trees. In particular, it is consistent with GCH that for some singular cardinal λ, there are no λ^+-Souslin trees. On the other hand, Jensen proved the following.

Theorem 3.7 (Jensen). *For a singular cardinal λ, $\mathrm{CH}_\lambda + \square_\lambda$ entails the existence of a λ^+-Souslin tree.*

Since $\square_\lambda \Rightarrow \square^*_\lambda$ and the latter still witnesses the existence of a λ^+-Aronszajn tree, the question which appears to be the agreed analogue of Question 7 is the following.

Question 14 (folklore). For a singular cardinal λ, does $\mathrm{GCH} + \square^*_\lambda$ imply the existence of a λ^+-Souslin tree?

A minor modification of Jensen's proof of Theorem 3.7 entails a positive answer to Question 14 provided that there exists a non-reflecting stationary subset of $E^{\lambda^+}_{\neq\mathrm{cf}(\lambda)}$. However, by Magidor and Ben-David [4], it is relatively consistent with the existence of a supercompact cardinal that the GCH holds, $\square^*_{\aleph_\omega}$ holds, and every stationary subset of $E^{\aleph_{\omega+1}}_{\neq\omega}$ reflects.

A few years ago, Schimmerling [48] suggested that the community should perhaps try to attack a softer version of Question 14, which is related to the following hierarchy of square principles.

Definition 3.8 (Schimmerling, [47]). For cardinals, μ,λ, $\square_{\lambda,<\mu}$ asserts the existence of a sequence $\langle \mathcal{C}_\alpha \mid \alpha<\lambda^+\rangle$ such that for all limit $\alpha<\lambda^+$:

- $0<|\mathcal{C}_\alpha|<\mu$;
- C is a club subset of α for all $C\in\mathcal{C}_\alpha$;
- if $\mathrm{cf}(\alpha)<\lambda$, then $\mathrm{otp}(C)<\lambda$ for all $C\in\mathcal{C}_\alpha$;
- if $C\in\mathcal{C}_\alpha$ and $\beta\in\mathrm{acc}(C)$, then $C\cap\beta\in\mathcal{C}_\beta$.

We also write $\square_{\lambda,\mu}$ for $\square_{\lambda,<\mu^+}$.

Question 15 (Schimmerling). Does $\mathrm{GCH} + \square_{\aleph_\omega,\omega}$ imply the existence of an $\aleph_{\omega+1}$-Souslin tree?

[13] See Definition 4.1 below.

In [1], Abraham, Shelah and Solovay showed that if $\mathrm{CH}_\lambda + \square_\lambda$ holds for a given strong limit singular cardinal, λ, then a principle which is called *square with built-in diamond* may be inferred. Then, they continued to show how to construct a λ^+-Souslin tree with a certain special property, based on this principle.

There are several variations of square-with-built-in-diamond principles (the first instance appearing in [24]), and several constructions of peculiar trees that utilizes principles of this flavor (see [5], [6], [30], [72]). Recalling the work of Abraham-Shelah-Solovay in [1], it seems reasonable to seek for a principle that ramifies the hypothesis of Question 15. Here is our humble suggestion.

Definition 3.9 ([43]). For cardinals, μ, λ, $\boxed{\Diamond}_{\lambda,<\mu}$ asserts the existence of two sequences, $\langle \mathcal{C}_\alpha \mid \alpha < \lambda^+\rangle$ and $\langle \varphi_\theta \mid \theta \in \Gamma\rangle$, such that all of the following holds:

- $\emptyset \neq \Gamma \subseteq \{\theta < \lambda^+ \mid \mathrm{cf}(\theta) = \theta\}$;
- $\langle \mathcal{C}_\alpha \mid \alpha < \lambda^+\rangle$ is a $\square_{\lambda,<\mu}$-sequence;
- $\varphi_\theta : \mathcal{P}(\lambda^+) \to \mathcal{P}(\lambda^+)$ is a function, for all $\theta \in \Gamma$;
- for every subset $A \subseteq \lambda^+$, every club $D \subseteq \lambda^+$, and every cardinal $\theta \in \Gamma$, there exists some $\alpha \in E^{\lambda^+}_\theta$ such that for all $C \in \mathcal{C}_\alpha$:

$$\sup\{\beta \in \mathrm{nacc}(\mathrm{acc}(C)) \cap D \mid \varphi_\theta(C \cap \beta) = A \cap \beta\} = \alpha.$$

We write $\boxed{\Diamond}_{\lambda,\mu}$ for $\boxed{\Diamond}_{\lambda,<\mu^+}$.

