SIGMA-PRIKRY FORCING III: DOWN TO $\aleph_\omega$

ALEJANDRO POVEDA, ASSAF RINOT, AND DIMA SINAPOVA

Abstract. We prove the consistency of the failure of the singular cardinals hypothesis at $\aleph_\omega$ together with the reflection of all stationary subsets of $\aleph_{\omega+1}$. This shows that two classical results of Magidor (from 1977 and 1982) can hold simultaneously.

1. Introduction

Many natural questions cannot be resolved by the standard mathematical axioms (ZFC); the most famous example being Hilbert's first problem, the continuum hypothesis (CH). At the late 1930's, Gödel constructed an inner model of set theory [Göd40] in which the generalized continuum hypothesis (GCH) holds, demonstrating, in particular, that CH is consistent with ZFC. Then, in 1963, Cohen invented the method of forcing [Coh63] and used it to prove that $\neg$CH is, as well, consistent with ZFC.

In an advance made by Easton [Eas70], it was shown that any reasonable behavior of the continuum function $\kappa \mapsto 2^\kappa$ for *regular* cardinals κ may be materialized. In a review on Easton's paper for *AMS Mathematical Reviews*, Azriel Lévy writes:

> The corresponding question concerning the singular $\aleph_\alpha$'s is still open, and seems to be one of the most difficult open problems of set theory in the post-Cohen era. It is, e.g., unknown whether for all $n(n < \omega \rightarrow 2^{\aleph_n} = \aleph_{n+1})$ implies $2^{\aleph_\omega} = \aleph_{\omega+1}$ or not.

A preliminary finding of Bukovský [Buk65] (and independently Hechler) suggested that singular cardinals may indeed behave differently, but it was only around 1975, with Silver's theorem [Sil75] and the pioneering work of Galvin and Hajnal [GH75], that it became clear that singular cardinals obey much deeper constraints. This lead to the formulation of the singular cardinals hypothesis (SCH) as a (correct) relativization of GCH to singular cardinals, and ultimately to Shelah's *pcf* theory [She92, She00]. Shortly after Silver's discovery, advances in inner model theory due to Jensen (see [DJ75]) provided a *covering lemma* between Gödel's original model of GCH and many other models of set theory, thus establishing that any consistent

Date: October 9, 2023.

failure of SCH must rely on an extension of ZFC involving large cardinals axioms.

Compactness is the phenomenon where if a certain property holds for every strictly smaller substructure of a given object, then it holds for the object itself. Countless results in topology, graph theory, algebra and logic demonstrate that the first infinite cardinal is compact. Large cardinals axioms are compactness postulates for the higher infinite.

A crucial tool for connecting large cardinals axioms with singular cardinals was introduced by Prikry in [Pri70]. Then Silver (see [Men76]) constructed a model of ZFC whose extension by Prikry's forcing gave the first universe of set theory with a singular strong limit cardinal κ such that $2^\kappa > \kappa^+$. Shortly after, Magidor [Mag77a] proved that the same may be achieved at level of the very first singular cardinal, that is, $\kappa = \aleph_\omega$. Finally, in 1977, Magidor answered the question from Lévy's review in the affirmative:

Theorem 1 (Magidor, [Mag77b])**.** *Assuming the consistency of a supercompact cardinal and a huge cardinal above it, it is consistent that $2^{\aleph_n} = \aleph_{n+1}$ for all $n < \omega$, and $2^{\aleph_\omega} = \aleph_{\omega+2}$.*

Later works of Gitik, Mitchell, and Woodin pinpointed the optimal large cardinal hypothesis required for Magidor's theorem (see [Git02, Mit10]).

Note that Theorem 1 is an incompactness result; the values of the powerset function are small below $\aleph_\omega$, and blow up at $\aleph_\omega$. In a paper from 1982, Magidor obtained a result of an opposite nature, asserting that *stationary reflection* — one of the most canonical forms of compactness — may hold at the level of the successor of the first singular cardinal:

Theorem 2 (Magidor, [Mag82])**.** *Assuming the consistency of infinitely many supercompact cardinals, it is consistent that every stationary subset of $\aleph_{\omega+1}$ reflects.*[1]

Ever since, it remained open whether Magidor's compactness and incompactness results may co-exist.

The main tool for obtaining Theorem 1 (and the failures of SCH, in general) is Prikry-type forcing (see Gitik's survey [Git10]), however, adding Prikry sequences at a cardinal κ typically implies the failure of reflection at κ^+. On the other hand, Magidor's proof of Theorem 2 goes through Lévy-collapsing ω-many supercompact cardinals to become the $\aleph_n$'s, and in any such model SCH would naturally hold at the supremum, $\aleph_\omega$.

Various partial progress to combine the two results was made along the way. Cummings, Foreman and Magidor [CFM01] investigated which sets can reflect in the classical Prikry generic extension. In his 2005 dissertation [Sha05], Sharon analyzed reflection properties of extender-based Prikry forcing (EBPF, due to Gitik and Magidor [GM94]) and devised a way to

[1] That is, for every subset $S \subseteq \aleph_{\omega+1}$, if for every ordinal $\alpha < \aleph_{\omega+1}$ (of uncountable cofinality), there exists a closed and unbounded subset of α disjoint from S, then there exists a closed and unbounded subset of $\aleph_{\omega+1}$ disjoint from S.

kill one non-reflecting stationary set, again in a Prikry-type fashion. He then described an iteration to kill all non-reflecting stationary sets, but the exposition was incomplete.

In the other direction, works of Solovay [Sol74], Foreman, Magidor and Shelah [FMS88], Veličković [Vel92], Todorčević [Tod93], Foreman and Todorčević [FT05], Moore [Moo06], Viale [Via06], Rinot [Rin08], Shelah [She08], Fuchino and Rinot [FR11], and Sakai [Sak15] add up to a long list of compactness principles that are sufficient to imply the SCH.

In [PRS21], we introduced a new class of Prikry-type forcing called Σ-*Prikry* and showed that many of the standard Prikry-type forcing for violating SCH at the level of a singular cardinal of countable cofinality fits into this class. In addition, we verified that Sharon's forcing for killing a single non-reflecting stationary set fits into this class. Then, in [PRS22], we devised a general iteration scheme for Σ-Prikry forcing. From this, we constructed a model of the failure of SCH at κ with stationary reflection at κ^+. The said model is a generic extension obtained by first violating the SCH using EBPF, and then carrying out an iteration of length κ^{++} of the Σ-Prikry posets to kill all non-reflecting stationary subsets of κ^+.

Independently, and around the same time, Ben-Neria, Hayut and Unger [BHU19] also obtained the consistency of the failure of SCH at κ with stationary reflection at κ^+. Their proof differs from ours in quite a few aspects; we mention just two of them. First, they cleverly avoid the need to carry out iterated forcing, by invoking iterated ultrapowers, instead. Second, instead of EBPF, they violate SCH by using Gitik's very recent forcing [Git19a] which is also applicable to cardinals of uncountable cofinality. In addition, the authors devote an entire section to the countable cofinality case, where they get the desired reflection pattern out of a partial supercompact cardinal. An even simpler proof was then given by Gitik in [Git22].

Still, in all of the above, the constructions are for a singular cardinal κ that is very high up; more precisely, κ is a limit of inaccessible cardinals. Obtaining a similar construction for $\kappa = \aleph_\omega$ is quite more difficult, as it involves interleaving collapses. This makes key parts of the forcing no longer closed, and closure is an essential tool to make use of the indestructibility of the supercompact cardinals when proving reflection.

In this paper, we extend the machinery developed in [PRS21, PRS22] to support interleaved collapses, and show that this new framework captures Gitik's EBPF with interleaved collapses [Git19b]. The new class is called $(\Sigma, \vec{\mathbb{S}})$-Prikry. Finally, by running our iteration of $(\Sigma, \vec{\mathbb{S}})$-Prikry forcings over a suitable ground model, we establish that Magidor's compactness and incompactness results can indeed co-exist:

Main Theorem. *Assuming the consistency of infinitely many supercompact cardinals, it is consistent that all of the following hold:*

(1) $2^{\aleph_n} = \aleph_{n+1}$ *for all* $n < \omega$*;*
(2) $2^{\aleph_\omega} = \aleph_{\omega+2}$*;*

(3) every stationary subset of $\aleph_{\omega+1}$ reflects.

1.1. **Road map.** Let us give a high-level overview of the proof of the Main Theorem, and how it differs from the theory developed in Parts I and II of this series. Here, we assume that the reader has some familiarity with [PRS21, PRS22].

- Recall that the class of Σ-Prikry forcing provably does not add bounded sets (due to the Complete Prikry Property, CPP) and typically does not collapse cardinals (due to Linked$_0$-property, which provides a strong chain condition). Since in this paper, we do want to collapse some cardinals, we introduce a bigger class that enables it. Similarly to what was done in the prequels, we shall ultimately carry out a transfinite iteration of posets belonging to this class, but to have control of the cardinal collapses, we will be limiting the collapses to take place only on the very first step of the iteration.

  This leads to the definition of our new class — $(\Sigma, \vec{\mathbb{S}})$-Prikry — in which the elements of $\vec{\mathbb{S}}$ are the sole responsible for adding bounded sets. This class is presented in Section 3. But, before we can get there, there is something more basic to address:
- Recall that the successive step of the iteration scheme for the Σ-Prikry class involves a functor that, to each Σ-Prikry poset $(\mathbb{P}, \ell_{\mathbb{P}}, c_{\mathbb{P}})$ and some 'problem' z produces a new Σ-Prikry poset $(\mathbb{A}, \ell_{\mathbb{A}}, c_{\mathbb{A}})$ that admits a *forking projection* to $(\mathbb{P}, \ell_{\mathbb{P}}, c_{\mathbb{P}})$ and 'solves' z. In particular, each layer $\mathbb{A}_n$ projects to $\mathbb{P}_n$, but while collapses got an entry permit, it is not the case that the $\mathbb{A}_n$'s admit a decomposition of the form (closed enough forcing) $\times$ (small collapsing forcing).

  This leads to the concept of *nice projections* that we study in Section 2. This concept plays a role in the very definition of the $(\Sigma, \vec{\mathbb{S}})$-Prikry class, and enables us to analyze the patterns of stationary reflection taking place in various models of interest. Specifically, a 4-cardinal *suitability for reflection* property is identified with the goal of, later on, linking two different notions of sparse stationary sets.
- In Section 4, we prove that Gitik's Extender Based Prikry Forcing with Collapses (EBPFC) fits into the $(\Sigma, \vec{\mathbb{S}})$-Prikry framework. We analyze the preservation of cardinals in the corresponding generic extension and, in Subsection 4.3, we show that EBPFC is suitable for reflection.
- Recall that in [PRS22, §2.3], we showed how the existence of a forking projection from $(\mathbb{A}, \ell_{\mathbb{A}}, c_{\mathbb{A}})$ to a Σ-Prikry poset $(\mathbb{P}, \ell_{\mathbb{P}}, c_{\mathbb{P}})$ can be used to infer that $(\mathbb{A}, \ell_{\mathbb{A}}, c_{\mathbb{A}})$ is Σ-Prikry. This went through arguing that a forking projection that has the *weak mixing property* inherits a winning strategy for a diagonalizability game (Property $\mathcal{D}$) and the CPP from $(\mathbb{P}, \ell_{\mathbb{P}}, c_{\mathbb{P}})$. In Section 5, we briefly extend these results to the $(\Sigma, \vec{\mathbb{S}})$-Prikry class, by imposing the niceness feature of Section 2

on the definition of forking projections. An additional stronger notion (*super nice*) of forking projections is defined, but is not used before we arrive at Section 7.

- The next step is to revisit the functor $\mathbb{A}(\cdot,\cdot)$ from [PRS22, §4.1] for killing one non-reflecting stationary set. As the elegant reduction obtained at end of [PRS21, §5] is untrue in the new context of $(\Sigma,\vec{\mathbb{S}})$-Prikry forcing, we take a step back and look at *fragile* stationary sets. In Subsection 6.1, we prove that, given a $(\Sigma,\vec{\mathbb{S}})$-Prikry poset $\mathbb{P}$ and a fragile stationary set $\dot{T}$, a variation of $\mathbb{A}(\cdot,\cdot)$ yields a $(\Sigma,\vec{\mathbb{S}})$-Prikry poset admitting a nice forking projection to $\mathbb{P}$ and killing the stationarity of $\dot{T}$. The new functor now uses collapses in the domains of the strategies. To appreciate this difference, compare the proof of Theorem 6.18 below with that of [PRS21, Theorem 6.8].

  Then, in Subsection 6.2, we present a sufficient condition for linking fragile stationary sets with non-reflecting stationary sets.
- In Section 7, we present an iteration scheme for $(\Sigma,\vec{\mathbb{S}})$-Prikry forcings. Luckily, the scheme from [PRS22, §3] is successful at accommodating the new class, once the existence of a coherent system of (super) nice projections is assumed.
- In Section 8, we construct the model witnessing our Main Theorem. We first arrange a ground model V of GCH with an increasing sequence $\langle\kappa_n\mid n<\omega\rangle$ of supercompact cardinals that are indestructible under κ_n-directed-closed notions of forcing that preserve the GCH. Setting $\kappa:=\sup_{n<\omega}\kappa_n$, we then carry out an iteration of length κ^{++} of $(\Sigma,\vec{\mathbb{S}})$-Prikry. The first step is Gitik's EBPFC to collapse κ to $\aleph_\omega$, getting GCH below $\aleph_\omega$ and $2^{\aleph_\omega}=\aleph_{\omega+2}$. Using the strong chain condition of the iteration, we fix a bookkeeping of all names of potential fragile stationary sets. Each successor stage $\mathbb{P}_{\alpha+1}$ of the iteration is obtained by invoking the functor $\mathbb{A}(\cdot,\cdot)$ with respect to $\mathbb{P}_\alpha$ and a name for one fragile stationary set given by the bookkeeping list. At limit stages $\mathbb{P}_\alpha$, we verify that the iteration remains $(\Sigma,\vec{\mathbb{S}})$-Prikry. At all stages $\alpha<\kappa^{++}$, we verify that the initial GCH is preserved and that this stage is suitable for reflection. From this, we infer that in the end model $V^{\mathbb{P}_{\kappa^{++}}}$, not only that there are no fragile stationary sets, but, in fact, there are no non-reflecting stationary sets. So, Clauses (1)–(3) all hold in $V^{\mathbb{P}_{\kappa^{++}}}$.

1.2. **Notation and conventions.** Our forcing convention is that $p\le q$ means that p extends q. We write $\mathbb{P}\downarrow q$ for $\{p\in\mathbb{P}\mid p\le q\}$. Denote $E^\mu_\theta:=\{\alpha<\mu\mid \mathrm{cf}(\alpha)=\theta\}$. The sets $E^\mu_{<\theta}$ and $E^\mu_{>\theta}$ are defined in a similar fashion. For a stationary subset S of a regular uncountable cardinal μ, we write $\mathrm{Tr}(S):=\{\delta\in E^\mu_{>\omega}\mid S\cap\delta \text{ is stationary in } \delta\}$. H_ν denotes the collection of all sets of hereditary cardinality less than ν. For every set of ordinals x, we denote $\mathrm{cl}(x):=\{\sup(x\cap\gamma)\mid \gamma\in\mathrm{Ord}, x\cap\gamma\neq\emptyset\}$, and

$\mathrm{acc}(x) := \{\gamma \in x \mid \sup(x \cap \gamma) = \gamma > 0\}$. We write CH_μ to denote $2^\mu = \mu^+$ and $\mathsf{GCH}_{<\nu}$ as a shorthand for CH_μ holds for every infinite cardinal $\mu < \nu$.

For a sequence of maps $\vec{\varpi} = \langle \varpi_n \mid n < \omega\rangle$ and yet a another map π such that $\mathrm{Im}(\pi) \subseteq \bigcap_{n<\omega} \mathrm{dom}(\varpi_n)$, we let $\vec{\varpi} \bullet \pi$ denote $\langle \varpi_n \circ \pi \mid n < \omega\rangle$.

2. Nice projections and reflection

Definition 2.1. Given a poset $\mathbb{P} = (P, \leq)$ with greatest element $\mathbb{1}$ and a map ϖ with $\mathrm{dom}(\varpi) \supseteq P$, we derive a poset $\mathbb{P}^\varpi := (P, \leq^\varpi)$ by letting

$$p \leq^\varpi q \text{ iff } (p = \mathbb{1} \text{ or } (p \leq q \text{ and } \varpi(p) = \varpi(q))).$$

Definition 2.2. For two notions of forcing $\mathbb{P} = (P, \leq)$ and $\mathbb{S} = (S, \preceq)$ with maximal elements $\mathbb{1}_\mathbb{P}$ and $\mathbb{1}_\mathbb{S}$, respectively, we say that a map $\varpi : P \to S$ is a *nice projection from* $\mathbb{P}$ *to* $\mathbb{S}$ iff all of the following hold:

(1) $\varpi(\mathbb{1}_\mathbb{P}) = \mathbb{1}_\mathbb{S}$;
(2) for any pair $q \leq p$ of elements of P, $\varpi(q) \preceq \varpi(p)$;
(3) for all $p \in P$ and $s \preceq \varpi(p)$, the set $\{q \in P \mid q \leq p \wedge \varpi(q) \preceq s\}$ admits a $\leq$-greatest element,[2] which we denote by $p + s$. Moreover, $p + s$ has the additional property that $\varpi(p + s) = s$;
(4) for every $q \leq p + s$, there is $p' \leq^\varpi p$ such that $q = p' + \varpi(q)$; In particular, the map $(p', s') \mapsto p' + s'$ constitutes a projection from $(\mathbb{P}^\varpi \downarrow p) \times (\mathbb{S} \downarrow s)$ onto $\mathbb{P} \downarrow p$.

Example 2.3. If $\mathbb{P}$ is a product of the form $\mathbb{S} \times \mathbb{T}$, then the map $(s, t) \mapsto s$ constitutes a nice projection from $\mathbb{P}$ to $\mathbb{S}$.

Note that the composition of nice projections is again a nice projection.

Definition 2.4. Let $\mathbb{P} = (P, \leq)$ and $\mathbb{S} = (S, \preceq)$ be two notions of forcing and $\varpi : P \to S$ be a nice projection. For an $\mathbb{S}$-generic filter H, we define *the quotient forcing* $\mathbb{P}/H := (P/H, \leq_{\mathbb{P}/H})$ as follows:

- $P/H := \{p \in P \mid \varpi(p) \in H\}$;
- for all $p, q \in P/H$, $q \leq_{\mathbb{P}/H} p$ iff there is $s \in H$ with $s \preceq \varpi(q)$ such that $q + s \leq p$.

Remark 2.5. In a slight abuse of notation, we tend to write $\mathbb{P}/\mathbb{S}$ when referring to a quotient as above, without specifying the generic for $\mathbb{S}$ or the map ϖ. By standard arguments, $\mathbb{P}$ is isomorphic to a dense subposet of $\mathbb{S} * \mathbb{P}/\mathbb{S}$ (see [Abr10, p. 337]).

Lemma 2.6. *Suppose that $\varpi : \mathbb{P} \to \mathbb{S}$ is a nice projection. Let $p \in P$ and set $s := \varpi(p)$. For any condition $a \in \mathbb{S} \downarrow s$, define an ordering $\leq_a$ over $\mathbb{P}^\varpi \downarrow p$ by letting $p_0 \leq_a p_1$ iff $p_0 + a \leq^\varpi p_1 + a$.*[3] *Then:*

(1) $(\mathbb{S} \downarrow a) \times ((\mathbb{P}^\varpi \downarrow p), \leq_a)$ projects to $\mathbb{P} \downarrow (p + a)$, and
(2) $((\mathbb{P}^\varpi \downarrow p), \leq_a)$ projects to $((\mathbb{P}^\varpi \downarrow p), \leq_{a'})$ for all $a' \preceq a$.[4]

[2]By convention, a greatest element, if exists, is unique.

[3]Strictly speaking, $\leq_a$ is reflexive and transitive, but not asymmetric. But this is also always the case, for instance, in iterated forcing.

[4]Taking $a = s$ we have in particular that $\mathbb{P}^\varpi \downarrow p$ projects to $((\mathbb{P}^\varpi \downarrow p), \leq_{a'})$.

(3) *If $\mathbb{P}^{\varpi}$ contains a δ-closed dense set, then so does $((\mathbb{P}^{\varpi}\downarrow p),\leq_a)$.*

Proof. Note that for p_0,p_1 in $\mathbb{P}^{\varpi}\downarrow p$:

- $p_0\leq_s p_1$ iff $p_0\leq^{\varpi} p_1$, and so $(\mathbb{P}^{\varpi}\downarrow p,\leq_s)$ is simply $\mathbb{P}^{\varpi}\downarrow p$;
- if $p_0\leq_a p_1$, then $p_0\leq_{a'} p_1$ for any $a'\preceq a$;
- in particular, if $p_0\leq^{\varpi} p_1$, then $p_0\leq_a p_1$ for any a in $\mathbb{S}\downarrow s$.

The first projection is given by $(a',r)\mapsto r+a'$, and the second projection is given by the identity.

For the last statement, denote $\mathbb{P}_a:=((\mathbb{P}^{\varpi}\downarrow p),\leq_a)$ and let D be a δ-closed dense subset of $\mathbb{P}^{\varpi}$. We claim that $D_a:=\{r\in\mathbb{P}^{\varpi}\downarrow p\mid r+a\in D\}$ is a δ-closed dense subset of $\mathbb{P}_a$. For the density, if $r\in\mathbb{P}^{\varpi}\downarrow p$, let $q\leq^{\varpi} r+a$ be in D. Then, by Clause (4) of Definition 2.2, $q=r'+a$ for some $r'\leq^{\varpi} r$, and so $r'\leq_a r$ and $r'\in D_a$. For the closure, suppose that $\langle p_i\mid i<\tau\rangle$ is a $\leq_{\mathbb{P}_a}$-decreasing sequence in D_a for some $\tau<\delta$. Setting $q_i:=p_i+a$, that means that $\langle q_i\mid i<\tau\rangle$ is a $\leq^{\varpi}$-decreasing sequence in D and so has a lower bound q. More precisely, $q\in D$ and for each i, $q\leq q_i$ and $\varpi(q)=\varpi(q_i)=a$. Let $p^*\leq^{\varpi} p$, be such that $p^*+a=q$. Here again we use Clause (4) of Definition 2.2. Then $p^*\in D_a$, which is the desired $\leq_{\mathbb{P}_a}$-lower bound. □

The next lemma clarifies the relationship between the different generic extensions that we will be considering:

Lemma 2.7. *Suppose that $\varpi:\mathbb{P}\to\mathbb{S}$, $p\in P$, $s:=\varpi(p)$ and $\leq_a$ for a in $\mathbb{S}\downarrow s$ are as in the above lemma. Let G be $\mathbb{P}$-generic with $p\in G$.*

Next, let $H\times G^*$ be $((\mathbb{S}\downarrow s)\times(\mathbb{P}^{\varpi}\downarrow p))/G$-generic over $V[G]$. For each $a\in H$, let G_a be the $((\mathbb{P}^{\varpi}\downarrow p),\leq_a)$-generic filter obtained from G^*. Then:

1. *For any $a\in H$, $V[G]\subseteq V[H\times G_a]\subseteq V[H\times G^*]$, and $G\supseteq G_a\supseteq G^*$;*
2. *For any pair $a'\preceq a$ of elements of H, $V[H\times G_{a'}]\subseteq V[H\times G_a]$, and $G_{a'}\supseteq G_a$;*
3. *$G\cap(\mathbb{P}^{\varpi}\downarrow p)=\bigcup_{a\in H}G_a$.*

Proof. For notational convenience, denote $\mathbb{P}^*:=\mathbb{P}^{\varpi}\downarrow p$ and $\mathbb{S}^*:=\mathbb{S}\downarrow s$.

The first two items follow from the corresponding choices of the projections in Lemma 2.6. For the third item, first note that $\bigcup_{a\in H}G_a\subseteq\mathbb{P}^*\cap G$. Suppose that $r^*\in G\cap\mathbb{P}^*$. In $V[H]$, define

$$D:=\{r\in P^*\mid(\exists a\in H)(r\leq_a r^*)\vee r\perp_{\mathbb{P}/H} r^*\}.^5$$

Claim 2.7.1. *D is a dense set in $\mathbb{P}^*$.*

Proof. Let $r\in P^*$. If $r\perp_{\mathbb{P}/H} r^*$, then $r\in D$, and we are done. So suppose that r and r^* are compatible in $\mathbb{P}/H$. Let $q\in P/H$ be such that, $q\leq_{\mathbb{P}/H} r$ and $q\leq_{\mathbb{P}/H} r^*$. Let $a\preceq\varpi(q)$ in H be such that $q+a\leq r,r^*$. By Clause (4) of Definition 2.2 for ϖ we may let $r'\leq^{\varpi} r$, be such that $r'+a=q+a$. In particular, $r'\leq_a r^*$, and so $r'\in D$. □

[5] Since $r\in P^*$ then $\varpi(r)=\varpi(p)\in H$ and thus r is a condition in $\mathbb{P}/H$.

Now let $r \in D \cap G^*$. Since both $r, r^* \in G$, it must be that $r \leq_a r^*$ for some $a \in H$. And since $r \in G^* \subseteq G_a$, we get that $r^* \in G_a$. $\square$

Let us recall the classical concept of *stationary reflection*:

Definition 2.8. For stationary subsets Δ, Γ of a regular uncountable cardinal μ, $\mathrm{Refl}(\Delta, \Gamma)$ asserts that for every stationary subset $T \subseteq \Delta$, there exists $\gamma \in \Gamma \cap E^\mu_{>\omega}$ such that $T \cap \gamma$ is stationary in γ.

In this section we establish a sufficient condition for $\mathrm{Refl}(\dots)$ to hold in generic extensions (see Lemma 2.12 below); this will play a crucial role at the end of Section 5. Our proof uses the instrumental notion of *suitability for reflection* of Definition 2.10 below. Before introducing this key concept we have to provide some context.

Suppose that $\pi\colon \mathbb{P} \to \mathbb{S} \times \mathbb{R}$ is a projection. Let ϖ be the projection from $\mathbb{P}$ to $\mathbb{S}$ induced by π via the fiber map from $\mathbb{S} \times \mathbb{R}$ to $\mathbb{S}$. In this section we shall work in a scenario where $\varpi\colon \mathbb{P} \to \mathbb{S}$ is in addition a nice projection. In this case we will have that $(\mathbb{S} \downarrow \varpi(p)) \times (\mathbb{P}^\varpi \downarrow p) \to \mathbb{P} \downarrow p$ given by $\langle s, q\rangle \mapsto q + s$ is a projection.

Notation 2.9. In future arguments we will have that for each $s \in S \setminus \{\mathbb{1}_{\mathbb{S}}\}$ the poset $\mathbb{S} \downarrow s$ is isomorphic to $\mathbb{Q} \times \mathrm{Col}(\delta, {<}\sigma)$ for some forcing $\mathbb{Q}$. In that scenario, given $p \in P$ with $\varpi(p) \neq \mathbb{1}_{\mathbb{S}}$, we shall denote by $\varpi_\aleph$ the nice projection $\varpi_\aleph\colon \mathbb{P} \downarrow p \to \mathbb{Q}$ induced by π and the above-mentioned isomorphism. Similarly, we denote $\varpi_\beth\colon \mathbb{P} \downarrow p \to (\mathrm{Col}(\delta, {<}\sigma) \times \mathbb{R})$.

Definition 2.10. For infinite cardinals $\tau < \sigma < \kappa < \mu$,[6] we say that $(\mathbb{P}, \mathbb{S}, \mathbb{R}, \varpi)$ is *suitable for reflection with respect to $\langle \tau, \sigma, \kappa, \mu\rangle$* iff all the following hold:

(1) $\varpi\colon \mathbb{P} \to \mathbb{S}$ is a nice projection which is the fiber map of a projection $\pi\colon \mathbb{P} \to \mathbb{S} \times \mathbb{R}$;
(2) $\mathbb{P}^\varpi$ contains a σ-directed-closed dense set;
(3) For every $p \in P$ such that $s := \varpi(p)$ is different from $\mathbb{1}_{\mathbb{S}}$, there is a cardinal δ with $\tau^+ < \delta < \sigma$, such that $\mathbb{S} \downarrow s \cong \mathbb{Q} \times \mathrm{Col}(\delta, {<}\sigma)$ for some forcing $\mathbb{Q}$ of size $< \delta$. Furthermore:
   (a) $|\mu| = \mathrm{cf}(\mu) = \kappa = \sigma^{++}$ holds in $V^{\mathbb{S} \times \mathbb{R}}$ and in $V^{(\mathbb{S} \downarrow s) \times (\mathbb{P}^\varpi \downarrow p)}$;
   (b) $(\mathbb{P}^{\varpi_\aleph} \downarrow p)/(\mathrm{Col}(\delta, {<}\sigma) \times \mathbb{R})$ contains a δ-closed dense set;
   (c) $((\mathbb{P}^\varpi \downarrow p) \times \mathrm{Col}(\delta, {<}\sigma))/(\mathbb{P}^{\varpi_\aleph} \downarrow p)$ contains a δ-closed dense set;
   (d) For every $m < \omega$, for every sequence $\langle p_i \mid i < m\rangle$ of conditions in $\mathbb{P}$ that are $\leq_{\mathbb{P}}$-below p, for every $r \in \mathrm{Col}(\delta, {<}\sigma) \times \mathbb{R}$, if
      - $\langle \varpi_\aleph(p_i) \mid i < m\rangle$ are pairwise $\mathbb{Q}$-incompatible, and
      - $r \leq_{\mathrm{Col}(\delta, <\sigma) \times \mathbb{R}} \varpi_\beth(p_i)$ for every $i < m$,

      then, then there exists $q \leq^{\varpi_\aleph} p$ with $\varpi_\beth(q) = r$ such that $q + \varpi_\aleph(p_i) \leq p_i$ for every $i < m$.

[6]In all cases of interest, σ will be a regular cardinal.

Remark 2.11. Regarding Clause (3b), note that $\mathbb{P}^{\varpi_\aleph}/(\mathrm{Col}(\delta,{<}\sigma)\times\mathbb{R})$ is well-defined in that $\varpi_\beth \restriction (\mathbb{P}^{\varpi_\aleph}\downarrow p)$ is a projection (this is a consequence of Clause (3d) with $m=1$). Similarly, for Clause (3c), there is a natural projection between $(\mathbb{P}^{\varpi}\downarrow p)\times\mathrm{Col}(\delta,{<}\sigma)$ and $\mathbb{P}^{\varpi_\aleph}\downarrow p$.

Let us state the main lemma of the section:

Lemma 2.12. *Suppose $(\mathbb{P},\mathbb{S},\mathbb{R},\varpi)$ is suitable for reflection with respect to a 4-tuple of cardinals $\langle\tau,\sigma,\kappa,\mu\rangle$. If σ is a supercompact cardinal indestructible under forcing with $\mathbb{P}^{\varpi}$, then $V^{\mathbb{P}}\models\mathrm{Refl}(E^{\mu}_{\leq\tau},E^{\mu}_{<\sigma^+})$.*

Towards the proof of this lemma we prove two auxiliary results about preservation of stationary sets. Henceforth we assume that $(\mathbb{P},\mathbb{S},\mathbb{R},\varpi)$ is suitable for reflection with respect to $\langle\tau,\sigma,\kappa,\mu\rangle$ as witnessed by a projection $\pi\colon\mathbb{P}\to\mathbb{S}\times\mathbb{R}$, which induces ϖ, $\varpi_\aleph$ and $\varpi_\beth$.

Lemma 2.13. *For every $\bar{p}\in\mathbb{P}$ with $\varpi(\bar{p})\neq\mathbb{1}_{\mathbb{S}}$, for every $\mathbb{P}$-generic filter G containing $\bar{p}$, $((\mathbb{P}^{\varpi_\aleph}\downarrow\bar{p})\times\mathbb{Q})/G$ preserves stationary subsets of $(E^{\kappa}_{<\delta})^{V[G]}$.*

Proof. Suppose otherwise and fix $\bar{p}\in P$ a witness for this. For the scope of the proof let us write $\mathbb{P}^*:=\mathbb{P}^{\varpi_\aleph}\downarrow\bar{p}$. Let $p^*\in\mathbb{P}^*$ and $a^*\in\mathbb{Q}$ be such that (p^*,a^*) $(\mathbb{P}^*\times\mathbb{Q})$-forces that a $\mathbb{P}$-name $\dot{T}$ is a $V[\dot{G}]$-stationary whose stationarity is destroyed. Since $\mathbb{Q}$ is small, we may let $\dot{C}$ be a $\mathbb{P}^*$-name such that

- $p^*\Vdash_{\mathbb{P}^*}$ "$\dot{C}$ is a club in κ", and
- $(p^*,a^*)\Vdash_{\mathbb{P}^*\times\mathbb{Q}}$ "$\dot{C}\cap\dot{T}=\emptyset$".

Let G^* be a $\mathbb{P}^*$-generic containing p^*, and let F be the induced V-generic for $\mathrm{Col}(\delta,{<}\sigma)\times\mathbb{R}$. Let $\langle H_i\mid i<|\mathbb{Q}|\rangle$ be mutually generics for $\mathbb{Q}\downarrow a^*$ such that $\prod^{\mathrm{fin}}_i H_i$ is generic for the finite support product $\prod^{\mathrm{fin}}_i\mathbb{Q}$ over $V[G^*]$. For each $i<|\mathbb{Q}|$, let G_i be the $\mathbb{P}$-generic obtained from $G^*\times H_i$. Namely, G_i is the upwards closure of the set

$$\{p+a\mid p\in G^*,\ a\in H_i\}$$

where the $+$ operation is computed with respect to $\varpi_\aleph$.

Claim 2.13.1. *For every $a\preceq a^*$ there exists $i<|\mathbb{Q}|$ such that $a\in H_i$.*

Proof. The set $D_a:=\{\vec{a}\in\prod^{\mathrm{fin}}_i\mathbb{Q}\mid\exists i\in\mathrm{supp}(\vec{a})\,[a_i=a]\}$ is dense, so there is $\vec{a}\in D_a\cap\prod^{\mathrm{fin}}_i H_i$. Let $i\in\mathrm{supp}(\vec{a})$ be such that $a_i=a$. Then $a\in H_i$. $\square$

Claim 2.13.2. *$\prod^{\mathrm{fin}}_i G_i$ is $V[F]$-generic for $\prod^{\mathrm{fin}}_i(\mathbb{P}/F)$.*

Proof. Clearly, $\prod^{\mathrm{fin}}_i G_i$ is a filter. For genericity, suppose $D\in V[F]$ is a dense open subset of $\prod^{\mathrm{fin}}_i(\mathbb{P}/F)$. Working in $V[G^*]$, define

$$E:=\{\vec{a}\in\textstyle\prod^{\mathrm{fin}}_i\mathbb{Q}\mid\exists p\in G^*,\langle p+a_i\mid i\in\mathrm{supp}(\vec{a})\rangle\in D\}.$$

We claim that E is a dense subset of $\prod^{\mathrm{fin}}_i\mathbb{Q}$. Fix $\vec{a}\in\prod^{\mathrm{fin}}_i\mathbb{Q}$ and note that the following set is in $V[F]$:

$$E_{\vec{a}}:=\{p\in\mathbb{P}^*/F\mid\exists\vec{b}\in\textstyle\prod^{\mathrm{fin}}_i\mathbb{Q}\,[\vec{b}\leq_{\prod^{\mathrm{fin}}_i\mathbb{Q}}\vec{a}\ \&\ \langle p+b_i\mid i\in\mathrm{supp}(\vec{b})\rangle\in D]\}.$$

Next we show that $E_{\vec{a}}$ is dense: Fix $p \in \mathbb{P}^*/F$. Since D is dense we may let $\vec{t} \in D$ below $\langle p+a_i \mid i \in \mathrm{supp}(\vec{a})\rangle \in \prod_i^{\mathrm{fin}}(\mathbb{P}/F)$. For each $i \in \mathrm{supp}(\vec{t})$, denote $b_i := \varpi_{\aleph}(t_i)$. By possibly $\mathbb{Q}$-strengthening if necessary, we may assume that the elements of $\langle b_i \mid i \in \mathrm{supp}(\vec{t})\rangle$ are pairwise incompatible. In addition, there is a lower bound $r \in F$ for the $\mathrm{Col}(\delta, {<}\sigma) \times \mathbb{R}$-parts of $\vec{t}$ (i.e., $\langle \varpi_{\beth}(t_i) \mid i \in \mathrm{supp}(\vec{t})\rangle$) because all of these are in F. Using Clause (3d) of suitability for reflection there is $q \leq^{\varpi_{\aleph}} p$ with $\varpi_{\beth}(q) = r \in F$ such that $q + b_i \leq t_i$ for all $i \in \mathrm{supp}(\vec{t})$. In particular, $\langle q + b_i \mid i \in \mathrm{supp}(\vec{b})\rangle \in \prod_i^{\mathrm{fin}} \mathbb{P}/F$ and is stronger than $\vec{t}$. So, by openness of D, $\langle q+b_i \mid i \in \mathrm{supp}(\vec{b})\rangle \in D$. Therefore, $q \in E_{\vec{a}}$. So,

$$E_{\vec{a}} \text{ is dense in } \mathbb{P}^*/F.$$

Since $E_{\vec{a}}$ is dense for $\mathbb{P}^*/F$ and G^* is generic over $V[F]$ there is $p \in E_{\vec{a}} \cap G^*$. Let $\vec{b}$ be a witness for $p \in E_{\vec{a}}$, so that $\vec{b} \in E$. This concludes the proof that E is dense.

