SOUSLIN TREES AT SUCCESSORS OF REGULAR CARDINALS

ASSAF RINOT

Abstract. We present a weak sufficient condition for the existence of Souslin trees at successors of regular cardinals. The result is optimal and simultaneously improves an old theorem of Gregory and a more recent theorem of the author.

Introduction

In [Gre76], Gregory proved that for every (regular) uncountable cardinal $\lambda = \lambda^{<\lambda}$, if $2^\lambda = \lambda^+$ and there exists a non-reflecting stationary subset of $E^{\lambda^+}_{<\lambda}$, then there exists a λ^+-Souslin tree. A special case of a result from [Rin17] asserts that for every uncountable cardinal $\lambda = \lambda^{<\lambda}$, if $2^\lambda = \lambda^+$ and $\square(\lambda^+)$ holds, then there exists a λ^+-Souslin tree. By results from inner model theory, Gregory's theorem implies that if GCH holds, and there are no $\aleph_2$-Souslin trees, then $\aleph_2$ is a Mahlo cardinal in L, and our theorem implies that if GCH holds, and there are no $\aleph_2$-Souslin trees, then $\aleph_2$ is a weakly compact cardinal in L. While the former corollary follows from the latter, the combinatorial theorem of Gregory does not follow from ours. The purpose of this note is to present a new combinatorial theorem that implies both:

Main Theorem. *Suppose that $\lambda = \lambda^{<\lambda}$ is an uncountable cardinal, and $2^\lambda = \lambda^+$.*
If there exists a $\square(\lambda^+, \lambda)$-sequence $\langle \mathcal{C}_\alpha \mid \alpha < \lambda^+\rangle$ for which $\{\alpha \in E^{\lambda^+}_{<\lambda} \mid |\mathcal{C}_\alpha| < \lambda\}$ is stationary, then there exists a λ^+-Souslin tree.

An immediate corollary to the Main Theorem is an optimal improvement to a result from [Rin17] that was promised in [BR19a]:

Corollary. *Suppose that λ is a regular uncountable cardinal.*
If GCH and $\square(\lambda^+, {<}\lambda)$ both hold, then there exists a λ^+-Souslin tree.[1]

The corollary is indeed optimal, since GCH implies that $\square(\lambda^+, \lambda)$ holds for every regular cardinal λ (in fact, with a witnessing sequence $\langle \mathcal{C}_\alpha \mid \alpha < \lambda^+\rangle$ satisfying $|\mathcal{C}_\alpha| = 1$ for all $\alpha \in E^{\lambda^+}_{\lambda}$), whereas by a recent striking result of Asperó and Golshani [AG18], modulo large cardinals, GCH is consistent with the non-existence of a λ^+-Souslin tree for any prescribed value of a regular uncountable λ.

Notation and conventions. Throughout this note, κ and λ stand for arbitrary regular uncountable cardinals. Write $[\kappa]^{<\lambda}$ for the collection of all subsets of κ of cardinality less than λ. Denote $E^\kappa_\lambda := \{\alpha < \kappa \mid \mathrm{cf}(\alpha) = \lambda\}$ and $E^\kappa_{<\lambda} := \{\alpha < \kappa \mid \mathrm{cf}(\alpha) < \lambda\}$.

Suppose that C and D are sets of ordinals. Write $C \sqsubseteq D$ iff there exists some ordinal β such that $C = D \cap \beta$. Write $\mathrm{acc}(C) := \{\alpha \in C \mid \sup(C \cap \alpha) = \alpha > 0\}$, $\mathrm{nacc}(C) := C \setminus \mathrm{acc}(C)$, and $\mathrm{acc}^+(C) := \{\alpha < \sup(C) \mid \sup(C \cap \alpha) = \alpha > 0\}$.

Date: January 14, 2019.
2010 *Mathematics Subject Classification.* Primary 03E05; Secondary 03E65.
This research was partially supported by the European Research Council (grant agreement ERC-2018-StG 802756) and by the Israel Science Foundation (grant agreement 2066/18).

[1] In fact, there exists λ^+-Souslin tree which is moreover λ-complete. This refinement will be detailed in [BR17b]

1. Square principles and Souslin trees

Definition 1.1. For any cardinal μ, $\square(\kappa,{<}\mu)$ asserts the existence of a sequence $\langle \mathcal{C}_\alpha \mid \alpha<\kappa\rangle$ such that:

(1) For every limit ordinal $\alpha<\kappa$:
- $\mathcal{C}_\alpha$ is a nonempty collection of club subsets of α, with $|\mathcal{C}_\alpha|<\mu$;
- for every $C\in\mathcal{C}_\alpha$ and $\bar\alpha\in\operatorname{acc}(C)$, we have $C\cap\bar\alpha\in\mathcal{C}_{\bar\alpha}$;

(2) For every club $D\subseteq\kappa$, there is $\alpha\in\operatorname{acc}(D)$ such that $D\cap\alpha\notin\mathcal{C}_\alpha$.