Notice that the above principle combines square, diamond and club guessing. The value of this definition is witnessed by the following.

Theorem 3.10 ([43]). *Suppose that λ is an uncountable cardinal.*
If $\boxed{\Diamond}_{\lambda,\lambda}$ holds, then there exists a λ^+-Souslin tree.

Remark. An interesting feature of the (easy) proof of the preceding theorem is that the construction does not depend on whether λ is a regular cardinal or a singular one.

It follows that if $\mathrm{GCH} + \square_{\aleph_\omega,\omega}$ entails $\boxed{\Diamond}_{\aleph_\omega,\aleph_\omega}$, then this would supply an affirmative answer to Question 15. However, so far, a ramification is available only for the case $\mu \leq \mathrm{cf}(\lambda)$.

Theorem 3.11 ([43]). *For cardinals $\lambda \geq \aleph_2$, and $\mu \leq \mathrm{cf}(\lambda)$, the following are equivalent:*

(a) $\square_{\lambda,<\mu} + \mathrm{CH}_\lambda$;
(b) $\boxed{\Diamond}_{\lambda,<\mu}$.

Remark. In the proof of (a)$\Rightarrow$(b), we obtain a $\boxed{\Diamond}_{\lambda,<\mu}$-sequence as in Definition 3.9 for which, moreover, Γ is a non-empty *final segment* of $\{\theta < \lambda \mid \mathrm{cf}(\theta) = \theta\}$.

Clearly, in the presence of a non-reflecting stationary set, one can push Theorem 3.11 much further (Cf. [43]). Thus, to see the difficulty of dealing with the case $\mu = \mathrm{cf}(\lambda)^+$, consider the following variation of club guessing.

Question 16. Suppose that λ is a singular cardinal, $\square_{\lambda,\mathrm{cf}(\lambda)}$ holds, and every stationary subset of λ^+ reflects.

Must there exist a regular cardinal θ with $\mathrm{cf}(\lambda) < \theta < \lambda$ and a $\square_{\lambda,\mathrm{cf}(\lambda)}$-sequence, $\langle \mathcal{C}_\alpha \mid \alpha < \lambda^+\rangle$, such that for every club $D \subseteq \lambda^+$, there exists some $\alpha \in E^{\lambda^+}_\theta$ satisfying $\sup(\mathrm{nacc}(C) \cap D) = \alpha$ for all $C \in \mathcal{C}_\alpha$?

To conclude this section, let us mention two questions that suggests an alternative generalizations of Theorems 3.1 and 3.7.

Question 17 (Juhász). Does $\clubsuit_{\omega_1}$ entail the existence of an $\aleph_1$-Souslin tree?

Question 18 (Magidor). For a singular cardinal λ, does $\square_\lambda$ entail the existence of a λ^+-Souslin tree?

Juhász's question is well-known and a description of its surrounding results deserves a survey paper of its own. Here, we just mention that most of these results may be formulated in terms of the parameterized diamond principles of [**41**]. For instance, see [**40**].

To answer Magidor's question, one needs to find a yet another GCH-free version of diamond which suggests some non-trivial guessing features. In [**66**], Shelah introduced a principle of this flavor, named *Middle Diamond*, and a corollary to the results of [**67**, §4] reads as follows (compare with Definitions 1.1 and 2.15.)