Let $\vec{a} \in E \cap \prod_i^{\mathrm{fin}} H_i$ and $p \in G^*$ witness it. Then, $\langle p + a_i \mid i \in \mathrm{supp}(\vec{a})\rangle$ belongs to $D \cap \prod_i^{\mathrm{fin}} G_i$, concluding the proof of the claim. □

For all $i < |\mathbb{Q}|$ and $a \preceq a^*$, denote

- $T_i := \dot{T}_{G_i}$;
- $T^*_a := \{\gamma < \kappa \mid \exists q^* \in G^* \, (q^* \leq^{\varpi_{\aleph}} p^* \wedge q^* + a \Vdash_{\mathbb{P}} \check{\gamma} \in \dot{T})\}$.

Note that $T_i \in V[G_i]$ and $T^*_a \in V[G^*]$.

Claim 2.13.3. *$\bigcup_{i<|\mathbb{Q}|} T_i = \bigcup_{a \preceq a^*} T^*_a$ holds in $V[G^* \times \prod^{\mathrm{fin}}_{i<|\mathbb{Q}|} H_i]$.*

Proof. Let $\gamma \in T_i$ and $q \in G_i$ be such that $q \Vdash_{\mathbb{P}} \check{\gamma} \in \dot{T}$. Recall that G_i is the upwards closure of $\{q^* + a \mid s \in H_i,\ q^* \in G^*\}$ so we may let $q^* \leq^{\varpi_{\aleph}} p^*$ in G^* and $a \in H_i$ with $q^* + a \leq q$. Then, $\gamma \in T^*_a$. Conversely, let $\gamma \in T^*_a$ and $q^* \in G^*$ witnessing it. By Claim 2.13.1, there is $i < |\mathbb{Q}|$ such that $a \in H_i$. Since $q^* + a \in G_i$ it follows that $\gamma \in T_i$. □

Denote $S := \bigcup_{i<|\mathbb{Q}|} T_i$. The next is the crucial claim:

Claim 2.13.4. *$S \in V[F]$.*

Proof. Let $\dot{S}$ be a $\prod_i^{\mathrm{fin}}(\mathbb{P}/F)$-name whose interpretation under the generic $\prod_i^{\mathrm{fin}} G_i$ yields S. We will show that

$$S = \{\gamma < \kappa \mid \mathbb{1} \Vdash^{V[F]}_{\prod_i^{\mathrm{fin}}((\mathbb{P}/F)\downarrow(p^*+a^*))} \gamma \in \dot{S}\}.$$

The right-to-left inclusion is obvious. Fix $\gamma \in S$ and suppose for a contradiction that the trivial condition of $\prod_i^{\mathrm{fin}}(\mathbb{P}/F)$ does not force "$\gamma \in \dot{S}$".

Let $\vec{t} \in \prod_i^{\mathrm{fin}}(\mathbb{P}/F)$ be such that $V[F] \models \vec{t} \Vdash_{\prod_i^{\mathrm{fin}} \mathbb{P}/F} \gamma \notin \dot{S}$ and

$$t_i \leq p^* + a^* \text{ for each } i \in \mathrm{supp}(\vec{t}).$$

By $\mathbb{Q}$-strengthening if necessary, assume that setting $b_i := \varpi_{\aleph}(t_i)$, we have

$$b_i \perp b_j \text{ for all } i \neq j \text{ in } \mathrm{supp}(\vec{t}).$$

Since $\gamma \in S$, let $\vec{r} \in \prod_i^{\mathrm{fin}} G_i$ be such that each $r_i \leq p^* + a^*$ and $\vec{r}$ forces (in $\prod_i^{\mathrm{fin}}(\mathbb{P}/F)$) that $\gamma \in \dot{S}$. Let $c_i = \varpi_{\aleph}(r_i)$ for $i \in \mathrm{supp}(\vec{r})$. By strengthening if necessary we may assume that for all $i \in \mathrm{supp}(\vec{r})$ and $j \in \mathrm{supp}(t)$ the conditions c_i and b_j are $\mathbb{Q}$-incompatible, and that for all $i < j$, $c_i \perp_{\mathbb{Q}} c_j$. Then Clause (3d) of suitability for reflection yields a condition in $\mathbb{P}^*/F$, $p \leq^{\varpi_{\aleph}} p^*$, such that for each $i \in \mathrm{supp}(\vec{t})$, $p + b_i \leq t_i$ and for each $i \in \mathrm{supp}(\vec{r})$, $p + c_i \leq r_i$. In particular,

$$\langle p + b_i \mid i \in \mathrm{supp}(\vec{t})\rangle \Vdash^{V[F]}_{\prod_i^{\mathrm{fin}} \mathbb{P}/F} \gamma \notin \dot{S}$$

and

$$\langle p + c_i \mid i \in \mathrm{supp}(\vec{r})\rangle \Vdash^{V[F]}_{\prod_i^{\mathrm{fin}} \mathbb{P}/F} \gamma \in \dot{S}.$$

Let $q \leq_{\mathbb{P}^*/F} p$ be such that $q \parallel_{\mathbb{P}^*/F} \gamma \in \dot{S}$. Note that we get a contradiction: On one hand, $q + c_i \leq p + c_i$, for all $i \in \mathrm{supp}(\vec{t})$; on the other hand, $q + b_i \leq p + b_i$ for all $i \in \mathrm{supp}(\vec{r})$. Thus two extensions of q decide the assertion "$\gamma \in \dot{S}$" in a contradictory way, which is not possible. □

Claim 2.13.5. *S is stationary in $V[F]$ and $S \cap \dot{C}_{G^*} = \emptyset$.*

Proof. Note that $T_i \subseteq S$ and that T_i is $V[G_i]$-stationary, so S is $V[G_i]$-stationary, and thus also stationary in $V[F]$. Also, $S \cap \dot{C}_{G^*} = \emptyset$. To show this let $\gamma < \kappa$ and $q \in G^*$ such that $q \Vdash_{\mathbb{P}^*} \gamma \in \dot{C}$. Since $(q, a^*) \Vdash_{\mathbb{P}^* \times \mathbb{Q}} \dot{C} \cap \dot{T} = \emptyset$ and $q + a^* \in G_i$ for all $i < |\mathbb{Q}|$, it follows that $\gamma \notin T_i$. Thus, $\gamma \notin S$. □

Finally we show that S is stationary in $V[G^*]$, which will contradict the previous claim and in turn will complete the proof of Lemma 2.13. Combining Clause (3a) of suitability for reflection with $|\mathbb{Q}| < \kappa$ we have that $V[F] \models$ "$\kappa = \sigma^{++}$". By [She91, Lemma 4.4], then,

$$S \subseteq (E^{\sigma^{++}}_{\leq\sigma})^{V[F]} \in I[\sigma^{++}]^{V[F]}.$$

In addition, $\mathbb{P}^{\varpi_{\aleph}}/F$ contains a δ-closed dense set (Clause (3a)), hence, by Shelah's theorem from [She79], forcing with it preserves the stationarity of S. □

Lemma 2.14. *For every $p \in \mathbb{P}$ with $\varpi(p) \neq \mathbb{1}_{\mathbb{S}}$, for every $\mathbb{P}$-generic filter G containing p, $((\mathbb{P}^{\varpi} \downarrow p) \times (\mathbb{S} \downarrow s))/G$ preserves stationary subsets of $(E^{\kappa}_{<\delta})^{V[G]}$.*

Proof. Let $p \in G$ for which $s := \varpi(p)$ strictly extends $\mathbb{1}_{\mathbb{S}}$. Let G be $\mathbb{P}$-generic with $p \in G$. In $V[G]$, let T be a stationary subset of $E^{\kappa}_{<\delta}$ as forced by p. Back in V, using the beginning of Clause (3) of Definition 2.10, fix a cardinal δ with $\tau^+ < \delta < \sigma$, a notion of forcing $\mathbb{Q}$ of size $< \delta$, and an isomorphism ι from $\mathbb{S} \downarrow s$ to $\mathbb{Q} \times \mathrm{Col}(\delta, {<}\sigma)$.

Let ϱ be the nice projection from $\mathbb{P} \downarrow p \to \mathrm{Col}(\delta, {<}\sigma)$ induced by ϖ and ι.

Claim 2.14.1.

(i) $(\mathbb{S}\downarrow s)\times(\mathbb{P}^{\varpi}\downarrow p)$ *projects onto* $\mathbb{Q}\times(\mathbb{P}^{\varpi_{\aleph}}\downarrow p)$ *which in turn projects onto* $\mathbb{P}\downarrow p$*;*

(ii) $(\mathbb{S}\downarrow s)\times(\mathbb{P}^{\varpi}\downarrow p)$ *projects onto* $\mathbb{Q}\times(\mathbb{P}^{\varpi_{\aleph}}\downarrow p)$ *which in turn projects onto* $\mathbb{P}^{\varpi_{\aleph}}\downarrow p$*;*

(iii) $(\mathbb{S}\downarrow s)\times(\mathbb{P}^{\varpi}\downarrow p)$ *projects onto* $\mathrm{Col}(\delta,{<}\sigma)\times(\mathbb{P}^{\varpi}\downarrow p)$ *which in turn projects onto* $\mathbb{P}^{\varpi_{\aleph}}\downarrow p$*.*

Proof. (i) For the first part, the map $(s',p')\mapsto(\iota_0(s'),p'+\iota_1(s'))$ is such a projection, where $+$ operation is computed with respect to the nice projection ϱ. For the second part, the map $(q',p')\mapsto p'+q'$ gives such a projection, where the $+$ operation is computed with respect to $\varpi_{\aleph}$.

(ii) For the second part, the map $(q',p')\mapsto p'$ is such a projection.

(iii) For the first part, the map $(s',p')\mapsto(\iota_1(s'),p')$ is such a projection. For the second part, the map $(c',p')\mapsto p'+c'$ is such a projection, where the $+$ operation is with respect to the nice projection ϱ. $\square$

By Definition 2.10(3a), in the forcing extensions with any of the three posets mentioned in Clause (i) of the preceding claim, κ is a cardinal which is the double successor of σ. But then, since $|\mathbb{Q}|<\kappa$, it follows from the second part of Clause (ii) that κ is the double successor of σ in forcing extensions by $\mathbb{P}^{\varpi_{\aleph}}\downarrow p$. Actually, in any forcing extension by $\mathbb{P}^{\varpi_{\aleph}}$.[7] Altogether, in all forcing extensions with posets from the preceding subclaim, κ is the double successor of σ.

Let $G_q\times G^*$ be $(\mathbb{Q}\times(\mathbb{P}^{\varpi_{\aleph}}\downarrow p))/G$-generic over $V[G]$. By Lemma 2.13, T remains stationary in $V[G_q\times G^*]$. As $\mathbb{Q}$ is small, we may fix $T'\subseteq T$ such that T' is in $V[G^*]$ and moreover stationary in $V[G^*]$. As established earlier, $V[G^*]\models$ "$T'\subseteq E^{\sigma^{++}}_{<\sigma^+}\in I[\sigma^{++}]$". By our assumption in Clause (3) of Definition 2.10, the quotient $(\mathrm{Col}(\delta,{<}\sigma)\times(\mathbb{P}^{\varpi}\downarrow p))/G^*$ contains a δ-closed dense set. So, again it follows that $(\mathrm{Col}(\delta,{<}\sigma)\times(\mathbb{P}^{\varpi}\downarrow p))/G^*$ preserves the stationarity of T'. Finally, since $\mathbb{S}\downarrow s\cong\mathbb{Q}\times\mathrm{Col}(\delta,{<}\sigma)$, the quotient

$$((\mathbb{S}\downarrow s)\times(\mathbb{P}^{\varpi}\downarrow p))/(\mathrm{Col}(\delta,{<}\sigma)\times(\mathbb{P}^{\varpi}\downarrow p))$$

is isomorphic to $\mathbb{Q}$, which is a small of forcing. Altogether, T' (and hence also T) remains stationary in the generic extension by $((\mathbb{P}^{\varpi}\downarrow p)\times(\mathbb{S}\downarrow s))/G$. $\square$

We are now in conditions to prove the main lemma of the section:

Proof of Lemma 2.12. Suppose σ is a supercompact cardinal indestructible under forcing with $\mathbb{P}^{\varpi}$. By Definitions 2.10(3a) and 2.2(4),it suffices to prove that

$$V^{\mathbb{P}}\models\mathrm{Refl}(E^{\kappa}_{\leq\tau},E^{\kappa}_{<\sigma^+}).$$

Let $p\in P$ be a condition such that $s:=\varpi(p)$ is not $\mathbb{1}_{\mathbb{S}}$ and that $\mathbb{P}$-forces that $\dot{T}$ is a stationary subset of $\dot{E}^{\kappa}_{\leq\tau}$. Let G a $\mathbb{P}$-generic filter with $p\in G$ and H be the generic filter for $\mathbb{S}$ induced by ϖ and G. Let G^* be such that

[7] Note that Subclaim 2.14.1 remains valid even if we replace p by any $p'\leq p$. For instance, regarding Clause (i), we will then have that $(\mathbb{S}\downarrow\varpi(p'))\times(\mathbb{P}^{\varpi_{\aleph}}\downarrow p')$ projects onto $(\mathbb{Q}\downarrow\varpi_{\aleph}(p'))\times(\mathbb{P}^{\varpi_{\aleph}}\downarrow p')$ and that this latter projects onto $\mathbb{P}\downarrow p'$.

the product $H \times G^*$ is generic for $(\mathbb{S}^* \times \mathbb{P}^*)/G$. By Lemma 2.13, T is still stationary in $V[G^*][H]$. Also, by Clause (3a) of suitability for reflection,

$$T \subseteq (E^{\sigma^{++}}_{\leq\tau})^{V[G^*][H]}.$$

Using that σ is a supercompact indestructible under $\mathbb{P}^{\varpi}$, let (in $V[G^*]$)

$$j : V[G^*] \to M$$

be a κ-supercompact embedding with $\mathrm{crit}(j) = \sigma$.

Work below the condition s that we fixed earlier. Recall that $\mathbb{S} \downarrow s \cong \mathbb{Q} \times \mathrm{Col}(\delta, {<}\sigma)$ for some poset $\mathbb{Q}$ of size $< \delta$ with $\tau^+ < \delta < \sigma$. So, H may be seen as a product of two corresponding generics, $H = H_0 \times H_1$. For the ease of notation, put $\mathbb{C} := \mathrm{Col}(\delta, {<}\sigma)$.

Since $\mathbb{Q}$ has size $< \delta < \mathrm{crit}(j)$, we can lift j to an embedding

$$j : V[G^*][H_0] \to M'.$$

Then we lift j again to get

$$j : V[G^*][H] \to N$$

in an outer generic extension of $V[G^*][H]$ by $j(\mathbb{C})/H_1$. Since $j(\mathbb{C})/H_1$ is δ-closed in $M'[H_1]$ and this latter model is closed under κ-sequences in $V[G^*][H]$, it follows that $j(\mathbb{C})/H_1$ is also δ-closed in $V[G^*][H]$.

Set $\gamma := \sup(j``\kappa)$. Clearly, $j(T) \cap \gamma = j``T$. Note that, by virtue of the collapse $j(\mathbb{C})$, $N \models$ "$|\kappa| = \delta$ & $\mathrm{cf}(\gamma) \leq \mathrm{cf}(|\kappa|) = \delta < j(\sigma)$".

Once again, [She91, Lemma 4.4] and Definition 2.10(3a) together yield

$$T \subseteq (E^{\sigma^{++}}_{\leq\tau})^{V[G^*][H]} \subseteq (E^{\sigma^{++}}_{<\sigma^+})^{V[G^*][H]} \in I[\sigma^{++}]^{V[G^*][H]}.$$

Then, Shelah's theorem from [She79] along with the δ-closedness of $j(\mathbb{C})/H_1$ in $V[G^*][H]$ imply that this latter forcing preserves the stationarity of T. Now, a standard argument shows that $j(T) \cap \gamma$ is stationary in N. Thus,

$$N \models \exists \alpha \in E^{j(\kappa)}_{<j(\sigma)}(j(T) \cap \alpha \text{ is stationary in } \alpha).$$

So, by elementarity, in $V[G^*][H]$, T reflects at a point of cofinality $< \sigma^+$.[8] Since reflection is downwards absolute, it follows that T reflects at a point of cofinality $< \sigma^+$ in $V[G]$. Finally, let $q \leq p$ be a condition forcing this, noting that this completes the proof. $\square$

3. $(\Sigma, \vec{\mathbb{S}})$-Prikry forcings

We commence by recalling a few concepts from [PRS22, §2].

Definition 3.1. A *graded poset* is a pair $(\mathbb{P}, \ell)$ such that $\mathbb{P} = (P, \leq)$ is a poset, $\ell : P \to \omega$ is a surjection, and, for all $p \in P$:

- For every $q \leq p$, $\ell(q) \geq \ell(p)$;
- There exists $q \leq p$ with $\ell(q) = \ell(p) + 1$.

[8] Actually, at a point of cofinality $< \sigma$.

Convention 3.2. For a graded poset as above, we denote $P_n := \{p \in P \mid \ell(p) = n\}$ and $\mathbb{P}_n := (P_n \cup \{\mathbb{1}\}, \leq)$. In turn, $\mathbb{P}_{\geq n}$ and $\mathbb{P}_{>n}$ are defined analogously. We also write $P^p_n := \{q \in P \mid q \leq p, \ell(q) = \ell(p) + n\}$, and sometimes write $q \leq^n p$ (and say that q is *an n-step extension* of p) rather than writing $q \in P^p_n$.

A subset $U \subseteq P$ is said to be 0*-open set* iff, for all $r \in U$, $P^r_0 \subseteq U$.

Now, we define the $(\Sigma, \vec{\mathbb{S}})$-Prikry class, a class broader than Σ-Prikry from [PRS22, Definition 2.3].

Definition 3.3. Suppose:

(α) $\Sigma = \langle \sigma_n \mid n < \omega\rangle$ is a non-decreasing sequence of regular uncountable cardinals, converging to some cardinal κ;
(β) $\vec{\mathbb{S}} = \langle \mathbb{S}_n \mid n < \omega\rangle$ is a sequence of notions of forcing, $\mathbb{S}_n = (S_n, \preceq_n)$, with $|S_n| < \sigma_n$;
(γ) $\mathbb{P} = (P, \leq)$ is a notion of forcing with a greatest element $\mathbb{1}$;
(δ) μ is a cardinal such that $\mathbb{1} \Vdash_{\mathbb{P}} \check{\mu} = \check{\kappa}^+$;
(ε) $\ell : P \to \omega$ and $c : P \to \mu$ are functions;[9]
(ζ) $\vec{\varpi} = \langle \varpi_n \mid n < \omega\rangle$ is a sequence of functions.

We say that $(\mathbb{P}, \ell, c, \vec{\varpi})$ is $(\Sigma, \vec{\mathbb{S}})$-*Prikry* iff all of the following hold:

(1) $(\mathbb{P}, \ell)$ is a graded poset;
(2) For all $n < \omega$, $\mathbb{P}_n := (P_n \cup \{\mathbb{1}\}, \leq)$ contains a dense subposet $\mathring{\mathbb{P}}_n$ which is countably-closed;[10]
(3) For all $p, q \in P$, if $c(p) = c(q)$, then $P^p_0 \cap P^q_0$ is non-empty;
(4) For all $p \in P$, $n, m < \omega$ and $q \leq^{n+m} p$, the set $\{r \leq^n p \mid q \leq^m r\}$ contains a greatest element which we denote by $m(p, q)$. In the special case $m = 0$, we shall write $w(p, q)$ rather than $0(p, q)$;[11]
(5) For all $p \in P$, the set $W(p) := \{w(p, q) \mid q \leq p\}$ has size $< \mu$;
(6) For all $p' \leq p$ in P, $q \mapsto w(p, q)$ forms an order-preserving map from $W(p')$ to $W(p)$;
(7) Suppose that $U \subseteq P$ is a 0-open set. Then, for all $p \in P$ and $n < \omega$, there is $q \leq^0 p$, such that, either $P^q_n \cap U = \emptyset$ or $P^q_n \subseteq U$;
(8) For all $n < \omega$, ϖ_n is a nice projection from $\mathbb{P}_{\geq n}$ to $\mathbb{S}_n$, such that, for any integer $k \geq n$, $\varpi_n \restriction \mathbb{P}_k$ is again a nice projection;
(9) For all $n < \omega$, if $\mathring{\mathbb{P}}_n$ is a witness for Clause (2) then $\mathring{\mathbb{P}}^{\varpi_n}_n$ is a dense and σ_n-directed-closed subposet of $\mathbb{P}^{\varpi_n}_n := (P_n \cup \{\mathbb{1}\}, \leq^{\varpi_n})$.[12]

Convention 3.4. We derive yet another ordering $\leq^{\vec{\varpi}}$ of the set P, letting $\leq^{\vec{\varpi}} := \bigcup_{n<\omega} \leq^{\varpi_n}$. Simply put, this means that $q \leq^{\vec{\varpi}} p$ iff $(p = \mathbb{1})$, or, $(q \leq^0 p$, $\ell(p) = \ell(q)$ and $\varpi_{\ell(p)}(p) = \varpi_{\ell(q)}(q))$.

---

[9] In some applications c will be a function from P to some canonical structure of size μ, such as H_μ (assuming $\mu^{<\mu} = \mu$).

[10] Recall that P_n stands for $\{p \in P \mid \ell(p) = n\}$.

[11] Note that $w(p, q)$ is the weakest extension of p above q.

[12] More verbosely, for every $p \in P_n$ there is $q \in \mathring{P}_n$ such that $q \leq^{\varpi_n} p$ (see Notation 2.1).

Convention 3.5. We say that $(\mathbb{P}, \ell, c)$ has the Linked$_0$-property if it witnesses Clause (3) above. Similarly, we will say that $(\mathbb{P}, \ell)$ has the Complete Prikry Property (CPP) if it witnesses Clause (7) above.

Any Σ-Prikry triple $(\mathbb{P}, \ell, c)$ can be regarded as a $(\Sigma, \vec{\mathbb{S}})$-Prikry forcing $(\mathbb{P}, \ell, c, \vec{\varpi})$ by letting $\vec{\mathbb{S}} := \langle (n, \{\mathbb{1}_{\mathbb{P}}\}) \mid n < \omega\rangle$ and $\vec{\varpi}$ be the sequence of trivial projections $p \mapsto \mathbb{1}_{\mathbb{P}}$. Conversely, any $(\Sigma, \vec{\mathbb{S}})$-Prikry quadruple $(\mathbb{P}, \ell, c, \vec{\varpi})$ with $\vec{\mathbb{S}}$ and $\vec{\varpi}$ as above witnesses that $(\mathbb{P}, \ell, c)$ is Σ-Prikry. In particular, all the forcings from [PRS21, §3] are examples of $(\Sigma, \vec{\mathbb{S}})$-Prikry forcings. In Section 4, we will add a new example to this list by showing that Gitik's EPBFC (The long Extender-Based Prikry forcing with Collapses [Git19b]) falls into the $(\Sigma, \vec{\mathbb{S}})$-Prikry framework. For the moment, and to help the reader digest the previous definition, let us exhibit the easiest (yet non-trivial) example of a $(\Sigma, \vec{\mathbb{S}})$-Prikry forcing - *the Prikry forcing with interleaved collapses.*

Example 3.6. Suppose κ is a measurable cardinal with $2^\kappa = \kappa^+$. Fix $\mathcal{U}$ a normal measure on κ and let $j\colon V \to M$ be the ultrapower embedding by it. Arguing as in [Cum10, Lemma 8.5], one can construct a *guiding generic* for $K \subseteq \mathrm{Col}(\kappa^{++}, {<}j(\kappa))^M$; namely, a set $K \in V$ which is M-generic for the previously-mentioned Lévy collapse. Using $\mathcal{U}$ and K one can define $\mathbb{P}$, the so-called *Prikry forcing with interleaved collapses.* Condition in $\mathbb{P}$ are vectors p of the form $\langle c^p_0, \alpha^p_1, c^p_1, \ldots, \alpha^p_{n_p}, c^p_{n_p}, A^p, H^p\rangle$ where:

- $\langle \alpha^p_i \mid i < n\rangle$ is an increasing sequence of inaccessibles $<\kappa$;
- $c^p_0 \in \mathrm{Col}(\omega_2, {<}\alpha^p_1)$ and $c^p_i \in \mathrm{Col}((\alpha^p_{i-1})^{++}, {<}\alpha^p_i)$;
- $A^p \in \mathcal{U}$ consists of inaccessible cardinals above α^p_n;
- H^p is a function with $\mathrm{dom}(H^p) = A^p$ such that $j(H^p)(\kappa) \in K$.

This poset fits naturally into the $(\Sigma, \vec{\mathbb{S}})$-Prikry framework. Let $\Sigma := \langle \kappa \mid n < \omega\rangle$, $\ell(p) := n_p$ and $c : P \to H(\kappa^+)$ given by $p \mapsto \langle c^p_0, \alpha^p_1, c^p_1, \ldots, \alpha^p_{n_p}, c^p_{n_p}\rangle$. Regarding the $\vec{\mathbb{S}}$-part, for each $n < \omega$, set $\mathbb{S}_n := \{(c^p_0, \ldots, c^p_n) \mid p \in P_n\}$ and let $\varpi_n : P_{\geq n} \to S_n$ be the map *extracting the collapses*; more formally, $p \mapsto \langle c^p_0, \ldots, c^p_n\rangle$. It can be verified that $(\mathbb{P}, \ell, c, \vec{\mathbb{S}}, \vec{\varpi})$ is a $(\Sigma, \vec{\mathbb{S}})$-Prikry quadruple. For instance, the use of the guiding generic K ensures that the putative *compatibility map* c in effect works.[13] Also, the dense subposet $\mathring{\mathbb{P}}_n$ of $\mathbb{P}_n$ mentioned in Clause (2) can simply be taken to be $\mathbb{P}_n$ because $\mathbb{P}^{\varpi_n}_n$ (the subposet of $\mathbb{P}_n$ where we *freeze the collapses*) is κ-closed.

Throughout the rest of the section, assume that $(\mathbb{P}, \ell, c, \vec{\varpi})$ is a $(\Sigma, \vec{\mathbb{S}})$-Prikry quadruple. We shall spell out some basic features of the components of the quadruple, and work towards proving Lemma 3.15 that explains how bounded sets of κ are added to generic extensions by $\mathbb{P}$.

Lemma 3.7 (The p-tree)**.** *Let $p \in P$.*

[13]It is unclear to the authors whether there is a compatibility map $c : \mathbb{P} \to H(\kappa^+)$ for the non-guiding-generic version of the forcing. In contrast, in §4 we show that Gitik's EBPF with collapses is $(\Sigma, \vec{\mathbb{S}})$-Prikry, in spite that the poset does not use guiding generics.

(1) *For every $n < \omega$, $W_n(p)$ is a maximal antichain in $\mathbb{P} \downarrow p$;*
(2) *Every two compatible elements of $W(p)$ are comparable;*
(3) *For any pair $q' \leq q$ in $W(p)$, $q' \in W(q)$;*
(4) *$c \restriction W(p)$ is injective.*

Proof. The proof of [PRS21, Lemma 2.8] goes through. $\square$

We commence by introducing the notion of coherent sequence of nice projections, which will be important in Section 6.

Definition 3.8. The sequence of nice projections $\vec{\varpi}$ is called *coherent* iff the two hold:

(1) for all $n < \omega$, if $p \in P_{\geq n}$ then ϖ_n"$W(p) = \{\varpi_n(p)\}$;
(2) for all $n \leq m < \omega$, ϖ_m factors through ϖ_n; i.e., there is a map $\pi_{m,n} \colon \mathbb{S}_m \to \mathbb{S}_n$ such that $\varpi_n(p) = \pi_{m,n}(\varpi_m(p))$ for all $p \in P_{\geq m}$.[14]

Lemma 3.9. *Let $p \in P$. Then for each $q \in W(p)$, $n \leq \ell(q)$ and $t \preceq_n \varpi_n(q)$,*

$$w(p, q+t) = w(p, q).$$

Proof. Note that $w(p, q+t)$ and $w(p,q)$ are two compatible conditions in $W(p)$ with the same length. In effect, Lemma 3.7(1) yields the desired. $\square$

Lemma 3.10. *Assume that $\vec{\varpi}$ is coherent.*

For all $n < \omega$, $p \in P_{\geq n}$ and $t \preceq_n \varpi_n(p)$, the following hold:

(1) *for each $q \in W(p+t)$, $q = w(p,q) + t$;*
(2) *for each $q \in W(p)$, $w(p+t, q+t) = q+t$;*
(3) *for each $m < \omega$, $W_m(p+t) = \{q + t \mid q \in W_m(p)\}$.*
(4) *$p + t = p + \varpi_{\ell(p)}(p+t)$;*

Proof. (1) Let $q \in W(p+t)$. By virtue of Definition 3.8(1), we have $\varpi_n(q) = \varpi_n(p+t) = t$. This, together with $q \leq^0 w(p,q)$, implies that $w(p,q)+t$ is well-defined and also that $q \leq^0 w(p,q)+t$. On the other hand, $q \leq^0 w(p,q) + t \leq p + t$, hence $w(p+t, w(p,q)+t)$ and q are two compatible conditions in $W(p+t)$ that have the same length. By Lemma 3.7(1) it follows that $q = w(p+t, w(p,q)+t)$, hence $w(p,q)+t \leq^0 q$, as desired.

(2) By Definition 3.8(1), $q \leq^{\varpi_n} p$, hence $q+t$ is well-defined and so $w(p+t, q+t)$ belongs to $W(p+t)$. Combining Clause (1) above with [PRS21, Lemma 2.9] we obtain the following chain of equalities:

$$w(p+t, q+t) = w(p, w(p+t, q+t)) + t = w(p, q+t) + t.$$

Now, combine Lemma 3.9 with $q \in W(p)$ to infer that $w(p, q+t) = q$. Altogether, this shows that $w(p+t, q+t) = q+t$.

(3) The left-to-right inclusion is given by (1) and the converse by (2).

(4) Note that $p + t \leq p + \varpi_{\ell(p)}(p+t)$. Conversely, by using Clause (2) of Definition 3.8 we have that $\varpi_n(p + \varpi_{\ell(p)}(p+t)) = \varpi_n(p+t) = t$. $\square$

[14]In most of the applications of the framework there is a natural $\pi_{m,n}$ (*the restriction*) which serves as a projection from $\mathbb{S}_m$ to $\mathbb{S}_n$. We refrain to make this extra assumption upon $\pi_{m,n}$ to avoid potential complications in the future arguments.

Proposition 3.11. *For every condition p in $\mathbb{P}$ and an ordinal $\alpha<\kappa$, there exists an extension $p'\le p$ such that $\sigma_{\ell(p')}>\alpha$.*

Proof. Let p and α be as above. Since $\alpha<\kappa=\sup_{n<\omega}\sigma_n$, we may find some $n<\omega$ such that $\alpha<\sigma_n$. By Definition 3.3(1), $(\mathbb{P},\ell)$ is a graded poset, so by possibly iterating the second bullet of Definition 3.1 finitely many times, we may find an extension $p'\le p$ such that $\ell(p')\ge n$. As Σ is non-decreasing, p' is as desired. $\square$

As in the context of Σ-Prikry forcings, also here, the CPP implies the Prikry Property (PP) and the Strong Prikry Property (SPP).

Lemma 3.12. *Let $p\in P$.*

(1) Suppose φ is a sentence in the language of forcing. Then there is $p'\le^0 p$, such that p' decides φ;

(2) Suppose $D\subseteq P$ is a 0-open set which is dense below p. Then there is $p'\le^0 p$, and $n<\omega$, such that $P^{p'}_n\subseteq D$.[15]

Moreover, we can let p' above to be a condition from $\mathring{\mathbb{P}}^{\varpi_{\ell(p)}}_{\ell(p)}\downarrow p$.

Proof. We only give the proof of (1), the proof of (2) is similar. Fix φ and p. Put $U^+_\varphi:=\{q\in P\mid q\Vdash_{\mathbb{P}}\varphi\}$ and $U^-_\varphi:=\{q\in P\mid q\Vdash_{\mathbb{P}}\neg\varphi\}$. Both of these are 0-open, so applying Clause (7) of Definition 3.3 twice, we get following:

Claim 3.12.1. *For all $q\in P$ and $n<\omega$, there is $q'\le^0 q$, such that either all $r\in P^{q'}_n$ decide φ the same way, or no $r\in P^{q'}_n$ decides φ.*

Now using the claim construct a $\le^0$ decreasing sequence $\langle p_n\mid n<\omega\rangle$ below p. By using Clause (2) of Definition 3.3 we may additionally assume that these are conditions in $\mathring{\mathbb{P}}_{\ell(p)}$. Letting p' be a $\le^0$-lower bound for this sequence we obtain $\le^0$-extension of p deciding φ. $\square$

Corollary 3.13. *Let $p\in P$ and $s\preceq_{\ell(p)}\varpi_{\ell(p)}(p)$.*

(1) Suppose φ is a sentence in the language of forcing. Then there is $p'\le^{\vec{\varpi}} p$ and $s'\preceq_{\ell(p)} s$ such that $p'+s'$ decides φ;

(2) Suppose $D\subseteq P$ is a 0-open set which is dense below p. Then there are $p'\le^{\vec{\varpi}} p$, $s'\preceq_{\ell(p)} s$ and $n<\omega$ such that $P^{p'+s'}_n\subseteq D$.

Moreover, we can let p' above to be a condition from $\mathring{\mathbb{P}}^{\varpi_{\ell(p)}}_{\ell(p)}\downarrow p$.

Proof. We only show (1) as (2) is similar. By Lemma 3.12, let $q\le^0 p+s$ deciding φ. By Definition 3.3(8) the map ϖ_n is a nice projection, hence there is $p'\le^{\vec{\varpi}} p$ and $s'\preceq_{\ell(p)} s$ such that $p'+s'=q$ (Definition 2.2(4)). The moreover part follows from density of $\mathring{\mathbb{P}}^{\varpi_{\ell(p)}}_{\ell(p)}$ in $\mathbb{P}^{\varpi_{\ell(p)}}_{\ell(p)}$ (Definition 3.3(9)). $\square$

Working a bit more we can obtain the following:

Lemma 3.14. *Let $p\in P$. Set $\ell:=\ell(p)$ and $s:=\varpi_n(p)$.*

[15] Note that if D is moreover open, then $P^q_m\subseteq D$ for all $m\ge n$.

(1) *Suppose φ is a sentence in the language of forcing. Then there is $q \leq^{\vec{\varpi}} p$ such that $D_{\varphi,q} := \{t \preceq_\ell s \mid (q+t \Vdash_{\mathbb{P}} \varphi)$ or $(q+t \Vdash_{\mathbb{P}} \neg\varphi)\}$ is dense in $\mathbb{S}_\ell \downarrow s$;*
(2) *Suppose $D \subseteq P$ is a 0-open set. Then there is $q \leq^{\vec{\varpi}} p$ such that $U_{D,q} := \{t \preceq_\ell s \mid \forall m < \omega \ (P^{q+t}_m \subseteq D$ or $P^{q+t}_m \cap D = \emptyset)\}$ is dense in $\mathbb{S}_\ell \downarrow s$.*
(3) *Suppose $D \subseteq P$ is a 0-open set which is dense below p. Then there is $q \leq^{\vec{\varpi}} p$ such that $U_{D,q} := \{t \preceq_\ell s \mid \exists m < \omega \ P^{q+t}_m \subseteq D\}$ is dense in $\mathbb{S}_\ell \downarrow s$.*

Moreover, q above belongs to $\mathring{\mathbb{P}}^{\varpi_\ell}_\ell \downarrow p$.

Proof. (1) By Definition 3.3(β), let us fix some cardinal $\theta < \sigma_\ell$ along with an injective enumeration $\langle s_\alpha \mid \alpha < \theta\rangle$ of the conditions in $\mathbb{S}_\ell \downarrow s$, such that $s_0 = s$. We will construct by recursion two sequences of conditions $\vec{p} = \langle p^\alpha \mid \alpha < \theta\rangle$ and $\vec{s} = \langle s^\alpha \mid \alpha < \theta\rangle$ for which all of the following hold:

(a) $\vec{p}$ is a $\leq^{\vec{\varpi}}$-decreasing sequence of conditions in $\mathring{\mathbb{P}}^{\varpi_\ell}_\ell$ below p;
(b) $\vec{s}$ is a sequence of conditions below s;
(c) for each $\alpha < \theta$, $s^\alpha \preceq_n s_\alpha$ and $p^\alpha + s^\alpha \parallel_{\mathbb{P}} \varphi$.

To see that this will do, assume for a moment that there are sequences $\vec{p}$ and $\vec{s}$ as above. Since $\theta < \sigma_\ell$, we may find a $\leq^{\vec{\varpi}}$-lower bound q for $\vec{p}$ in $\mathring{\mathbb{P}}^{\varpi_\ell}_\ell$. In particular, $q \leq^{\vec{\varpi}} p$. We claim that $D_{\varphi,q}$ is dense in $\mathbb{S}_\ell \downarrow s$. To this end, let $s' \preceq_\ell s$ be arbitrary. Find $\alpha < \theta$ such that $s' = s_\alpha$. By the hypothesis, $s^\alpha \preceq_\ell s_\alpha$ and $p^\alpha + s^\alpha$ decides φ, hence $q + s^\alpha$ also decides it. In particular, s^α is an extension of s' belonging to $D_{\varphi,q}$.