Remark. (1) Note that there are no restrictions on $\operatorname{otp}(C)$ for $C\in\mathcal{C}_\alpha$.
(2) We write $\square(\kappa,\mu)$ for $\square(\kappa,{<}\mu^+)$, and write $\square(\kappa)$ for $\square(\kappa,1)$.

To connect Gregory's theorem with the Main Theorem, let us point out the following.

Proposition 1.2. *Suppose that $\lambda^{<\lambda}=\lambda$ and there exists a non-reflecting stationary subset of $E^{\lambda^+}_{<\lambda}$. Then there exists a $\square(\lambda^+,\lambda)$-sequence $\langle\mathcal{C}_\alpha\mid\alpha<\lambda^+\rangle$ for which $\{\alpha\in E^{\lambda^+}_{<\lambda}\mid |\mathcal{C}_\alpha|<\lambda\}$ is stationary.*

Proof. Fix a subset $S\subseteq E^{\lambda^+}_{<\lambda}$ which is stationary and non-reflecting. We now define $\vec{\mathcal{C}}:=\langle\mathcal{C}_\alpha\mid\alpha<\lambda^+\rangle$, as follows:

▶ Let $\mathcal{C}_0:=\{\emptyset\}$.

▶ For all $\alpha<\lambda^+$, let $\mathcal{C}_{\alpha+1}:=\{\{\alpha\}\}$.

▶ For all $\alpha\in S\cup E^{\lambda^+}_{\lambda}$, since S is non-reflecting, we may fix a club C_α in α of order-type $\operatorname{cf}(\alpha)$ which is disjoint from S. Now, let $\mathcal{C}_\alpha:=\{C_\alpha\}$.

▶ For all $\alpha\in\operatorname{acc}(\lambda^+)\setminus(S\cup E^{\lambda^+}_{\lambda})$, let $\mathcal{C}_\alpha$ be the collection of all clubs C in α such that $\operatorname{otp}(C)<\lambda$ and $C\cap S=\emptyset$. As $\operatorname{cf}(\alpha)<\lambda$ and as S is non-reflecting, we know that $\mathcal{C}_\alpha$ is nonempty. As $\lambda^{<\lambda}=\lambda$, we also know that $|\mathcal{C}_\alpha|\le\lambda$.

Let us verify that $\vec{\mathcal{C}}$ is as sought:

- Evidently, $\{\alpha\in E^{\lambda^+}_{<\lambda}\mid|\mathcal{C}_\alpha|<\lambda\}$ covers the stationary set S.
- Fix arbitrary $\alpha\in\operatorname{acc}(\lambda^+)$, $C\in\mathcal{C}_\alpha$ and $\bar\alpha\in\operatorname{acc}(C)$. There are two options:
  ▶ If $\alpha\in S\cup E^{\lambda^+}_{\lambda}$, then $\operatorname{cf}(\bar\alpha)\le\operatorname{otp}(C\cap\bar\alpha)<\operatorname{otp}(C)=\operatorname{cf}(\alpha)\le\lambda$. Also, $C=C_\alpha$ is disjoint from S, so that, altogether, $\bar\alpha\in\operatorname{acc}(\lambda^+)\setminus(S\cup E^{\lambda^+}_{\lambda})$. It now follows from the definition of $\mathcal{C}_{\bar\alpha}$ that $C\cap\bar\alpha\in\mathcal{C}_{\bar\alpha}$.
  ▶ Otherwise, C is a club α such that $\operatorname{cf}(\bar\alpha)\le\operatorname{otp}(C\cap\bar\alpha)<\operatorname{otp}(C)<\lambda$ and $C\cap S=\emptyset$. It again follows from the definition of $\mathcal{C}_{\bar\alpha}$ that $C\cap\bar\alpha\in\mathcal{C}_{\bar\alpha}$.
- Given any club D in λ^+, pick $\alpha\in D$ such that $\operatorname{otp}(D\cap\alpha)=\lambda+\omega$. Then $\alpha\in\operatorname{acc}(D)$ and $D\cap\alpha\notin\mathcal{C}_\alpha$. □

As mentioned earlier, even in the presence of GCH, $\square(\kappa,{<}\kappa)$ does not imply the existence of a κ-Souslin tree. For this, Brodsky and the author have introduced the following slight strengthening of $\square(\kappa,{<}\kappa)$:

Definition 1.3 ([BR19b])**.** $\boxtimes^*(\kappa)$ asserts the existence of a sequence $\langle\mathcal{C}_\alpha\mid\alpha<\kappa\rangle$ such that:

(1) For every limit ordinal $\alpha<\kappa$:
- $\mathcal{C}_\alpha$ is a nonempty collection of club subsets of α, with $|\mathcal{C}_\alpha|<\kappa$;
- for every $C\in\mathcal{C}_\alpha$ and $\bar\alpha\in\operatorname{acc}(C)$, we have $C\cap\bar\alpha\in\mathcal{C}_{\bar\alpha}$;

(2) For every cofinal $X\subseteq\kappa$, there is $\alpha\in\operatorname{acc}(\kappa)$ such that $\sup(\operatorname{nacc}(C)\cap X)=\alpha$ for all $C\in\mathcal{C}_\alpha$.