Theorem 3.12 (Shelah, [**67**]). *For every cardinal $\lambda \geq \beth_{\omega_1}$, there exist a finite set $\mathfrak{d} \subseteq \beth_{\omega_1}$, and a sequence $\langle (C_\alpha, A_\alpha) \mid \alpha < \lambda^+ \rangle$ such that:*

- *for all limit α, C_α is a club in α, and $A_\alpha \subseteq C_\alpha$;*
- *if Z is a subset of λ^+, then for every regular cardinal $\kappa \in \beth_{\omega_1} \setminus \mathfrak{d}$, the following set is stationary:*

$$\{\alpha \in E^{\lambda^+}_\kappa \mid Z \cap C_\alpha = A_\alpha\}.$$

For more information on the middle diamond, consult [**45**].

4. Saturation of the Nonstationary Ideal

Definition 4.1 (folklore). Suppose that S is a stationary subset of a cardinal, λ^+. We say that $\mathrm{NS}_{\lambda^+} \restriction S$ is *saturated* iff for any family $\mathcal{F}$ of λ^{++} many stationary subsets of S, there exists two distinct sets $S_1, S_2 \in \mathcal{F}$ such that $S_1 \cap S_2$ is stationary.

Of course, we say that NS_{λ^+} is saturated iff $\mathrm{NS}_{\lambda^+} \restriction \lambda^+$ is saturated.

Now, suppose that $\diamondsuit_S$ holds, as witnessed by $\langle A_\alpha \mid \alpha \in S\rangle$. For every subset $Z \subseteq \lambda^+$, consider the set $\mathcal{G}_Z := \{\alpha \in S \mid Z \cap \alpha = A_\alpha\}$. Then $\mathcal{G}_Z$ is stationary and $|\mathcal{G}_{Z_1} \cap \mathcal{G}_{Z_2}| < \lambda^+$ for all distinct $Z_1, Z_2 \in \mathcal{P}(\lambda^+)$. Thus, $\diamondsuit_S$ entails that $\mathrm{NS}_{\lambda^+} \restriction S$ is non-saturated. For stationary subsets of $E^{\lambda^+}_{<\lambda}$, an indirect proof of this last observation follows from Theorem 4.3 below. For this, we first remind our reader that a set $\mathcal{X} \subseteq \mathcal{P}(\lambda^+)$ is said to be *stationary* (in the generalized sense) iff for any function $f : [\lambda^+]^{<\omega} \to \lambda^+$ there exists some $X \in \mathcal{X}$ with $f``[X]^{<\omega} \subseteq X$.

Definition 4.2 (Gitik-Rinot, [**22**]). For an infinite cardinal λ and a stationary set $S \subseteq \lambda^+$, consider the following two principles.

- ► $(1)_S$ asserts that there exists a stationary $\mathcal{X} \subseteq [\lambda^+]^{<\lambda}$ such that:
  - the sup-function on $\mathcal{X}$ is 1-to-1;
  - $\{\sup(X) \mid X \in \mathcal{X}\} \subseteq S$.
- ► $(\lambda)_S$ asserts that there exists a stationary $\mathcal{X} \subseteq [\lambda^+]^{<\lambda}$ such that:
  - the sup-function on $\mathcal{X}$ is $(\leq \lambda)$-to-1;
  - $\{\sup(X) \mid X \in \mathcal{X}\} \subseteq S$.

Theorem 4.3. *For an uncountable cardinal λ, and a stationary set $S \subseteq E^{\lambda^+}_{<\lambda}$, the implication* (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) *holds:*

(1) *$\diamondsuit_S$;*
(2) *$(1)_S$;*
(3) *$(\lambda)_S$;*
(4) *$\clubsuit^-_S$;*
(5) *$\mathrm{NS}_{\lambda^+} \restriction S$ is non-saturated.*

PROOF. For a proof of the implication (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4), see [**22**]. The proof of the last implication appears in [**44**], building on the arguments of [**12**]. $\square$

Note that by Theorem 1.19, the first four items of the preceding theorem coincide assuming CH_λ. In particular, the next question happens to be the contrapositive version of Question 1.

QUESTION 19. Suppose that λ is a singular cardinal. Does CH_λ entail the existence of a stationary $\mathcal{X} \subseteq [\lambda^+]^{<\lambda}$ on which $X \mapsto \sup(X)$ is an injective map from $\mathcal{X}$ to $E^{\lambda^+}_{\mathrm{cf}(\lambda)}$?