Claim 3.14.1. *There are sequences $\vec{p}$ and $\vec{s}$ as above.*

Proof. We construct the two sequences by recursion on $\alpha < \theta$. For the base case, appeal to Corollary 3.13(1) with p and s, and retrieve $p^0 \leq^{\vec{\varpi}} p$ and $s^0 \preceq_n s$ such that $p_0 \in \mathring{P}^{\varpi_\ell}_\ell$ and $p^0 + s^0$ indeed decides φ.

► Assume $\alpha = \beta + 1$ and that $\langle p^\gamma \mid \gamma \leq \beta\rangle$ and $\langle s^\gamma \mid \gamma \leq \beta\rangle$ have been already defined. Since $s_\alpha \preceq_\ell s = \varpi_\ell(p^\beta)$, it follows that $p^\beta + s_\alpha$ is a legitimate condition in P_ℓ. Appealing to Corollary 3.13(1) with p^β and s_α, let $p^\alpha \leq^{\vec{\varpi}} p^\beta$ and $s^\alpha \preceq_\ell s_\alpha$ be such that $p^\alpha \in \mathring{P}^{\varpi_\ell}_\ell$ and $p^\alpha + s^\alpha$ decides φ.

► Assume $\alpha \in \operatorname{acc}(\theta)$ and that the sequences $\langle p^\beta \mid \beta < \alpha\rangle$ and $\langle s^\beta \mid \beta < \alpha\rangle$ have already been defined. Appealing to Definition 3.3(9), let p^* be a $\leq^{\vec{\varpi}}$-lower bound for $\langle p^\beta \mid \beta < \alpha\rangle$. Finally, obtain $p^\alpha \in D$ and s^α by appealing to Corollary 3.13(1) with respect to p^* and s_α. □

This completes the proof of Clause (1). The proof of Clauses (2) and (3) is similar by amending suitably Clause (c) above. For instance, for Clause (2) we require the following in Clause (c): for each $\alpha < \theta$ and $n < \omega$, $s^\alpha \preceq_n s_\alpha$ and either $P^{p^\alpha + s^\alpha}_n \subseteq D$ or $P^{p^\alpha + s^\alpha}_n \cap D = \emptyset$. For the verification of this new requirement we combine Clauses (2), (7) and (8) of Definition 3.3 with Definition 2.2(4). Similarly, to prove Clause (3) of the lemma one uses Clause (2) of Corollary 3.13. □

We now arrive at the main result of the section:

Lemma 3.15 (Analysis of bounded sets)**.**

(1) If $p \in P$ forces that $\dot{a}$ is a $\mathbb{P}$-name for a bounded subset a of $\sigma_{\ell(p)}$, then a is added by $\mathbb{S}_{\ell(p)}$.[16] *In particular, if $\dot{a}$ is a $\mathbb{P}$-name for a bounded subset a of κ, then, for any large enough $n < \omega$, a is added by $\mathbb{S}_n$;*

(2) $\mathbb{P}$ preserves κ. Moreover, if κ is a strong limit, it remains so;

(3) For every regular cardinal $\nu \geq \kappa$, if there exists $p \in P$ for which $p \Vdash_{\mathbb{P}} \mathrm{cf}(\nu) < \kappa$, then there exists $q \leq^{\vec{\varpi}} p$ with $|W(q)| \geq \nu$;[17]

(4) Suppose $\mathbb{1} \Vdash_{\mathbb{P}}$ "κ is singular". Then $\mu = \kappa^+$ if and only if, for all $p \in P$, $|W(p)| \leq \kappa$.

Proof. (1) The "in particular" part follows from the first part together with Proposition 3.11. Thus, let us suppose that p is a given condition forcing that $\dot{a}$ is a name for a subset a of some cardinal $\theta < \sigma_{\ell(p)}$.

For each $\alpha < \theta$, denote the sentence "$\check{\alpha} \in \dot{a}$" by φ_α. Set $n := \ell(p)$ and $s := \varpi_n(p)$. Combining Definition 3.3(9) with Lemma 3.14(1), we may recursively obtain a $\leq^{\varpi_n}$-decreasing sequence of conditions $\vec{p} = \langle p^\alpha \mid \alpha < \theta\rangle$ with a lower bound, such that, for each $\alpha < \theta$, $p^\alpha \leq^{\varpi_n} p$ and $D_{\varphi_\alpha, p^\alpha}$ is dense in $\mathbb{S}_n \downarrow s$. Then let $q \in P_n$ be $\leq^{\varpi_n}$-below all elements of $\vec{p}$. It follows that for every $\alpha < \theta$,

$$D_{\varphi_\alpha, q} = \{t \preceq_n s \mid (q + t \Vdash_{\mathbb{P}} \varphi_\alpha) \text{ or } (q + t \Vdash_{\mathbb{P}} \neg\varphi_\alpha)\}$$

is dense in $\mathbb{S}_n \downarrow s$.

Now, let G be a $\mathbb{P}$-generic filter with $q \in G$. Let H_n be the $\mathbb{S}_n$-generic filter induced by ϖ_n from G, and work in $V[H_n]$. It follows that, for every $\alpha < \theta$, for some $t \in H_n$, either $(q + t \Vdash_{\mathbb{P}} \check{\alpha} \in \dot{a})$ or $(q + t \Vdash_{\mathbb{P}} \check{\alpha} \notin \dot{a})$. Set

$$b := \{\alpha < \theta \mid \exists t \in H_n [q + t \Vdash_{\mathbb{P}} \check{\alpha} \in \dot{a}]\}.$$

As $q \leq^{\vec{\varpi}} p$, we infer that $\varpi_n(q) = \varpi_n(p) = s \in H_n$, so that $q \in P/H_n$.

Claim 3.15.1. $q \Vdash_{\mathbb{P}/H_n} b = \dot{a}_{H_n}$.

Proof. Clearly, $q \Vdash_{\mathbb{P}/H_n} b \subseteq \dot{a}_{H_n}$. For the converse, let $\alpha < \theta$ and $r \leq_{\mathbb{P}/H_n} q$ be such that $r \Vdash_{\mathbb{P}/H_n} \check{\alpha} \in \dot{a}_{H_n}$. By the very Definition 2.4, there is $t_0 \in H_n$ with $t_0 \preceq_n \varpi_n(r)$ such that $r + t_0 \leq q$. By extending t if necessary, we may moreover assume that $r + t_0 \Vdash_{\mathbb{P}} \check{\alpha} \in \dot{a}$. Set $q_0 := r + t_0$.

By the choice of q, there is $t_1 \in H_n$ such that $q + t_1 \parallel_{\mathbb{P}} \check{\alpha} \in \dot{a}$. Set $q_1 := q + t_1$. Let $t \in H_n$ be such that $t \preceq_n t_0, t_1$. Recalling Definition 3.3(9), ϖ_n is nice, so $t \preceq_n \varpi_n(q_0), \varpi_n(q_1)$. By Definition 2.2(4), $q_0 + t$ witnesses the compatibility of q_0 and q_1, hence $q + t_1 \Vdash_{\mathbb{P}} \check{\alpha} \in \dot{a}$, and thus $\alpha \in b$. $\square$

[16] More explicitly, there is $q \leq^{\varpi_{\ell(p)}} p$ forcing (in $\mathbb{P}/\dot{\mathbb{S}}_{\ell(p)}$) $\dot{a}$ to equal a $\mathbb{S}_{\ell(p)}$-name $\dot{b}$.

[17] For future reference, we point out that this fact relies only on clauses (1), (5), (7), (8) and (9) of Definition 3.3.

Altogether, $\dot{a}_G \in V[H_n]$.

(2) If κ were to be collapsed, then, by Clause (1), it would have been collapsed by $\mathbb{S}_n$ for some $n < \omega$. However, $\mathbb{S}_n$ is a notion of forcing of size $< \sigma_n \leq \kappa$.

Next, suppose towards a contradiction that κ is strong limit cardinal, and yet, for some $\mathbb{P}$-generic filter G, for some $\theta < \kappa$, $V[G] \models 2^\theta \geq \kappa$. For each $n < \omega$, let H_n be the $\mathbb{S}_n$-generic filter induced by ϖ_n from G. Using Clause (1), for every $a \in \mathcal{P}^{V[G]}(\theta)$, we fix $n_a < \omega$ such that $a \in V[H_{n_a}]$.

► If κ is regular, then there must exist some $n < \omega$ for which $|\{a \in \mathcal{P}^{V[G]}(\theta) \mid n_a = n\}| \geq \kappa$. However $\mathbb{S}_n$ is a notion of forcing of some size $\lambda < \kappa$, and so by counting nice names, we see it cannot add more than θ^λ many subsets to θ, contradicting the fact that κ is strong limit.

► If κ is not regular, then Σ is not eventually constant, and $\mathrm{cf}(\kappa) = \omega$, so that, by König's lemma, $V[G] \models 2^\theta \geq \kappa^+$. It follows that exists some $n < \omega$ for which $|\{a \in \mathcal{P}^{V[G]}(\theta) \mid n_a = n\}| > \kappa$, leading to the same contradiction.[18]

(3) Suppose θ, ν are regular cardinals with $\theta < \kappa \leq \nu$, $\dot{f}$ is a $\mathbb{P}$-name for a function from θ to ν, and $p \in P$ is a condition forcing that the image of $\dot{f}$ is cofinal in ν. Denote $n := \ell(p)$ and $s := \varpi_n(p)$. By Proposition 3.11, we may extend p and assume that $\sigma_n > \theta$.

For all $\alpha < \theta$, set $D_\alpha := \{r \leq p \mid \exists \beta < \nu, r \Vdash_{\mathbb{P}} \dot{f}(\check{\alpha}) = \check{\beta}\}$. As D_α is 0-open and dense below p, by combining Lemma 3.14(3) with the σ_n-directed closure of $\mathring{\mathbb{P}}_n^{\varpi_n}$ (see Definition 3.3(9)), we may recursively define a $\leq^{\vec{\varpi}}$-decreasing sequence of conditions $\langle q^\alpha \mid \alpha \leq \theta \rangle$ below p such that, for every $\alpha < \theta$, U_{D_α, q^α} is dense in $\mathbb{S}_n \downarrow s$ (here U_{D_α, q_α} is as in Lemma 3.14(2)).

Set $q := q^\theta$, and note that

$$U_{D_\alpha, q} := \{t \preceq_n s \mid \exists m < \omega [P_m^{q+t} \subseteq D_\alpha]\}$$

is dense in $\mathbb{S}_n \downarrow s$ for all $\alpha < \theta$. In particular, the above sets are non-empty. For each $\alpha < \theta$, let us fix $t_\alpha \in U_{D_\alpha, q}$ and $m_\alpha < \omega$ witnessing this. We now show that $|W(q)| \geq \nu$. Let $A_\alpha := \{\beta < \nu \mid \exists r \in P_{m_\alpha}^{q+t_\alpha} [r \Vdash_{\mathbb{P}} \dot{f}(\check{\alpha}) = \check{\beta}]\}$. Since $W_{m_\alpha}(q + t_\alpha) \subseteq P_{m_\alpha}^{q+t_\alpha} \subseteq D_\alpha$, it is fairly easy to check that

$$A_\alpha = \{\beta < \nu \mid \exists r \in W_{m_\alpha}(q + t_\alpha) \, [r \Vdash_{\mathbb{P}} \dot{f}(\check{\alpha}) = \check{\beta}]\}.$$

Let $A := \bigcup_{\alpha < \theta} A_\alpha$. Then,

$$|A| \leq \sum_{m < \omega, t \preceq_n s} |W_m(q + t)| \leq \max\{\aleph_0, |S_n|\} \cdot |W(q)|.$$ [19]

---

[18] The referee pointed out that a stronger result should hold provided $\pi_{m,n} \colon \mathbb{S}_m \to \mathbb{S}_n$ is assumed to be a projection such that $\mathbb{S}_m^{\pi_{m,n}}$ is σ_n-closed, namely, for every cardinal $\lambda < \sigma_n$ every member of $\mathcal{P}(\lambda)$ is added by $\mathbb{S}_n$. The idea is to reduce any $\mathbb{P}$-name $\dot{a}$ for a subset of λ to a $\mathbb{S}_m$-name $\dot{b}$, for some suitable $m < \omega$. Next, using $\pi_{m,n}$ and the closure of $\mathbb{S}_m^{\pi_{m,n}}$ one can get that this name is evaluated the same way as an $\mathbb{S}_n$-name. This is the case in most applications, and is part of the analysis of the cardinal structure required for Gitik's theorem (Theorem 4.1).

[19] Observe that, for each $t \preceq_n s$, $|W(q + t)| \leq |W(q)|$.

Also, by clauses (α) and (β) of Definition 3.3 and our assumption on ν, $\max\{\aleph_0, |S_n|\} < \sigma_n < \nu$. It follows that if $|W(q)| < \nu$, then $|A| < \nu$, and so $\sup(A) < \nu$. Thus, q forces that the range of $\dot{f}$ is bounded below ν, which leads us to a contradiction. Therefore, $|W(q)| \geq \nu$, as desired.

(4) The left-to-right implication is obvious using Definition 3.3(5). Next, suppose that, for all $p \in P$, $|W(p)| \leq \kappa$. Towards a contradiction, suppose that there exist $p \in P$ forcing that κ^+ is collapsed. Denote $\nu := \kappa^+$. As by assumption $\mathbb{1} \Vdash_{\mathbb{P}}$ "κ is singular", this means that $p \Vdash_{\mathbb{P}} \mathrm{cf}(\nu) < \kappa$, contradicting Clause (3) of this lemma. □

We end this section recalling the concept of *property $\mathcal{D}$*. This notion was introduced in [PRS22, §2] and usually captures how various forcings satisfy the Complete Prirky Property (i.e., Clause (7) of Definition 3.3):

Definition 3.16. We say that $\vec{r} = \langle r_\xi \mid \xi < \chi\rangle$ is a *good enumeration* of a set A iff χ is a cardinal and $\vec{r}$ is a bijection from χ to A.

Definition 3.17 (Diagonalizability game)**.** Given $p \in P$, $n < \omega$, and a good enumeration $\vec{r} = \langle r_\xi \mid \xi < \chi\rangle$ of $W_n(p)$, we say that $\vec{q} = \langle q_\xi \mid \xi < \chi\rangle$ is *diagonalizable* (with respect to $\vec{r}$) iff the two hold:

(a) $q_\xi \leq^0 r_\xi$ for every $\xi < \chi$;
(b) there is $p' \leq^0 p$ such that for every $q' \in W_n(p')$, $q' \leq^0 q_\xi$, where ξ is the unique index to satisfy $r_\xi = w(p, q')$.

Besides, if D is a dense subset of $\mathbb{P}_{\ell_{\mathbb{P}}(p)+n}$, $\eth_{\mathbb{P}}(p, \vec{r}, D)$ is a game of length χ between two players **I** and **II**, defined as follows:

- At stage $\xi < \chi$, **I** plays a condition $p_\xi \leq^0 p$ compatible with r_ξ, and then **II** plays $q_\xi \in D$ such that $q_\xi \leq p_\xi$ and $q_\xi \leq^0 r_\xi$;
- **I** wins the game iff the resulting sequence $\vec{q} = \langle q_\xi \mid \xi < \chi\rangle$ is diagonalizable.

In the special case that D is all of $\mathbb{P}_{\ell_{\mathbb{P}}(p)+n}$, we omit it, writing $\eth_{\mathbb{P}}(p, \vec{r})$.

Definition 3.18 (Property $\mathcal{D}$)**.** We say that a graded poset $(\mathbb{P}, \ell_{\mathbb{P}})$ has *property $\mathcal{D}$* iff for any $p \in P$, $n < \omega$ and any good enumeration $\vec{r} = \langle r_\xi \mid \xi < \chi\rangle$ of $W_n(p)$, **I** has a winning strategy for the game $\eth_{\mathbb{P}}(p, \vec{r})$.

Convention 3.19. In a mild abuse of terminology, we shall say that $(\mathbb{P}, \ell, c, \vec{\varpi})$ has property $\mathcal{D}$ whenever the pair $(\mathbb{P}, \ell)$ has property $\mathcal{D}$.

4. Extender Based Prikry Forcing with collapses

In this section we present Gitik's notion of forcing from [Git19b], and analyze its properties. Gitik came up with this notion of forcing in September 2019, during the week of the *15th International Workshop on Set Theory in Luminy*, after being asked by the second author whether it is possible to interleave collapses in the Extender Based Prikry Forcing (EBPF) with long extenders [GM94, §3]. The following theorem summarizes the main properties of the generic extensions by Gitik's forcing $\mathbb{P}$:

Theorem 4.1 (Gitik). *Suppose the assumptions of the forthcoming Setup 4 hold. Then all of the following properties hold in $V^{\mathbb{P}}$:*

1. *All cardinals $\geq \kappa$ are preserved;*
2. *$\kappa = \aleph_\omega$, $\mu = \aleph_{\omega+1}$ and $\lambda = \aleph_{\omega+2}$;*
3. *$\aleph_\omega$ is a strong limit cardinal;*
4. *$\mathsf{GCH}_{<\aleph_\omega}$, provided that $V \models \mathsf{GCH}_{<\kappa}$;*
5. *$2^{\aleph_\omega} = \aleph_{\omega+2}$, hence the $\mathsf{SCH}_{\aleph_\omega}$ fails.*

For people familiar with [Git19b], some of the proofs in this section can be skipped. Yet, since this forcing notion is fairly new, we do include some proofs. Most notably, for us it is important to verify the existence of various nice projections and reflections properties in Corollary 4.32 and Lemma 4.34 below. In addition, unlike the exposition of this forcing from [Git19b], the exposition here shall not assume the GCH.

Setup 4. Throughout this section our setup will be as follows:

- $\vec{\kappa} = \langle \kappa_n \mid n < \omega \rangle$ is a strictly increasing sequence of cardinals;
- $\kappa_{-1} := \aleph_0$, $\kappa := \sup_{n<\omega} \kappa_n$, $\mu := \kappa^+$ and $\lambda := \mu^+$;
- $\mu^{<\mu} = \mu$ and $\lambda^{<\lambda} = \lambda$;
- for each $n < \omega$, κ_n is $(\lambda + 1)$-strong;
- $\Sigma := \langle \sigma_n \mid n < \omega \rangle$, where, for each $n < \omega$, $\sigma_n := (\kappa_{n-1})^+$;[20]

In particular, we are assuming that, for each $n < \omega$, there is a $(\kappa_n, \lambda+1)$-extender E_n whose associated embedding $j_n : V \to M_n$ is such that M_n is a transitive class, ${}^{\kappa_n}M_n \subseteq M_n$, $V_{\lambda+1} \subseteq M_n$ and $j_n(\kappa_n) > \lambda$.

For each $n < \omega$, and each $\alpha < \lambda$, set

$$E_{n,\alpha} := \{X \subseteq \kappa_n \mid \alpha \in j_n(X)\}.$$

Note that $E_{n,\alpha}$ is a non-principal κ_n-complete ultrafilter over κ_n, provided that $\alpha \geq \kappa_n$. Moreover, in the particular case of $\alpha = \kappa_n$, E_{n,κ_n} is also normal. For ordinals $\alpha < \kappa_n$ the measures $E_{n,\alpha}$ are principal so the only reason to consider them is for a more neat presentation.

For each $n < \omega$, we shall consider an ordering $\leq_{E_n}$ over λ, as follows:

Definition 4.2. For each $n < \omega$, set

$$\leq_{E_n} := \{(\beta, \alpha) \in \lambda \times \lambda \mid \beta \leq \alpha, \wedge \exists f \in {}^{\kappa_n}\kappa_n \; j_n(f)(\alpha) = \beta\}.$$

It is routine to check that $\leq_{E_n}$ is reflexive, transitive and antisymmetric, hence $(\lambda, \leq_{E_n})$ is a partial order. In case $\beta \leq_{E_n} \alpha$, we shall fix in advance a witnessing map $\pi_{\alpha,\beta} : \kappa_n \to \kappa_n$. Also, in the special case where $\alpha = \beta$, by convention, $\pi_{\alpha,\alpha}$ is the identity map id. Observe that $\leq_{E_n} \restriction (\kappa_n \times \kappa_n)$ is exactly the $\in$-order over κ_n so that when we refer to $\leq_{E_n}$ we will really be speaking about the restriction of this order to $\lambda \setminus \kappa_n$. The most notable property of the poset $(\lambda, \leq_{E_n})$ is that it is κ_n-directed: that is, for every $a \in [\lambda]^{<\kappa_n}$ there is $\alpha < \lambda$ such that $\beta \leq_{E_n} \alpha$ for all $\beta \in a$.[21] This and other

[20] In particular, $\sigma_0 = \aleph_1$.

[21] In fact, according to [Git10, p. 1366], $(\lambda, \leq_{E_n})$ is κ_n^{++}-directed.

nice features of $(\lambda, \leq_{E_n})$ are proved at the beginning of [Git10, §2] under the GCH. A proof without the GCH can be found in [Pov20, §10.2].

Remark 4.3. For future reference, it is worth mentioning that all the relevant properties of $(\lambda, \leq_{E_n})$ reflect down to $(\mu, \leq_{E_n}\restriction \mu\times\mu)$. In particular, it is true that every $a\in[\lambda]^{<\kappa_n}$ may be enlarged to an a^* such that $\kappa_n,\mu\in a^*$ and $a^*\cap\mu$ contains a $\leq_{E_n}$-greatest element. For details, see [Git10, §2].

4.1. **The forcing.** Before giving the definition of Gitik's forcing we shall first introduce the basic building block modules $\mathbb{Q}_{n0}$ and $\mathbb{Q}_{n1}$. To that effect, for each $n<\omega$, let us fix a map $s_n\colon\kappa_n\to\kappa_n$ representing μ using the *normal generator*, κ_n. Specifically, for each $n<\omega$, $j_n(s_n)(\kappa_n)=\mu$.

Definition 4.4. For each $n<\omega$, define $\mathbb{Q}_{n1}$, $\mathbb{Q}_{n0}$ and $\mathbb{Q}_n$ as follows:

$(0)_n$ $\mathbb{Q}_{n0} := (Q_{n0}, \leq_{n0})$ is the set of $p := (a^p, A^p, f^p, F^{0p}, F^{1p}, F^{2p})$, where:
  1. (a^p, A^p, f^p) is in the $n0$-module Q^*_{n0} from the Extender Based Prikry Forcing (EBPF) as defined in [Git10, Definition 2.6]. Moreover, we require that $\kappa_n,\mu\in a^p$ and that $a^p\cap\mu$ contains a $\leq_{E_n}$-greatest element denoted by $\mathrm{mc}(a^p\cap\mu)$;[22]
  2. for $i<3$, $\mathrm{dom}(F^{ip})=\pi_{\mathrm{mc}(a^p),\mathrm{mc}(a^p\cap\mu)}[A^p]$, and for $\nu\in\mathrm{dom}(F^{ip})$, setting $\nu_0:=\pi_{\mathrm{mc}(a^p\cap\mu),\kappa_n}(\nu)$, we have:
     (a) $F^{0p}(\nu)\in\mathrm{Col}(\sigma_n,{<}\nu_0)$;
     (b) $F^{1p}(\nu)\in\mathrm{Col}(\nu_0,s_n(\nu_0))$;
     (c) $F^{2p}(\nu)\in\mathrm{Col}(s_n(\nu_0)^{++},{<}\kappa_n)$.

  The ordering $\leq_{n0}$ is defined as follows: $q\leq_{n0}p$ iff $(a^q,A^q,f^q)\leq_{\mathbb{Q}^*_{n0}}(a^p,A^p,f^p)$ as in [Git10, Definition 2.7], and for each $\nu\in\mathrm{dom}(F^{iq})$, $F^{iq}(\nu)\supseteq F^{ip}(\nu')$, where $\nu':=\pi_{\mathrm{mc}(a^q\cap\mu),\mathrm{mc}(a^p\cap\mu)}(\nu)$.

$(1)_n$ $\mathbb{Q}_{n1} := (Q_{n1}, \leq_{n1})$ is the set of $p := (f^p, \rho^p, h^{0p}, h^{1p}, h^{2p})$, where:
  1. f^p is a function from some $x\in[\lambda]^{\leq\kappa}$ to κ_n;
  2. $\rho^p<\kappa_n$ inaccessible;
  3. $h^{0p}\in\mathrm{Col}(\sigma_n,{<}\rho^p)$;
  4. $h^{1p}\in\mathrm{Col}(\rho^p,s_n(\rho^p))$;
  5. $h^{2p}\in\mathrm{Col}(s_n(\rho^p)^{++},{<}\kappa_n)$.

  The ordering $\leq_{n1}$ is defined as follows: $q\leq_{n1}p$ iff $f^q\supseteq f^p$, $\rho^p=\rho^q$, and for $i<3$, $h^{iq}\supseteq h^{ip}$.

$(2)_n$ Set $\mathbb{Q}_n := (Q_{n0}\cup Q_{n1}, \leq_n)$ where the ordering $\leq_n$ is defined as follows: for each $p,q\in Q_n$, $q\leq_n p$ iff
  1. either $p,q\in Q_{ni}$, some $i\in\{0,1\}$, and $q\leq_{ni}p$, or
  2. $q\in Q_{n1}$, $p\in Q_{n0}$ and, for some $\nu\in A^p$, $q\leq_{n1}p^\frown\langle\nu\rangle$, where

$$p^\frown\langle\nu\rangle := (f^p\cup\{\langle\beta,\pi_{\mathrm{mc}(a^p),\beta}(\nu)\rangle\mid\beta\in a^p\}, \bar\nu_0, F^{0p}(\bar\nu), F^{1p}(\bar\nu), F^{2p}(\bar\nu)),$$

  and $\bar\nu=\pi_{\mathrm{mc}(a^p),\mathrm{mc}(a^p\cap\mu)}(\nu)$.

[22] Recall that $(a^p,A^p,f^p)\in Q^*_{n0}$ in particular implies that a^p contains a $\leq_{E_n}$-greatest element, which is typically denoted by $\mathrm{mc}(a^p)$. Note that since $\mu\in a^p$ then $\mathrm{mc}(a^p)$ is always strictly $\leq_{E_n}$-larger than $\mathrm{mc}(a^p\cap\mu)$.

Remark 4.5. For each $n<\omega$,

$$\{\rho<\kappa_n \mid (\kappa_{n-1})^+<\rho<s_n(\rho)<\kappa_n \ \&\ \rho \text{ inaccessible}\}\in E_{n,\kappa_n}.$$

Similarly, for $a\in[\lambda]^{<\kappa_n}$ as in $(0)_n(1)$ above and $A\in E_{n,\mathrm{mc}(a)}$,

$(\star)$ $\{\rho\in\pi_{\mathrm{mc}(a),\mathrm{mc}(a\cap\mu)}\text{``}A \mid |\{\nu\in A\mid \bar\nu_0=\rho_0\}|\leq s_n(\rho_0)^+\}\in E_{n,\mathrm{mc}(a\cap\mu)}$.

In what follows we assume that the above is always the case for all $\rho<\kappa_n$ that we ever consider. Similarly, we may also assume that $s_n(\rho)$ is regular (actually the successor of a singular) and that $s_n(\rho^p)^{<\rho^p}=s_n(\rho^p)$.

The reason we consider conditions witnessing Clause $(\star)$ above is related with the verification of property $\mathcal{D}$ and CPP (cf. Lemmas 4.21 and 4.22). Essentially, when we describe the moves of **I** and **II** we would like to be able to take lower bounds of the top-most collapsing maps appearing in conditions played by **II**. Namely, we would like to take lower bounds of the h^{2q_ξ}'s. Assuming $(\star)$ we will have that the number of maps that need to be amalgamated is at most $s_n(\nu_0)^+$, hence less than the completeness of the top-most Lévy collapse $\mathrm{Col}(s_n(\nu_0)^{++},{<}\kappa_{n+1})$.

Remark 4.6. The reason Gitik makes F^{ip}_n dependent on the partial extender $E_n\restriction\mu$ rather than on the full extender E_n is related with the verification of the chain condition (see [Git19b, Lemma 2.6]). Indeed, in that way the triple $\langle F^{0p}_n,F^{1p}_n,F^{2p}_n\rangle$ will represent three (partial) collapsing in the ultrapower by the measure $E_{n,\mathrm{mc}(a^p_n\cap\mu)}$. This will guarantee that the map c given in Definition 4.10 below will have H_μ as a range (see Remark 4.11).

Having all necessary building blocks, we can now define the poset $\mathbb{P}$.

Definition 4.7. The Extender Based Prikry Forcing with collapses (EBPFC) is the poset $\mathbb{P}:=(P,\leq)$ defined by the following clauses:

- Conditions in P are sequences $p=\langle p_n\mid n<\omega\rangle\in\prod_{n<\omega}Q_n$.
- For all $p\in P$,
  - There is $n<\omega$ such that $p_n\in Q_{n0}$;
  - For every $n<\omega$, if $p_n\in Q_{n0}$ then $p_m\in Q_{m0}$ and $a^{p_n}\subseteq a^{p_m}$, for every $m\geq n$.
- For all $p,q\in P$, $p\leq q$ iff $p_n\leq_n q_n$, for every $n<\omega$.

Definition 4.8. $\ell:P\to\omega$ is defined by letting for all $p=\langle p_n\mid n<\omega\rangle$,

$$\ell(p):=\min\{n<\omega\mid p_n\in Q_{n0}\}.$$

Notation 4.9. Given $p\in P$, $p=\langle p_n\mid n<\omega\rangle$, we will typically write $p_n=(f^p_n,\rho^p_n,h^{0p}_n,h^{1p}_n,h^{2p}_n)$ for $n<\ell(p)$, and $p_n=(a^p_n,A^p_n,f^p_n,F^{0p}_n,F^{1p}_n,F^{2p}_n)$ for $n\geq\ell(p)$. Also, for each $n\geq\ell(p)$, we shall denote $\alpha_{p_n}:=\mathrm{mc}(a^p_n\cap\mu)$.

We already have $(\mathbb{P},\ell)$ and we will eventually check that $\mathbb{1}\Vdash_{\mathbb{P}}\check\mu=\check\kappa^+$ (Corollary 4.25). Next, we introduce sequences $\vec{\mathbb{S}}=\langle\mathbb{S}_n\mid n<\omega\rangle$ and $\vec\varpi=\langle\varpi_n\mid n<\omega\rangle$, and a map $c:P\to H_\mu$ such that $(\mathbb{P},\ell,c,\vec\varpi)$ will be a $(\Sigma,\vec{\mathbb{S}})$-Prikry forcing having property $\mathcal{D}$.

As $\mu^\kappa = \mu$ and $2^\mu = \lambda$, using the Engelking-Karłowicz theorem, we fix a sequence of functions $\langle e^i \mid i < \mu\rangle$ from λ to μ such that, for all $x \in [\lambda]^\kappa$ and every function $e : x \to \mu$, there exists $i < \mu$ with $e \subseteq e^i$.

Definition 4.10. For every condition $p = \langle p_n \mid n < \omega\rangle$ in $\mathbb{P}$, define a sequence of indices $\langle i(p_n) \mid n < \omega\rangle$ as follows:[23]

$$i(p_n) := \begin{cases} \min\{i < \mu \mid f \subseteq e^i\}, & \text{if } n < \ell(p); \\ \min\{i < \mu \mid e^i \restriction a^p_n = 0 \ \&\ e^i \restriction \mathrm{dom}(f^p_n) = f^p_n + 1\}, & \text{if } n \geq \ell(p). \end{cases}$$

Define a map $c : P \to H_\mu$, by letting for any condition $p = \langle p_n \mid n < \omega\rangle$,

$$\begin{aligned} c(p) \quad := \quad & (\ell(p), \langle \rho^p_n \mid n < \ell(p)\rangle, \langle i(p_n) \mid n < \omega\rangle, \\ & \langle \vec{h}^p_n \mid n < \ell(p)\rangle, \langle \alpha_{p_n} \mid n \geq \ell(p)\rangle, \langle \vec{G}^p_n \mid n \geq \ell(p)\rangle), \end{aligned}$$

where $\vec{h}^p_n := \langle h^{ip}_n \mid i < 3\rangle$ and $\vec{G}^p_n := \langle j_{n,\alpha^p_n}(F^{ip}_n)([\mathrm{id}]_{E_{n,\alpha_{p_n}}}) \mid i < 3\rangle$.

Remark 4.11. Note that c is well-defined: all the entries appearing in $c(p)$ are clearly in H_μ with, perhaps, the only exception of the latter one, $\vec{G}^p_n$. However, all of these are conditions in Lévy collapses over $\mathrm{Ult}(V, E_{n,\alpha^p_n})$, an ultrapower by a κ_n-complete measure on κ_n. In particular, these collapses have V-cardinality $|j_n(\kappa_n)| < \kappa_{n+1} < \mu$. Hence, $\mathrm{Im}(c) \subseteq H_\mu$.

Definition 4.12. For each $n < \omega$, set

$$S_n := \begin{cases} \{\mathbb{1}\}, & \text{if } n = 0; \\ \{\langle(\rho^p_k, h^{0p}_k, h^{1p}_k, h^{2p}_k) \mid k < n\rangle \mid p \in P_n\}, & \text{if } n \geq 1. \end{cases}$$

For $n \geq 1$ and $s, t \in S_n$, write $s \preceq_n t$ iff there are $p, q \in P_n$ with $p \leq q$ witnessing, respectively, that s and t are in S_n.

Denote $\mathbb{S}_n := (S_n, \preceq_n)$ and set $\vec{\mathbb{S}} := \langle \mathbb{S}_n \mid n < \omega\rangle$.

Remark 4.13. Observe that $|S_n| < \sigma_n$. Moreover, for each $s \in S_n \setminus \{\mathbb{1}_{\mathbb{S}_n}\}$, $\mathbb{S}_n \downarrow s \cong \mathrm{Col}(\delta, {<}\kappa_{n-1}) \times \mathbb{Q}$, where $\mathbb{Q}$ is a notion of a forcing of size $< \delta$ such that $\sigma_{n-1} < \delta < \kappa_{n-1}$. Specifically, if $p \in P_n$ is the condition from which s arises, then $\delta = s_{n-1}(\rho^p_{n-1})^{++}$ and $\mathbb{Q}$ is a product

$$\mathbb{G} \times \mathrm{Col}(\sigma_{n-1}, {<}\rho^p_{n-1}) \times \mathrm{Col}(\rho^p_{n-1}, s_{n-1}(\rho^p_{n-1})),$$

where $\mathbb{G}$ is a notion of forcing of size $\leq \kappa_{n-2}$.[24] Also, by combining Easton's lemma with a counting of nice names, if the GCH holds below κ then $\mathbb{S}_n \downarrow s$ preserves this behavior of the power set function for each $s \in S_n \setminus \{\mathbb{1}_{\mathbb{S}_n}\}$.

On another note, observe that the the map $(q, s) \mapsto q + s$ yields an isomorphism between $(\mathbb{S}_n \downarrow \varpi_n(p)) \times (\mathbb{P}^{\varpi_n}_n \downarrow p)$ and $\mathbb{P}_n \downarrow p$.[25]

---

[23] In the next formula, 0 stands for the constant map with value 0.

[24] In the particular case where $n = 1$ the poset $\mathbb{R}$ is trivial.

[25] In general terms the above map simply defines a projection (see Definition 2.2(4)) but in the particular case of the EBPFC it moreover gives an isomorphism.

Definition 4.14. For each $n<\omega$, define $\varpi_n\colon P_{\geq n}\to S_n$ as follows:

$$\varpi_n(p):=\begin{cases}\{\mathbb{1}\}, & \text{if } n=0;\\ \langle(\rho^p_k,h^{0p}_k,h^{1p}_k,h^{2p}_k)\mid k<n\rangle, & \text{if } n\geq 1.\end{cases}$$

Set $\vec{\varpi}:=\langle\varpi_n\mid n<\omega\rangle$.

The next lemma collects some useful properties about the $n0$-modules of the EBPFC (i.e, the $\mathbb{Q}_{n0}$'s) and reveals some of their connections with the corresponding modules of the EBPF (i.e, the $\mathbb{Q}^*_{n0}$'s).

Lemma 4.15. *Let $n<\omega$. All of the following hold:*

(1) $\mathbb{P}_n$ projects to $\mathbb{Q}_{n0}$, and this latter projects to $\mathbb{Q}^*_{n0}$.
(2) $\mathbb{Q}^*_{n0}$ is κ_n-directed-closed, while $\mathbb{Q}_{n0}$ is σ_n-directed-closed.
(3) $\mathbb{S}_n$ satisfies the (κ_{n-1})-cc.

Proof. (1) The map $p\mapsto(a^p_n,A^p_n,f^p_n,F^{0p}_n,F^{1p}_n,F^{2p}_n)$ is a projection between $\mathbb{P}_n$ and $\mathbb{Q}_{n0}$. Similarly, $(a,A,f,F^0,F^1,F^2)\mapsto(a,A,f)$ defines a projection between $\mathbb{Q}_{n0}$ and $\mathbb{Q}^*_{n0}$.

(2) The argument for the κ_n-directed-closedness of $\mathbb{Q}^*_{n0}$ is given in [Pov20, Lemma 10.2.40]. Let $D\subseteq\mathbb{Q}_{n0}$ be a directed set of size $<\sigma_n$ and denote by ϱ_n the projection between $\mathbb{Q}_{n0}$ and $\mathbb{Q}^*_{n0}$ given in the proof of item (1). Clearly, $\varrho_n[D]$ is a directed subset of $\mathbb{Q}^*_{n0}$ of size $<\sigma_n$, so that we may let (a,A,f) be a $\leq_{\mathbb{Q}^*_{n0}}$-lower bound for it. By $\leq_{\mathbb{Q}^*_{n0}}$-extending (a,A,f) we may assume that $\kappa_n,\mu\in a$ and that $a\cap\mu$ contains a $\leq_{E_n}$-greatest element. Set $\alpha:=\mathrm{mc}(a\cap\mu)$. For each $i<3$ and each $\nu\in\pi_{\mathrm{mc}(a)\alpha}[A]$, define $F^i(\nu):=\bigcup_{p\in D}F^{ip}(\pi_{\alpha,\alpha_p}(\nu))$. Finally, (a,A,f,F^0,F^1,F^2) is a condition in $\mathbb{Q}_{n0}$ extending every $p\in D$.