In this paper, we shall not construct Souslin trees (we refer the reader to [BR17a] for background and definitions); all we need is encapsulated in the following fact.

Fact 1.4 ([BR19b])**.** $\boxtimes^*(\kappa)+\diamondsuit(\kappa)$ *entails the existence of a κ-Souslin tree.*

2. Proof of the Main Theorem

Definition 2.1 ([BR19a])**.** Suppose that D is a club in κ. Define a function $\Phi_D : \mathcal{P}(\kappa) \to \mathcal{P}(\kappa)$ by letting, for all $x \in \mathcal{P}(\kappa)$,

$$\Phi_D(x) := \begin{cases} \{\sup(D \cap \eta) \mid \eta \in x \ \& \ \eta > \min(D)\} & \text{if } \sup(D \cap \sup(x)) = \sup(x); \\ x \setminus \sup(D \cap \sup(x)) & \text{otherwise.} \end{cases}$$

Lemma 2.2. *Suppose that:*

- $\lambda < \kappa$*;*
- $\vec{\mathcal{C}} = \langle \mathcal{C}_\alpha \mid \alpha < \kappa \rangle$ *is a* $\square(\kappa, \kappa)$*-sequence;*
- S *is a stationary subset of* $\{\alpha \in \operatorname{acc}(\kappa) \mid |\mathcal{C}_\alpha| < \lambda\}$*.*

Then there exists a club $D \subseteq \kappa$ *such that for every club* $E \subseteq \kappa$*, there exists* $\alpha \in S$ *such that* $\sup(\operatorname{nacc}(\Phi_D(C)) \cap E) = \alpha$ *for all* $C \in \mathcal{C}_\alpha$*.*

Remark. All ingredients for the upcoming proof may already be found in [BR19a]. For completeness, we give here a self-contained proof that avoids various concepts that appear in [BR19a].

Proof of Lemma. Suppose not. Then, for every club $D \subseteq \kappa$, we may find a club $E^D \subseteq \kappa$ such that, for every $\delta \in S$, there is $C^D_\delta \in \mathcal{C}_\delta$ with

$$\sup(\operatorname{nacc}(\Phi_D(C^D_\delta)) \cap E^D) < \delta.$$

Define a sequence $\langle E_i \mid i \leq \lambda \rangle$ of clubs in κ, by recursion, as follows:

- Set $E_0 := \kappa$;
- For all $i < \lambda$, set $E_{i+1} := E^{E_i} \cap E_i$;
- For all $i \in \operatorname{acc}(\lambda + 1)$, set $E_i := \bigcap_{j<i} E_j$.

Write $E := E_\lambda$. For each $\delta \in S$, since $\{C^{E_i}_\delta \mid i < \lambda\} \subseteq \mathcal{C}_\delta$, and $\lambda = \operatorname{cf}(\lambda) > |\mathcal{C}_\delta|$, we may pick $C_\delta \in \mathcal{C}_\delta$ such that $I_\delta := \{i < \lambda \mid C^{E_i}_\delta = C_\delta\}$ is cofinal in λ. Now, there are three cases to consider, each leading to a contradiction:

Case 1. Suppose that there exists $\delta \in S \cap E^\kappa_{>\omega}$ for which $\sup(E \cap \delta \setminus C_\delta) = \delta$.