Back to non-saturation, since $\mathrm{ZFC} \vdash \clubsuit^-_S$ for every stationary subset $S \subseteq E^{\lambda^+}_{\neq\mathrm{cf}(\lambda)}$, one obtains the following analogue of Theorem 1.8.

COROLLARY 4.4 (Shelah, [**59**]). *If λ is an uncountable cardinal, and S is a stationary subset of $E^{\lambda^+}_{\neq\mathrm{cf}(\lambda)}$, then $\mathrm{NS}_{\lambda^+} \restriction S$ is non-saturated.*

Thus, as in diamond, we are led to focus our attention on the saturation of $\mathrm{NS}_{\lambda^+} \restriction S$ for stationary sets S which concentrates on the set of critical cofinality.

Kunen [**33**] was the first to establish the consistency of an abstract saturated ideal on ω_1. As for the saturation of the ideal NS_{ω_1}, this has been obtained first by Steel and Van Wesep by forcing over a model of determinacy.

THEOREM 4.5 (Steel-Van Wesep, [**70**]). *Suppose that V is a model of "$\mathrm{ZF} + \mathrm{AD}_\mathbb{R} + \Theta$ is regular". Then, there is a forcing extension satisfying $\mathrm{ZFC} + \mathrm{NS}_{\omega_1}$ is saturated.*

Woodin [**73**] obtained the same conclusion while weakening the hypothesis to the assumption "$V = L(\mathbb{R}) + \mathrm{AD}$". Several years later, in [**18**], Foreman, Magidor and Shelah introduced *Martin's Maximum*, MM, established its consistency from a supercompact cardinal, and proved that MM entails that NS_{ω_1} is saturated, and remains as such in any *c.c.c.* extension of the universe.

Then, in [**58**], Shelah established the consistency of the saturation of NS_{ω_1} from just a Woodin cardinal. Finally, recent work of Jensen and Steel on the existence of the core model below a Woodin cardinal yields the following definite resolution.

THEOREM 4.6 (Shelah, Jensen-Steel). *The following are equiconsistent:*

(1) ZFC + *"there exist a woodin cardinal";*
(2) ZFC + *"NS_{ω_1} is saturated".*

However, none of these results serves as a complete analogue of Theorem 1.4 in the sense that the following is still open.

QUESTION 20 (folklore). Is CH consistent with NS_{ω_1} being saturated?

REMARK. By [**43**], "$\mathrm{CH} + \mathrm{NS}_{\omega_1}$ is saturated" entails $\boxed{\diamondsuit}_{\omega_1,\omega_1}$.

Recalling that $\mathrm{CH} \Rightarrow \Phi_{\omega_1}$, it is worth pointing out that while the saturation of NS_{ω_1} is indeed an anti-$\diamondsuit_{\omega_1}$ principle, it is not an anti-Φ_{ω_1} principle. To exemplify this, start with a model of MM and add $\aleph_{\omega_1}$ many Cohen reals over this model; then as a consequence of Theorem 2.7 and the fact that Cohen forcing is *c.c.c.*, one obtains a model in which NS_{ω_1} is still saturated, while Φ_{ω_1} holds.

Let us consider a strengthening of saturation which does serve as an anti-Φ_{λ^+} principle.

DEFINITION 4.7 (folklore). Suppose that S is a stationary subset of a cardinal, λ^+. We say that $\mathrm{NS}_{\lambda^+} \restriction S$ is *dense* iff there exists a family $\mathcal{F}$ of λ^+ many stationary subsets of S, such that for any stationary subset $S_1 \subseteq S$, there exists some $S_2 \in \mathcal{F}$ such that $S_2 \setminus S_1$ is non-stationary.

Of course, we say that NS_{λ^+} is dense iff $\mathrm{NS}_{\lambda^+} \restriction \lambda^+$ is dense.