(3) This is immediate from the definition of $\mathbb{S}_n$ (Definition 4.12). □

4.2. **EBPFC is $(\Sigma,\vec{\mathbb{S}})$-Prikry.** We verify that $(\mathbb{P},\ell,c,\vec{\varpi})$ is $(\Sigma,\vec{\mathbb{S}})$-Prikry having property $\mathcal{D}$. To this end, we go over the clauses of Definition 3.3.

Convention 4.16. For every sequence $\{A_k\}_{i\leq k\leq j}$ such that each A_k is a subset of κ_k, we shall identify $\prod_{k=i}^{j}A_k$ with its subset consisting only of the sequences that are moreover increasing.

Definition 4.17. Let $p=\langle p_n\mid n<\omega\rangle\in P$. Define:

- $p^\frown\emptyset:=p$;
- For every $\nu\in A^p_{\ell(p)}$, $p^\frown\langle\nu\rangle$ is the unique condition $q=\langle q_n\mid n<\omega\rangle$, such that for each $n<\omega$:
$$q_n=\begin{cases}p_n, & \text{if } n\neq\ell(p);\\ {p_{\ell(p)}}^\frown\langle\nu\rangle, & \text{otherwise.}\end{cases}$$
- Inductively, for all $m\geq\ell(p)$ and $\vec{\nu}=\langle\nu_{\ell(p)},\ldots,\nu_m,\nu_{m+1}\rangle\in\prod_{n=\ell(p)}^{m+1}A^p_n$, set $p^\frown\vec{\nu}:=(p^\frown\vec{\nu}\restriction(m+1))^\frown\langle\nu_{m+1}\rangle$.

Fact 4.18. *Let $p,q\in P$.*

- *$q\leq^0 p$ iff $\ell(p)=\ell(q)$ and $q\leq_n p$, for each $n<\omega$;*

- *$q \leq p$ iff there is $\vec{\nu} \in \prod_{n=\ell(p)}^{\ell(q)-1} A^p_n$ such that $q \leq^0 p^\frown\vec{\nu}$;*
- *The sequence $\vec{\nu}$ above is uniquely determined by q. Specifically, for each $n \in [\ell(p), \ell(q))$, $\nu_n = f^q_n(\mathrm{mc}(a^p_n))$.*

By the very definition of the EBPFC (Definition 4.7) and the function ℓ (Definition 4.8), $(\mathbb{P}, \ell)$ is a graded poset, hence $(\mathbb{P}, \ell, c, \vec{\varpi})$ witnesses Clause (1). Also, combining Lemma 4.15(2) with the fact that all of the Lévy collapses considered are at least $\aleph_1$-closed, Clause (2) follows:

Lemma 4.19. *For all $n < \omega$, $\mathbb{P}_n$ is $\aleph_1$-closed.* □

We now verify that the map of Definition 4.10 witnesses Clause (3):

Lemma 4.20. *For all $p, q \in P$, if $c(p) = c(q)$, then $P^p_0 \cap P^q_0$ is non-empty.*

Proof. Let $p, q \in P$ and assume that $c(p) = c(q)$. By Definition 4.10, we have $\ell(p) = \ell(q)$ and $\rho^p_n = \rho^q_n$ for all $n < \ell(p)$. Set $\ell := \ell(p)$ and $\rho_n := \rho^p_n$ for each $n < \ell$. Also, $c(p) = c(q)$ yields $\vec{h}^p_n = \vec{h}^q_n$ for each $n < \ell$, and $\alpha_{p_n} = \alpha_{q_n}$ and $\vec{G}^p_n = \vec{G}^q_n$ for each $n \geq \ell$. Put $\vec{h}_n := \vec{h}^p_n$ and write $\vec{h}_n = (h^0_n, h^1_n, h^2_n)$. Denote by α_n the common value $\alpha_{p_n} = \alpha_{q_n}$. We now define $r \in P^p_0 \cap P^q_0$.

▶ If $n < \ell$ then $c(p) = c(q)$ implies $i = i(p_n) = i(q_n)$, and so $f^p_n \cup f^q_n \subseteq e^i$. Set $r_n := (\rho_n, f^p_n \cup f^q_n, h^0_n, h^1_n, h^2_n)$. Clearly, $r_n \in Q_{n1}$.

▶ For $n \geq \ell$ put $a^r_{\ell-1} := \emptyset$ and $\alpha_{\ell-1} := 0$ and argue by recursion towards defining a^r_n. We assume by induction that $\mathrm{mc}(a^r_m \cap \mu) = \alpha_m$ for $m \in [\ell-1, n)$.

Since $i(p_n) = i(q_n)$, arguing as in [Pov20, Lemma 10.2.41] we can make (a^p_n, A^p_n, f^p_n) and (a^q_n, A^q_n, f^q_n) compatible by taking the triple

$$(a^r_{n-1} \cup a^p_n \cup a^q_n \cup \{\alpha^*\}, A^*, f^p_n \cup f^q_n),$$

where α^* is some ordinal in $\lambda \setminus \bigcup_{\ell \leq m \leq n}(\mathrm{dom}(f^p_m) \cup \mathrm{dom}(f^q_m))$ such that α^* is $\leq_{E_n}$-above the ordinals in $a^r_{n-1} \cup a^p_n \cup a^q_n$. Also, A^* is some suitable E_{n,α^*}-large set. Since we can pick such an α^* as large as we wish (below λ) we may assume that it is actually above μ. In particular, putting

$$a^r_n := a^r_{n-1} \cup a^p_n \cup a^q_n \cup \{\alpha^*\}$$

we have that a^r_n has a $\leq_{E_n}$-maximal element and $\mathrm{mc}(a^r_n \cap \mu) = \alpha_n$.

Let us now to define the F-component of r_n. Since for each $i \leq 2$ $j_{n,\alpha_n}(F^{pi}_n)([\mathrm{id}]_{E_{n,\alpha_n}}) = j_{n,\alpha_n}(F^{qi}_n)([\mathrm{id}]_{E_{n,\alpha_n}})$ we have a E_{n,α_n}-measure one set B_i for which $F^{pi}_n \restriction B = F^{qi}_n \restriction B$. Let $B := B_0 \cap B_1 \cap B_2$ and $A^r_n := A^* \cap \pi^{-1}_{\alpha^*,\alpha_n}$“$B$. Clearly, $A^r_n \in E_{n,\alpha^*}$. Finally, define $F^{ri}_n := F^{pi}_n \restriction \pi_{\alpha^*,\alpha_n}$“$A^r_n$ and put

$$r_n = (a^r_n, A^r_n, f^p_n \cup f^q_n, F^{r0}_n, F^{r1}_n, F^{r2}_n).$$

After this recursive definition we obtain a condition $r = \langle r_n \mid n < \omega\rangle$ witnessing $P^p_0 \cap P^q_0 \neq \emptyset$. □

The verification of Clauses (4), (5) and (6) is the same as in [Pov20, Lemma 10.2.45, 10.2.46 and 10.2.47], respectively. It is worth saying that regarding Clause (5) we actually have that $|W(p)| \leq \kappa$ for each $p \in P$.

We now show that $(\mathbb{P},\ell)$ has property $\mathcal{D}$ and that it satisfies Clause (7).

Lemma 4.21. *$(\mathbb{P},\ell)$ has property $\mathcal{D}$.*

Proof. Let $p\in P$, $n<\omega$ and $\vec{r}$ be a good enumeration of $W_n(p)$. Our aim is to show that **I** has a winning strategy in the game $\Game_{\mathbb{P}}(p,\vec{r})$. To enlighten the exposition we just give details for the case when $n=1$. The general argument can be composed using the very same ideas.

Write $p=\langle (f_n,\rho_n,h^0_n,h^1_n,h^2_n)\mid n<\ell\rangle^\frown\langle (a_n,A_n,f_n,F^0_n,F^1_n,F^2_n)\mid n\geq\ell\rangle$. By Fact 4.18, we can identify $\vec{r}$ with $\langle \nu_\xi\mid \xi<\kappa_\ell\rangle$, a good enumeration of A_ℓ. Specifically, for each $\xi<\kappa_\ell$ we have that $r_\xi=p^\frown\langle\nu_\xi\rangle$. Using this enumeration we define a sequence $\langle (p_\xi,q_\xi)\mid\xi<\kappa_\ell\rangle$ of moves in $\Game_{\mathbb{P}}(p,\vec{r})$.

To begin with, **I** plays $p_0:=p$ and in response **II** plays some $q_0\leq^0 r_0$ with $q_0\leq p_0$. Note that this move is possible, as p_0 and r_0 are compatible.

Suppose by induction that we have defined a sequence $\langle (p_\eta,q_\eta)\mid\eta<\xi\rangle$ of moves in $\Game_{\mathbb{P}}(p,\vec{r})$ which moreover satisfies the following:

1. For each $n<\ell$ the following hold:
   (a) for all $\eta<\xi$, $\rho_n^{p_\xi}=\rho_n$, $h_n^{0p_\xi}=h^0_n$, $h_n^{1p_\xi}=h^1_n$, $h_n^{2p_\xi}=h^2_n$;
   (b) for all $\zeta<\eta<\xi$, $f_n^{q_\zeta}\subseteq f_n^{p_\eta}$;
2. For all $\zeta<\eta<\xi$ and $n>\ell$, $(q_\eta)_n\leq_{n0}(p_\eta)_n\leq_{n0}(q_\zeta)_n$;
3. For all $\eta<\xi$:
   (a) $a_\ell^{p_\xi}=a_\ell$, $A_\ell^{p_\xi}=A_\ell$, $F_\ell^{0p_\xi}=F^0_\ell$ and $F_\ell^{1p_\xi}=F^1_\ell$;
   (b) for each $\zeta<\eta$, if $(\bar{\nu}_\zeta)_0=(\bar{\nu}_\eta)_0$ then $h_\ell^{2q_\zeta}\subseteq h_\ell^{2q_\eta}$.

Let us show how to define the ξ^{th} move of **I**:

<u>Successor case:</u> Suppose $\xi=\eta+1$. Then put $p_\xi:=\langle (p_\xi)_n\mid n<\omega\rangle$, where

$$(p_\xi)_n:=\begin{cases}(f_n^{q_\eta},\rho_n,h^0_n,h^1_n,h^2_n), & \text{if } n<\ell;\\ (a_n,A_n,f_n^{q_\eta}\setminus a_n,F^0_n,F^1_n,F^{2\xi}), & \text{if } n=\ell;\\ (q_\eta)_n, & \text{if } n>\ell.\end{cases}$$

Here $F^{2\xi}$ denotes the map with domain $\pi_{\mathrm{mc}(a_\ell),\alpha_{p_\ell}}``A_\ell$ defined as follows:

$$F^{\xi,2}(\bar{\nu}):=\begin{cases}F^2_\ell(\bar{\nu})\cup\bigcup\{h_\ell^{q_\zeta,2}\mid\zeta<\xi,\ (\bar{\nu}_\zeta)_0=(\bar{\nu}_\xi)_0\}, & \text{if } \nu=\nu_\xi;\\ F^2_\ell(\bar{\nu}), & \text{otherwise.}\end{cases}$$

By Clauses (3) of the induction hypothesis and our comments in Remark 4.5, $F^{2\xi}$ is a function. A moment's reflection makes it clear that p_ξ is a condition in $\mathbb{P}$ witnessing (1) and (3)(a) above. Also, $p_\xi\leq^0 p$ and p_ξ is compatible with r_ξ, hence it is a legitimate move for **I**.[26] In response, **II** plays $q_\xi\leq^0 r_\xi$ such that $q_\xi\leq p_\xi$. In particular, for each $n>\ell$, $(q_\xi)_n\leq_{n0}(p_\xi)_n\leq_{n0}(q_\eta)_n$, and also $F^{2\xi}(\bar{\nu}_\xi)\subseteq h_\ell^{2q_\xi}$. This combined with the induction hypothesis yield Clause (2) and (3)(b), which completes the successor case.

[26]Note that $p_\xi{}^\frown\langle\nu_\xi\rangle\leq p_\xi,r_\xi$.

Limit case: In the limit case we put $p_\xi := \langle (p_\xi)_n \mid n < \omega\rangle$, where

$$(p_\xi)_n := \begin{cases} (\bigcup_{\eta<\xi} f_n^{q_\eta}, \rho_n, h_n^0, h_n^1, h_n^2), & \text{if } n < \ell; \\ (a_n, A_n, \bigcup_{\eta<\xi}(f_n^{q_\eta} \setminus a_n), F_n^0, F_n^1, F^{2\xi}), & \text{if } n = \ell; \\ (q_\xi^*)_n, & \text{if } n > \ell. \end{cases}$$

Here, $F^{2\xi}$ is defined as before and $(q_\xi^*)_n$ is a lower bound for the sequence $\langle (p_\eta)_n \mid \eta < \xi\rangle$. Note that this choice is possible because the orderings $\leq_{n0}$ are $\sigma_{\ell+1}$-directed-closed. Once again, p_ξ is a legitimate move for **I** and, in response, **II** plays q_ξ. It is routine to check that (1)–(3) above hold.

After this process we get a sequence $\langle (p_\xi, q_\xi) \mid \xi < \kappa_\ell\rangle$. We next show how to form a condition $p^* \leq^0 p$ diagonalizing $\langle q_\xi \mid \xi < \kappa_\ell\rangle$.

Note that by shrinking A_ℓ to some A'_ℓ we may assume that there are maps $\langle (h_n^{*0}, h_n^{*1}, h_n^{*2}) \mid n < \ell\rangle$ such that $h_n^{iq_\xi} = h_n^{*i}$ for all $\nu_\xi \in A'_\ell$ and $i < 3$. Next, define a map t with domain A'_ℓ such that $t(\nu) := \langle h_\ell^{0q_\nu}, h_\ell^{1q_\nu}\rangle$.[27] Since $j_\ell(t)(\mathrm{mc}(a_\ell)) \in V_{\kappa+1}^{M_{E_\ell}}$ we can argue as in [Git19b, Claim 1] that there is $\alpha < \mu$ and a map t' such that $j_\ell(t)(\mathrm{mc}(a_\ell)) = j_\ell(t')(\alpha)$. Now let a_ℓ^* be such that $a_\ell \cup \{\alpha\} \subseteq a_\ell^*$ witnessing Clause (1) of Definition 4.4(0)$_n$. Then,

$$A := \{\nu < \kappa_\ell \mid t \circ \pi_{\mathrm{mc}(a_\ell^*),\mathrm{mc}(a_\ell)}(\nu) = t' \circ \pi_{\mathrm{mc}(a_\ell^*\cap\mu),\alpha} \circ \pi_{\mathrm{mc}(a_\ell^*),\mathrm{mc}(a_\ell^*\cap\mu)}(\nu)\}$$

is $E_{\ell,\mathrm{mc}(a_\ell^*)}$-large. Set $A_\ell^* := A \cap \pi^{-1}_{\mathrm{mc}(a_\ell^*),\mathrm{mc}(a_\ell)} A'_\ell$ and

$$\hat{t} := (t' \circ \pi_{\mathrm{mc}(a_\ell^*\cap\mu),\alpha}) \restriction \pi_{\mathrm{mc}(a_\ell^*),\mathrm{mc}(a_\ell^*\cap\mu)} “A_\ell^*.$$

Note that $\pi_{\mathrm{mc}(a_\ell^*),\mathrm{mc}(a_\ell)} “A_\ell^* \subseteq A'_\ell \subseteq A_\ell$. Also, for each $\nu \in A_\ell^*$,

$$\hat{t}(\pi_{\mathrm{mc}(a_\ell^*),\mathrm{mc}(a_\ell^*\cap\mu)}(\nu)) = t(\tilde{\nu}) = \langle h_\ell^{0q_{\tilde{\nu}}}, h_\ell^{1q_{\tilde{\nu}}}\rangle,$$

where $\tilde{\nu} := \pi_{\mathrm{mc}(a_\ell^*),\mathrm{mc}(a_\ell)}(\nu)$. For each $i < 2$, define a map $F_\ell^{*,i}$ with domain $\pi_{\mathrm{mc}(a_\ell^*),\mathrm{mc}(a_\ell)} “A_\ell^*$, such that for each $\nu \in A_\ell^*$,

$$F_\ell^{*,i}(\pi_{\mathrm{mc}(a_\ell^*),\mathrm{mc}(a_\ell^*\cap\mu)}(\nu)) := h_\ell^{iq_{\tilde{\nu}}}.$$

Similarly, define $F_\ell^{*,2}$ by taking lower bounds over the stages of the inductive construction mentioning ordinals $\nu_\eta \in \pi_{\mathrm{mc}(a_\ell^*),\mathrm{mc}(a_\ell)} “A_\ell^*$; i.e.,

$$F_\ell^{*,2}(\pi_{\mathrm{mc}(a_\ell^*),\mathrm{mc}(a_\ell^*\cap\mu)}(\nu)) := \bigcup\{h^{2q_{\nu_\eta}} \mid \nu_\eta \in \pi_{\mathrm{mc}(a_\ell^*),\mathrm{mc}(a_\ell)} “A_\ell^*,\ (\bar{\tilde{\nu}})_0 = (\bar{\nu}_\eta)_0\}.$$[28]

Next, define $p^* := \langle p_n^* \mid n < \omega\rangle$, where

$$p_n^* := \begin{cases} (\bigcup_{\xi<\kappa_\ell} f_n^{q_\xi}, \rho_n, h_n^{*0}, h_n^{*1}, h_n^{*2}), & \text{if } n < \ell; \\ (a_n^*, A_n^*, f_n^p \cup \bigcup_{\xi<\kappa_\ell}(f_n^{q_\eta} \setminus a_n^*), F_n^{*0}, F_n^{*1}, F_n^{*2}), & \text{if } n = \ell; \\ q_n^*, & \text{if } n > \ell. \end{cases}$$

and q_n^* is a $\leq_{n0}$-lower bound for $\langle (q_\xi)_n \mid \xi < \kappa_\ell\rangle$.

---

[27] In a slight abuse of notation, here we are identifying q_ν with q_ξ, where $\nu = \nu_\xi$.

[28] Once again, this choice is legitimate in that there are not too many ν with the same projection to the normal measure, ν_0 (see Remark 4.5).

Claim 4.21.1. *p^* is a condition in $\mathbb{P}$ diagonalizing $\langle q_\xi \mid \xi < \kappa_\ell\rangle$.*

Proof. Clearly, $p^* \in P$ and it is routine to check that $p^* \leq^0 p$.

Let $s \in W_1(p^*)$ and $\nu \in A^*_\ell$ be with $s = p^{*\frown}\langle\nu\rangle$. Since $\pi_{\mathrm{mc}(a^*_\ell),\mathrm{mc}(a_\ell)}$"$A^*_\ell$ is contained in A_ℓ there is some $\xi < \kappa_\ell$ such that $\tilde{\nu} = \nu_\xi$. Note that $w(p,s) = p^\frown\langle\nu_\xi\rangle$, hence we need to prove that $p^{*\frown}\langle\nu\rangle \leq^0 q_\xi$. Note that for this it is enough to show that $h^{iq_\xi}_\ell \subseteq F^{*i}_\ell(\pi_{\mathrm{mc}(a^*_\ell),\mathrm{mc}(a^*_\ell\cap\mu)}(\nu))$ for $i < 3$. And, of course, this follows from our definition of F^{*i}_ℓ and the fact that $\tilde{\nu} = \nu_\xi$. $\square$

The above shows that **I** has a winning strategy for the game $\Game_{\mathbb{P}}(p,\vec{r})$. $\square$

Lemma 4.22. *$(\mathbb{P},\ell)$ has the* CPP.

Proof. Fix $p \in P$, $n < \omega$ and U a 0-open set. Set $\ell := \ell(p)$.

Claim 4.22.1. *There is $q \leq^0 p$ such that if $r \in P^q \cap U$ then $w(q,r) \in U$.*

Proof. For each $n < \omega$ and a good enumeration $\vec{r} := \langle r^n_\xi \mid \xi < \chi\rangle$ of $W_n(p)$ appeal to Lemma 4.21 and find $p_n \leq^0 p$ such that p_n diagonalizes a sequence $\langle q^n_\xi \mid \xi < \chi\rangle$ of moves for **II** which moreover satisfies

$$P_0^{r^n_\xi} \cap U \neq \emptyset \implies q^n_\xi \in U.$$

Appealing iteratively to Lemma 4.21 we arrange $\langle p_n \mid n < \omega\rangle$ to be $\leq^0$-decreasing, and by Definition 3.32 we find $q \leq^0 p$ a lower bound for it.

Let $r \leq q$ be in D and set $n := \ell(r) - \ell(q)$. Then, $r \leq^n p_n$ and so $r \leq^0 w(p_n, r) \leq^0 q^n_\xi$ for some ξ. This implies that $q^n_\xi \in U$. Finally, since $w(q,r) \leq^0 w(p_n,r) \leq^0 q^n_\xi$ we infer from 0-openess of U that $w(q,r) \in U$. $\square$

Let $q \leq^0 p$ be as in the conclusion of Claim 4.22.1. We define by induction a $\leq^0$-decreasing sequence of conditions $\langle q_n \mid n < \omega\rangle$ such that for each $n < \omega$

$$(\star)_n \;\; W_n(q_n) \subseteq U \text{ or } W_n(q_n) \cap U = \emptyset.$$

The cases $n \leq 1$ are easily handled and the cases $n \geq 3$ are similar to the case $n = 2$. So, let us simply describe how do we proceed in this latter case. Suppose that q_1 has been defined. For each $\nu \in A^{q_1}_\ell$, define

$$A^+_\nu := \{\delta \in A^{q_1}_{\ell+1} \mid q_1{}^\frown\langle\nu,\delta\rangle \in U\} \text{ and } A^-_\nu := A^{q_1}_{\ell+1} \setminus A^+_\nu.$$

If A^+_ν is large then set $A_\nu := A^+_\nu$.[29] Otherwise, define $A_\nu := A^-_\nu$. Put $A^+ := \{\nu \in A^{q_1}_\ell \mid A_\nu = A^+_\nu\}$, and $A^- := \{\nu \in A^{q_1}_\ell \mid A_\nu = A^-_\nu\}$. If A^+ is large we let $A_\ell := A^+$ and otherwise $A_\ell := A^-$. Finally, let $q_2 \leq^0 q_1$ be such that $A^{q_2}_\ell := A_\ell$ and $A^{q_2}_{\ell+1} := \bigcap_{\nu\in A^{q_1}_\ell} A_\nu$ Then, q_2 witnesses $(\star)_2$.

Once we have defined $\langle q_n \mid n < \omega\rangle$, let q_ω be a $\leq^0$-lower bound for it (cf. Definition 3.3(2)). It is routine to check that, for each $n < \omega$, the condition q_ω witnesses property $(\star)_n$, hence it is as desired. $\square$

Let us dispose with the verification of Clauses (8) and (9):

[29] More explicitly, $E_{\ell+1,\mathrm{mc}(a^{q_1}_{\ell+1})}$-large.

Lemma 4.23. *For all $n<\omega$, the map ϖ_n is a nice projection from $\mathbb{P}_{\geq n}$ to $\mathbb{S}_n$ such that, for all $k\geq n$, $\varpi_n\restriction\mathbb{P}_k$ is again a nice projection to $\mathbb{S}_n$.*

Moreover, the sequence of nice projections $\vec{\varpi}$ is coherent.[30]

Proof. Fix some $n<\omega$. By definition, $\varpi_n(\mathbb{1}_{\mathbb{P}})=\mathbb{1}_{\mathbb{S}_n}$ and it is not hard to check that it is order-preserving. Let $p\in P_{\geq n}$ and $s\preceq_n\varpi_n(p)$.

Then $s=\langle(\rho^p_k,h^0_k,h^1_k,h^2_k)\mid k<n\rangle$, and we define $r:=\langle r_k\mid k<\omega\rangle$ as

$$r_k:=\begin{cases}(\rho^p_k,f^p_k,h^0_k,h^1_k,h^2_k), & \text{if } k<n;\\ p_k, & \text{otherwise.}\end{cases}$$

One can check that $r\leq^0 p$ and $\varpi_n(r)=s$. Actually, r is the greatest such condition, hence $r=p+s$. This yields Clause (3) of Definition 2.2.

For the verification of Clause (4) of Definition 2.2, let $q\leq^0 p+s$ and define a sequence $p':=\langle p'_n\mid n<\omega\rangle$ as follows:

$$p'_k:=\begin{cases}(\rho^p_k,f^p_k,h^{0p}_k,h^{1p}_k,h^{2p}_k), & \text{if } k<n;\\ q_k, & \text{otherwise.}\end{cases}$$

Note that $p'\in P$ and $p'+\varpi_n(q)=q$. Thus, Clause (4) follows.

Altogether, the above shows that ϖ_n is a nice projection. Similarly, one shows that $\varpi_n\restriction\mathbb{P}_k$ is a nice projection for each $k\geq n$. Finally, the moreover part of the lemma follows from the definition of ϖ_n and the fact that $W(p)=\{p^\frown\vec{\nu}\mid\vec{\nu}\in\prod_{k=\ell(p)}^{\ell(p)+|\vec{\nu}|-1}A^p_k\}$. $\square$

Lemma 4.24. *For each $n<\omega$, $\mathbb{P}^{\varpi_n}_n$ is σ_n-directed-closed.*[31]

Proof. Since $\mathbb{P}^{\varpi_\aleph}_0=\{\mathbb{1}\}$ the result is clearly true for $n=0$.

Let $n\geq 1$ and $D\subseteq\mathbb{P}^{\varpi_n}_n$ be a directed set of size $<\sigma_n$. By definition,

$$\varpi_n[D]=\{\langle(\rho^p_k,h^{0p}_k,h^{1p}_k,h^{2p}_k)\mid k<n\rangle\},$$

for some (all) $p\in D$. By taking intersection of the measure one sets and unions on the other components of the conditions of D one can easily form a condition q which is a $\leq^{\vec{\varpi}}$-lower bound fo D. $\square$

Finally, the proof of the next is identical to [Pov20, Corollary 10.2.53].

Corollary 4.25. $\mathbb{1}_{\mathbb{P}}\Vdash_{\mathbb{P}}\check{\mu}=\kappa^+$. $\square$

Combining all the previous lemmas we finally arrive at the desired result:

Corollary 4.26. $(\mathbb{P},\ell,c,\vec{\varpi})$ *is a $(\Sigma,\vec{\mathbb{S}})$-Prikry forcing that has property $\mathcal{D}$.*

Moreover, the sequence $\vec{\varpi}$ is coherent. $\square$

---

[30] See Definition 3.8.

[31] In particular, taking $\mathring{\mathbb{P}}_n:=\mathbb{P}_n$ Clause (9) follows.

4.3. EBPFC is suitable for reflection.

Definition 4.27. For each $n < \omega$ denote by $\mathbb{R}_n$ the poset with universe $\{\langle f_0^p, \dots, f_{n-1}^p, (a_n^p, A_n^p, f_n^p), \dots\rangle \mid p \in P_n\}$ and whose forcing ordering $\leq_{\mathbb{R}_n}$is the one induced by $\mathbb{P}_n$. Given $p \in P_n$ we shall denote

$$\varpi_{n,\sqsupseteq}(p) \ = \langle f_0^p, \dots, f_{n-1}^p, (a_n^p, A_n^p, f_n^p), \dots\rangle.$$

In this section we show that $(\mathbb{P}_n, \mathbb{S}_n, \mathbb{R}_n, \varpi_n)$ is suitable for reflection with respect to a relevant sequence of cardinals. Our setup will be the same as the one on page 22 and we will also rely on the notation established on page 24. The main result of the section is Corollary 4.32, which will be preceded by a series of technical lemmas. The first one is essentially due to Sharon:

Lemma 4.28 ([Sha05])**.** *For each $n < \omega$, $V^{\mathbb{Q}^*_{n0}} \models |\mu| = \mathrm{cf}(\mu) = \kappa_n$.*

Proof. By Lemma 4.15, $\mathbb{Q}^*_{n0}$ preserves cofinalities $\leq\kappa_n$, and by the Linked$_0$-property [Pov20, Lemma 10.2.41] it preserves cardinals $\geq\mu^+$.

Next we show that $\mathbb{Q}^*_{n0}$ collapses μ to κ_n. For each condition $p \in \mathbb{Q}^*_{n0}$, denote $p := (a^p, A^p, f^p)$. Let G be $\mathbb{Q}^*_{n0}$-generic and set $a := \bigcup_{p\in G} a^p$. By a density argument, $\sup(a) = \mu$ and $\mathrm{otp}(a \cap \mu) = \kappa_n$. Hence, μ is collapsed. Finally, by a result of Shelah [She82, Lemma 4.9 on page 440], this implies that $V^{\mathbb{Q}^*_{n0}} \models \mathrm{cf}(|\mu|) = \mathrm{cf}(\mu)$. $\square$

Lemma 4.29. *For each $n < \omega$, $V^{\mathbb{Q}_{n0}} \models |\mu| = \mathrm{cf}(\mu) = \kappa_n = (\sigma_n)^+$.*

Proof. For each $p \in Q_{n0}$, F^{0p}, F^{1p} and F^{2p} can be seen (respectively) as representatives of conditions in the collapses $\mathrm{Col}(\sigma_n, {<}\kappa_n)^{M^*_n}$, $\mathrm{Col}(\kappa_n, \kappa^+)^{M^*_n}$ and $\mathrm{Col}(\kappa^{+3}, {<}j_n(\kappa_n))^{M^*_n}$, where $M^*_n \cong \mathrm{Ult}(V, E_n \restriction \mu)$.[32] Also, observe that the first of these forcings is nothing but $\mathrm{Col}(\sigma_n, {<}\kappa_n)^V$.[33] Set $C_n := \{\langle F^{1p}, F^{2p}\rangle \mid p \in Q_{n0}\}$ and define $\sqsubseteq$ as follows:

$$\langle F^{1p}, F^{2p}\rangle \sqsubseteq \langle F^{1q}, F^{2q}\rangle \ \text{ iff } \ \forall i \in \{1,2\}\ j_n(F^{ip})(\alpha_p) \supseteq j_n(F^{iq})(\alpha_q).$$

Clearly, $\sqsubseteq$ is transitive, so that $\mathbb{C}_n := (C_n, \sqsubseteq)$ is a forcing poset.

For each condition c in $\mathbb{C}_n$ let us denote by α_c the ordinal α_{p_c} relative to a condition p_c in $\mathbb{Q}_{n0}$ witnessing that $c \in C_n$. The following is a routine verification:

Claim 4.29.1. *$\mathbb{C}_n$ is κ_n-directed closed. Furthermore, if $D \subseteq \mathbb{C}_n$ is a directed set of size $< \kappa_n$ and $\alpha < \mu$ is $\leq_{E_n}$-above all $\{\alpha_c \mid c \in D\}$. Then, there is $\sqsubseteq$-lower bound $\langle F^1, F^2\rangle$ for D with $\mathrm{dom}(F^1) = \mathrm{dom}(F^2) \in E_{n,\alpha}$.*

Let G be a $\mathbb{Q}_{n0}$-generic filter over V and denote by G^* the $\mathbb{Q}^*_{n0}$-generic induced by G and the projection ϱ_n of Lemma 4.15(1). By Lemma 4.28, $V[G^*] \models |\mu| = \mathrm{cf}(\mu) = \kappa_n$, hence it lefts to check that κ_n is preserved and

[32] Despite the fact that $\mathrm{dom}(F^{ip}) \in E_{n,\alpha_{p_n}}$ there is a natural factor embedding between $M_{n,\alpha_{p_n}} \simeq \mathrm{Ult}(V, E_{n,\alpha_{p_n}})$ and M^*_n that enable us to see $j_n(F^{ip})(\alpha_{p_n})$ as a member of the said collapses; namely, the embedding defined by $k(j_{n,\alpha_{p_n}}(F^{ip})(\alpha_{p_n})) = j_n(F^{ip})(\alpha_{p_n})$.

[33] Here we use that $V_{\kappa_n} \subseteq M^*_n$.

turned into $(\sigma_n)^+$. We prove this in two steps, being the proof of the first one a routine verification.

Claim 4.29.2. *The map* $e\colon \mathbb{Q}_{n0}/G^* \to \mathrm{Col}(\sigma_n, {<}\kappa_n)^V \times \mathbb{C}_n$ *defined in* $V[G^*]$ *by* $p \mapsto \langle j_n(F^{0p})(\alpha_p), \langle F^{1p}, F^{2p}\rangle\rangle$ *is a dense embedding.*

Claim 4.29.3. $V[G] \models |\mu| = \mathrm{cf}(\mu) = \kappa_n = (\sigma_n)^+$.

Proof. Since $\mathbb{Q}^*_{n0}$ is κ_n-directed closed, $\mathrm{Col}(\sigma_n, {<}\kappa_n)^V = \mathrm{Col}(\sigma_n, {<}\kappa_n)^{V[G^*]}$. Note that $\mathbb{C}_n$ is still κ_n-directed closed over $V[G^*]$ and that in any generic extension by $\mathrm{Col}(\sigma_n, {<}\kappa_n)^V$ over $V[G^*]$, "$|\mu| = \kappa_n = (\sigma_n)^+$" holds.

Appealing to Easton's lemma, $\mathbb{C}_n$ is κ_n-distributive in any extension of $V[G^*]$ by $\mathrm{Col}(\sigma_n, {<}\kappa_n)^V$. Thus, forcing with $\mathrm{Col}(\sigma_n, {<}\kappa_n)^V \times \mathbb{C}_n$ (over $V[G^*]$) yields a generic extension where "$|\mu| = \kappa_n = (\sigma_n)^+$" holds. Since $(\mu^+)^V$ is preserved, a theorem of Shelah (see [CFM01, Fact 4.5]) yields "$\mathrm{cf}(\mu) = \mathrm{cf}(|\mu|)$" in the above generic extension. Thus, $\mathrm{Col}(\sigma_n, {<}\kappa_n)^V \times \mathbb{C}_n$ forces (over $V[G^*]$) that "$|\mu| = \mathrm{cf}(\mu) = \kappa_n = (\sigma_n)^+$" holds.

The result now follows using Claim 4.29.2, as it in particular implies that $\mathrm{Col}(\sigma_n, {<}\kappa_n)^V \times \mathbb{C}_n$ and $\mathbb{Q}_{n0}/G^*$ are forcing equivalent over $V[G^*]$. $\square$

This completes the proof. $\square$

Lemma 4.30. *For all non-zero* $n < \omega$, $\prod_{i<n} \mathbb{Q}_{i1}$ *is isomorphic to a product of* $\mathbb{S}_n$ *with some* μ*-directed-closed forcing.*

Proof. The map $p \mapsto \langle\langle(\rho^{p_i}, h^{0p_i}, h^{1p_i}, h^{2p_i}) \mid i < n\rangle, \langle f^{p_i} \mid i < n\rangle\rangle$ yields the desired isomorphism. $\square$

Lemma 4.31. *For each* $n < \omega$, $V^{\mathbb{P}_n} \models |\mu| = \mathrm{cf}(\mu) = \kappa_n = (\sigma_n)^+$.

Proof. Observe that $\mathbb{P}_n$ is a dense subposet of $\prod_{i<n} \mathbb{Q}_{i1} \times \prod_{i\geq n} \mathbb{Q}_{i0}$, hence both forcing produce the same generic extension. By virtue of Lemma 4.29 we have $V^{\mathbb{Q}_{n0}} \models |\mu| = \mathrm{cf}(\mu) = \kappa_n = (\sigma_n)^+$. Also, $\mathbb{Q}_{n0}$ is σ_n-directed-closed, hence Easton's lemma, Lemma 4.15(3) and Lemma 4.30 combined imply that $\mathbb{Q}_{n0}$ forces $\prod_{i<n} \mathbb{Q}_{i1}$ to be (κ_{n-1})-cc and $\prod_{i>n} \mathbb{Q}_{i0}$ to be κ_n-distributive. Thereby, forcing with $\prod_{i<n} \mathbb{Q}_{i1} \times \prod_{i>n} \mathbb{Q}_{i0}$ over $V^{\mathbb{Q}_{n0}}$ preserves the above cardinal configuration and thus the result follows. $\square$

As a consequence of the above we get the main result of the section:

Corollary 4.32. *For each* $n \geq 2$, $(\mathbb{P}_n, \mathbb{S}_n, \mathbb{R}_n, \varpi_n)$ *is suitable for reflection with respect to the sequence* $\langle \sigma_{n-1}, \kappa_{n-1}, \kappa_n, \mu\rangle$.

Proof. We go over the clauses of Definition 2.10. Clause (1) follows from Lemma 4.23 being $\pi\colon p \mapsto \langle \varpi_n(p), \varpi_{n,\sqsupseteq}(p)\rangle$. Clause (2) holds since $\mathbb{P}_n^{\varpi_n}$ is σ_n-directed closed by Lemma 4.24. The beginning of Clause (3) follows from the comments in Remark 4.13. Clause (3a) holds because $\mathbb{P}_n$ forces "$|\mu| = \mathrm{cf}(\mu) = \kappa_n = (\sigma_n)^+$" by Lemma 4.31, hence the last paragraph of Remark 4.13 implies that $\mathbb{S}_n \times \mathbb{P}_n^{\varpi_n}$ forces the same.[34] Clause (3b) is true since

[34]Recall that $\sigma_n := (\kappa_{n-1})^+$ (Setup 4).