Fix such δ and let $\{i_n \mid n < \omega\}$ be the increasing enumeration of some subset of I_δ. Since $\langle E_i \mid i < \lambda \rangle$ is a $\subseteq$-decreasing sequence, for all $n < \omega$, we have in particular that $E_{i_{n+1}} \subseteq E_{i_n+1} \subseteq E^{E_{i_n}}$, so that $\alpha_n := \sup(\operatorname{nacc}(\Phi_{E_{i_n}}(C_\delta)) \cap E_{i_{n+1}})$ is $< \delta$. Put $\alpha := \sup_{n<\omega} \alpha_n$. As $\operatorname{cf}(\delta) > \omega$, we have $\alpha < \delta$. Fix $\beta \in (E \cap \delta) \setminus C_\delta$ above α. Put $\gamma := \min(C_\delta \setminus \beta)$. Then $\delta > \gamma > \beta > \alpha$, and for all $i < \lambda$, since $\beta \in E \subseteq E_i$, we infer that $\sup(E_i \cap \gamma) \geq \beta$. So it follows from the definition of $\Phi_{E_i}(C_\delta)$ that $\min(\Phi_{E_i}(C_\delta) \setminus \beta) = \sup(E_i \cap \gamma)$ for all $i < \lambda$. Since $\langle E_{i_n} \mid n < \omega \rangle$ is an infinite $\subseteq$-decreasing sequence, let us fix some $n < \omega$ such that $\sup(E_{i_n} \cap \gamma) = \sup(E_{i_{n+1}} \cap \gamma)$. Then $\min(\Phi_{E_{i_n}}(C_\delta) \setminus \beta) = \min(\Phi_{E_{i_{n+1}}}(C_\delta) \setminus \beta)$, and in particular, $\beta^* := \min(\Phi_{E_{i_n}}(C_\delta) \setminus \beta)$ is in $E_{i_{n+1}} \setminus (\alpha + 1)$. Now, there are two options, each leading to a contradiction:

- ▶ If $\beta^* \in \operatorname{nacc}(\Phi_{E_{i_n}}(C_\delta))$, then we get a contradiction to the fact that $\beta^* > \alpha \geq \alpha_n$.
- ▶ If $\beta^* \in \operatorname{acc}(\Phi_{E_{i_n}}(C_\delta))$, then $\beta^* = \beta$ and $\beta^* \in \operatorname{acc}(C_\delta)$, contradicting the fact that $\beta \notin C_\delta$.

Case 2. Suppose that there exists $\delta \in S \cap E^\kappa_\omega$ for which $\sup(E \cap \delta \setminus C_\delta) = \delta$.

Fix such δ, and note that, for all $i \in I_\delta$, the ordinal $\alpha_i := \sup(\operatorname{nacc}(\Phi_{E_i}(C_\delta)) \cap E_{i+1})$ is $< \delta$. So, as $\operatorname{cf}(\delta) \neq \omega_1$, let $\{i_\nu \mid \nu < \omega_1\}$ be the increasing enumeration of some subset of I_δ, for which $\alpha := \sup_{\nu<\omega_1} \alpha_{i_\nu}$ is $< \delta$. Fix $\beta \in (E \cap \delta) \setminus C_\delta$ above α. Put $\gamma := \min(C_\delta \setminus \beta)$. Then $\delta > \gamma > \beta > \alpha$, and $\min(\Phi_{E_i}(C_\delta) \setminus \beta) = \sup(E_i \cap \gamma)$ for all $i < \lambda$. Fix some $\nu < \omega_1$ such that $\sup(E_{i_\nu} \cap \gamma) = \sup(E_{i_{\nu+1}} \cap \gamma)$. Then $\beta^* := \min(\Phi_{E_{i_\nu}}(C_\delta) \setminus \beta)$ is in $E_{i_{\nu+1}} \setminus (\alpha + 1)$, and as in the previous case, each of the two possible options leads to a contradiction.

Case 3. Suppose that $\sup(E \cap \delta \setminus C_\delta) < \delta$ for all $\delta \in S$.

Fix $\epsilon < \kappa$ for which $S' := \{\delta \in S \mid \sup(E \cap \delta \setminus C_\delta) = \epsilon\}$ is stationary. Put $B := \mathrm{acc}(E \setminus \epsilon)$, and note that, for every $\delta \in S'$, we have $B \cap \delta \subseteq \mathrm{acc}(C_\delta)$. Let $\{\beta_\alpha \mid \alpha < \kappa\}$ denote the increasing enumeration of the club $\{0\} \cup B$. For all $\alpha < \kappa$, put:

$$T_\alpha := \{C_\delta \cap \beta_\alpha \mid \delta \in S', \beta_\alpha < \delta\}.$$

Claim 2.2.1. *$(\bigcup_{\alpha<\kappa} T_\alpha, \sqsubseteq)$ is a tree whose α^{th} level is T_α, and $|T_\alpha| \leq |\mathcal{C}_{\beta_\alpha}|$ for all $\alpha < \kappa$.*

Proof. We commence by pointing out that $T_\alpha \subseteq \mathcal{C}_{\beta_\alpha}$ for all $\alpha < \kappa$. Clearly, $T_0 = \{\emptyset\} = \mathcal{C}_0 = \mathcal{C}_{\beta_0}$. Thus, consider an arbitrary nonzero $\alpha < \kappa$ along with some $t \in T_\alpha$. Fix $\delta \in S'$ above β_α such that $t = C_\delta \cap \beta_\alpha$. Then $\beta_\alpha \in B \cap \delta \subseteq \mathrm{acc}(C_\delta)$, so that $C_\delta \cap \beta_\alpha \in \mathcal{C}_{\beta_\alpha}$. That is, $t \in \mathcal{C}_{\beta_\alpha}$.