It is not hard to see that if $\mathrm{NS}_{\lambda^+} \restriction S$ is dense, then it is also saturated. The above discussion and the next theorem entails that these principles do not coincide.

THEOREM 4.8 (Shelah, [**57**]). *If Φ_{ω_1} holds, then NS_{ω_1} is not dense.*

Improving Theorem 4.5, Woodin proved:

THEOREM 4.9 (Woodin, [**74**]). *Suppose that V is a model of "$V = L(\mathbb{R})+\mathrm{AD}$". Then there is a forcing extension of ZFC in which NS_{ω_1} is dense.*

The best approximation for a positive answer to Question 20 is, as well, due to Woodin, who proved that CH is consistent together with $\mathrm{NS}_{\omega_1} \restriction S$ being dense for some stationary $S \subseteq \omega_1$. Woodin also obtained an approximation for a negative answer to the very same question. By [**74**], if NS_{ω_1} is saturated and there exists a measurable cardinal, then CH must fail.

As for an analogue of Theorem 1.9 — the following is completely open:

QUESTION 21 (folklore). Is it consistent that $\mathrm{NS}_{\omega_2} \restriction E^{\omega_2}_{\omega_1}$ is saturated?

A major, related, result is the following unpublished theorem of Woodin (for a proof, see [**19**, §8].)

THEOREM 4.10 (Woodin). *Suppose that λ is an uncountable regular cardinal and κ is a huge cardinal above it. Then there exists a $<\lambda$-closed notion of forcing $\mathbb{P}$, such that in $V^{\mathbb{P}}$ the following holds:*

(1) $\kappa = \lambda^+$;
(2) *there exists a stationary $S \subseteq E^{\lambda^+}_{\lambda}$ such that $\mathrm{NS}_{\lambda^+} \restriction S$ is saturated.*

Moreover, if GCH holds in the ground model, then GCH holds in the extension.

Foreman, elaborating on Woodin's proof, established the consistency of the saturation of $\mathrm{NS}_{\lambda^+} \restriction S$ for some stationary set $S \subseteq E^{\lambda^+}_{\lambda}$ and a supercompact cardinal, λ, and showed that it is then possible to collapse λ to $\aleph_\omega$, while preserving saturation. Thus, yielding:

THEOREM 4.11 (Foreman). *It is relatively consistent with the existence of a supercompact cardinal and an almost huge cardinal above it, that the GCH holds, and $\mathrm{NS}_{\aleph_{\omega+1}} \restriction S$ is saturated for some stationary $S \subseteq E^{\aleph_{\omega+1}}_{\omega}$.*

Since the stationary set S was originally a subset of $E^{\lambda^+}_{\lambda}$, it is a non-reflecting stationary set. This raises the following question.

QUESTION 22 (folklore). Suppose that λ is a singular cardinal, and $S \subseteq E^{\lambda^+}_{\mathrm{cf}(\lambda)}$ reflects stationarily often, must $\mathrm{NS}_{\lambda^+} \restriction S$ be non-saturated?

Recently, the author [**44**] found several partial answers to Question 22. To start with, as a consequence of Theorem 1.21 and Theorem 4.3, we have:

THEOREM 4.12 ([**44**]). *Suppose $S \subseteq \lambda^+$ is a stationary set, for a singular cardinal λ. If $I[S;\lambda]$ contains a stationary set, then $\mathrm{NS}_{\lambda^+} \restriction S$ is non-saturated.*

In particular, SAP_λ (and hence $\square^*_\lambda$) impose a positive answer to Question 22. Recalling Theorem 1.30, we also obtain the following.

THEOREM 4.13 ([**44**]). *If λ is a singular cardinal of uncountable cofinality and $S \subseteq \lambda^+$ is a stationary set such that $\mathrm{NS}_{\lambda^+} \restriction S$ is saturated, then for every regular cardinal θ with $\mathrm{cf}(\lambda) < \theta < \lambda$, at least one of the two holds:*

(1) *$R_2(\theta, \mathrm{cf}(\lambda))$ fails;*
(2) *$\mathrm{Tr}(S) \cap E^{\lambda^+}_\theta$ is nonstationary.*

Next, to describe an additional aspect of Question 22, we remind our reader that a set $T \subseteq \lambda^+$ is said to *carry a weak square sequence* iff there exists sequence $\langle C_\alpha \mid \alpha \in T\rangle$ such that:

(1) C_α is a club subset of α of order-type $\leq \lambda$, for all limit $\alpha \in T$;
(2) $|\{C_\alpha \cap \gamma \mid \alpha \in T\}| \leq \lambda$ for all $\gamma < \lambda^+$.