$(\mathbb{P}_n^{\varpi_n,\aleph}\downarrow p)/\operatorname{Col}(\delta,{<}\kappa_{n-1})\times\mathbb{R}_n$ is isomorphic to $\operatorname{Col}(\delta,{<}\kappa_{n-1})\times\mathbb{R}_n$, which is δ-closed.[35] Clause (3c) is true since $((\mathbb{P}_n^{\varpi_n}\downarrow p)\times\operatorname{Col}(\delta,{<}\kappa_{n-1}))/(\mathbb{P}^{\varpi_n,\aleph}\downarrow p)$ is isomorphic to $\mathbb{P}_n^{\varpi_n,\aleph}\downarrow p$, which is δ-closed as well. Finally, clause (3d) is trivial. □

We conclude this section, establishing two more facts that will be needed for the proof of the Main Theorem in Section 8.

Definition 4.33. For every $n<\omega$, let $\mathbb{T}_n:=\mathbb{S}_n\times\operatorname{Col}(\sigma_n,{<}\kappa_n)$, and let $\psi_n:\mathbb{P}_n\to\mathbb{T}_n$ be the map defined via

$$\psi_n(p):=\begin{cases}\langle\varpi_n(p),j_n(F^{0p_n})(\alpha_{p_n})\rangle, & \text{if } \ell(p)>0;\\ \langle\mathbb{1}_{\mathbb{S}_n},\emptyset\rangle, & \text{otherwise.}\end{cases}$$

Lemma 4.34. *Let $n<\omega$.*

(1) $\mathbb{T}_n$ is a κ_n-cc poset of size κ_n;
(2) ψ_n defines a nice projection;
(3) $\mathbb{P}_n^{\psi_n}$ is κ_n-directed-closed;
(4) for each $p\in P_n$, $\mathbb{P}_n\downarrow p$ and $(\mathbb{T}_n\downarrow\psi_n(p))\times(\mathbb{P}_n^{\psi_n}\downarrow p)$ are isomorphic. In particular, both are forcing equivalent.

Proof. (1) This is obvious.

(2) Let us go over the clauses of Definition 2.2. Clearly, $\psi_n(\mathbb{1}_{\mathbb{P}})=\langle\mathbb{1}_{\mathbb{S}_n},\emptyset\rangle$, so Clause (1) holds. Likewise, using that ϖ_n is order-preserving it is routine to check that so is ψ_n. Thus, Clause (2) holds, as well.

Let us now prove Clause (3). Let $p\in P_n$ and $t\le_{\mathbb{S}_n\times\operatorname{Col}(\sigma_n,<\kappa_n)}\psi_n(p)$. Putting $t=:\langle s,c\rangle$ we have $s\preceq_n\varpi_n(p)$ and $c\supseteq j_n(F^{0p_n})(\alpha_{p_n})$. On one hand, since ϖ_n is a nice projection, $q:=p+s$ is a condition in $\mathbb{P}_n$. On the other hand, there is a function F and $\beta<\mu$ with $\operatorname{dom}(F)\in E_{n,\beta}$ and $j_n(F)(\beta)=c$.[36] By possibly enlarging a^{q_n} we may actually assume that $\beta=\alpha_q$ and also that $\operatorname{dom}(F)=\pi_{\mathrm{mc}(a^{q_n}),\alpha_{q_n}}``A^{q_n}$. Let r be the condition in $\mathbb{P}_n$ with the same entries as q but with $F^{0r_n}:=F$. By shrinking the measure one set of F if necessary we have that $r\le q\le p$. Also, by the way r is defined, $\psi_n(r)=\langle\varpi_n(r),c\rangle=\langle\varpi_n(q),c\rangle=\langle s,c\rangle=t$.

Note that if $u\in P_n$ is such that $u\le p$ and $\psi_n(u)=t$, then $u\le r$. Altogether, $r=p+t$, which yields Clause (3).[37] Finally, for Clause (4) one argues in the same lines as in Lemma 4.23.

(3) Let $D\subseteq\mathbb{P}_n^{\psi_n}$ be a directed set of size $<\kappa_n$. Then, $\psi_n[D]=\{\langle s,c\rangle\}$ for some $\langle s,c\rangle\in\mathbb{S}_n\times\operatorname{Col}(\sigma_n,{<}\kappa_n)$. Thus, for each $p\in D$, $j_n(F^{0p_n})(\alpha_{p_n})=c$. Arguing as usual, let $a\in[\lambda]^{<\kappa_n}$ be such that both $a\cap\mu$ and a have $\le_{E_n}$-greatest elements α and β, respectively, and $a\supseteq\bigcup_{p\in D}a^{p_n}$. Then, for each

[35] Recall that $\varpi_{n,\aleph}$ stands for the projection from $\mathbb{P}_n\downarrow p$ to $\mathbb{Q}$ induced by ϖ_n and the isomorphism $\mathbb{S}_n\downarrow\varpi_n(p)\simeq\mathbb{Q}\times\operatorname{Col}(\delta,{<}\sigma)$.

[36] Here we use that $\operatorname{Col}(\sigma_n,{<}\kappa_n)^V=\operatorname{Col}(\sigma_n,{<}\kappa_n)^{M_n^*}$, where $M_n^*\cong\operatorname{Ult}(V,E_n\restriction\mu)$.

[37] The $+$ here is regarded with respect to the map ψ_n.

$p, q \in D$, $B_{p,q} := \{\nu < \kappa_n \mid F^{0p_n}(\pi_{\alpha,\alpha_{p_n}}(\nu)) = F^{0q_n}(\pi_{\alpha,\alpha_{q_n}}(\nu))\} \in E_{n,\alpha}$ and, by κ_n-completeness of $E_{n,\alpha}$, $B := \bigcap\{B_{p,q} \mid p, q \in D\} \in E_{n,\alpha}$.

Set $A := \pi^{-1}_{\beta,\alpha}$ "B. By shrinking A if necessary, we may further assume $\pi_{\beta,\mathrm{mc}(a^p_n)}$ "$A \subseteq A^{p_n}$ for each $p \in D$. Since $\psi_n \restriction D$ is constant the map $\varpi_n \colon p \mapsto \langle(\rho^p_k, h^{0p}_k, h^{1p}_k, h^{2p}_k) \mid k < n\rangle$ is so. Let $\langle(\rho_k, h^0_k, h^1_k, h^2_k) \mid k < n\rangle$ be such constant value. For each $k < \omega$, set $f_k := \bigcup_{p\in D} f^p_k$ and F^0 be such that $\mathrm{dom}(F^0) = B$ and $F^0(\nu) := F^{0p}(\pi_{\alpha,\alpha_{p_n}}(\nu))$ for some $p \in D$.

Observe that $\{\langle F^{1p}_n, F^{2p}_n\rangle \mid p \in D\}$ forms a directed subset of $\mathbb{C}_n$ of size $< \kappa_n$ (Lemma 4.29). Using the κ_n-directed-closedness of $\mathbb{C}_n$ we may let $\langle F^1, F^2\rangle \in C_n$ be a $\sqsubseteq$-lower bound. Actually, by using the moreover clause of Lemma 4.29 we may assume that $\mathrm{dom}(F^1) = \mathrm{dom}(F^2) \in E_{n,\alpha}$. Thus, by shrinking A and B if necessary we may assume $\mathrm{dom}(F^1) = \mathrm{dom}(F^2) = B$.

Define $p^* := \langle p^*_k \mid k < \omega\rangle$ as follows:

$$p^*_k := \begin{cases} (f_k, \rho_k, h^0_k, h^1_k, h^2_k), & \text{if } k < n; \\ (a, A, f_k, F^0, F^1, F^2), & \text{if } k = n; \\ (a_k, A_k, f_k, F^0_k, F^1_k, F^2_k), & \text{if } k > n, \end{cases}$$

where $(a_k, A_k, f_k, F^0_k, F^1_k, F^2_k)$ is constructed as described in Lemma 4.20. Clearly, $p^* \in Q_{n0}$ and it gives a $\leq^{\psi_n}$-lower bound for D.

(4) By Item (2) of this lemma, $(\mathbb{T}_n \downarrow \psi_n(p)) \times (\mathbb{P}^{\psi_n}_n \downarrow p)$ projects onto $\mathbb{P}_n \downarrow p$.[38] Actually both posets are easily seen to be isomorphic. □

Lemma 4.35. *Assume* GCH. *Let $n < \omega$.*

(1) $\mathbb{P}_n$ is μ^+-Linked;
(2) $\mathbb{P}_n$ forces CH_θ for any cardinal $\theta \geq \sigma_n$;
(3) $\mathbb{P}^{\varpi_n}_n$ preserves the GCH.

Proof. (1) By Definition 4.10, Lemma 4.20 and the fact that $|H_\mu| = \mu$.

(2) As $\mathbb{P}_n$ has size $\leq \mu^+$, Clause (1) together with a counting-of-nice-names argument implies that $2^\theta = \theta^+$ for any cardinal $\theta \geq \mu^+$. By Lemma 4.31, in any generic extension by $\mathbb{P}_n$, $|\mu| = \mathrm{cf}(\mu) = \kappa_n = (\sigma_n)^+$. It thus left to verify that $\mathbb{P}_n$ forces $2^\theta = \theta^+$ for $\theta \in \{\sigma_n, \kappa_n\}$.

► By Clauses (1), (3) and (4) of Lemma 4.34, together with Easton's lemma, $\mathbb{P}_n$ forces CH_{σ_n} if and only if $\mathbb{T}_n$ forces CH_{σ_n}. By Clause (1) of Lemma 4.34, $\mathbb{T}_n$ is a κ_n-cc poset of size κ_n, so, the number of $\mathbb{T}_n$-nice names for subsets of σ_n is at most $\kappa_n^{<\kappa_n} = \kappa_n = \sigma_n^+$, as wanted.

► The number of $\mathbb{P}_n$-nice names for subsets of κ_n is $((\mu^+)^\mu)^{\kappa_n} = \mu^+$, and hence CH_{κ_n} is forced by $\mathbb{P}_n$.

(3) By Lemma 4.24, $\mathbb{P}^{\varpi_n}_n$ preserves GCH below σ_n. Also, since $\mathbb{P}_n \simeq (\mathbb{P}^{\varpi_n}_n \times \mathbb{S}_n)$ (Remark 4.13) and $|\mathbb{S}_n| < \sigma_n$, we infer from Item (2) above that GCH holds at cardinals $\geq \sigma_n$, as well. □

[38] See Definition 2.2(4).

5. Nice forking projections

In this short section we introduce the notions of *nice* and *super nice forking projection*. These provide a natural (and necessary) strengthening of the corresponding key concept from Part I of this series. The reader familiar with [PRS22, §2] may want to skip some of the details here and, instead, consult the new concepts (Definitions 5.2, 5.4, 5.9) and results (Lemma 5.6).

Definition 5.1 ([PRS21, §4])**.** Suppose that $(\mathbb{P}, \ell_{\mathbb{P}}, c_{\mathbb{P}}, \vec{\varpi})$ is a $(\Sigma, \vec{\mathbb{S}})$-Prikry forcing,[39] $\mathbb{A} = (A, \trianglelefteq)$ is a notion of forcing, and $\ell_{\mathbb{A}}$ and $c_{\mathbb{A}}$ are functions with $\mathrm{dom}(\ell_{\mathbb{A}}) = \mathrm{dom}(c_{\mathbb{A}}) = A$. A pair of functions $(\pitchfork, \pi)$ is said to be a *forking projection* from $(\mathbb{A}, \ell_{\mathbb{A}})$ to $(\mathbb{P}, \ell_{\mathbb{P}})$ iff all of the following hold:

(1) π is a projection from $\mathbb{A}$ onto $\mathbb{P}$, and $\ell_{\mathbb{A}} = \ell_{\mathbb{P}} \circ \pi$;
(2) for all $a \in A$, $\pitchfork(a)$ is an order-preserving function from $(\mathbb{P} \downarrow \pi(a), \leq)$ to $(\mathbb{A} \downarrow a, \trianglelefteq)$;
(3) for all $p \in P$, $\{a \in A \mid \pi(a) = p\}$ admits a greatest element, which we denote by $\lceil p \rceil^{\mathbb{A}}$;
(4) for all $n, m < \omega$ and $b \trianglelefteq^{n+m} a$, $m(a, b)$ exists and satisfies:
$$m(a, b) = \pitchfork(a)(m(\pi(a), \pi(b)));$$
(5) for all $a \in A$ and $r \leq \pi(a)$, $\pi(\pitchfork(a)(r)) = r$;
(6) for all $a \in A$ and $r \leq \pi(a)$, $a = \lceil \pi(a) \rceil^{\mathbb{A}}$ iff $\pitchfork(a)(r) = \lceil r \rceil^{\mathbb{A}}$;
(7) for all $a \in A$, $a' \trianglelefteq^0 a$ and $r \leq^0 \pi(a')$, $\pitchfork(a')(r) \trianglelefteq \pitchfork(a)(r)$.

The pair $(\pitchfork, \pi)$ is said to be a forking projection from $(\mathbb{A}, \ell_{\mathbb{A}}, c_{\mathbb{A}})$ to $(\mathbb{P}, \ell_{\mathbb{P}}, c_{\mathbb{P}})$ iff, in addition to all of the above, the following holds:

(8) for all $a, a' \in A$, if $c_{\mathbb{A}}(a) = c_{\mathbb{A}}(a')$, then $c_{\mathbb{P}}(\pi(a)) = c_{\mathbb{P}}(\pi(a'))$ and, for all $r \in P_0^{\pi(a)} \cap P_0^{\pi(a')}$, $\pitchfork(a)(r) = \pitchfork(a')(r)$.

Definition 5.2. A pair of functions $(\pitchfork, \pi)$ is said to be a *nice forking projection* from $(\mathbb{A}, \ell_{\mathbb{A}}, \vec{\varsigma})$ to $(\mathbb{P}, \ell_{\mathbb{P}}, \vec{\varpi})$ iff all of the following hold:

(a) $(\pitchfork, \pi)$ is a forking projection from $(\mathbb{A}, \ell_{\mathbb{A}})$ to $(\mathbb{P}, \ell_{\mathbb{P}})$;
(b) $\vec{\varsigma} = \vec{\varpi} \bullet \pi$, that is, $\varsigma_n = \varpi_n \circ \pi$ for all $n < \omega$. Also, for each n, ς_n is a nice projection from $\mathbb{A}_{\geq n}$ to $\mathbb{S}_n$, and for each $k \geq n$, $\varsigma_n \restriction \mathbb{A}_k$ is again a nice projection.

The pair $(\pitchfork, \pi)$ is said to be a *nice forking projection* from $(\mathbb{A}, \ell_{\mathbb{A}}, c_{\mathbb{A}}, \vec{\varsigma})$ to $(\mathbb{P}, \ell_{\mathbb{P}}, c_{\mathbb{P}}, \vec{\varpi})$ if, in addition, Clause (8) of Definition 5.1 is satisfied.

Remark 5.3. If $(\mathbb{P}, \ell_{\mathbb{P}}, c_{\mathbb{P}})$ is a Σ-Prikry forcing then a pair of maps $(\pitchfork, \pi)$ is a forking projection from $(\mathbb{P}, \ell_{\mathbb{P}})$ to $(\mathbb{A}, \ell_{\mathbb{A}})$ iff it is a nice forking projection from $(\mathbb{P}, \ell_{\mathbb{P}}, \vec{\varpi})$ to $(\mathbb{A}, \ell_{\mathbb{A}}, \vec{\varsigma})$. In a similar vein, the same applies to forking projections from $(\mathbb{A}, \ell_{\mathbb{A}}, c_{\mathbb{A}}, \vec{\varsigma})$ to $(\mathbb{P}, \ell_{\mathbb{P}}, c_{\mathbb{P}}, \vec{\varpi})$.

As we will see, most of the theory of iterations of $(\Sigma, \vec{\mathbb{S}})$-Prikry forcings can be developed starting from the concept of nice forking projection.

[39] In [PRS21], the following notion is formulated in terms of Σ-Prikry forcings. However the same notion is meaningful in the more general context of $(\Sigma, \vec{\mathbb{S}})$-Prikry forcings.

Nonetheless, to be successful at limit stages, one needs nice forking projections yielding canonical witnesses for niceness of the ς_n's. Roughly speaking, we want that whenever p' is a witness for niceness of some ϖ_n then there is a witness a' for niceness of ς_n which *lifts* p'. This leads to the concept of *super nice forking projection* that we now turn to introduce:

Definition 5.4. A nice forking projection $(\pitchfork, \pi)$ from $(\mathbb{A}, \ell_{\mathbb{A}}, \vec{\varsigma})$ to $(\mathbb{P}, \ell_{\mathbb{P}}, \vec{\varpi})$ is called *super nice* if for each $n < \omega$ the following property holds:

Let $a, a' \in A_{\geq n}$ and $s \in S_n$ such that $a' \trianglelefteq a + s$. Then, for each $p^* \in P_{\geq n}$ such that $p^* \leq^{\varpi_n} \pi(a)$ and $\pi(a') = p^* + \varsigma_n(a')$, there is $a^* \trianglelefteq^{\varsigma_n} a$ with

$$\pi(a^*) = p^* \text{ and } a' = a^* + \varsigma_n(a').$$

The notion of *super nice forking projection* from $(\mathbb{A}, \ell_{\mathbb{A}}, c_{\mathbb{A}}, \vec{\varsigma})$ to $(\mathbb{P}, \ell_{\mathbb{P}}, c_{\mathbb{P}}, \vec{\varpi})$ is defined in a similar fashion.

Remark 5.5. The notion of super nice forking projection will not be needed in the current section. Its importance will only become clear in Section 7, where we present our iteration scheme. The purpose for drawing this distinction is to emphasize that at successor stages of the iteration *nice* forking projection suffice to establishing $(\Sigma, \vec{\mathbb{S}})$-Prikryness of the different iterates. In other words, the iteration machinery introduced in [PRS22, §3] is essentially successful. Nevertheless, *super nice* forking projection will play a crucial role in the verifications at limit stages. For an example, see Claim 7.4.1, p. 62.

Next we show that if $(\pitchfork, \pi)$ is a forking projection (not necessarily nice) and $\vec{\varsigma} = \pi \circ \vec{\varpi}$ then ς_n satisfies Definition 2.2(3) for each $n < \omega$.

Lemma 5.6. *Let $(\pitchfork, \pi)$ be a forking projection from $(\mathbb{A}, \ell_{\mathbb{A}})$ to $(\mathbb{P}, \ell_{\mathbb{P}})$ and suppose that $\vec{\varsigma} = \pi \circ \vec{\varpi}$. Then, for all $a \in A$, $n \leq \ell_{\mathbb{A}}(a)$ and $s \preceq_n \varsigma_n(a)$,*

$$a + s = \pitchfork(a)(\pi(a) + s).$$

Proof. Combining Clauses (2) and (5) of Definition 5.1 with Clause (b) of Definition 5.2 it follows that $\pitchfork(a)(\pi(a) + s) \trianglelefteq a$ and $\varsigma_n(\pitchfork(a)(\pi(a) + s)) = s$.

Let $b \in A$ such that $b \trianglelefteq^0 a$ and $\varsigma_n(b) \preceq_n s$. Then, $\pi(b) \leq \pi(a) + s$. By [PRS22, Lemma 2.17], $b = \pitchfork(b)(\pi(b))$, hence Clauses (2) and (7) of Definition 5.1 yield $b \trianglelefteq^0 \pitchfork(a)(\pi(b)) \trianglelefteq^0 \pitchfork(a)(\pi(a) + s)$.

We are not done yet with establishing that $a + s = \pitchfork(a)(\pi(a) + s)$, as we have just dealt with $b \trianglelefteq^0 a$. However we can further argue as follows. Let $b \trianglelefteq a$ be with $\varsigma_n(b) \preceq_n s$. Put, $b' := \pitchfork(w(a, b))(w(\pi(a), \pi(b)) + \varsigma_n(b))$. It is easy to check that $b \trianglelefteq b' \trianglelefteq^0 a$ and that $\varsigma_n(b') = \varsigma_n(b) \preceq_n s$. Hence, applying the previous argument we arrive at $b \trianglelefteq b' \trianglelefteq^0 \pitchfork(a)(\pi(a) + s)$. ☐

In [PRS22, §2], we drew a map of connections between Σ-Prikry forcings and forking projection, demonstrating that this notion is crucial to define a viable iteration scheme for Σ-Prikry posets. However, to be successful in iterating the Σ-Prikry forcings used in this paper, forking projections need to be accompanied with *types*, which are key to establish the CPP and property $\mathcal{D}$ for $(\mathbb{A}, \ell_{\mathbb{A}})$.

Definition 5.7 ([PRS22, §2])**.** A *type over a forking projection* $(\pitchfork, \pi)$ is a map $\mathrm{tp}\colon A \to {}^{<\mu}\omega$ having the following properties:

(1) for each $a \in A$, either $\mathrm{dom}(\mathrm{tp}(a)) = \alpha + 1$ for some $\alpha < \mu$, in which case we define $\mathrm{mtp}(a) := \mathrm{tp}(a)(\alpha)$, or $\mathrm{tp}(a)$ is empty, in which case we define $\mathrm{mtp}(a) := 0$;
(2) for all $a, b \in A$ with $b \trianglelefteq a$, $\mathrm{dom}(\mathrm{tp}(a)) \leq \mathrm{dom}(\mathrm{tp}(b))$ and for each $i \in \mathrm{dom}(\mathrm{tp}(a))$, $\mathrm{tp}(b)(i) \leq \mathrm{tp}(a)(i)$;
(3) for all $a \in A$ and $q \leq \pi(a)$, $\mathrm{dom}(\mathrm{tp}(\pitchfork(a)(q))) = \mathrm{dom}(\mathrm{tp}(a))$;
(4) for all $a \in A$, $\mathrm{tp}(a) = \emptyset$ iff $a = \lceil \pi(a) \rceil^{\mathbb{A}}$;
(5) for all $a \in A$ and $\alpha \in \mu \setminus \mathrm{dom}(\mathrm{tp}(a))$, there exists a *stretch of a to α*, denoted $a^{\curvearrowright \alpha}$, and satisfying the following:
   (a) $a^{\curvearrowright \alpha} \trianglelefteq^{\pi} a$;
   (b) $\mathrm{dom}(\mathrm{tp}(a^{\curvearrowright \alpha})) = \alpha + 1$;
   (c) $\mathrm{tp}(a^{\curvearrowright \alpha})(i) \leq \mathrm{mtp}(a)$ whenever $\mathrm{dom}(\mathrm{tp}(a)) \leq i \leq \alpha$;
(6) for all $a, b \in A$ with $\mathrm{dom}(\mathrm{tp}(a)) = \mathrm{dom}(\mathrm{tp}(b))$, for every $\alpha \in \mu \setminus \mathrm{dom}(\mathrm{tp}(a))$, if $b \trianglelefteq a$, then $b^{\curvearrowright \alpha} \trianglelefteq a^{\curvearrowright \alpha}$;
(7) For each $n < \omega$, the poset $\mathring{\mathbb{A}}_n$ is dense in $\mathbb{A}_n$, where $\mathring{\mathbb{A}}_n := (\mathring{A}_n, \trianglelefteq)$ and $\mathring{A}_n := \{a \in A_n \mid \pi(a) \in \mathring{P}_n \ \&\ \mathrm{mtp}(a) = 0\}$.

Remark 5.8. Note that Clauses (2) and (3) imply that for all $m, n < \omega$, $a \in \mathring{A}_m$ and $q \leq \pi(a)$, if $q \in \mathring{P}_n$ then $\pitchfork(a)(q) \in \mathring{A}_n$.

In the more general context of $(\Sigma, \vec{\mathbb{S}})$-Prikry forcings – where the pair $(\pitchfork, \pi)$ needs to be a nice forking projection – we need to require a bit more:

Definition 5.9. A *nice type over a nice forking projection* $(\pitchfork, \pi)$ is a type over $(\pitchfork, \pi)$ which moreover satisfies the following:

(8) For each $n < \omega$, the poset $\mathring{\mathbb{A}}_n^{\varsigma_n}$ is dense in $\mathbb{A}_n^{\varsigma_n}$.[40]

Remark 5.10. If $(\mathbb{P}, \ell_{\mathbb{P}}, c_{\mathbb{P}})$ is Σ-Prikry then any forking projection $(\pitchfork, \pi)$ is nice and $\mathring{\mathbb{A}}_n^{\varsigma_n} = \mathring{\mathbb{A}}_n$ for all $n < \omega$. In particular, any type over $(\pitchfork, \pi)$ is nice.

We now turn to collect sufficient conditions — assuming the existence of a nice forking projection $(\pitchfork, \pi)$ from $(\mathbb{A}, \ell_{\mathbb{A}}, c_{\mathbb{A}}, \vec{\varsigma})$ to $(\mathbb{P}, \ell_{\mathbb{P}}, c_{\mathbb{P}}, \vec{\varpi})$ — for $(\mathbb{A}, \ell_{\mathbb{A}}, c_{\mathbb{A}}, \vec{\varsigma})$ to be $(\Sigma, \vec{\mathbb{S}})$-Prikry on its own, and then address the problem of ensuring that the $\mathbb{A}_n$'s be suitable for reflection. This study will be needed in Section 6, most notably, in the proof of Theorem 6.17.

Setup 5. Throughout the rest of this section, we suppose that:

- $\mathbb{P} = (P, \leq)$ is a notion of forcing with a greatest element $\mathbb{1}_{\mathbb{P}}$;
- $\mathbb{A} = (A, \trianglelefteq)$ is a notion of forcing with a greatest element $\mathbb{1}_{\mathbb{A}}$;

[40]Here $\mathring{\mathbb{A}}_n$ is the forcing from Definition 5.7(7).

- $\Sigma = \langle \sigma_n \mid n < \omega \rangle$ is a non-decreasing sequence of regular uncountable cardinals, converging to some cardinal κ, and μ is a cardinal such that $\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}} \check{\mu} = \check{\kappa}^+$;
- $\vec{\mathbb{S}} = \langle \mathbb{S}_n \mid n < \omega \rangle$ is a sequence of notions of forcing, $\mathbb{S}_n = (S_n, \preceq_n)$, with $|S_n| < \sigma_n$;
- $\ell_{\mathbb{P}}$, $c_{\mathbb{P}}$ and $\vec{\varpi} = \langle \varpi_n \mid n < \omega \rangle$ are witnesses for $(\mathbb{P}, \ell_{\mathbb{P}}, c_{\mathbb{P}}, \vec{\varpi})$ being $(\Sigma, \vec{\mathbb{S}})$-Prikry;
- $\ell_{\mathbb{A}}$ and $c_{\mathbb{A}}$ are functions with $\mathrm{dom}(\ell_{\mathbb{A}}) = \mathrm{dom}(c_{\mathbb{A}}) = A$, and $\vec{\varsigma} = \langle \varsigma_n \mid n < \omega \rangle$ is a sequence of functions.
- $(\pitchfork, \pi)$ is a nice forking projection from $(\mathbb{A}, \ell_{\mathbb{A}}, c_{\mathbb{A}}, \vec{\varsigma})$ to $(\mathbb{P}, \ell_{\mathbb{P}}, c_{\mathbb{P}}, \vec{\varpi})$.

Theorem 5.11. *Under the assumptions of Setup 5, $(\mathbb{A}, \ell_{\mathbb{A}}, c_{\mathbb{A}}, \vec{\varsigma})$ satisfies all the clauses of Definition 3.3, with the possible exception of* (2), (7) *and* (9). *Moreover, if $\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}}$ "$\check{\kappa}$ is singular", then $\mathbb{1}_{\mathbb{A}} \Vdash_{\mathbb{A}} \check{\mu} = \check{\kappa}^+$.*

Proof. Clauses (1) and (3) follow respectively from [PRS21, Lemmas 4.3 and 4.7]. Clause (4) holds by virtue of Clause (4) of Definition 5.1. Clauses (5) and (6) are respectively proved in [PRS21, Lemmas 4.7 and 4.10], and Clause (8) follows from Clause (b) of Definition 5.2. Finally, Lemma 3.15(3) yields the moreover part. For more details, see [PRS21, Corollary 4.13]. $\square$

Next, we give sufficient conditions in order for $(\mathbb{A}, \ell_{\mathbb{A}})$ to satisfy the CPP. In Part II of this series we proved that CPP follows from property $\mathcal{D}$ of $(\mathbb{A}, \ell_{\mathbb{A}})$:

Lemma 5.12 ([PRS22, Lemma 2.21])**.** *Suppose that $(\mathbb{A}, \ell_{\mathbb{A}})$ has property $\mathcal{D}$. Then it has the* CPP.

Therefore, everything amounts to find sufficient conditions for $(\mathbb{A}, \ell_{\mathbb{A}})$ to satisfy property $\mathcal{D}$. The following concept will be useful on that respect:

Definition 5.13 (Weak Mixing property)**.** The forking projection $(\pitchfork, \pi)$ is said to have the *weak mixing property* iff it admits a type tp satisfying that for every $n < \omega$, $a \in A$, $\vec{r}$, and $p' \leq^0 \pi(a)$, and for every function $g : W_n(\pi(a)) \to \mathbb{A} \downarrow a$, if there exists an ordinal ι such that all of the following hold:

(1) $\vec{r} = \langle r_\xi \mid \xi < \chi \rangle$ is a good enumeration of $W_n(\pi(a))$;
(2) $\langle \pi(g(r_\xi)) \mid \xi < \chi \rangle$ is diagonalizable with respect to $\vec{r}$, as witnessed by p';[41]
(3) for every $\xi < \chi$:[42]
    - if $\xi < \iota$, then $\mathrm{dom}(\mathrm{tp}(g(r_\xi)) = 0$;
    - if $\xi = \iota$, then $\mathrm{dom}(\mathrm{tp}(g(r_\xi)) \geq 1$;
    - if $\xi > \iota$, then $(\sup_{\eta<\xi} \mathrm{dom}(\mathrm{tp}(g(r_\eta))) + 1 < \mathrm{dom}(\mathrm{tp}(g(r_\xi))$;

[41]Recall Definition 3.17.

[42]The role of the ι is to keep track of the support when we apply the weak mixing lemma in the iteration (see, e.g. [PRS22, Lemma 3.10]).

(4) for all $\xi \in (\iota, \chi)$ and $i \in [\mathrm{dom}(\mathrm{tp}(a)), \sup_{\eta<\xi} \mathrm{dom}(\mathrm{tp}(g(r_\eta)))]$,

$$\mathrm{tp}(g(r_\xi))(i) \leq \mathrm{mtp}(a),$$

(5) $\sup_{\xi<\chi} \mathrm{mtp}(g(r_\xi)) < \omega$,

then there exists $b \trianglelefteq^0 a$ with $\pi(b) = p'$ such that, for all $q' \in W_n(p')$,

$$\pitchfork(b)(q') \trianglelefteq^0 g(w(\pi(a), q')).$$

Remark 5.14. We would like to emphasize that the above notion make sense even when both $(\pitchfork, \pi)$ and tp are not nice. This is simply because the above clauses do not involve the maps ς_n's nor the forcings $\mathring{\mathbb{A}}_n^{\varsigma_n}$.

As shown in [PRS22, §2], the weak mixing property is the key to ensure that $(\mathbb{A}, \ell_{\mathbb{A}})$ has property $\mathcal{D}$. In this respect, the following lemma gathers the results proved in Lemma 2.27 and Corollary 2.28 of [PRS22]:

Lemma 5.15. *Suppose that $(\pitchfork, \pi)$ has the weak mixing property and that $(\mathbb{P}, \ell_{\mathbb{P}})$ has property $\mathcal{D}$. Then $(\mathbb{A}, \ell_{\mathbb{A}})$ has property $\mathcal{D}$, as well.*

In particular, if $(\mathbb{P}, \ell_{\mathbb{P}})$ has property $\mathcal{D}$ and $(\pitchfork, \pi)$ has the weak mixing property, then $(\mathbb{A}, \ell_{\mathbb{A}})$ has both property $\mathcal{D}$ and the CPP.

We still need to verify Clause (2) and (9) of Definition 3.3. Arguing similarly to [PRS22, Lemma 2.29] we can prove the following:

Lemma 5.16. *Suppose that $(\pitchfork, \pi)$ is as in Setup 5 or, just a pair of maps satisfying Clauses* (1), (2), (5) *and* (7) *of Definition 5.1.*

Let $n < \omega$. If $(\pitchfork, \pi)$ admits a type, and $\mathring{\mathbb{A}}_n$ is defined according to the last clause of Definition 5.7, if $\mathring{\mathbb{A}}_n^\pi$ is $\aleph_1$-directed-closed, then so is $\mathring{\mathbb{A}}_n$. Similarly, if $\mathring{\mathbb{A}}_n^\pi$ is σ_n-directed-closed, then so is $\mathring{\mathbb{A}}_n^{\varsigma_n}$.

If in addition $(\pitchfork, \pi)$ admits a nice type then $(\mathbb{A}, \ell_{\mathbb{A}}, c_{\mathbb{A}}, \vec{\varsigma})$ satisfies Clauses (2) *and* (9) *of Definition 3.3.*

Additionally, a routine verification gives the following:

Lemma 5.17. *Suppose that $(\pitchfork, \pi)$ is as in Setup 5. Then, if $\vec{\varpi}$ is a coherent sequence of nice projections then so is $\vec{\varsigma}$.* □

6. Stationary Reflection and Killing a Fragile Stationary Set

In this section, we isolate a natural notion of a *fragile set* and study two aspects of it. In the first subsection, we prove that, given a $(\Sigma, \vec{\mathbb{S}})$-Prikry poset $\mathbb{P}$ and an $r^\star$-fragile stationary set $\dot{T}$, a tweaked version of Sharon's functor $\mathbb{A}(\cdot, \cdot)$ from [PRS22, §4.1] yields a $(\Sigma, \vec{\mathbb{S}})$-Prikry poset $\mathbb{A}(\mathbb{P}, \dot{T})$ admitting a (super) nice forking projection to $\mathbb{P}$ and killing the stationarity of $\dot{T}$. In the second subsection, we make the connection between fragile stationary sets, suitability for reflection and non-reflecting stationary sets. The two subsections can be read independently of each other.

Setup 6. As a setup for the whole section, we assume that $(\mathbb{P}, \ell, c, \vec{\varpi})$ is a given $(\Sigma, \vec{\mathbb{S}})$-Prikry forcing such that $(\mathbb{P}, \ell)$ satisfies property $\mathcal{D}$. Denote $\mathbb{P} = (P, \leq)$, $\Sigma = \langle \sigma_n \mid n < \omega \rangle$, $\vec{\varpi} = \langle \varpi_n \mid n < \omega \rangle$, $\vec{\mathbb{S}} = \langle \mathbb{S}_n \mid n < \omega \rangle$. Also, define κ and μ as in Definition 3.3, and assume that $\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}}$ "$\check{\kappa}$ is singular" and that $\mu^{<\mu} = \mu$. For each $n < \omega$, we denote by $\mathring{\mathbb{P}}_n$ the countably-closed dense subposet of $\mathbb{P}_n$ given by Clause (2) of Definition 3.3. Recall that by virtue of Clause (9), $\mathring{\mathbb{P}}_n^{\varpi_n}$ is a σ_n-directed-closed dense subforcing of $\mathring{\mathbb{P}}_n$. We often refer to $\mathring{\mathbb{P}}_n$ as *the ring* of $\mathbb{P}_n$. In addition, we will assume that $\vec{\varpi}$ is a coherence sequence of nice projections (see Definition 3.8).

The following concept is implicit in the proof of [CFM01, Theorem 11.1]:

Definition 6.1. Suppose $r^\star \in P$ forces that $\dot{T}$ is a $\mathbb{P}$-name for a stationary subset T of μ. We say that $\dot{T}$ is $r^\star$*-fragile* if, looking for each $n < \omega$ at $\dot{T}_n := \{(\check{\alpha}, p) \mid (\alpha, p) \in \mu \times P_n \ \&\ p \Vdash_{\mathbb{P}} \check{\alpha} \in \dot{T}\}$, then, for every $q \leq r^\star$, $q \Vdash_{\mathbb{P}_{\ell(q)}}$ "$\dot{T}_{\ell(q)}$ is nonstationary".

6.1. **Killing one fragile set.** Let $r^\star \in P$ and $\dot{T}$ be a $\mathbb{P}$-name for an $r^\star$-fragile stationary subset of μ. Let $I := \omega \setminus \ell(r^\star)$. By Definition 6.1, for all $q \leq r^\star$, $q \Vdash_{\mathbb{P}_{\ell(q)}}$ "$\dot{T}_{\ell(q)}$ is nonstationary". Thus, for each $n \in I$, we may pick a $\mathbb{P}_n$-name $\dot{C}_n$ for a club subset of μ such that, for all $q \leq r^\star$,

$$q \Vdash_{\mathbb{P}_{\ell(q)}} \dot{T}_{\ell(q)} \cap \dot{C}_{\ell(q)} = \emptyset.$$

Consider the following binary relation:

$$R := \{(\alpha, q) \in \mu \times P \mid q \leq r^\star \ \&\ \forall r \leq q\, (r \Vdash_{\mathbb{P}_{\ell(r)}} \check{\alpha} \in \dot{C}_{\ell(r)})\},$$

and define $\dot{T}^+ := \{(\check{\alpha}, p) \mid (\alpha, p) \in \mu \times P \ \&\ \forall q \leq p, q \Vdash_{\mathbb{P}_{\ell(q)}} \check{\alpha} \notin \dot{C}_{\ell(q)}\}$.[43]

Lemma 6.2.