This shows that for all $t \in \bigcup_{\alpha<\kappa} T_\alpha$:

$$t \in T_\alpha \text{ iff } \sup(t) = \beta_\alpha.$$

Next, consider arbitrary $\alpha < \kappa$ and $t \in T_\alpha$, and let $t_\downarrow := \{s \in \bigcup_{\alpha'<\kappa} T_{\alpha'} \mid s \sqsubseteq t, s \neq t\}$ be the set of predecessors of t. Fix $\delta \in S'$ above β_α such that $t = C_\delta \cap \beta_\alpha$. We claim that $t_\downarrow = \{C_\delta \cap \beta_{\alpha'} \mid \alpha' < \alpha\}$, from which it follows that $(t_\downarrow, \sqsubseteq) \cong (\alpha, \in)$.

Consider $\alpha' < \alpha$. Then $\beta_{\alpha'} < \beta_\alpha < \delta$, so that $s := C_\delta \cap \beta_{\alpha'}$ is in $T_{\alpha'}$, and it is clear that s is a proper initial segment of t. That is, $s \in t_\downarrow$.

Conversely, consider $s \in t_\downarrow$. Fix $\alpha' < \kappa$ such that $s \in T_{\alpha'}$. By our earlier observation, $\sup(s) = \beta_{\alpha'}$, so that since $s \sqsubseteq t$, $s \neq t$, and $\sup(t) = \beta_\alpha$, we must have $\beta_{\alpha'} < \beta_\alpha$, and therefore $\alpha' < \alpha$. Thus, $s = t \cap \beta_{\alpha'} = (C_\delta \cap \beta_\alpha) \cap \beta_{\alpha'} = C_\delta \cap \beta_{\alpha'}$, as required. ⊟

Consider the stationary set $S'' := \{\alpha \in S' \mid \alpha = \beta_\alpha\}$. For each $\alpha \in S''$, we have $|T_\alpha| \leq |\mathcal{C}_\alpha| < \lambda$, so $T := \bigcup_{\alpha \in S''} T_\alpha$ ordered by $\sqsubseteq$ is a κ-tree each of whose levels has cardinality less than λ. Now, by a lemma of Kurepa (see [Kan03, Proposition 7.9]), $(T, \sqsubseteq)$ admits a cofinal branch, i.e., a chain $C \subseteq T$ (with respect to $\sqsubseteq$) that satisfies $|C \cap T_\alpha| = 1$ for all $\alpha \in S''$. Put $D := \bigcup C$ and note that D is a club in κ. As $\vec{\mathcal{C}}$ is a $\square(\kappa, \kappa)$-sequence, let us pick $\beta \in \mathrm{acc}(D)$ such that $D \cap \beta \notin \mathcal{C}_\beta$. Now, by definition of D, let us pick $t \in C$ such that $D \cap \beta \sqsubseteq t$. Then, as $t \in T$, let us pick $\delta \in S'$ above $\sup(t)$ such that $t \sqsubseteq C_\delta$. So $D \cap \beta \sqsubseteq C_\delta$. As $\beta \in \mathrm{acc}(D)$, we have $\beta \in \mathrm{acc}(C_\delta)$, and hence $D \cap \beta = C_\delta \cap \beta \in \mathcal{C}_\beta$, contradicting the choice of β. □

Corollary 2.3. *Suppose that $\vec{\mathcal{C}} = \langle \mathcal{C}_\alpha \mid \alpha < \lambda^+ \rangle$ is a $\square(\lambda^+, \lambda)$-sequence for which $\{\alpha \in E^{\lambda^+}_{<\lambda} \mid |\mathcal{C}_\alpha| < \lambda\}$ is stationary. Then there exists a $\square(\lambda^+, \lambda)$-sequence $\vec{\mathcal{C}^\bullet} = \langle \mathcal{C}^\bullet_\alpha \mid \alpha < \lambda^+ \rangle$ such that, for every club $E \subseteq \lambda^+$, there exists $\alpha \in \mathrm{acc}(\lambda^+) \cap E^{\lambda^+}_{<\lambda}$ with $|\mathcal{C}^\bullet_\alpha| < \lambda$ such that $\sup(\mathrm{nacc}(y) \cap E) = \alpha$ for all $y \in \mathcal{C}^\bullet_\alpha$.*