FACT 4.14 ([**44**]). *Suppose λ is a singular cardinal, and $S \subseteq \lambda^+$ is a given stationary set. If some stationary subset of $\mathrm{Tr}(S)$ carries a weak square sequence, then $I[S;\lambda]$ contains a stationary set, and in particular, $\mathrm{NS}_{\lambda^+} \restriction S$ is non-saturated.*

The consistency of the existence of a stationary set that does not carry a weak square sequence is well-known, and goes back to Shelah's paper [**52**]. However, the following question is still open.

QUESTION 23. Suppose that λ is a singular cardinal. Must there exist a stationary subset of $E^{\lambda^+}_{>\mathrm{cf}(\lambda)}$ that carries a partial weak square sequence?

REMARK. The last question is closely related to a conjecture of Foreman and Todorcevic from [**21**, §6]. Note that by Fact 4.14 and Theorems 1.19, 1.21, a positive answer imposes a negative answer on Question 1.

Back to Question 22, still, there are a few ZFC results; the first being:

THEOREM 4.15 (Gitik-Shelah, [**23**]). *If λ is a singular cardinal, then $\mathrm{NS}_{\lambda^+} \restriction E^{\lambda^+}_{\mathrm{cf}(\lambda)}$ is non-saturated.*

Gitik and Shelah's proof utilizes the ZFC fact that a certain weakening of the club guessing principle from Theorem 3.4(2) holds for all singular cardinal, λ. Then, they show that if $\mathrm{NS}_{\lambda^+} \restriction E^{\lambda^+}_{\mathrm{cf}(\lambda)}$ were saturated, then their club guessing principle may be strengthened to a principle that combines their variation of 3.4(2), together with 3.4(3). However, as they show, this strong combination is already inconsistent.

In [**32**], Krueger pushed further the above argument, yielding the following generalization.

THEOREM 4.16 (Krueger, [**32**]). *If λ is a singular cardinal and $S \subseteq \lambda^+$ is a stationary set such that $\mathrm{NS}_{\lambda^+} \restriction S$ is saturated, then S is co-fat.*[14]

[14]Here, a set $T \subseteq \lambda^+$ is *fat* iff for every cardinal $\kappa < \lambda$ and every club $D \subseteq \lambda^+$, $T \cap D$ contains some closed subset of order-type κ.

To conclude this section, we mention two complementary results to the Gitik-Shelah argument.

Theorem 4.17 (Foreman-Komjáth, [**20**]). *Suppose that λ is an uncountable regular cardinal and κ is an almost huge cardinal above it. Then there exists a notion of forcing $\mathbb{P}$, such that in $V^{\mathbb{P}}$ the following holds:*

1. $\kappa = \lambda^+$;
2. *there exists a stationary $S \subseteq E^{\lambda^+}_{\lambda}$ such that $\mathrm{NS}_{\lambda^+} \restriction S$ is saturated;*
3. $E^{\lambda^+}_{\mu}$ *carries a strong club guessing sequence for any regular $\mu \leq \lambda$.*

Remark. By *strong club guessing*, we refer to the principle appearing in Theorem 3.4(3).

Theorem 4.18 (Woodin, [**74**]). *Assuming $\mathrm{AD}^{L(\mathbb{R})}$, there exists a forcing extension of $L(\mathbb{R})$ in which:*

1. NS_{ω_1} *is saturated;*
2. *there exists a strong club guessing sequence on $E^{\omega_1}_{\omega}$.*

For interesting variations of Woodin's theorem, we refer the reader to [**35**].

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