(1) $r^\star \Vdash_{\mathbb{P}} \dot{T} \subseteq \dot{T}^+$;
(2) If $(\alpha, q) \in R$ *then* $q \Vdash_{\mathbb{P}} \alpha \notin \dot{T}^+$*:*
(3) If $(\alpha, q) \in R$ *and* $q' \leq q$ *then* $(\alpha, q') \in R$.

Proof. (1) Let $q \leq r^\star$ and $\alpha < \mu$ be such that $q \Vdash_{\mathbb{P}} \alpha \in \dot{T}$. We claim that $(\alpha, q) \in \dot{T}^+$ and so that $q \Vdash_{\mathbb{P}} \alpha \in \dot{T}^+$. Let $r \leq q$ be arbitrary. Since $r \Vdash_{\mathbb{P}} \alpha \in \dot{T}$, by definition, $(\alpha, r) \in \dot{T}_{\ell(r)}$. Hence, $r \Vdash_{\mathbb{P}_{\ell(r)}} \alpha \in \dot{T}_{\ell(r)}$. Since $r \leq r^\star$ in fact $r \Vdash_{\mathbb{P}_{\ell(r)}} \alpha \notin \dot{C}_{\ell(r)}$. This yields $(\alpha, q) \in \dot{T}^+$, as wished.

(2) Suppose otherwise. Let $q' \leq q$ so that $(\alpha, q') \in \dot{T}^+$. Notice that this contradicts our assumption that $(\alpha, q) \in R$ because for all $r \leq q$ (in particular, for all $r \leq q'$) $r \Vdash_{\mathbb{P}_{\ell(r)}} \alpha \in \dot{C}_{\ell(r)}$.

(3) This follows from the very definition of R. $\square$

[43]In [PRS22, §4] and in a previous version of this manuscript, the definition of $\dot{T}^+$ was $\{(\check{\alpha}, p) \mid (\alpha, p) \in \mu \times P \ \&\ p \Vdash_{\mathbb{P}_{\ell(p)}} \check{\alpha} \notin \dot{C}_{\ell(p)}\}$. We are grateful to the referee for pointing out that the previous definition was not sufficient to comply with Clauses (1) and (2) of the upcoming Lemma 6.2. This issue applies to [PRS22, Lemma 4.6], as well, and is resolved in the very same way, by taking the new definition of $\dot{T}^+$.

As in [PRS22, §4.1], in this section we attempt to kill the stationarity of the bigger set $\dot{T}^+$ in place of T. The reason for this was explained in [PRS22], but we briefly reproduce it for the reader's benefit: for each $n<\omega$ let τ_n be the $\mathbb{P}_n$-name $\{(\check{\alpha},p)\in\dot{T}^+\mid \alpha\in\mu\ \&\ p\in P_n\}$. Intuitively speaking, τ_n is the *trace* of $\dot{T}^+$ to a $\mathbb{P}_n$-name. The two key properties of τ_n are:

- $\tau_n\subseteq\dot{T}^+$
- $p\Vdash_{\mathbb{P}_n}\tau_n=(\check{\mu}\setminus\dot{C}_n)$ for all $p\in P_n$.

These two features of τ_n were crucially used in [PRS22, Lemma 4.28] when we verified the density of the *ring poset* $(\mathring{\mathbb{P}}_\delta)_n$ in $(\mathbb{P}_\delta)_n$, for $\delta\in\operatorname{acc}(\mu^++1)$.

The next upcoming lemma is a generalization of [PRS21, Claim 5.6.1]:

Lemma 6.3. *For all $p\le r^\star$ and $\gamma<\mu$, there is an ordinal $\bar{\gamma}\in(\gamma,\mu)$ and $\bar{p}\le^{\vec{\varpi}}p$, such that $(\bar{\gamma},\bar{p})\in R$.*

Proof. We begin by proving the following auxiliary claim:

Claim 6.3.1. *For all $p\le r^\star$ and $\gamma<\mu$, there is an ordinal $\bar{\gamma}\in(\gamma,\mu)$ and $\bar{p}\le^{\vec{\varpi}}p$, such that for all $q\le\bar{p}$, $q\Vdash_{\mathbb{P}_{\ell(q)}}\dot{C}_{\ell(q)}\cap(\check{\gamma},\check{\bar{\gamma}})\neq\check{\emptyset}$.*

Proof. Let γ and p be as above. Set $\ell:=\ell(p)$, $s:=\varpi_\ell(p)$ and put

$$D_{p,\gamma}:=\{q\in P\mid q\le p\ \&\ \exists\gamma'>\gamma\ (q\Vdash_{\mathbb{P}_{\ell(q)}}\check{\gamma}'\in\dot{C}_{\ell(q)})\}.$$

Clearly, $D_{p,\gamma}$ is 0-open. Hence appealing to Clause (2) of Lemma 3.14 we obtain a condition $\bar{p}\le^{\vec{\varpi}}p$ with the property that the set

$$U_{D_{p,\gamma}}:=\{t\preceq_\ell s\mid\forall n<\omega\ (P_n^{\bar{p}+t}\subseteq D_{p,\gamma}\text{ or }P_n^{\bar{p}+t}\cap D_{p,\gamma}=\emptyset)\}$$

is dense in $\mathbb{S}_\ell\downarrow s$.

We note that $U_{D_{p,\gamma}}=\{t\preceq_\ell s\mid\forall n<\omega\ (P_n^{\bar{p}+t}\subseteq D_{p,\gamma})\}$: Fix $t\in U_{D_{p,\gamma}}$ and $n<\omega$. Let $q\le^n\bar{p}+t$ be arbitrary. Since $q\le\bar{p}\le p\le r^\star$ we have that $q\Vdash_{\mathbb{P}_{\ell(q)}}$ "$\dot{C}_{\ell(q)}$ is a club in μ". Thus, there is $q'\le^0 q$ and $\gamma'\in(\gamma,\mu)$ such that $q'\Vdash_{\mathbb{P}_{\ell(q)}}\gamma'\in\dot{C}_{\ell(q)}$. By definition, $q'\in D_{p,\gamma}\cap P_n^{\bar{p}+t}$, so that this latter intersection is non-empty. Since we picked $t\in U_{D_{p,\gamma}}$ it follows that $P_n^{\bar{p}+t}\subseteq D_{p,\gamma}$. This proves the above-displayed equality.

For each condition $r\in W(\bar{p}+t)$ pick some ordinal $\gamma_r\in(\gamma,\mu)$ witnessing that $r\in D_{p,\gamma}$, and put

$$\bar{\gamma}:=\sup\{\gamma_r\mid r\in W(\bar{p}+t)\ \&\ t\preceq_\ell s\}+1.$$

Combining Clauses (β) and (5) of Definition 3.3 we infer that $\bar{\gamma}<\mu$.

We claim that $\bar{p}$ is as desired. Otherwise, let $q\le\bar{p}$ forcing the negation of the claim. By virtue of Clause (8) of Definition 3.3, ϖ_ℓ is a nice projection from $\mathbb{P}_{\ge\ell}$ to $\mathbb{S}_\ell$, hence Definition 2.2(4) applied to this map yields $q=\bar{q}+\varpi_\ell(q)$, for some $\bar{q}\le^{\varpi_\ell}\bar{p}$. Putting $t:=\varpi_\ell(q)$, it is clear that $t\preceq_\ell s$. By extending t if necessary, we may freely assume that $t\in U_{D_{p,\gamma}}$.

On the other hand, $q\le^0 w(\bar{p},q)$, hence Lemma 3.9 and Clause (3) of Lemma 3.10 yield $q\le^0 w(\bar{p},\bar{q}+t)+t=w(\bar{p},\bar{q})+t\in W(\bar{p}+t)$. This clearly yields a contradiction with our choice of q. $\square$

Now we take advantage of the previous claim to prove the lemma. So, let $p \le r^\star$ and $\gamma < \mu$. Applying the above claim inductively, we find a $\le^{\vec{\varpi}}$-decreasing sequence $\langle p_n \mid n < \omega\rangle$ and an increasing sequence of ordinals below μ, $\langle \gamma_n \mid n < \omega\rangle$, such that $p_0 := p$, $\gamma_0 := \gamma$, and such that for every $n < \omega$, the pair (p_{n+1}, γ_{n+1}) witnesses the conclusion of Claim 6.3.1 when putting $(p, \gamma) := (p_n, \gamma_n)$. Moreover, density of $\mathring{\mathbb{P}}^{\varpi_\ell}_\ell$ in $\mathbb{P}^{\varpi_\ell}_\ell$ (Definition 3.3(9)) enable us to assume that the $p_n \in \mathring{\mathbb{P}}^{\varpi_\ell}_\ell \downarrow p$, In particular, there is $\bar{p}$ a $\le^{\vec{\varpi}}$-lower bound for the sequence. Setting $\bar{\gamma} := \sup_{n<\omega} \gamma_n$ we have that $(\bar{\gamma}, \bar{p}) \in R$. $\square$

6.1.1. *Definition of the functor and basic properties.*

Definition 6.4. Let p be a condition in $\mathbb{P}$. A *labeled* $\langle p, \vec{\mathbb{S}}\rangle$*-tree* is a function $S\colon \operatorname{dom}(S) \to [\mu]^{<\mu}$, where

$$\operatorname{dom}(S) = \{(q, t) \mid q \in W(p) \ \&\ t \in \bigcup_{\ell(p) \le n \le \ell(q)} \mathbb{S}_n \downarrow \varpi_n(q)\},$$

and such that for all $(q, t) \in \operatorname{dom}(S)$ the following hold:

1. $S(q, t)$ is a closed bounded subset of μ;
2. $S(q', t') \supseteq S(q, t)$ whenever $q' + t' \le q + t$;
3. $q + t \Vdash_{\mathbb{P}} S(q, t) \cap \dot{T}^+ = \emptyset$;
4. there is $m < \omega$ such that for any $q \in W(p)$ and $(q', t') \in \operatorname{dom}(S)$ with $q' \le q$, if $S(q', t') \ne \emptyset$ and $\ell(q) \ge \ell(p) + m$, then $(\max(S(q', t')), q) \in R$. The least such m is denoted by $m(S)$.

Remark 6.5. By Clause (4) and the definition of R, for any $(q', t'), (q, t)$ in $\operatorname{dom}(S)$ with $q' + t' \le q + t$, if q is incompatible with $r^\star$ then $S(q', t') = \emptyset$.

Definition 6.6. For $p \in P$, we say that $\vec{S} = \langle S_i \mid i \le \alpha\rangle$ is a $\langle p, \vec{\mathbb{S}}\rangle$*-strategy* iff all of the following hold:

1. $\alpha < \mu$;
2. for all $i \le \alpha$, S_i is a labeled $\langle p, \vec{\mathbb{S}}\rangle$-tree;
3. for every $i < \alpha$ and $(q, t) \in \operatorname{dom}(S_i)$, $S_i(q, t) \sqsubseteq S_{i+1}(q, t)$;
4. for every $i < \alpha$ and pairs $(q, t), (q', t')$ in $\operatorname{dom}(S_i)$ with $q' + t' \le q + t$, $S_{i+1}(q, t) \setminus S_i(q, t) \sqsubseteq S_{i+1}(q', t') \setminus S_i(q', t')$;
5. for every limit $i \le \alpha$ and $(q, t) \in \operatorname{dom}(S_i)$, $S_i(q, t)$ is the ordinal closure of $\bigcup_{j<i} S_j(q, t)$.

Definition 6.7. Let $\mathbb{A}(\mathbb{P}, \vec{\mathbb{S}}, \dot{T})$ be the notion of forcing $\mathbb{A} := (A, \trianglelefteq)$, where:

1. $(p, \vec{S}) \in A$ iff $p \in P$ and either $\vec{S} = \emptyset$ or $\vec{S}$ is a $\langle p, \vec{\mathbb{S}}\rangle$-strategy;
2. $(p', \vec{S'}) \trianglelefteq (p, \vec{S})$ iff:
   (a) $p' \le p$;
   (b) $\operatorname{dom}(\vec{S'}) \ge \operatorname{dom}(\vec{S})$;
   (c) for each $i \in \operatorname{dom}(\vec{S})$ and $(q, t) \in \operatorname{dom}(S'_i)$,

$$S'_i(q, t) = S_i(w(p, q), t_q),$$

where $t_q := \varpi_{\ell(q)}(q+t)$.[44]

For all $p \in P$, denote $\lceil p \rceil^{\mathbb{A}} := (p, \emptyset)$.

Definition 6.8 (The maps)**.**

(1) Let $\ell_{\mathbb{A}} := \ell \circ \pi$ and $\vec{\varsigma} := \vec{\varpi} \bullet \pi$, where $\pi : \mathbb{A} \to \mathbb{P}$ is defined via $\pi(p, \vec{S}) := p$;

(2) Define $c_{\mathbb{A}} : A \to H_\mu$, by letting, for all $(p, \vec{S}) \in A$,
$$c_{\mathbb{A}}(p, \vec{S}) := (c(p), \{(i, c(q), S_i(q, \cdot)) \mid i \in \mathrm{dom}(\vec{S}), q \in W(p)\}),$$
where $S_i(q, \cdot)$ denotes the map $t \mapsto S_i(q, t)$;

(3) Let $a = (p, \vec{S}) \in A$. The map $\pitchfork(a) : \mathbb{P} \downarrow p \to A$ is defined by letting $\pitchfork(a)(p') := (p', \vec{S'})$, where $\vec{S'}$ is a sequence such that $\mathrm{dom}(\vec{S'}) = \mathrm{dom}(\vec{S})$, and for all $i \in \mathrm{dom}(\vec{S'})$ the following are true:
  (a) $\mathrm{dom}(S'_i) = \{(r, t) \mid r \in W(p') \ \&\ t \in \bigcup_{\ell(p') \le n \le \ell(r)} \mathbb{S}_n \downarrow \varpi_n(r)\}$,
  (b) for all $(q, t) \in \mathrm{dom}(S'_i)$,

$$(*) \qquad S'_i(q, t) = S_i(w(p, q), t_q).$$

Remark 6.9. If $(\mathbb{P}, \ell, c)$ is Σ-Prikry then a moment's reflection reveal that we arrive at the corresponding notions from [PRS22, §4].

We next show that $(\pitchfork, \pi)$ defines a super nice forking projection from $(\mathbb{A}, \ell_{\mathbb{A}}, c_{\mathbb{A}}, \vec{\varsigma})$ to $(\mathbb{P}, \ell, c, \vec{\varpi})$. The next lemma takes care partially of this task by showing that $(\pitchfork, \pi)$ is a forking projection from $(\mathbb{A}, \ell_{\mathbb{A}}, c_{\mathbb{A}})$ to $(\mathbb{P}, \ell, c)$.

Lemma 6.10 (Forking projection)**.** *The pair $(\pitchfork, \pi)$ is a forking projection between $(\mathbb{A}, \ell_{\mathbb{A}}, c_{\mathbb{A}})$ and $(\mathbb{P}, \ell, c)$.*

Proof. We just give some details for the verification of Clause (2). The rest can be proved arguing similarly to [PRS21, Lemma 6.13]. Here it goes:

(2) Let $a = (p, \vec{S})$ and $p' \le \pi(a)$. We just proved that $\pitchfork(a)$ is well-defined. The argument for $\pitchfork(a)$ being order-preserving is very similar to the one from [PRS21, Lemma 6.13(2)]. If $\vec{S} = \emptyset$, then Definition 6.8(*) yields $\pitchfork(a)(p') = (p', \emptyset) \in A$. So, suppose that $\mathrm{dom}(\vec{S}) = \alpha + 1$. Let $(p', \vec{S'}) := \pitchfork(a)(p')$ and $i \le \alpha$. We shall first verify that S'_i is a $\langle p', \mathbb{S}\rangle$-labeled tree. Let $(q, t) \in \mathrm{dom}(S'_i)$ and let us go over the clauses of Definition 6.4. Since the verification of Clauses (3) and (4) are on the same lines of that of (2) we just give details for the latter.

(2): Let $(q', t') \in \mathrm{dom}(S'_i)$ be such that $q' + t' \le q + t$. By Clause (6) of Definition 3.3, $w(p, q' + t') \le w(p, q + t)$. Also, combining [PRS21, Lemma 2.9] and Lemma 3.9 we have the following chain of equalities:
$$w(p, q' + t') = w(p, w(p', q' + t')) = w(p, w(p', q')) = w(p, q').$$
Similarly one shows $w(p, q + t) = w(p, q)$. Thus, $w(p, q') \le w(p, q)$.

[44] Note that $t_q \preceq_n \varpi_n(w(p, q))$ for some $n \in [\ell(p'), \ell(q)]$. Thus, $(w(p, q), t_q) \in \mathrm{dom}(S_i)$. And if $t \preceq_{\ell(q)} \varpi_{\ell(q)}(q)$, then $t = t_q$.

By coherency of $\vec{\varpi}$ (Definition 3.8(2)) we have

$$\varpi_{\ell(q)}(w(p,q')+t'_{q'}) = \pi_{\ell(q'),\ell(q)}(\varpi_{\ell(q')}(w(p,q')+t'_{q'}))$$
$$= \pi_{\ell(q'),\ell(q)}(t'_{q'}) = \pi_{\ell(q'),\ell(q)}(\varpi_{\ell(q')}(q'+t')) = \varpi_{\ell(q)}(q'+t')$$

Since $q'+t' \leq q+t$ we also have $\varpi_{\ell(q)}(q'+t') \preceq_{\ell(q)} \varpi_{\ell(q)}(q+t) = t_q$.

Thereby, combining both things we arrive at

$$w(p,q')+t'_{q'} \leq w(p,q)+t_q.$$

Finally, use Clause (2) for the labeled $\langle p, \vec{\mathbb{S}}\rangle$-tree S_i to get that

$$S'_i(q',t') = S_i(w(p,q'),t'_{q'}) \supseteq S_i(w(p,q),t_q) = S'_i(q,t).$$

To prove that $(p', \vec{S}') \in A$ it is left to argue that $\vec{S}'$ fulfils the requirements described in Clauses (3), (4) and (5) of Definition 6.6. Indeed, each of these clauses follow from the corresponding ones for $\vec{S}$. There is just one delicate point in Clause (4), where one needs to argue that $w(p,q')+t'_{q'} \leq w(p,q)+t_q$. This is again done as in the verification of Clause (2) above.

Finally, it is clear that $\pitchfork(a)(p') = (p', \vec{S}') \trianglelefteq (p, \vec{S})$ (see Definition 6.7). This concludes the verification of Clause (2). $\square$

Lemma 6.11. *For each $n < \omega$, ς_n is a nice projection from $\mathbb{A}_{\geq n}$ to $\mathbb{S}_n$, and for each $k \geq n$, $\varsigma_n \restriction \mathbb{A}_k$ is again a nice projection.*

Proof. We go over the clauses of Definition 2.2. Clauses (1) and (2) follow from the fact that ς_n is the composition of the projections ϖ_n and π and Clause (3) follows from Lemma 5.6 and Lemma 6.10. To complete the argument we prove the following, which, in particular, yields Clause (4).

Claim 6.11.1. *Let $a, a' \in A_{\geq n}$ and $s \preceq_n \varsigma_n(a)$ with $a' \trianglelefteq a+s$. Then, for each $p^* \in P_{\geq n}$ such that $p^* \leq^{\varpi_n} \pi(a)$ and $\pi(a') = p^* + \varsigma_n(a')$ there is $a^* \in A_{\geq n}$ such that $a^* \trianglelefteq^{\varsigma_n} a$ with $\pi(a^*) = p^*$ and $a' = a^* + \varsigma_n(a')$.*

In particular, ς_n satisfies Clause (4).

Proof. Let $a = (p, \vec{S})$, $a' = (p', \vec{S}')$ and $s \preceq_n \varsigma_n(a)$ be as above.

By Lemma 5.6, $a' \trianglelefteq a+s = \pitchfork(a)(p+s)$, hence $p' \leq p+s$. Since ϖ_n is a nice projection from $\mathbb{P}_{\geq n}$ to $\mathbb{S}_n$, Definition 2.2(4) yields the existence of a condition $p^* \in P_{\geq n}$ such that $p^* \leq^{\varpi_n} p$ and $p' = p^* + \varsigma_n(a')$. So, let p^* be some such condition and set $t := \varpi_n(p')$. We have that $\varsigma_n(a') = t$. Our aim is to find a sequence $\vec{S}^*$ such that $a^* := (p^*, \vec{S}^*)$ is a condition in $\mathbb{A}_{\geq n}$ with the property that $a^* \trianglelefteq^{\varsigma_n} a$ and $a^* + t = a'$.

As $n \leq \ell(p^*)$, coherency of $\vec{\varpi}$ yields ϖ_n“$W(p^*) = \{\varpi_n(p)\}$ (see Definition 3.8(1)), hence $q+t$ is well-defined for all $q \in W(p^*)$. Moreover, by virtue of Lemma 3.10(3), $q+t \in W(p^*+t) = W(p')$ for every $q \in W(p^*)$.

Put $\vec{S} := \langle S_i \mid i \leq \alpha\rangle$ and $\vec{S}' := \langle S'_i \mid i \leq \beta\rangle$. Let $\vec{S}^* := \langle S_i \mid i \leq \beta\rangle$ be the sequence where for each $i \leq \beta$, S^*_i is the function with domain $\{(q,u) \mid q \in W(p^*) \,\&\, u \in \bigcup_{\ell(p^*) \leq m \leq \ell(q)} \mathbb{S}_m \downarrow \varpi_m(q)\}$ defined according to the following casuistic:

(a) If $\vec{S}$ is the empty sequence, then

$$S_i^*(q,u) := \begin{cases} S_i'(q+t, u_q), & \text{if } q+u \leq q+t; \\ \emptyset, & \text{otherwise.} \end{cases}$$

(b) If $\vec{S}$ is non-empty then there are two more cases to consider:
  (1) If $\alpha < i \leq \beta$, then

$$S_i^*(q,u) := \begin{cases} S_i'(q+t, u_q), & \text{if } q+u \leq q+t; \\ S_\alpha(w(p,q), u_q), & \text{otherwise.} \end{cases}$$

  (2) Otherwise, $S_i^*(q,u) := S_i(w(p,q), u_q)$.

By Lemma 3.10(4), $q+u = q+u_q$ for all $(q,u) \in \operatorname{dom}(S_i^*)$ and $i \leq \beta$.

We next show that $\vec{S}^*$ is a $\langle p^*, \vec{\mathbb{S}}\rangle$-strategy. To this aim we go over the clauses of Definition 6.6. Clause (1) is indeed obvious. As for the rest we just provide details for (2). The reason for this choice is the following: first, the verification of (2) makes a crucial use of the notion of *coherent system of projections* (Definition 3.8) and of the t_q component in the labeled trees (see Definition 6.7(c));[45] second, this verification contains all the key ingredients to assist the reader in the verification of the rest of clauses.

So, let us verify that S_i^* is a labeled $\langle p^*, \vec{S}\rangle$-tree for all $i \leq \beta$. Fix some $i \leq \beta$ and let us go over the clauses of Definition 6.4.

(1): This is obvious.

(2): Let $(q',u'), (q,u) \in \operatorname{dom}(S_i^*)$ such that $q'+u' \leq q+u$.

Case (a): We need to distinguish two subcases:

► If $q+u \not\leq q+t$, then $S_i^*(q,u) = \emptyset$ and so $S_i^*(q,u) \subseteq S_i^*(q',u')$.

► Otherwise, $q+u \leq q+t$ and so $S_i^*(q,u) = S_i'(q+t, u_q)$.[46] On the other hand, since $q'+u' \leq q+u \leq q+t$, we have that $\varpi_n(q'+u') \preceq_n \varpi_n(q+t) = t$. In particular, $q'+u' \leq q'+t$ and so $S_i^*(q',u') = S_i'(q'+t, u'_{q'})$.

Now, it is routine to check that $(q+t)+u_q = q+u_q = q+u$.[47] Similarly, the same applies to q' and $u'_{q'}$. Appealing to Clause (2) for S_i' we get $S_i^*(q,u) \subseteq S_i^*(q',u')$.

Case (b)(1): There are several cases to consider:

► Assume $q+u \not\leq q+t$. Then $S_i^*(q,u) = S_\alpha(w(p,q), u_q)$.

►► Suppose $q'+u' \not\leq q+t$. Then $S_i^*(q',u') = S_\alpha(w(p,q'), u'_{q'})$. On one hand, $w(p, q'+u') \leq w(p, q+u)$ (see Definition 3.3(6)). Combining [PRS21, Lemma 2.9] with Lemma 3.9, we have $w(p, q'+u') = w(p, w(p^*, q'+u')) = w(p, w(p^*, q')) = w(p,q')$. Similarly, one shows that $w(p, q+u) = w(p,q)$. Thus, $w(p,q') \leq w(p,q)$. Also, arguing as in page 45, one can prove that

[45] This is yet another wrinkle compared with the non-collapses scenario.

[46] This follows from the definition of the forcing order.

[47] Note that the coherency of $\vec{\varpi}$ is used at this point (recall Definition 3.8).

$w(p,q') + u'_{q'} \leq w(p,q) + u_q$. This finally yields

$$S^*_i(q,u) = S_\alpha(w(p,q),u_q) \subseteq S_\alpha(w(p,q'),u'_{q'}) = S^*_i(q',u').$$

►► Otherwise, $q' + u' \leq q + t$, and so $S^*_i(q',u') = S'_i(q'+t,u'_{q'})$.

Since $\alpha < i$ and $b \trianglelefteq a$, Clauses (3) and (5) of Definition 6.6 for $\vec{S}'$ yield

$$S_\alpha(w(p,q'+t),u^*) = S'_\alpha(q'+t,u'_{q'}) \subseteq S'_i(q'+t,u'_{q'}),$$

where $u^* := \varpi_{\ell(q')}((q'+t)+u'_{q'})$. One can check that $(q'+t)+u'_{q'} = q'+u'_{q'}$, hence $u^* = u'_{q'}$, and thus $S_\alpha(w(p,q'+t),u'_{q'}) \subseteq S'_i(q'+t,u'_{q'})$.

Arguing as in the previous case, $w(p,q'+t) = w(p,q')$. Therefore,

$$S_\alpha(w(p,q'),u'_{q'}) \subseteq S'_i(q'+t,u'_{q'}) = S^*_i(q',u').$$

Once again, $w(p,q') + u'_{q'} \leq w(p,q) + u_q$. Hence, Clause (2) for S_α yields

$$S^*_i(q,u) = S_\alpha(w(p,q),u_q) \subseteq S_\alpha(w(p,q'),u'_{q'}) \subseteq S^*_i(q',u').$$

► Assume $q + u \leq q + t$. Then $q' + u' \leq q + t$, as well. In particular,

$$S^*_i(q,u) = S'_i(q+t,u_q) \subseteq S'_i(q'+t,u'_{q'}) = S^*_i(q',u'),$$

where the above follows from Clause (2) for S'_i.[48]

Case (b)(2): This is clear using Clause (2) for S_i.

After the above verification we infer that $a^* = (p^*, \vec{S}^*) \in \mathbb{A}_{\geq n}$. One can check that $a^* \trianglelefteq^{\varsigma_n} a$. So we are left with checking that $a^* + t = a'$.[49]

By Lemma 5.6, $a^* + t = \pitchfork(a^*)(p^* + t)$. Thus, since $p^* + t = p'$, $a^* + t = \pitchfork(a^*)(p')$. We will be done by showing that $\pitchfork(a^*)(p') = a'$.

Put $\pitchfork(a^*)(p') = (p', \vec{Q})$. Let $i \leq \beta$ and $(q,u) \in \mathrm{dom}(Q_i)$. By virtue of Definition 6.8(3) we have that $q \leq p'$ and $u \preceq_m \varpi_m(q)$. Using coherency of $\vec{\varpi}$ (particularly, Definition 3.8(2)) one can check that $q + u \leq q + t$.

Case (a): In this case we have the following chain of equalities:

$$Q_i(q,u) = S^*_i(w(p^*,q),u_q) = S'_i(w(p^*,q)+t,u_q) = S'_i(q,u_q) = S'_i(q,u).$$

The first equality follows from Definition 6.8(3)(*); the third from Lemma 3.10(1); and the right-most combining Definition 6.4(2) with $q + u_q = q + u$.

Case (b): If $\alpha < i \leq \beta$ then arguing as before $Q_i(q,u) = S'_i(q,u)$.

Otherwise, $i \leq \alpha$ and we have the following chain of equalities

$$Q_i(q,u) = S^*_i(w(p^*,q),u_q) = S_i(w(p,q),u_q) = S'_i(q,u_q) = S'_i(q,u),$$

Note that for the third equality we used $a' \trianglelefteq a$ and $u_q = \varpi_{\ell(q)}(q + u_q)$. □

The above claim finishes the proof of the lemma. □

Corollary 6.12. *The pair* $(\pitchfork, \pi)$ *is a super nice forking projection from* $(\mathbb{A}, \ell_{\mathbb{A}}, c_{\mathbb{A}}, \vec{\varsigma})$ *to* $(\mathbb{P}, \ell, c, \vec{\varpi})$.

[48] See also the argument of Case (a) above.

[49] Recall that $t = \varpi_n(p') = \varsigma_n(a')$.

Proof. First, $(\pitchfork, \pi)$ is a forking projection from $(\mathbb{A}, \ell_{\mathbb{A}}, c_{\mathbb{A}}, \vec{\varsigma})$ to $(\mathbb{P}, \ell, c, \vec{\varpi})$ by virtue of Lemma 6.10. Second, $\vec{\varsigma} = \vec{\varpi} \bullet \pi$ by our choice in Definition 6.8. Besides, Lemma 6.11 shows that for each $n < \omega$, ς_n is a nice projection from $\mathbb{A}_{\geq n}$ and $\mathbb{S}_n$. Moreover, Claim 6.11.1 actually shows that $(\pitchfork, \pi)$ is super nice (see Definition 5.4). This completes the proof. □

Next, we introduce a map tp and we will later prove that it indeed defines a nice type over $(\pitchfork, \pi)$. Afterwards, we will also show that tp witnesses the pair $(\pitchfork, \pi)$ to satisfy the weak mixing property.

Definition 6.13. Define a map $\mathrm{tp} : A \to {}^{<\mu}\omega$, as follows.

Given $a = (p, \vec{S})$ in A, write $\vec{S}$ as $\langle S_i \mid i < \beta\rangle$, and then let

$$\mathrm{tp}(a) := \langle m(S_i) \mid i < \beta\rangle.$$

We shall soon verify that tp is a nice type, but will use the mtp notation of Definition 5.7 from the outset. In particular, for each $n < \omega$, we will have $\mathring{\mathbb{A}}_n := (\mathring{A}_n, \trianglelefteq)$, with $\mathring{A}_n := \{a \in A \mid \pi(a) \in \mathring{P}_n \,\&\, \mathrm{mtp}(a) = 0\}$. We will often refer to the $m(S_i)$'s as the **delays** of the strategy $\vec{S}$.

Arguing along the lines of [PRS21, Lemma 6.15] one can prove the following:

Fact 6.14. *For each $n < \omega$, $\mathring{\mathbb{A}}^{\pi}_n$ is a μ-directed closed forcing.*

Lemma 6.15. *The map* tp *is a nice type over* $(\pitchfork, \pi)$.

Proof. The verification of Clauses (1)–(6) of Definition 5.9 is essentially the same as in [PRS22, Lemma 4.19]. A moment's reflection makes it clear that it suffices to prove Clause (8) to complete the lemma.

Let $a = (p, \vec{S}) \in A$ and to avoid trivialities, let us assume that $\vec{S} \neq \emptyset$.

► Suppose that p is incompatible with $r^\star$. Then, by Remark 6.5, for all $i < \mathrm{dom}(\mathrm{tp}(a))$ and all $(q, t) \in \mathrm{dom}(S_i)$, $S_i(q, t) = \emptyset$. Therefore, $\mathrm{mtp}(a) = 0$. Using Definition 3.3(9), let $p' \leq^{\vec{\varpi}} p$ be in $\mathring{P}_{\ell(p)}$ and set $b := \pitchfork(a)(p')$. Combining Clauses (2) and (3) of Definition 5.7 with $\mathrm{mtp}(a) = 0$ it is immediate that $\mathrm{mtp}(b) = 0$. Also, $\pi(b) = p' \in \mathring{P}_{\ell(p)}$. Thus, $b \in \mathring{A}_{\ell_{\mathbb{A}}(a)}$ and $b \trianglelefteq^{\vec{\varsigma}} a$.

► Suppose $p \leq r^\star$. Appealing to Clause (5) of Definition 3.3 let $\gamma < \mu$ be above $\sup_{i<\mathrm{dom}(\vec{S})}\{S_i(q, s) \mid (q, s) \in \mathrm{dom}(S_i)\}$ and $\mathrm{dom}(\vec{S})$. By Lemma 6.3, let $\bar{\gamma} \in (\gamma, \mu)$ and $\bar{p} \leq^{\vec{\varpi}} p$ such that $(\bar{\gamma}, \bar{p}) \in R$. Using Definition 3.3(9) we may further assume that $\bar{p}$ belongs to $\mathring{P}_{\ell(p)}$.

Next, define a sequence $\vec{T} = \langle T_i \mid i \leq \bar{\gamma}\rangle$ with

$$\mathrm{dom}(T_i) := \{(q, u) \mid q \in W(\bar{p}) \,\&\, u \in \bigcup_{\ell(p^*) \leq m \leq \ell(q)} \mathbb{S}_m \downarrow \varpi_m(q)\},$$

as follows

$$T_i(q, u) := \begin{cases} S_i(w(p, q), u_q), & \text{if } i < \mathrm{dom}(\vec{S}), \\ S_{\max(\mathrm{dom}(\vec{S}))}(w(p, q), u_q) \cup \{\bar{\gamma}\}, & \text{otherwise.} \end{cases}$$

Arguing as in Claim 6.11.1 one shows that $(\bar{p}, \vec{T})$ is a condition in $\mathring{\mathbb{A}}_{\ell(p)}$. Clearly $b \trianglelefteq^{\vec{\varsigma}} a$, so $\mathring{\mathbb{A}}^{\varsigma_n}_n$ is dense in $\mathbb{A}^{\varsigma_n}_n$. □

We now check that the pair $(⫛, \pi)$ has the weak mixing property, as witnessed by the type tp given in Definition 6.13 (see Definition 5.13).

Lemma 6.16. *The pair* $(⫛, \pi)$ *has the weak mixing property as witnessed by the type* tp *from Definition 6.13.*

Proof. Let a, $\vec{r}$, $p' \leq^0 \pi(a)$, $g\colon W_n(\pi(a)) \to \mathbb{A} \downarrow a$ and ι be as in the statement of the Weak Mixing Property (see Definition 5.13). More precisely, $\vec{r} = \langle r_\xi \mid \xi < \chi\rangle$ is a good enumeration of $W_n(\pi(a))$, $\langle \pi(g(r_\xi)) \mid \xi < \chi\rangle$ is diagonalizable with respect to $\vec{r}$ (as witnessed by p') and g is a function witnessing Clauses (3)–(5) of Definition 5.13 with respect to the type tp.

Put $a := (p, \vec{S})$ and for each $\xi < \chi$, set $(p_\xi, \vec{S}^\xi) := g(r_\xi)$.

Claim 6.16.1. *If* $\iota \geq \chi$ *then there is a condition* b *in* $\mathbb{A}$ *as in the conclusion Definition 5.13.*

Proof. If $\iota \geq \chi$ then Clause (3) yields $\mathrm{dom}(\mathrm{tp}(g(r_\xi)) = 0$ for all $\xi < \chi$. Hence, Clause (4) of Definition 5.7 yields $g(r_\xi) = \lceil p_\xi \rceil^{\mathbb{A}}$ for all $\xi < \chi$. Since $g(r_\xi) \trianglelefteq a$ this in particular implies that $a = \lceil p \rceil^{\mathbb{A}}$.

Set $b := \lceil p' \rceil^{\mathbb{A}}$. Clearly, $\pi(b) = p'$ and $b \trianglelefteq^0 a$. Let $q' \in W_n(p')$. By Clause (2) of Definition 5.13, $q' \leq^0 p_\xi$, where ξ is such that $r_\xi = w(p, q')$. Finally, Definition 5.1(6) yields $⫛(b)(q') = \lceil q' \rceil^{\mathbb{A}} \trianglelefteq^0 \lceil p_\xi \rceil^{\mathbb{A}} = g(r_\xi)$. □

So, hereafter let us assume that $\iota < \chi$. For each $\xi \in [\iota, \chi)$, Clause (3) of Definition 5.13 and Definition 6.13 together imply that $\mathrm{dom}(\vec{S}^\xi) = \alpha_\xi + 1$ for some $\alpha_\xi < \mu$. Moreover, Clause (3) yields $\sup_{\iota \leq \eta < \xi} \alpha_\eta < \alpha_\xi$ for all $\xi \in (\iota, \chi)$. Also, the same clause implies that $g(r_\xi) = \lceil p_\xi \rceil^{\mathbb{A}}$, hence $\vec{S}^\xi = \emptyset$, for all $\xi < \iota$.

Let $\langle s_\tau \mid \tau < \theta\rangle$ be the good enumeration of $W_n(p')$. By Definition 3.3(5), $\theta < \mu$. For each $\tau < \theta$, set $r_{\xi_\tau} := w(p, s_\tau)$.