Proof. Appeal to Lemma 2.2 with $\kappa := \lambda^+$, $\vec{\mathcal{C}}$, and $S := \{\alpha \in \mathrm{acc}(\kappa) \cap E^\kappa_{<\lambda} \mid |\mathcal{C}_\alpha| < \lambda\}$, and let $D \subseteq \lambda^+$ be the outcome club. Define $\vec{\mathcal{C}^\bullet} = \langle \mathcal{C}^\bullet_\alpha \mid \alpha < \lambda^+ \rangle$ as follows:

- ▶ $\mathcal{C}^\bullet_0 := \{\emptyset\}$.
- ▶ For all $\alpha < \lambda^+$, $\mathcal{C}^\bullet_{\alpha+1} := \{\{\alpha\}\}$.
- ▶ For all $\alpha \in \mathrm{acc}(\lambda^+)$, let $\mathcal{C}^\bullet_\alpha := \{\Phi_D(C) \mid C \in \mathcal{C}_\alpha\}$.

By [BR19a, Lemma 2.2], for all $\alpha \in \mathrm{acc}(\lambda^+)$ and $C \in \mathcal{C}_\alpha$, $\Phi_D(C)$ is a club in α satisfying that, for all $\bar\alpha \in \mathrm{acc}(\Phi_D(C))$, $\bar\alpha \in \mathrm{acc}(C)$ and $\Phi_D(C) \cap \bar\alpha = \Phi_D(C \cap \bar\alpha) \in \mathcal{C}^\bullet_{\bar\alpha}$. In addition, by the choice of the club D, we know that for every club $E \subseteq \lambda^+$, there exists $\alpha \in \mathrm{acc}(\lambda^+) \cap E^{\lambda^+}_{<\lambda}$ with $|\mathcal{C}^\bullet_\alpha| \leq |\mathcal{C}_\alpha| < \lambda$ such that $\sup(\mathrm{nacc}(y) \cap E) = \alpha$ for all $y \in \mathcal{C}^\bullet_\alpha$.

Finally, given an arbitrary club D' in λ^+, consider the club $E := \mathrm{acc}(D')$, and fix $\alpha \in \mathrm{acc}(\lambda^+)$ such that $\sup(\mathrm{nacc}(y) \cap E) = \alpha$ for all $y \in \mathcal{C}^\bullet_\alpha$. It follows that, for all $y \in \mathcal{C}^\bullet_\alpha$, $\mathrm{nacc}(y) \cap \mathrm{acc}(D' \cap \alpha) \neq \emptyset$, let alone $y \neq D' \cap \alpha$. □

We now arrive at the heart of the matter.

Theorem 2.4. *Suppose that $\lambda^{<\lambda} = \lambda$, $2^\lambda = \lambda^+$, and there exists a $\square(\lambda^+, \lambda)$-sequence $\langle \mathcal{C}_\alpha \mid \alpha < \lambda^+ \rangle$ for which $\{\alpha \in E^{\lambda^+}_{<\lambda} \mid |\mathcal{C}_\alpha| < \lambda\}$ is stationary. Then $\boxtimes^*(\lambda^+)$ holds.*

Proof. Let $\vec{\mathcal{C}^\bullet} = \langle \mathcal{C}^\bullet_\alpha \mid \alpha < \lambda^+ \rangle$ be given by Corollary 2.3. Fix a bijection $\pi : \lambda^+ \times \lambda \leftrightarrow \lambda^+$. Also, for each $\beta < \lambda^+$, fix a bijection $g_\beta : \lambda \leftrightarrow E^{\beta+1}_{<\lambda} \times \lambda$.

By [Gre76, Lemma 2.1], $\lambda^{<\lambda} = \lambda$ and $2^\lambda = \lambda^+$ imply together that $\diamondsuit^*(E^{\lambda^+}_{<\lambda})$ holds. This means that we may fix a matrix $\langle Z_{\beta,j} \mid \beta \in E^{\lambda^+}_{<\lambda}, j < \lambda \rangle$ such that, for every $Z \subseteq \lambda^+$, for some club $D \subseteq \lambda^+$, we have

$$D \cap E^{\lambda^+}_{<\lambda} \subseteq \{\beta \in E^{\lambda^+}_{<\lambda} \mid \exists j < \lambda (Z \cap \beta = Z_{\beta,j})\}.$$

As $\lambda^{<\lambda} = \lambda$, the main result of [EK65] provides us with a sequence $\langle f_i \mid i < \lambda \rangle$ of functions from λ^+ to λ, such that, for every function $f : e \to \lambda$ with $e \in [\lambda^+]^{<\lambda}$, for some $i < \lambda$, we have $f \subseteq f_i$.