By Definition 5.13(1),

$$s_\tau \leq^0 \pi(g(w(p, s_\tau))) = \pi(g(r_{\xi_\tau})) = p_{\xi_\tau},$$

for each $\tau < \theta$. Set $\alpha' := \sup_{\iota \leq \xi < \chi} \alpha_\xi$ and $\alpha := \sup(\mathrm{dom}(\vec{S}))$.[50] By regularity of μ and Definition 5.13(3) it follows that $\alpha < \alpha' < \mu$. Our goal is to define a sequence $\vec{T} = \langle T_i \mid i \leq \alpha'\rangle$, with $\mathrm{dom}(T_i) := \{(q, u) \mid q \in W(p') \,\&\, u \in \bigcup_{\ell(p') \leq m \leq \ell(q)} \mathbb{S}_m \downarrow \varpi_m(q)\}$ for $i \leq \alpha'$, such that $b := (p', \vec{T})$ is a condition in $\mathbb{A}$ satisfying the conclusion of the weak mixing property.

As $\langle s_\tau \mid \tau < \theta\rangle$ is a good enumeration of the n^{th}-level of the p'-tree $W(p')$, Lemma 3.7(2) entails that, for each $q \in W(p')$, there is a unique ordinal $\tau_q < \theta$, such that q is comparable with s_{τ_q}. It thus follows from Lemma 3.7(3) that, for all $q \in W(p')$, $\ell(q) - \ell(p') \geq n$ iff $q \in W(s_{\tau_q})$.

[50]Note that a might be $\lceil p \rceil^{\mathbb{A}}$, so we are allowing $\alpha = 0$.

Moreover, for each $q \in W_{\geq n}(p')$, $q \leq s_{\tau_q} \leq^0 p_{\xi_{\tau_q}}$, hence $w(p_{\xi_{\tau_q}}, q)$ is well-defined. Now, for all $i \leq \alpha'$ and $q \in W(p')$, let:

$$T_i(q,u) := \begin{cases} S^{\xi_{\tau_q}}_{\min\{i,\alpha_{\xi_{\tau_q}}\}}(w(p_{\xi_{\tau_q}}, q), u_q), & \text{if } q \in W(s_{\tau_q}) \ \&\ \iota \leq \xi_{\tau_q}; \\ S_{\min\{i,\alpha\}}(w(p,q), u_q), & \text{if } q \notin W(s_{\tau_q}) \ \&\ \alpha > 0; \\ \emptyset, & \text{otherwise.} \end{cases}$$

Next, we show that $\vec{T}$ is a $\langle p', \vec{\mathbb{S}}\rangle$-strategy. The only non-routine part is to show that T_i is a labeled $\langle p', \vec{\mathbb{S}}\rangle$-tree for all $i \leq \alpha'$ (i.e., Clause (2) in Definition 6.6).

Claim 6.16.2. *Let $i \leq \alpha'$. Then T_i is a labeled $\langle p', \vec{\mathbb{S}}\rangle$-tree.*

Proof. Fix $(q,u) \in \mathrm{dom}(T_i)$ and let us go over the Clauses of Definition 6.4. The verification of (1), (2) and (3) are similar to that of [PRS21, Claim 6.16.1] and, actually, also to that of Claim 6.11.1 above. The reader is thus referred there for more details. We just elaborate on Clause (4).

For each $i < \alpha'$, set $\xi(i) := \min\{\xi \in [\iota, \chi) \mid i \leq \alpha_\xi\}$.

Subclaim 6.16.2.1. *If $i < \alpha'$, then*

$$m(T_i) \leq n + \max\{\mathrm{mtp}(a), \sup_{\iota \leq \eta < \xi(i)} \mathrm{mtp}(g(r_\eta)), \mathrm{tp}(g(r_{\xi(i)}))(i)\}.$$

Proof. Let $q \in W_k(p')$ and (q', u') be a pair in $\mathrm{dom}(T_i)$ with $q' \leq q$, where

$$k \geq n + \max\{\mathrm{mtp}(a), \sup_{\iota \leq \eta < \xi(i)} \mathrm{mtp}(g(r_\eta)), \mathrm{tp}(g(r_{\xi(i)}))(i)\}.$$

Suppose that $T_i(q', u') \neq \emptyset$. Denote $\tau := \tau_{q'}$ and $\delta := \max(T_i(q', u'))$. Since $\ell(q) \geq \ell(p') + n$, note that $q, q' \in W(s_\tau)$. Also, $\iota \leq \xi_\tau$, as otherwise $T_i(q', u') = \emptyset$. Thus, we fall into the first option of the casuistic getting

$$T_i(q', u') = S^{\xi_\tau}_{\min\{i, \alpha_{\xi_\tau}\}}(w(p_{\xi_\tau}, q'), u'_{q'}).$$

▶ Assume that $\xi_\tau < \xi(i)$. Then, $\alpha_{\xi_\tau} < i$ and so

$$T_i(q', u') = S^{\xi_\tau}_{\alpha_{\xi_\tau}}(w(p_{\xi_\tau}, q'), u'_{q'}).$$

We have that $w(p_{\xi_\tau}, q') \leq w(p_{\xi_\tau}, q)$ is a pair in $W_{k-n}(p_{\xi_\tau})$ and that the set $S^{\xi_\tau}_{\alpha_{\xi_\tau}}(w(p_{\xi_\tau}, q'), u'_{q'})$ is non-empty. Also, $k - n \geq \mathrm{mtp}(g(r_{\xi_\tau})) = m(S^{\xi_\tau}_{\alpha_{\xi_\tau}})$. So, by Clause (4) for $S^{\xi_\tau}_{\alpha_{\xi_\tau}}$, we have that $(\delta, w(p_{\xi_\tau}, q)) \in R$. Finally, since $q \leq w(p_{\xi_\tau}, q)$, we have $(\delta, q) \in R$, as desired.

▶ Assume that $\xi(i) \leq \xi_\tau$. Then $i \leq \alpha_{\xi(i)} \leq \alpha_{\xi_\tau}$, and thus

$$T_i(q', u') = S^{\xi_\tau}_i(w(p_{\xi_\tau}, q'), u'_{q'}).$$

If $\mathrm{dom}(\mathrm{tp}(a)) \leq i \leq \sup_{\iota \leq \eta < \xi(i)} \alpha_\eta$, by Clause (4) of Definition 5.13,

$$\mathrm{tp}(g(r_{\xi_\tau}))(i) \leq \mathrm{mtp}(a).$$

Otherwise, if $\sup_{\iota \leq \eta < \xi(i)} \alpha_\eta < i \leq \alpha_{\xi(i)}$, again by Definition 5.13(4)

$$\mathrm{tp}(g(r_{\xi_\tau}))(i) \leq \max\{\mathrm{mtp}(a), \mathrm{tp}(g(r_{\xi(i)}))(i)\}.$$

In either case, $w(p_{\xi_\tau}, q) \in W_{k-n}(p_{\xi_\tau})$ and $k - n \geq \mathrm{tp}(g(r_{\xi_\tau}))(i) = m(S_i^{\xi_\tau})$. By Clause (4) of $S_i^{\xi_\tau}$ we get that $(\delta, w(p_{\xi_\tau}, q)) \in R$, hence $(\delta, q) \in R$. $\square$

Subclaim 6.16.2.2. $m(T_{\alpha'}) \leq n + \sup_{\iota \leq \xi < \chi} \mathrm{mtp}(g(r_\xi))$.

Proof. Let $q \in W_k(p')$ and $(q', u') \in \mathrm{dom}(T_i)$ with $q' \leq q$, where $k \geq n + \sup_{\iota \leq \xi < \chi} \mathrm{mtp}(g(r_\xi))$. Suppose that $T_{\alpha'}(q', u') \neq \emptyset$ and denote $\tau := \tau_{q'}$ and $\delta := \max(T_{\alpha'}(q', u'))$. Since $k \geq n$, $q, q' \in W(s_\tau)$. Also, $\iota \leq \xi_\tau$, as otherwise $T_{\alpha'}(q', u') = \emptyset$. So, $T_{\alpha'}(q', u') = S^{\xi_\tau}_{\alpha_{\xi_\tau}}(w(p_{\xi_\tau}, q'), u'_{q'})$. Then $w(p_{\xi_\tau}, q') \leq w(p_{\xi_\tau}, q)$ is a pair in $W_{k-n}(p_{\xi_\tau})$ with $k - n \geq \mathrm{mtp}(g(r_{\xi_\tau})) = m(S^{\xi_\tau}_{\alpha_{\xi_\tau}})$. So, Definition 6.4(4) regarded with respect to $S^{\xi_\tau}_{\alpha_{\xi_\tau}}$ yields $(\delta, w(p_{\xi_\tau}, q)) \in R$. Once again, it follows that $(\delta, q) \in R$, as wanted. $\square$

The combination of the above subclaims yield Clause (4) for T_i. $\square$

Thereby we establish that $b := (p', \vec{T})$ is a legitimate condition in $\mathbb{A}$. Next, we show that b satisfies the requirements for $(\pitchfork, \pi)$ to have the weak mixing property. By definition, $\pi(b) = p'$, and it is easy to show that $b \trianglelefteq^0 a$.

Claim 6.16.3. *Let $\tau < \theta$. For each $q \in W_n(s_\tau)$, $w(p', q) = w(s_\tau, q) = q$.* $\square$

Claim 6.16.4. *For each $\tau < \theta$, $\pitchfork(b)(s_\tau) \trianglelefteq^0 g(r_{\xi_\tau})$.*[51]

Proof. Let $\tau < \theta$ and denote $\pitchfork(b)(s_\tau) = (s_\tau, \vec{T}_\tau)$. By Lemma 6.10(5) we have that $\pi(\pitchfork(b)(s_\tau)) = s_\tau \leq^0 p_{\xi_\tau}$, so Clause (a) of Definition 6.7 holds.

If $\xi_\tau < \iota$, then $\pitchfork(b)(s_\tau) \trianglelefteq^0 \lceil p_{\xi_\tau} \rceil^{\mathbb{A}} = g(r_{\xi_\tau})$, and we are done. So, let us assume that $\iota \leq \xi_\tau$. Let $i \leq \alpha_{\xi_\tau}$ and $q \in W(s_\tau)$. By Definition 6.8(*), $T_i^\tau(q, u) = T_i(w(p', q), u_q)$ and by Claim 6.16.3, $w(p', q) = w(s_\tau, q) = q$, hence $T_i^\tau(q, u) = T_i(q, u_q) = T_i(q, u)$.[52] Also $r_{\xi_{\tau_q}} = w(p, s_{\tau_q}) = w(p, s_\tau) = r_{\xi_\tau}$, where the second equality follows from $q \in W(s_\tau)$. Therefore,

$$T_i^\tau(q, u) = S^{\xi_\tau}_{\min\{i, \alpha_{\xi_\tau}\}}(w(p_{\xi_\tau}, q), u_q) = S_i^{\xi_\tau}(w(p_{\xi_\tau}, q), u_q).$$

Altogether, $\pitchfork(b)(s_\tau) \trianglelefteq^0 g(r_{\xi_\tau})$, as wanted. $\square$

The combination of the above claims yield the proof of the lemma.

Let us sum up what we have shown so far:

Corollary 6.17. *$(\pitchfork, \pi)$ is a super nice forking projection from $(\mathbb{A}, \ell_{\mathbb{A}}, c_{\mathbb{A}}, \vec{\varsigma})$ to $(\mathbb{P}, \ell, c, \vec{\varpi})$ having the weak mixing property.*

In particular, $(\mathbb{A}, \ell_{\mathbb{A}}, c_{\mathbb{A}}, \vec{\varsigma})$ is a $(\Sigma, \vec{\mathbb{S}})$-Prikry, $(\mathbb{A}, \ell_{\mathbb{A}})$ has property $\mathcal{D}$, $\mathbb{1}_{\mathbb{A}} \Vdash_{\mathbb{A}} \mu = \check{\kappa}^+$ and $\vec{\varsigma}$ is a coherent sequence of nice projections.

Proof. The first part follows from Corollary 6.12 and Lemma 6.16. Likewise, $(\mathbb{A}, \ell_{\mathbb{A}})$ has property $\mathcal{D}$ by virtue of Lemma 5.15, and $\vec{\varsigma}$ is coherent by virtue of Lemma 5.17 (see also Setup 6). Thus, we are left with arguing that $(\mathbb{A}, \ell_{\mathbb{A}}, c_{\mathbb{A}}, \vec{\varsigma})$ is $(\Sigma, \vec{\mathbb{S}})$-Prikry and that $\mathbb{1}_{\mathbb{A}} \Vdash_{\mathbb{A}} \mu = \check{\kappa}^+$. All the

[51]Recall that $\langle s_\tau \mid \tau < \theta \rangle$ was a good enumeration of $W_n(p')$.

[52]For the second equality we use Definition 6.4(2) for T_i and, again, that $q + u = q + u_q$.

Clauses of Definition 3.3 with the possible exception of (2), (7) and (9) follow from Theorem 5.11. First, from this latter result and the assumptions in Setup 6, $\mathbb{1}_{\mathbb{A}} \Vdash_{\mathbb{A}} \mu = \check{\kappa}^+$; Second, Clauses (2) and (9) follow from Lemma 5.16, Fact 6.14 and Lemma 6.15. Finally, Clause (7) is an immediate consequence of Lemma 5.15. $\square$

Our next task is to show that, after forcing with $\mathbb{A}$ the $\mathbb{P}$-name, $\dot{T}^+$ ceases to be stationary (Recall our blanket assumptions from page 41).

Lemma 6.18. $\lceil r^\star \rceil^{\mathbb{A}} \Vdash_{\mathbb{A}}$ *"$\dot{T}^+$ is nonstationary".*

Proof. Let G be $\mathbb{A}$-generic over V, with $\lceil r^\star \rceil^{\mathbb{A}} \in G$. Work in $V[G]$. Let $\bar{G}$ (resp. H_n) denote the generic filter for $\mathbb{P}$ (resp. $\mathbb{S}_n$) induced by π (resp. ς_n) and G. For all $a = (p, \vec{S}) \in G$ and $i \in \mathrm{dom}(\vec{S})$ write

$$d_a^i := \bigcup \{S_i(q,t) \mid q \in \bar{G} \cap W(p) \,\&\, \exists n \in [\ell(p), \ell(q)]\, (t \preceq_n \varpi_n(q_n) \wedge t \in H_n)\},$$

where $\langle q_n \mid n \geq \ell(p) \rangle$ is the increasing enumeration of $\bar{G} \cap W(p)$ (see Lemma 3.7).

Then, let

$$d_a := \begin{cases} d_a^{\max(\mathrm{dom}(\vec{S}))}, & \text{if } \vec{S} \neq \emptyset; \\ \emptyset, & \text{otherwise.} \end{cases}$$

Claim 6.18.1. *Suppose that $a = (p, \vec{S}) \in G$. In $V[\bar{G}]$, for all $i \in \mathrm{dom}(\vec{S})$, the ordinal closure $\mathrm{cl}(d_a^i)$ of d_a^i is disjoint from $(\dot{T}^+)_G$.*

Proof. To avoid trivialities we shall assume that $\vec{S} \neq \emptyset$. We prove the claim by induction on $i \in \mathrm{dom}(\vec{S})$. The base case $i = 0$ is trivial, as $S_0 \colon W(p) \to \{\emptyset\}$ (see Definition 6.6(5)). So, let us assume by induction that $\mathrm{cl}(d_a^j)$ is disjoint from $(\dot{T}^+)_G$ for every $0 \leq j < i$.

Let $\gamma \in \mathrm{cl}(d_a^i) \setminus \bigcup_{j<i} \mathrm{cl}(d_a^j)$. By virtue of Clause (2) of Definition 6.4 applied to S_i, we may further assume that $\gamma \notin d_a^i$.

<u>Succesor case:</u> Suppose that $i = j+1$. There are two cases to discuss:

► Assume $\mathrm{cf}^{V[\bar{G}]}(\gamma) = \omega$. Working in $V[\bar{G}]$, we have $\gamma = \sup_{n<\omega} \gamma_n$, where for each $n < \omega$, there is (q_n, t_n), such that $q_n \in \bar{G} \cap W(p)$, $t_n \preceq_k \varpi_k(q_n)$ with $t_n \in H_k$ for some $k \in [\ell(p), \ell(q_n)]$, and $\gamma_n \in S_{j+1}(q_n, t_n) \setminus S_j(q_n, t_n)$. Strengthening if necessary, we may assume $q_n + t_n \leq q_m + t_m$ for $m \leq n$.[53]

For each $n < \omega$ set $\delta_n := \max(S_{j+1}(q_n, t_n))$. Clearly, $\gamma \leq \sup_{n<\omega} \delta_n$.

We claim that $\gamma = \sup_{n<\omega} \delta_n$: Assume to the contrary that this is not the case. Then, there is $n_0 < \omega$ such that $\gamma_m < \delta_{n_0}$ for all $m < \omega$. Let $m \geq n_0$. Then, $\gamma_m \in S_{j+1}(q_m, t_m) \setminus S_j(q_m, t_m)$. Also, since $\gamma_{n_0} \notin S_j(q_{n_0}, t_{n_0})$, Definition 6.6(3) for $\vec{S}$ yields $\delta_{n_0} \in S_{j+1}(q_{n_0}, t_{n_0}) \setminus S_j(q_{n_0}, t_{n_0})$. By virtue of Clause (4) of Definition 6.6 we also have

$$S_{j+1}(q_{n_0}, t_{n_0}) \setminus S_j(q_{n_0}, t_{n_0}) \sqsubseteq S_{j+1}(q_m, t_m) \setminus S_j(q_m, t_m) \ni \gamma_m.$$

[53] For this we use Definition 6.4(2).

Thus, as $\gamma_m < \delta_0$, we have that γ_m belongs to the left-hand-side set.

Since m above was arbitrary we get $\gamma \in S_{j+1}(q_{n_0}, t_{n_0}) \subseteq d^i_a$. This yields a contradiction with our original assumption that $\gamma \notin d^i_a$.

So, $\gamma = \sup_{n<\omega} \delta_n$. Now, let $n^\star < \omega$ such that $\ell(q_{n^\star}) \geq \ell(p) + m(S_{j+1})$. Then, for all $n \geq n^\star$, Clause (4) of Definition 6.4 yields $(\delta_n, q_{n^\star}) \in R$. In particular, $(\gamma, q_{n^*}) \in R$ and thus $q_{n^*} \Vdash_{\mathbb{P}} \check{\gamma} \notin \dot{T}^+$ (see page 41). Finally, since $q_{n^*} \in \bar{G}$, we conclude that $\gamma \notin (\dot{T}^+)_G$, as desired.

► Assume $\mathrm{cf}^{V[\bar{G}]}(\gamma) \geq \omega_1$. Working in $V[\bar{G}]$, we have $\gamma = \sup_{\alpha<\mathrm{cf}(\gamma)} \gamma_\alpha$, where for each $\alpha < \mathrm{cf}(\gamma)$, there is $t_\alpha \in H_n$ with $t_\alpha \preceq_n \varpi_n(q)$ such that $\gamma_\alpha \in S_{j+1}(q, t_\alpha) \setminus S_j(q, t_\alpha)$. Here, $q \in \bar{G} \cap W(p)$ and $n \in [\ell(p), \ell(q)]$.[54] By strengthening q if necessary, we may also assume that $q \in W_{\geq m(S_{j+1})}(p)$.[55]

For each $\alpha < \mathrm{cf}(\gamma)$, set $\delta_\alpha := \max(S_{j+1}(q, t_\alpha))$. Clearly, $\gamma \leq \sup_{\alpha<\mathrm{cf}(\gamma)} \delta_\alpha$.

We claim that $\gamma = \sup_{\alpha<\mathrm{cf}(\gamma)} \delta_\alpha$: Otherwise, suppose $\alpha^* < \mathrm{cf}(\gamma)$ is such that $\gamma_\beta < \delta_{\alpha^*}$, for all $\beta < \mathrm{cf}(\gamma)$. Fix $\beta \geq \alpha^*$ and let $t \in H_n$ be such that $t \preceq_n t_\beta, t_{\alpha^*}$. Then, $q + t \leq q + t_\beta$, so Definition 6.4(2) for S_{j+1} yields $\gamma_\beta \in S_{j+1}(q, t_\beta) \subseteq S_{j+1}(q, t)$. Hence, we have that $\gamma_\beta \in S_{j+1}(q, t) \setminus S_j(q, t)$. Also, arguing as in the previous case we have $\delta_{\alpha^*} \in S_{j+1}(q, t_{\alpha^*}) \setminus S_j(q, t_{\alpha^*})$. Finally, combining Clause (4) of Definition 6.6 with $\gamma_\beta < \delta_{\alpha^*}$ we conclude that $\gamma_\beta \in S_{j+1}(q, t_{\alpha^*})$. Since $S_{j+1}(q, t_{\alpha^*})$ is a closed set, we get $\gamma \in S_{j+1}(q, t_{\alpha^*})$, which contradicts our assumption that $\gamma \notin d^i_a$.

So, $\gamma = \sup_{\alpha<\mathrm{cf}(\gamma)} \delta_\alpha$. Mimicking the argument of the former case we infer that $q \Vdash_{\mathbb{P}} \check{\gamma} \notin \dot{T}^+$, which yields $\gamma \notin (\dot{T}^+)_G$.

Limit case: Suppose that i is limit. If $\mathrm{cf}(i) \neq \mathrm{cf}(\gamma)$, then $\gamma \in \mathrm{cl}(d^j_a)$ for some $j < i$, and we are done. Thus, suppose $\mathrm{cf}(i) = \mathrm{cf}(\gamma)$. For simplicity assume $i = \mathrm{cf}(i)$, as the general argument is analogous. We have two cases.

► Assume $\mathrm{cf}^{V[\bar{G}]}(\gamma) = \omega$. Working in $V[\bar{G}]$, we have $\gamma = \sup_{n<\omega} \gamma_n$, where for each $n < \omega$, there is (q_n, t_n), such that $q_n \in \bar{G} \cap W(p)$, $t_n \preceq_k \varpi_k(q_n)$ with $t_n \in H_k$ for some $k \in [\ell(p), \ell(q_n)]$, and $\gamma_n \in S_\omega(q_n, t_n)$. Strengthening if necessary, we may further assume $q_n + t_n \leq q_m + t_m$ for $m \leq n$.

For each $n < \omega$ set $\delta_n := \max(S_\omega(q_n, t_n))$. Clearly, $\gamma \leq \sup_{n<\omega} \delta_n$.

As in the previous cases, we claim that $\gamma = \sup_{n<\omega} \delta_n$: Suppose otherwise and let $n_0 < \omega$ such that $\gamma_m < \delta_{n_0}$ for all $m < \omega$. Actually $\gamma < \delta_{n_0}$, as otherwise $\gamma \in S_\omega(q_{n_0}, t_{n_0}) \subseteq d^\omega_a$, which would yield a contradiction.

By Clause (5) of Definition 6.6, $\delta_{n_0} = \sup_{k<\omega} \max(S_k(q_{n_0}, t_{n_0}))$, hence there is some $k_0 < \omega$ such that $\gamma < \max(S_{k_0}(q_{n_0}, t_{n_0}))$.

Fix $m \geq n_0$. Since $q_m + t_m \leq q_{n_0} + t_{n_0}$, Clause (2) of Definition 6.4 yields

$$\gamma < \max(S_{k_0}(q_{n_0}, t_{n_0})) \leq \max(S_{k_0}(q_m, t_m)).$$

Also, Clause (3) of Definition 6.6 implies that

$$S_{k_0}(q_m, t_m) \sqsubseteq S_\omega(q_m, t_m) \ni \gamma_m,$$

[54] Note that this is the case in that $W(p)$ is a tree with height ω and $\mathrm{cf}(\gamma) \geq \omega_1$.

[55] Note that increasing q would only increase $S_{j+1}(q, t_\alpha)$.

so that, $\gamma_m \in S_{k_0}(q_m, t_m)$. Since m was arbitrary, we infer that $\gamma \in \mathrm{cl}(d_a^{k_0})$, which yields a contradiction with our original assumption.

So, $\gamma = \sup_{n<\omega} \delta_n$. Arguing as in previous cases conclude that $\gamma \notin (\dot{T}^+)_G$.

► Assume $\mathrm{cf}^{V[\bar{G}]}(\gamma) \geq \omega_1$. Working in $V[\bar{G}]$, we have $\gamma = \sup_{\alpha<i} \gamma_\alpha$, where for each $\alpha < i$, there is $t_\alpha \in H_n$ with $t_\alpha \preceq_n \varpi_n(q)$ such that $\gamma_\alpha \in S_i(q, t_\alpha)$. As before, here both q and n are fixed and $q \in W_{\geq m(S_i)}(p)$.

For each $\alpha < i$, set $\delta_\alpha := \max(S_i(q, t_\alpha))$. Once again, we aim to show that $\gamma = \sup_{\alpha<i} \delta_\alpha$: Suppose that this is not the case, and let $\alpha^* < i$ such that $\gamma_\alpha < \delta_{\alpha^*}$ for all $\alpha < i$. As before, $\gamma \neq \delta_{\alpha^*}$, so there is some $\bar{\alpha} < i$ such that $\gamma < \max(S_{\bar{\alpha}}(q, t_{\alpha^*}))$ Now, let $\alpha < i$ be arbitrary and find $s_\alpha \preceq_n t_{\alpha^*}, t_\alpha$ in H_n. Then, $\gamma_\alpha \in S_i(q, t_\alpha) \subseteq S_i(q, s_\alpha)$. Also, $S_{\bar{\alpha}}(q, t_{\alpha^*}) \subseteq S_{\bar{\alpha}}(q, s_\alpha)$ and so $\max(S_{\bar{\alpha}}(q, s_\alpha)) > \gamma$. By Clause (3) of Definition 6.6 we have that $S_{\bar{\alpha}}(q, s_\alpha) \sqsubseteq S_i(q, s_\alpha)$, hence $\gamma_\alpha \in S_{\bar{\alpha}}(q, s_\alpha)$.

The above shows that $\gamma \in \mathrm{cl}(d_a^{\bar{\alpha}})$, which is a contradiction.

So, $\gamma = \sup_{\alpha<i} \delta_\alpha$. Now proceed as in previous cases, invoking Clause (4) of Definition 6.4, and infer that $\gamma \notin (\dot{T}^+)_G$.

□

Claim 6.18.2. *Suppose $a = (p, \vec{S}) \in A$, where $p \leq r^\star$. For every $\gamma < \mu$, there exists $\bar{\gamma} \in (\gamma, \mu)$ and $(\bar{p}, \vec{T}) \trianglelefteq (p, \vec{S})$, such that $\max(\mathrm{dom}(\vec{T})) = \alpha$ and for all $(q, t) \in \mathrm{dom}(T_\alpha)$, $\max(T_\alpha(q, t)) = \bar{\gamma}$.*

Proof. This is indeed what the argument of Lemma 6.15 shows. □

Working in $V[G]$, the above claim yields an unbounded set $I \subseteq \mu$ such that for each $\gamma \in I$ there is $a_\gamma = (p_\gamma, \vec{S}^\gamma) \in G$ with $\max(\mathrm{dom}(\vec{S}^\gamma)) = \gamma$ and $\max(S^\gamma_\gamma(q, t)) = \gamma$ for all $(q, t) \in \mathrm{dom}(S^\gamma_\gamma)$. For each $\gamma \in I$, set $D_\gamma := \mathrm{cl}(d_{a_\gamma})$.

Claim 6.18.3. *For each $\gamma < \gamma'$ both in I, $D_\gamma \sqsubseteq D_{\gamma'}$.*

Proof. Let $\gamma < \gamma'$ be in I. It is enough to prove that $d_{a_\gamma} \sqsubseteq d_{a_{\gamma'}}$; namely, we show that $d_{a_\gamma} = d_{a_{\gamma'}} \cap (\gamma + 1)$. Fix $b = (r, \vec{R}) \in G$ be such that $b \trianglelefteq a_\gamma, a_{\gamma'}$.

For the first direction, suppose that $\delta \in d_{a_\gamma}$ and let $(q, t) \in \mathrm{dom}(S^\gamma_\gamma)$ be a pair witnessing this. By strengthening q and t if necessary, we may further assume that $\ell(q) \geq \ell(r)$ and $t \in H_n$, $t \preceq_n \varpi_n(q)$, where $n := \ell(q)$.[56]

Let $r' \in W(r) \cap \bar{G}$ be the unique condition with $\ell(r') = n$. Also, let $t' \in H_n$ be such that $t' \preceq_n \varpi_n(r'), t$. Then $w(p_\gamma, r') = q$, $t' = \varpi_n(r' + t')$ and $q + t' \leq q + t$. So, by $b \trianglelefteq a_\gamma$ and $b \trianglelefteq a_{\gamma'}$, we get:

$$\delta \in S^\gamma_\gamma(q, t) \subseteq S^\gamma_\gamma(q, t') = R_\gamma(r', t') = S^{\gamma'}_\gamma(w(p_\gamma, r'), t') \subseteq d_{a_{\gamma'}}.$$

For the other direction, suppose that $\delta \in d_{a_{\gamma'}} \cap (\gamma + 1)$ and let (q, t) be a pair in $\mathrm{dom}(S^{\gamma'}_{\gamma'})$ witnessing this. Again, by strengthening q, we may assume that $\ell(q) \geq \ell(r)$ and $t \in H_n$, $t \preceq_n \varpi_n(q_n)$, where $n := \ell(q)$. Similarly as

[56] Suppose that (q, t) is the pair we are originally given and that $q' \in W_n(p) \cap \bar{G}$, where $n \geq \ell(r)$. Setting $t' := \varpi_n(q' + t)$ it is immediate that $q' + t' \leq q + t$, hence $\delta \in S^\gamma_\gamma(q', t')$. Also, it is not hard to check that $q' + t \in G$, hence $t' \preceq_n \varpi_n(q')$ and $t' \in H_n$.

above, let $r' \in W(r) \cap \bar{G}$ be with $\ell(r') = n$, and $t' \in H_n$ be such that $t' \preceq_n \varpi_n(r'), t$. Then $w(p_{\gamma'}, r') = q$, $t' = \varpi_n(r' + t')$, $q + t' \leq q + t$ and:

(1) $R_{\gamma'}(r', t') = S^{\gamma'}_{\gamma'}(q, t')$, since $b \trianglelefteq a_{\gamma'}$;
(2) $R_{\gamma}(r', t') = S^{\gamma}_{\gamma}(w(p_{\gamma}, r'), t')$, and so $\gamma = \max(R_{\gamma}(r', t'))$;
(3) $R_{\gamma}(r', t') \sqsubseteq R_{\gamma'}(r', t')$, by Clause (3) of Definition 6.6 for $\vec{R}$.

Combining all three, we get that

$$\delta \in S^{\gamma'}_{\gamma'}(q, t) \cap (\gamma + 1) \subseteq S^{\gamma'}_{\gamma'}(q, t') \cap (\gamma + 1) =$$
$$R_{\gamma'}(r', t') \cap (\gamma + 1) = R_{\gamma}(r', t') = S^{\gamma}_{\gamma}(w(p_{\gamma}, r'), t') \subseteq d_{a_{\gamma}},$$

as desired. □

Let $D := \bigcup_{\gamma \in I} D_{\gamma}$. By Claims 6.18.1 and 6.18.3, D is disjoint from $(T^+)_G$. Additionally, Claim 6.18.3 implies that D is closed and, since $I \subseteq D$, it is also unbounded. So, $(\dot{T}^+)_G$ is nonstationary in $V[G]$. □

Remark 6.19. Note that Lemma 6.18 together with $r^\star \Vdash_{\mathbb{P}} \dot{T} \subseteq \dot{T}^+$ (see page 41) imply that $\lceil r^\star \rceil^{\mathbb{A}} \Vdash_{\mathbb{A}}$ "$\dot{T}$ is nonstationary".

The next corollary sums up the content of Subsection 6.1:

Corollary 6.20. *Suppose that $(\Sigma, \vec{\mathbb{S}})$-Prikry quadruple $(\mathbb{P}, \ell, c, \vec{\varpi})$ such that, $\mathbb{P} = (P, \leq)$ is a subset of H_{μ^+}, $(\mathbb{P}, \ell)$ has property $\mathcal{D}$, $\vec{\varpi}$ is a coherent sequence of nice projections, $\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}} \check{\mu} = \check{\kappa}^+$ and $\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}}$ "κ is singular".*

For every $r^\star \in P$ and a $\mathbb{P}$-name z for an $r^\star$-fragile stationary subset of μ, there are a $(\Sigma, \vec{\mathbb{S}})$-Prikry quadruple $(\mathbb{A}, \ell_{\mathbb{A}}, c_{\mathbb{A}}, \vec{\varsigma})$ having property $\mathcal{D}$, and a pair of maps $(\pitchfork, \pi)$ such that all the following hold:

(a) $(\pitchfork, \pi)$ is a super nice forking projection from $(\mathbb{A}, \ell_{\mathbb{A}}, c_{\mathbb{A}}, \vec{\varsigma})$ to $(\mathbb{P}, \ell, c, \vec{\varpi})$ that has the weak mixing property;
(b) $\vec{\varsigma}$ is a coherent sequence of nice projections;
(c) $\mathbb{1}_{\mathbb{A}} \Vdash_{\mathbb{A}} \check{\mu} = \check{\kappa}^+$;
(d) $\mathbb{A} = (A, \trianglelefteq)$ is a subset of H_{μ^+};
(e) For every $n < \omega$, $\mathring{\mathbb{A}}^{\pi}_n$ is a μ-directed-closed;
(f) $\lceil r^\star \rceil^{\mathbb{A}}$ forces that z is nonstationary.

Proof. Since all the assumptions of Setup 6 are valid we obtain from Definitions 6.7 and 6.8, a notion of forcing $\mathbb{A} = (A, \trianglelefteq)$ together with maps $\ell_{\mathbb{A}}$ and $c_{\mathbb{A}}$, and a sequence $\vec{\varsigma}$ such that, by Corollary 6.17, $(\mathbb{A}, \ell_{\mathbb{A}}, c_{\mathbb{A}}, \vec{\varsigma})$ is a $(\Sigma, \vec{\mathbb{S}})$-Prikry quadruple having property $\mathcal{D}$ and Clauses (a)–(c) above hold. Clause (d) easily follows from the definition of $\mathbb{A} = (A, \trianglelefteq)$ (see, e.g. [PRS21, Lemma 6.6]), Clause (e) is Fact 6.14 and Clause (f) is Lemma 6.18 together with Remark 6.19. □

6.2. **Fragile sets vs non-reflecting stationary sets.** For every $n < \omega$, denote $\Gamma_n := \{\alpha < \mu \mid \mathrm{cf}^V(\alpha) < \sigma_{n-2}\}$, where, by convention, we define σ_{-2} and σ_{-1} to be $\aleph_0$.

The next lemma is an analogue of [PRS21, Lemma 6.1] and will be crucial for the proof of reflection in the model of the Main Theorem.

Lemma 6.21. *Suppose that:*

(i) for every $n<\omega$, $V^{\mathbb{P}_n}\models \mathrm{Refl}(E^{\mu}_{<\sigma_{n-2}},E^{\mu}_{<\sigma_n})$;

(ii) $r^\star$ is a condition in $\mathbb{P}$;

(iii) $\dot T$ is a nice $\mathbb{P}$-name for a subset of $\Gamma_{\ell(r^\star)}$;

(iv) $r^\star$ $\mathbb{P}$-forces that $\dot T$ is a non-reflecting stationary set.

Then $\dot T$ is $r^\star$-fragile.

Proof. Suppose that $\dot T$ is not $r^\star$-fragile (see Definition 6.1), and let q be an extension of $r^\star$ witnessing that. Set $n:=\ell(q)$, so that

$$q\Vdash_{\mathbb{P}_n}\text{“}\dot T_n\text{ is stationary”}.$$

Since $\dot T$ is a nice $\mathbb{P}$-name for a subset of $\Gamma_{\ell(r^\star)}$, it altogether follows that q $\mathbb{P}_n$-forces that $\dot T_n$ is a stationary subset of $E^{\mu}_{<\sigma_{n-2}}$.

Let G_n be $\mathbb{P}_n$-generic containing q. By Clause (i), we have that $T_n:=(\dot T_n)_{G_n}$ reflects at some ordinal γ of $(V[G_n]$-)cofinality $<\sigma_n$. Since ϖ_n is a nice projection, we have that $\mathbb{P}_n^{\varpi_n}\times\mathbb{S}_n$ projects to $\mathbb{P}_n$.[57] Then by $|S_n|<\sigma_n$ and the fact that $\mathbb{P}_n^{\varpi_n}$ contains a σ_n-directed-closed dense subset, it follows that $\theta:=\mathrm{cf}^V(\gamma)$ is $<\sigma_n$. In V, fix a club $C\subseteq\gamma$ of order-type θ.

Work in $V[G_n]$. Set $A:=T_n\cap C$, and note that A is a stationary subset of γ of size $\le\theta$. Let H_n be the $\mathbb{S}_n$-generic filter induced from G_n by ϖ_n.

Again, since $\mathbb{P}_n^{\varpi_n}$ contains a σ_n-directed-closed dense subset, it cannot have added A. So, $A\in V[H_n]$. Let $\langle\alpha_i\mid i<\theta\rangle$ be some enumeration (possibly with repetitions) of A, and let $\langle\dot\alpha_i\mid i<\theta\rangle$ be a sequence of $\mathbb{S}_n$-name for it. Pick a condition r in $\mathbb{P}_n/H_n$ such that $r\Vdash_{\mathbb{P}_n}\dot A\subseteq\dot T_n\cap\gamma$ and such that $\varpi_n(r)\Vdash_{\mathbb{S}_n}\dot A=\{\dot\alpha_i\mid i<\theta\}$. Denote $s:=\varpi_n(r)$ and note that $s\in H_n$. We now go back and work in V.