Now, let $i < \lambda$ be arbitrary. First, define a coloring $c_i : [\lambda^+]^2 \to \lambda^+$ by letting, for all $\eta < \beta < \lambda^+$,

$$c_i(\eta, \beta) := \min\{\xi \in (\eta, \beta] \mid \xi = \beta \text{ or } \pi(\xi, i) \in Z_{g_\beta(f_i(\beta))}\}.$$

Then, for every $y \in \mathcal{P}(\lambda^+)$, let

$$y_i := \mathrm{acc}(y) \cup \{c_i(\sup(y \cap \beta), \beta) \mid \beta \in \mathrm{nacc}(y)\}.$$

Finally, for every $\alpha \in \mathrm{acc}(\lambda^+)$, let $\mathcal{C}^i_\alpha := \{y_i \mid y \in \mathcal{C}^\bullet_\alpha\}$. Also, let $\mathcal{C}^i_0 := \{\emptyset\}$, and let $\mathcal{C}^i_{\alpha+1} := \{\{\alpha\}\}$ for all $\alpha < \lambda^+$.

Claim 2.4.1. *Suppose that $\alpha \in \mathrm{acc}(\lambda^+)$ and $C \in \mathcal{C}^i_\alpha$. Then:*

1. *C is a club in α;*
2. *For all $\bar\alpha \in \mathrm{acc}(C)$, $C \cap \bar\alpha \in \mathcal{C}^i_{\bar\alpha}$.*

Proof. Fix a club $y \in \mathcal{C}^\bullet_\alpha$ such that $C = y_i$.

(1) It is easy to see that for any two successive elements $\eta < \beta$ of the club y, we have that $C \cap (\eta, \beta]$ is a singleton. Consequently, $\sup(C) = \sup(y) = \alpha$, and $\mathrm{acc}^+(C) \subseteq \mathrm{acc}(y)$. But, by definition of $C = y_i$, we also have $\mathrm{acc}(y) \subseteq C$, so, C is a club in α.

(2) Let $\bar\alpha \in \mathrm{acc}(C)$ be arbitrary. By the above analysis, $\bar\alpha \in \mathrm{acc}(y)$, so that $y \cap \bar\alpha \in \mathcal{C}_{\bar\alpha}$. But $C \cap \bar\alpha = y_i \cap \bar\alpha = (y \cap \bar\alpha)_i$, and hence $C \cap \bar\alpha \in \mathcal{C}^i_{\bar\alpha}$. ⊟

Claim 2.4.2. *There exists $i < \lambda$ for which $\langle \mathcal{C}^i_\alpha \mid \alpha < \lambda^+ \rangle$ witnesses $\boxtimes^*(\lambda^+)$.*

Proof. Suppose not. It follows from Claim 2.4.1 that for each $i < \lambda$, we may pick some cofinal subset $X_i \subseteq \lambda^+$ such that, for all $\alpha \in \mathrm{acc}(\lambda^+)$, for some $C \in \mathcal{C}^i_\alpha$, we have $\sup(\mathrm{nacc}(C) \cap X_i) < \alpha$.

Let $Z := \pi``\bigcup_{i<\lambda}(X_i \times \{i\})$, and then fix a club D in λ^+ such that for all $\beta \in D$:

- $\pi[\beta \times \lambda] = \beta$;
- $\sup(X_i \cap \beta) = \beta$ for all $i < \lambda$;
- if $\mathrm{cf}(\beta) < \lambda$, then there exists $j < \lambda$ with $Z \cap \beta = Z_{\beta,j}$.

Consider the club $E := \mathrm{acc}(D)$. By the choice of $\vec{\mathcal{C}^\bullet}$, we may now pick $\alpha \in \mathrm{acc}(\lambda^+) \cap E^{\lambda^+}_{<\lambda}$ with $|\mathcal{C}^\bullet_\alpha| < \lambda$ such that $\sup(\mathrm{nacc}(y) \cap E) = \alpha$ for all $y \in \mathcal{C}^\bullet_\alpha$. Since

$\mathrm{cf}(\alpha)<\lambda$ and $|\mathcal{C}^\bullet_\alpha|<\lambda$, let us fix some $e\in[E\cap\alpha]^{<\lambda}$ such that $\sup(\mathrm{nacc}(y)\cap e)=\alpha$ for all $y\in\mathcal{C}^\bullet_\alpha$. Define a function $f:e\rightarrow\lambda$ by letting, for all $\beta\in e$,

$$f(\beta):=\begin{cases}\min g_\beta^{-1}\{(\gamma,j)\in\{\beta\}\times\lambda\mid Z\cap\beta=Z_{\beta,j}\} & \text{if } \mathrm{cf}(\beta)<\lambda;\\ \min g_\beta^{-1}\left\{(\gamma,j)\in(D\cap E^\beta_{<\lambda})\times\lambda \;\middle|\; \begin{array}{l} Z\cap\gamma=Z_{\gamma,j}, \text{ and for all } y\in\mathcal{C}^\bullet_\alpha,\\ \beta\in\mathrm{nacc}(y)\implies\sup(y\cap\beta)<\gamma\end{array}\right\} & \text{if } \mathrm{cf}(\beta)=\lambda.\end{cases}$$

Fix $i<\lambda$ such that $f\subseteq f_i$. By the choice of X_i, let us fix $C\in\mathcal{C}^i_\alpha$ such that $\sup(\mathrm{nacc}(C)\cap X_i)<\alpha$. Find $y\in\mathcal{C}^\bullet_\alpha$ such that $C=y_i$. Fix a large enough $\beta\in\mathrm{nacc}(y)\cap e$ such that $\eta:=\sup(y\cap\beta)$ is greater than $\sup(\mathrm{nacc}(C)\cap X_i)$. In particular, $c_i(\eta,\beta)$ must be an element of $\mathrm{nacc}(C)\setminus X_i$. Now, there are two cases to consider, each leading to a contradiction:

▶ If $\mathrm{cf}(\beta)<\lambda$, then for some $j<\lambda$, we have $g_\beta(f(\beta))=(\beta,j)$ and $Z\cap\beta=Z_{\beta,j}$. But $\beta\in e\subseteq E\subseteq D$, so that $g_\beta(f_i(\beta))=(\beta,j)$, $\pi[\beta\times\lambda]=\beta$, and

$$\{\xi<\beta\mid\pi(\xi,i)\in Z_{g_\beta(f_i(\beta))}\}=\{\xi<\beta\mid\pi(\xi,i)\in Z\cap\beta\}=X_i\cap\beta.$$

As $\beta\in D$, we have $\sup(X_i\cap\beta)=\beta$, so, $c_i(\eta,\beta)\in X_i\cap\beta$. This is a contradiction.

▶ If $\mathrm{cf}(\beta)=\lambda$, then let $(\gamma,j):=g_\beta(f(\beta))$, so that $\gamma\in D$ and $Z\cap\gamma=Z_{\gamma,j}$. In particular, $\{\xi<\gamma\mid\pi(\xi,i)\in Z_{g_\beta(f_i(\beta))}\}=X_i\cap\gamma$ and $\sup(X_i\cap\gamma)=\gamma$. Since $\beta\in\mathrm{nacc}(y)$, it also follows that $\eta=\sup(y\cap\beta)<\gamma<\beta$. Consequently, $c_i(\eta,\beta)\in X_i\cap\beta$. This is a contradiction. ⊟ □

We are now ready to derive the Main Theorem.

Proof of the Main Theorem. By [Gre76, Lemma 2.1], $\lambda^{<\lambda}=\lambda$ and $2^\lambda=\lambda^+$ imply together that $\diamondsuit^*(E^{\lambda^+}_{<\lambda})$ holds. In particular, as λ is uncountable, $\diamondsuit(\lambda^+)$ holds. In addition, by Theorem 2.4, $\boxtimes^*(\lambda^+)$ holds. So, by Fact 1.4, there exists a λ^+-Souslin tree. □

References

[AG18] David Asperó and Mohammad Golshani. The special aronszajn tree property at $\aleph_2$ and GCH. *arXiv preprint arXiv:1809.07638*, 2018.

[BR17a] Ari Meir Brodsky and Assaf Rinot. A microscopic approach to Souslin-tree constructions. Part I. *Ann. Pure Appl. Logic*, 168(11):1949–2007, 2017.

[BR17b] Ari Meir Brodsky and Assaf Rinot. A remark on Schimmerling's question. submitted November 2017. `http://assafrinot.com/paper/32`

[BR19a] Ari Meir Brodsky and Assaf Rinot. Distributive Aronszajn trees. *Fund. Math.*, to appear 2019. `http://doi.org/10.4064/fm542-4-2018`

[BR19b] Ari Meir Brodsky and Assaf Rinot. More notions of forcing add a Souslin tree. *Notre Dame J. Form. Log.*, to appear 2019. `http://assafrinot.com/paper/26`

[EK65] R. Engelking and M. Karłowicz. Some theorems of set theory and their topological consequences. *Fund. Math.*, 57:275–285, 1965.

[Gre76] John Gregory. Higher Souslin trees and the generalized continuum hypothesis. *J. Symbolic Logic*, 41(3):663–671, 1976.

[Kan03] Akihiro Kanamori. *The higher infinite*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, second edition, 2003. Large cardinals in set theory from their beginnings.

[Rin17] Assaf Rinot. Higher Souslin trees and the GCH, revisited. *Adv. Math.*, 311(C):510–531, 2017.

Department of Mathematics, Bar-Ilan University, Ramat-Gan 5290002, Israel.
URL: `http://www.assafrinot.com`