Claim 6.21.1. *Let $i<\theta$ and $\alpha<\gamma$. For all $r'\le^{\varpi_n}r$ and $s'\preceq_n s$, if $s'\Vdash_{\mathbb{S}_n}\dot\alpha_i=\check\alpha$, then there are $r''\le^{\varpi_n}r'$ and $s''\preceq_n s'$ such that $r''+s''\Vdash_{\mathbb{P}}\check\alpha\in\dot T$.*

Proof. Suppose r',s' are as above. As r' extends r and s' extends s, it follows that $r'+s'\Vdash_{\mathbb{P}_n}\check\alpha\in\dot T_n$ and $s'\Vdash_{\mathbb{S}_n}\check\alpha\in\dot A$. Let $p\le^0 r'+s'$ be such that $(\check\alpha,p)\in\dot T_n$. By the definition of the name $\dot T_n$, we have that $p\Vdash_{\mathbb{P}}\check\alpha\in\dot T$. Now, since ϖ_n is a nice projection Definition 2.2(4) gives $s''\preceq_n s'$ and $r''\le^{\varpi_n}r'$ such that $r''+s''=p$. So $r''+s''\Vdash_{\mathbb{P}}\check\alpha\in\dot T$, as desired. □

Fix an injective enumeration $\langle(i_\xi,s_\xi)\mid\xi<\chi\rangle$ of $\theta\times(\mathbb{S}_n\downarrow s)$. Note that $\chi<\sigma_n$. Using that $\mathbb{P}_n^{\varpi_n}$ is σ_n-strategically-closed (in $V[H_n]$), build a $\le^{\varpi_n}$-decreasing sequence of conditions $\langle r_\xi\mid\xi\le\chi\rangle$, such that, for every $\xi<\chi$, $r_\xi\le^{\vec\varpi}r$, and, for any $\alpha<\gamma$, if $s_\xi\Vdash_{\mathbb{S}_n}\dot\alpha_{i_\xi}=\check\alpha$ (and $s_\xi\in H_n$), then there is $s^\xi\preceq_n s_\xi$ (with $s^\xi\in H_n$) such that $r_\xi+s^\xi\Vdash_{\mathbb{P}}\check\alpha\in\dot T$. Finally, let $r^*:=r_\chi$. Note that $\varpi_n(r^*)=\varpi_n(r)=s\in H_n$, so that $r^*\in P/H_n$.

[57] More precisely, $(\mathbb{P}_n^{\varpi_n}\downarrow q)\times(\mathbb{S}_n\downarrow\varpi_n(q))$ projects onto $\mathbb{P}_n\downarrow q$.

Claim 6.21.2. $r^* \Vdash_{\mathbb{P}/H_n} A \subseteq \dot{T} \cap \check{\gamma}$.

Proof. Let $r \leq_{\mathbb{P}/H_n} r^*$ and $i < \theta$. By extending $\varpi_n(r)$ if necessary we may assume that $\varpi_n(r) \in H$ and that it decices (in $\mathbb{S}_n$) $\dot{\alpha}_i$ to be some ordinal $\alpha < \gamma$. Fix $\xi < \chi$ such that $(i_\xi, s_\xi) = (i, s')$. By the construction, there is $s^\xi \preceq_n s_\xi$, $s^\xi \in H_n$ such that $r_\xi + s^\xi \Vdash_{\mathbb{P}} \check{\alpha} \in \dot{T}$. Hence, $r^* + s^\xi \Vdash_{\mathbb{P}} \check{\alpha} \in \dot{T} \cap \check{\gamma}$, and thus $r + s^\xi$ forces the same. Since $r + s^\xi \in P/H_n$ we are done. $\square$

Finally, since $(\mathbb{P}, \ell, c, \vec{\varpi})$ is $(\Sigma, \vec{\mathbb{S}})$-Prikry, Lemma 3.15(1) implies that $\mathbb{P}/H_n$ does not add any new subsets of θ and, incidentally, no new subsets of C. Hence $\mathbb{P}/H_n$ preserves the stationarity of A, and thus the stationarity of $T \cap \gamma$. This contradicts hypothesis (iv) of the lemma. $\square$

7. Iteration scheme

In this section, we define an iteration scheme for $(\Sigma, \vec{\mathbb{S}})$-Prikry forcings, following closely and expanding the work from [PRS22, §3]. The reader familiar with our iteration machinery may opt for reading just Lemma 7.4. There we use in a crucial way a new feature of the forking projections; namely, super niceness.

Setup 7. The blanket assumptions for this section are as follows:

- μ is some cardinal satisfying $\mu^{<\mu} = \mu$, so that $|H_\mu| = \mu$;
- $\langle (\sigma_n, \sigma_n^*) \mid n < \omega \rangle$ is a sequence of pairs of regular uncountable cardinals, such that, for every $n < \omega$, $\sigma_n \leq \sigma_n^* \leq \mu$ and $\sigma_n \leq \sigma_{n+1}$;
- $\vec{\mathbb{S}} = \langle \mathbb{S}_n \mid n < \omega \rangle$ is a sequence of notions of forcing, $\mathbb{S}_n = (S_n, \preceq_n)$, with $|S_n| < \sigma_n$;
- $\Sigma := \langle \sigma_n \mid n < \omega \rangle$ and $\kappa := \sup_{n<\omega} \sigma_n$.

The following convention will be applied hereafter:

Convention 7.1. For a pair of ordinals $\gamma \leq \alpha \leq \mu^+$:

(1) $\emptyset_\alpha := \alpha \times \{\emptyset\}$ denotes the α-sequence with constant value $\emptyset$;
(2) For a γ-sequence p and an α-sequence q, $p * q$ denotes the unique α-sequence satisfying that for all $\beta < \alpha$:
$$(p * q)(\beta) = \begin{cases} q(\beta), & \text{if } \gamma \leq \beta < \alpha; \\ p(\beta), & \text{otherwise.} \end{cases}$$
(3) Let $\mathbb{P}_\alpha := (P_\alpha, \leq_\alpha)$ and $\mathbb{P}_\gamma := (P_\gamma, \leq_\gamma)$ be forcing posets such that $P_\alpha \subseteq {}^\alpha H_{\mu^+}$ and $P_\gamma \subseteq {}^\gamma H_{\mu^+}$. Also, assume $p \mapsto p \restriction \gamma$ defines a projection between $\mathbb{P}_\alpha$ and $\mathbb{P}_\gamma$. We denote by $i^\alpha_\gamma : V^{\mathbb{P}_\gamma} \to V^{\mathbb{P}_\alpha}$ the map defined by recursion over the rank of each $\mathbb{P}_\gamma$-name σ as follows:
$$i^\alpha_\gamma(\sigma) := \{(i^\alpha_\gamma(\tau), p * \emptyset_\alpha) \mid (\tau, p) \in \sigma\}.$$

Our iteration scheme requires three building blocks:

Building Block I. We are given a $(\Sigma, \vec{\mathbb{S}})$-Prikry forcing $(\mathbb{Q}, \ell, c, \vec{\varpi})$ such that $(\mathbb{Q}, \ell)$ satisfies property $\mathcal{D}$. We moreover assume that $\mathbb{Q} = (Q, \leq_Q)$

is a subset of H_{μ^+}, $\mathbb{1}_{\mathbb{Q}} \Vdash_{\mathbb{Q}}$ "$\check{\mu} = \check{\kappa}^+ \,\&\, \kappa$ is singular" and $\vec{\varpi}$ is a coherent sequence. To streamline the matter, we also require that $\mathbb{1}_{\mathbb{Q}}$ be equal to $\emptyset$.

Building Block II. Suppose that $(\mathbb{P}, \ell_{\mathbb{P}}, c_{\mathbb{P}}, \vec{\varpi})$ is a $(\Sigma, \vec{\mathbb{S}})$-Prikry quadruple having property $\mathcal{D}$ such that $\mathbb{P} = (P, \leq)$ is a subset of H_{μ^+}, $\vec{\varpi}$ is a coherent sequence of nice projections, $\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}}$ "$\check{\mu} = \check{\kappa}^+$" and $\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}}$ "$\check{\kappa}$ is singular".

For every $r^\star \in P$, and a $\mathbb{P}$-name $z \in H_{\mu^+}$, we are given a corresponding $(\Sigma, \vec{\mathbb{S}})$-Prikry quadruple $(\mathbb{A}, \ell_{\mathbb{A}}, c_{\mathbb{A}}, \vec{\varsigma})$ having property $\mathcal{D}$ such that:

(a) there is a super nice forking projection $(\pitchfork, \pi)$ from $(\mathbb{A}, \ell_{\mathbb{A}}, c_{\mathbb{A}}, \vec{\varsigma})$ to $(\mathbb{P}, \ell_{\mathbb{P}}, c_{\mathbb{P}}, \vec{\varpi})$ that has the weak mixing property;
(b) $\vec{\varsigma}$ is a coherent sequence of nice projections;
(c) for every $n < \omega$, $\mathring{\mathbb{A}}^\pi_n$ is σ^*_n-directed-closed;[58]
(d) $\mathbb{1}_{\mathbb{A}} \Vdash_{\mathbb{A}} \check{\mu} = \check{\kappa}^+$;
(e) $\mathbb{A} = (A, \trianglelefteq)$ is a subset of H_{μ^+};

By [PRS22, Lemma 2.18], we may also require that:

(f) each element of A is a pair (x, y) with $\pi(x, y) = x$;
(g) for every $a \in A$, $\lceil \pi(a) \rceil^{\mathbb{A}} = (\pi(a), \emptyset)$;
(h) for every $p, q \in P$, if $c_{\mathbb{P}}(p) = c_{\mathbb{P}}(q)$, then $c_{\mathbb{A}}(\lceil p \rceil^{\mathbb{A}}) = c_{\mathbb{A}}(\lceil q \rceil^{\mathbb{A}})$.

Building Block III. We are given a function $\psi : \mu^+ \to H_{\mu^+}$.

Goal 7.2. Our goal is to define a system $\langle (\mathbb{P}_\alpha, \ell_\alpha, c_\alpha, \vec{\varpi}_\alpha, \langle \pitchfork_{\alpha,\gamma} \mid \gamma \leq \alpha \rangle) \mid \alpha \leq \mu^+ \rangle$ in such a way that for all $\gamma \leq \alpha \leq \mu^+$:

(i) $\mathbb{P}_\alpha$ is a poset $(P_\alpha, \leq_\alpha)$, $P_\alpha \subseteq {}^\alpha H_{\mu^+}$, and, for all $p \in P_\alpha$, $|B_p| < \mu$, where $B_p := \{\beta + 1 \mid \beta \in \mathrm{dom}(p) \;\&\; p(\beta) \neq \emptyset\}$;
(ii) The map $\pi_{\alpha,\gamma} : P_\alpha \to P_\gamma$ defined by $\pi_{\alpha,\gamma}(p) := p \restriction \gamma$ forms an projection from $\mathbb{P}_\alpha$ to $\mathbb{P}_\gamma$ and $\ell_\alpha = \ell_\gamma \circ \pi_{\alpha,\gamma}$;
(iii) $\mathbb{P}_0$ is a trivial forcing, $\mathbb{P}_1$ is isomorphic to $\mathbb{Q}$ given by Building Block I, and $\mathbb{P}_{\alpha+1}$ is isomorphic to $\mathbb{A}$ given by Building Block II when invoked with respect to $(\mathbb{P}_\alpha, \ell_\alpha, c_\alpha, \vec{\varpi}_\alpha)$ and a pair $(r^\star, z)$ which is decoded from $\psi(\alpha)$;
(iv) If $\alpha > 0$, then $(\mathbb{P}_\alpha, \ell_\alpha, c_\alpha, \vec{\varpi}_\alpha)$ is a $(\Sigma, \vec{\mathbb{S}})$-Prikry notion of forcing satisfying property $\mathcal{D}$, whose greatest element is $\emptyset_\alpha$, $\ell_\alpha = \ell_1 \circ \pi_{\alpha,1}$ and $\emptyset_\alpha \Vdash_{\mathbb{P}_\alpha} \check{\mu} = \check{\kappa}^+$. Moreover, $\vec{\varpi}_\alpha$ is a coherent sequence of nice projections such that $\vec{\varpi}_\alpha = \vec{\varpi}_\gamma \bullet \pi_{\alpha,\gamma}$ for every $\gamma \leq \alpha$;
(v) If $0 < \gamma < \alpha \leq \mu^+$, then $(\pitchfork_{\alpha,\gamma}, \pi_{\alpha,\gamma})$ is a nice forking projection from $(\mathbb{P}_\alpha, \ell_\alpha, \vec{\varpi}_\alpha)$ to $(\mathbb{P}_\gamma, \ell_\gamma, \vec{\varpi}_\gamma)$; in case $\alpha < \mu^+$, $(\pitchfork_{\alpha,\gamma}, \pi_{\alpha,\gamma})$ is furthermore a nice forking projection from $(\mathbb{P}_\alpha, \ell_\alpha, c_\alpha, \vec{\varpi}_\alpha)$ to $(\mathbb{P}_\gamma, \ell_\gamma, c_\gamma, \vec{\varpi}_\gamma)$, and in case $\alpha = \gamma + 1$, $(\pitchfork_{\alpha,\gamma}, \pi_{\alpha,\gamma})$ is super nice and has the weak mixing property;
(vi) If $0 < \gamma \leq \beta \leq \alpha$, then, for all $p \in P_\alpha$ and $r \leq_\gamma p \restriction \gamma$, $\pitchfork_{\beta,\gamma}(p \restriction \beta)(r) = (\pitchfork_{\alpha,\gamma}(p)(r)) \restriction \beta$.

[58] $\mathring{A}_n$ is the poset given in Definition 5.7(7) defined with respect to the type map witnessing Clause (a) above.

7.1. **Defining the iteration.** For every $\alpha<\mu^+$, fix an injection $\phi_\alpha:\alpha\to\mu$. As $|H_\mu|=\mu$, by the Engelking-Karłowicz theorem, we may also fix a sequence $\langle e^i\mid i<\mu\rangle$ of functions from μ^+ to H_μ such that for every function $e:C\to H_\mu$ with $C\in[\mu^+]^{<\mu}$, there is $i<\mu$ such that $e\subseteq e^i$.

The upcoming definition is by recursion on $\alpha\le\mu^+$, and we continue as long as we are successful.

▶ Let $\mathbb{P}_0:=(\{\emptyset\},\le_0)$ be the trivial forcing. Let ℓ_0 and c_0 be the constant function $\{(\emptyset,\emptyset)\}$ and $\vec{\varpi}_0=\langle\{(\emptyset,\mathbb{1}_{\mathbb{S}_n})\}\mid n<\omega\rangle$. Finally, let $\pitchfork_{0,0}$ be the constant function $\{(\emptyset,\{(\emptyset,\emptyset)\})\}$, so that $\pitchfork_{0,0}(\emptyset)$ is the identity map.

▶ Let $\mathbb{P}_1:=(P_1,\le_1)$, where $P_1:={}^1Q$ and $p\le_1 p'$ iff $p(0)\le_Q p'(0)$. Evidently, $p\overset{\iota}{\mapsto}p(0)$ form an isomorphism between $\mathbb{P}_1$ and $\mathbb{Q}$, so we naturally define $\ell_1:=\ell\circ\iota$, $c_1:=c\circ\iota$ and $\vec{\varpi}_1:=\vec{\varpi}\bullet\iota$. Hereafter, the sequence $\vec{\varpi}_1$ is denoted by $\langle\varpi^1_n\mid n<\omega\rangle$. For all $p\in P_1$, let $\pitchfork_{1,0}(p):\{\emptyset\}\to\{p\}$ be the constant function, and let $\pitchfork_{1,1}(p)$ be the identity map.

▶ Suppose $\alpha<\mu^+$ and that $\langle(\mathbb{P}_\beta,\ell_\beta,c_\beta,\vec{\varpi}_\beta,\langle\pitchfork_{\beta,\gamma}|\ \gamma\le\beta\rangle)\mid\beta\le\alpha\rangle$ has already been defined. We now define $(\mathbb{P}_{\alpha+1},\ell_{\alpha+1},c_{\alpha+1},\vec{\varpi}_{\alpha+1})$ and $\langle\pitchfork_{\alpha+1,\gamma}|\ \gamma\le\alpha+1\rangle$.

▶▶ If $\psi(\alpha)$ happens to be a triple (β,r,σ), where $\beta<\alpha$, $r\in P_\beta$ and σ is a $\mathbb{P}_\beta$-name, then we appeal to Building Block II with $(\mathbb{P}_\alpha,\ell_\alpha,c_\alpha,\vec{\varpi}_\alpha)$, $r^\star:=r*\emptyset_\alpha$ and $z:=i^\alpha_\beta(\sigma)$ to get a corresponding $(\Sigma,\vec{\mathbb{S}})$-Prikry quadruple $(\mathbb{A},\ell_{\mathbb{A}},c_{\mathbb{A}},\vec{\varsigma})$.

▶▶ Otherwise, we obtain $(\mathbb{A},\ell_{\mathbb{A}},c_{\mathbb{A}},\vec{\varsigma})$ by appealing to Building Block II with $(\mathbb{P}_\alpha,\ell_\alpha,c_\alpha,\vec{\varpi}_\alpha)$, $r^\star:=\emptyset_\alpha$ and $z:=\emptyset$.

In both cases, we obtain a super nice forking projection $(\pitchfork,\pi)$ from $(\mathbb{A},\ell_{\mathbb{A}},c_{\mathbb{A}},\vec{\varsigma})$ to $(\mathbb{P}_\alpha,\ell_\alpha,c_\alpha,\vec{\varpi}_\alpha)$. Furthermore, each condition in $\mathbb{A}=(A,\trianglelefteq)$ is a pair (x,y) with $\pi(x,y)=x$, and, for every $p\in P_\alpha$, $\lceil p\rceil^{\mathbb{A}}=(p,\emptyset)$. Now, define $\mathbb{P}_{\alpha+1}:=(P_{\alpha+1},\le_{\alpha+1})$ by letting $P_{\alpha+1}:=\{x^\frown\langle y\rangle\mid(x,y)\in A\}$, and then letting $p\le_{\alpha+1}p'$ iff $(p\restriction\alpha,p(\alpha))\trianglelefteq(p'\restriction\alpha,p'(\alpha))$. Put $\ell_{\alpha+1}:=\ell_1\circ\pi_{\alpha+1,1}$ and define $c_{\alpha+1}:P_{\alpha+1}\to H_\mu$ via $c_{\alpha+1}(p):=c_{\mathbb{A}}(p\restriction\alpha,p(\alpha))$.

Let $\vec{\varpi}_\alpha=\langle\varpi^\alpha_n\mid n<\omega\rangle$ be defined in the natural way, i.e., for each $n<\omega$ and $x^\frown\langle y\rangle\in(P_\alpha)_{\ge n}$, we set $\varpi^\alpha_n(x^\frown\langle y\rangle):=\varsigma_n(x,y)$.

Next, let $p\in P_{\alpha+1}$, $\gamma\le\alpha+1$ and $r\le_\gamma p\restriction\gamma$ be arbitrary; we need to define $\pitchfork_{\alpha+1,\gamma}(p)(r)$. For $\gamma=\alpha+1$, let $\pitchfork_{\alpha+1,\gamma}(p)(r):=r$, and for $\gamma\le\alpha$, let

$$\pitchfork_{\alpha+1,\gamma}(p)(r):=x^\frown\langle y\rangle\text{ iff }\pitchfork(p\restriction\alpha,p(\alpha))(\pitchfork_{\alpha,\gamma}(p\restriction\alpha)(r))=(x,y).$$

▶ Suppose $\alpha\le\mu^+$ is a nonzero limit ordinal, and that the sequence $\langle(\mathbb{P}_\beta,\ell_\beta,c_\beta,\vec{\varpi}_\beta,\langle\pitchfork_{\beta,\gamma}|\ \gamma\le\beta\rangle)\mid\beta<\alpha\rangle$ has already been defined according to Goal 7.2.

Define $\mathbb{P}_\alpha:=(P_\alpha,\le_\alpha)$ by letting P_α be all α-sequences p such that $|B_p|<\mu$ and $\forall\beta<\alpha(p\restriction\beta\in P_\beta)$. Let $p\le_\alpha q$ iff $\forall\beta<\alpha(p\restriction\beta\le_\beta q\restriction\beta)$. Let $\ell_\alpha:=\ell_1\circ\pi_{\alpha,1}$. Next, we define $c_\alpha:P_\alpha\to H_\mu$, as follows.

▶▶ If $\alpha<\mu^+$, then, for every $p\in P_\alpha$, let

$$c_\alpha(p):=\{(\phi_\alpha(\gamma),c_\gamma(p\restriction\gamma))\mid\gamma\in B_p\}.$$

►► If $\alpha = \mu^+$, then, given $p \in P_\alpha$, first let $C := \mathrm{cl}(B_p)$, then define a function $e : C \to H_\mu$ by stipulating:

$$e(\gamma) := (\phi_\gamma[C \cap \gamma], c_\gamma(p \restriction \gamma)).$$

Then, let $c_\alpha(p) := i$ for the least $i < \mu$ such that $e \subseteq e^i$. Set $\vec{\varpi}_\alpha := \vec{\varpi}_1 \bullet \pi_{\alpha,1}$.

Finally, let $p \in P_\alpha$, $\gamma \leq \alpha$ and $r \leq_\gamma p \restriction \gamma$ be arbitrary; we need to define $\pitchfork_{\alpha,\gamma}(p)(r)$. For $\gamma = \alpha$, let $\pitchfork_{\alpha,\gamma}(p)(r) := r$, and for $\gamma < \alpha$, let $\pitchfork_{\alpha,\gamma}(p)(r) := \bigcup\{\pitchfork_{\beta,\gamma}(p \restriction \beta)(r) \mid \gamma \leq \beta < \alpha\}$.

7.2. **Verification.** Our next task is to verify that for all $\alpha \leq \mu^+$, the tuple $(\mathbb{P}_\alpha, \ell_\alpha, c_\alpha, \vec{\varpi}_\alpha, \langle \pitchfork_{\alpha,\gamma} \mid \gamma \leq \alpha \rangle)$ fulfills requirements (i)–(vi) of Goal 7.2. It is obvious that Clauses (i) and (iii) hold, so we focus on verifying the rest.

The next fact deals with an expanded version of Clause (vi). For the proof we refer the reader to [PRS22, Lemma 3.5]:

Fact 7.3. *For all $\gamma \leq \alpha \leq \mu^+$, $p \in P_\alpha$ and $r \in P_\gamma$ with $r \leq_\gamma p \restriction \gamma$, if we let $q := \pitchfork_{\alpha,\gamma}(p)(r)$, then:*

1. *$q \restriction \beta = \pitchfork_{\beta,\gamma}(p \restriction \beta)(r)$ for all $\beta \in [\gamma, \alpha]$;*
2. *$B_q = B_p \cup B_r$;*
3. *$q \restriction \gamma = r$;*
4. *If $\gamma = 0$, then $q = p$;*
5. *$p = (p \restriction \gamma) * \emptyset_\alpha$ iff $q = r * \emptyset_\alpha$;*
6. *for all $p' \leq^0_\alpha p$, if $r \leq^0_\gamma p' \restriction \gamma$, then $\pitchfork_{\alpha,\gamma}(p')(r) \leq_\alpha \pitchfork_{\alpha,\gamma}(p)(r)$.*

We move on to Clause (ii) and Clause (v):

Lemma 7.4. *Suppose that $\alpha \leq \mu^+$ is such that for all nonzero $\gamma < \alpha$, $(\mathbb{P}_\gamma, c_\gamma, \ell_\gamma, \vec{\varpi}_\gamma)$ is $(\Sigma, \vec{\mathbb{S}})$-Prikry. Then:*

- *for all nonzero $\gamma \leq \alpha$, $(\pitchfork_{\alpha,\gamma}, \pi_{\alpha,\gamma})$ is a nice forking projection from $(\mathbb{P}_\alpha, \ell_\alpha, \vec{\varpi}_\alpha)$ to $(\mathbb{P}_\gamma, \ell_\gamma, \vec{\varpi}_\gamma)$, where $\pi_{\alpha,\gamma}$ is defined as in Goal 7.2(ii);*
- *if $\alpha < \mu^+$, then $(\pitchfork_{\alpha,\gamma}, \pi_{\alpha,\gamma})$ is furthermore a nice forking projection from $(\mathbb{P}_\alpha, \ell_\alpha, c_\alpha, \vec{\varpi}_\alpha)$ to $(\mathbb{P}_\gamma, \ell_\gamma, c_\gamma, \vec{\varpi}_\gamma)$*
- *if $\alpha = \gamma + 1$, then $(\pitchfork_{\alpha,\gamma}, \pi_{\alpha,\gamma})$ is super nice and has the weak mixing property.*

Proof. The above items with the exception of the niceness requirement can be proved as in [PRS22, Lemma 3.6]. It thus suffices to prove the following:

Claim 7.4.1. *For all nonzero $\gamma \leq \alpha$, $\vec{\varpi}_\alpha = \vec{\varpi}_\gamma \bullet \pi_{\alpha,\gamma}$. Also, for each n, ϖ^α_n is a nice projection from $(\mathbb{P}_\alpha)_{\geq n}$ to $\mathbb{S}_n$ and for each $k \geq n$, $\varpi^\alpha_n \restriction (\mathbb{P}_\alpha)_k$ is again a nice projection.*

Proof. By induction on $\alpha \leq \mu^+$:

► The case $\alpha = 1$ is trivial, since then, $\gamma = \alpha$ and $\vec{\varpi}_1 = \vec{\varpi} \bullet \iota$.

► Suppose $\alpha = \alpha' + 1$ and the claim holds for α'. Recall that $\mathbb{P}_\alpha = \mathbb{P}_{\alpha'+1}$ was defined by feeding $(\mathbb{P}_{\alpha'}, \ell_{\alpha'}, c_{\alpha'}, \vec{\varpi}_{\alpha'})$ into Building Block II, thus obtaining a $(\Sigma, \vec{\mathbb{S}})$-Prikry forcing $(\mathbb{A}, \ell_\mathbb{A}, c_\mathbb{A}, \vec{\varsigma})$ along with the pair $(\pitchfork, \pi)$. Also, we have that $(x, y) \in A$ iff $x^\frown\langle y \rangle \in P_\alpha$.

By niceness of $(\pitchfork, \pi)$ and our recursive definition,

$$\varpi^\alpha_n(x^\frown\langle y\rangle) = \varsigma_n(x,y) = \varpi^{\alpha'}_n(\pi(x,y)) = \varpi^{\alpha'}_n(\pi_{\alpha,\alpha'}(x^\frown\langle y\rangle)),$$

for all $n<\omega$ and $x^\frown\langle y\rangle \in (P_\alpha)_{\geq n}$. Hence, $\vec{\varpi}_\alpha = \vec{\varpi}_{\alpha'} \bullet \pi_{\alpha,\alpha'}$. Using the induction hypothesis for $\vec{\varpi}_{\alpha'}$ we arrive at $\vec{\varpi}_\alpha = \vec{\varpi}_\gamma \bullet \pi_{\alpha,\gamma}$.

Let us now address the second part of the claim. We just show that for every $n<\omega$, the map ϖ^α_n is a nice projection from $(\mathbb{P}_\alpha)_{\geq n}$ to $\mathbb{S}_n$. The statement that $\varpi^\alpha_n \restriction (\mathbb{P}_\alpha)_k$ is a nice projection can be proved similarly.

So, let us go over the clauses of Definition 2.2. Clauses (1) and (2) are evident and Clause (3) follows from Lemma 5.6 applied to $(\pitchfork_{\alpha,\alpha'}, \pi_{\alpha,\alpha'})$.

(4): Let $p,q \in (P_\alpha)_{\geq n}$ and $s \preceq_n \varpi^\alpha_n(p)$ be such that $q \leq_\alpha p+s$. Then, $(q\restriction\alpha', q(\alpha')) \trianglelefteq (p\restriction\alpha', p(\alpha'))+s$. By Clause (a) of Building Block II we have that ς_n is a nice projection from $\mathbb{A}_{\geq n}$ to $\mathbb{S}_n$, hence there is $(x,y)\in A$ such that $(x,y) \trianglelefteq^{\varsigma_n} (p\restriction\alpha', p(\alpha'))$ and $(q\restriction\alpha', q(\alpha')) = (x,y)+\varsigma_n((q\restriction\alpha', q(\alpha')))$. Setting $p' := x^\frown\langle y\rangle$ it is immediate that $p' \leq^{\varpi^\alpha_n}_\alpha p$ and

$$q = p' + \varpi^{\alpha'}_n(q\restriction\alpha') = p' + \varpi^\alpha_n(q).$$

► For $\alpha \in \mathrm{acc}(\mu^+ + 1)$, the first part follows from $\vec{\varpi}_\alpha := \vec{\varpi}_1 \circ \pi_{\alpha,1}$ and the induction hypothesis. About the verification of the Clauses of Definition 2.2, Clauses (1) and (2) are automatic and Clause (3) follows from Lemma 5.6 applied to $(\pitchfork_{\alpha,1}, \pi_{\alpha,1})$. About Clause (4) we argue as follows.

Fix $p,q \in (\mathbb{P}_\alpha)_{\geq n}$ and $s \preceq_n \varpi^\alpha_n(p)$ be such that $q \leq_\alpha p+s$. The goal is to find a condition $p' \in (P_\alpha)_{\geq n}$ such that $p' \leq^{\varpi^\alpha_n} p$ and $q = p' + \varpi^\alpha_n(q)$.

Let $\langle \gamma_\tau \mid \tau \leq \theta\rangle$ be the increasing enumeration of the closure of B_q.[59] For every $\tau \in \mathrm{nacc}(\theta+1)$, γ_τ is a successor ordinal, so we let β_τ denote its predecessor. By recursion on $\tau \leq \theta$, we shall define a sequence of conditions $\langle p'_\tau \mid \tau \leq \theta\rangle \in \prod_{\tau\leq\theta}(P_{\gamma_\tau})$ such that $p'_\tau \leq^{\varpi^{\gamma_\tau}_n}_{\gamma_\tau} p\restriction\gamma_\tau$ and $q\restriction\gamma_\tau = p'_\tau + \varpi^{\gamma_\tau}_n(q\restriction\gamma_\tau)$.

In order to be able to continue with the construction at limits stages we shall moreover secure that $\langle p'_\tau \mid \tau \leq \theta\rangle$ is coherent: i.e., $p'_\tau \restriction \gamma_{\tau'} = p_{\tau'}$ for all $\tau' \leq \tau$. Also, note that $\langle \varpi^{\gamma_\tau}_n(q\restriction\gamma_\tau) \mid \tau \leq \theta\rangle$ is a constant sequence, so hereafter we denote by t its constant value.

To form the first member of the sequence we argue as follows. First note that $q\restriction 1 \leq_1 p\restriction 1 + s$, so that appealing to Definition 2.2(4) for ϖ^1_n we get a condition $p'_{-1} \in P_1$ such that $p'_{-1} \leq^{\varpi^1_n}_1 p\restriction 1$ and $q\restriction 1 = p'_{-1} + t$.

Now, let $r_0 := \pitchfork_{\gamma_0,1}(p\restriction\gamma_0)(p'_{-1})$. A moment's reflection makes it clear that $r_0 + s$ is well-defined and also $q\restriction\gamma_0 \leq_{\gamma_0} r_0 + s$. So, appealing to Definition 2.2(4) for $\varpi^{\gamma_0}_n$ we get a condition $p'_0 \in P_{\gamma_0}$ such that $p'_0 \leq^{\varpi^{\gamma_0}_n}_{\gamma_0} r_0$ and $q\restriction\gamma_0 = p'_0 + t$. Since $p'_{-1} \leq^{\varpi^1_n}_1 p\restriction 1$ and $\varpi^{\gamma_0}_n = \varpi^1_n \circ \pi_{\gamma_0,1}$ we have $\varpi^{\gamma_0}_n(r_0) = \varpi^{\gamma_0}_n(p\restriction\gamma_0)$. This completes the first step of the induction.

Let us suppose that we have already constructed $\langle p_{\tau'} \mid \tau' < \tau\rangle$.

<u>τ is successor:</u> Suppose that $\tau = \tau'+1$. Then, set $r_\tau := \pitchfork_{\gamma_\tau,\gamma_{\tau'}}(p\restriction\gamma_\tau)(p'_{\tau'})$. Using the induction hypothesis it is easy to see that $q\restriction\gamma_\tau \leq_{\gamma_\tau} r_\tau + s$. Instead

[59]Recall that $B_q := \{\beta+1 \mid \beta \in \mathrm{dom}(q)\ \&\ q(\beta) \neq \emptyset\}$.

of outright invoking the niceness of $\varpi_n^{\gamma_\tau}$ we would like to use that $(\pitchfork_{\gamma_\tau,\beta_\tau}, \pi_{\gamma_\tau,\beta_\tau})$ is a super nice forking projection (see Definition 5.4). This will secure that the future condition p'_τ will be coherent with $p'_{\tau'}$, and therefore with all the conditions constructed so far.

Applying the definition of $\pitchfork_{\gamma_\tau,\gamma_{\tau'}}$ given at page 59 we have

$$r_\tau = \pitchfork_{\gamma_\tau,\beta_\tau}(p \restriction \gamma_\tau)(\pitchfork_{\beta_\tau,\gamma_{\tau'}}(p \restriction \beta_\tau)(p'_{\tau'})).$$

Since $p \restriction \beta_\tau = p \restriction \gamma_{\tau'} * \emptyset_{\beta_\tau}$, Clause (6) of Fact 7.3 yields

$$\pitchfork_{\beta_\tau,\gamma_{\tau'}}(p \restriction \beta_\tau)(p'_{\tau'}) = p'_{\tau'} * \emptyset_{\beta_\tau}.$$

So, $r_\tau = \pitchfork_{\gamma_\tau,\beta_\tau}(p \restriction \gamma_\tau)(p'_{\tau'} * \emptyset_{\beta_\tau})$.

Subclaim 7.4.1.1. *$p'_{\tau'} * \emptyset_{\beta_\tau} \leq^{\varpi_n^{\beta_\tau}}_{\beta_\tau} p \restriction \beta_\tau$ and $q \restriction \beta_\tau = (p'_{\tau'} * \emptyset_{\beta_\tau}) + t$.*

Proof. The first part follows immediately from $p'_{\tau'} \leq^{\varpi_n^{\gamma'_\tau}}_{\gamma_{\tau'}} p \restriction \gamma_{\tau'}$. For the second part note that $q \restriction \beta_\tau = q \restriction \gamma_{\tau'} * \emptyset_{\beta_\tau}$, hence Fact 7.4(5) combined with the induction hypothesis yield

$$q \restriction \beta_\tau = \pitchfork_{\beta_\tau,\gamma_{\tau'}}(q \restriction \beta_\tau)(q \restriction \gamma_{\tau'}) = \pitchfork_{\beta_\tau,\gamma_{\tau'}}(q \restriction \beta_\tau)(p'_{\tau'} + t) = (p'_{\tau'} + t) * \emptyset_{\beta_\tau}.$$

On the other hand, using Lemma 5.6 with respect to $(\pitchfork_{\beta_\tau,\gamma_{\tau'}}, \pi_{\beta_\tau,\gamma_{\tau'}})$,

$$(p'_{\tau'} * \emptyset_{\beta_\tau}) + t = \pitchfork_{\beta_\tau,\gamma_{\tau'}}(p'_{\tau'} * \emptyset_{\beta_\tau})(p'_{\tau'} + t) = (p'_{\tau'} + t) * \emptyset_{\beta_\tau},$$

where the last equality follows again from Fact 7.4(5).

Combining the above expressions we arrive at $q \restriction \beta_\tau = (p'_{\tau'} * \emptyset_{\beta_\tau}) + t$. □

By Clause (f) of Building Block II and the definition of the iteration at successor stage (see page 59), the pair $(\pitchfork_{\gamma_\tau,\beta_\tau}, \pi_{\gamma_\tau,\beta_\tau})$ is a super nice forking projection from $(\mathbb{P}_{\gamma_\tau}, \ell_{\gamma_\tau}, \vec{\varpi}_{\gamma_\tau})$ to $(\mathbb{P}_{\beta_\tau}, \ell_{\beta_\tau}, \vec{\varpi}_{\beta_\tau})$. Combining this with the above subclaim we find a condition $p'_\tau \leq^{\varpi_n^{\gamma_\tau}}_{\gamma_\tau} r_\tau$ such that $p'_\tau \restriction \beta_\tau = p'_{\tau'} * \emptyset_{\beta_\tau}$ and $q \restriction \gamma_\tau = p'_\tau + t$. Clearly, p'_τ witnesses the desired property.

<u>τ is limit:</u> Put $p'_\tau := \bigcup_{\tau'<\tau} p'_{\tau'}$. Thanks to the induction hypothesis it is evident that $p'_\tau \leq^{\varpi_n^{\gamma_\tau}}_{\gamma_\tau} p \restriction \gamma_\tau$. Also, combining the induction hypothesis with Lemma 5.6 for $(\pitchfork_{\gamma_\tau,1}, \pi_{\gamma_\tau,1})$ we obtain the following chain of equalities:

$$q \restriction \gamma_\tau = \bigcup_{\tau'<\tau} (p'_{\tau'} + t) = \bigcup_{\tau'<\tau} \pitchfork_{\gamma_\tau,1}(p'_{\tau'})(p'_{\tau'} \restriction 1 + t) = \bigcup_{\tau'<\tau} \pitchfork_{\gamma_{\tau'},1}(p'_{\tau'})(p'_\tau \restriction 1 + t).$$ [60]

Using the definition of the pitchfork at limit stages (see page 60) we get

$$\bigcup_{\tau'<\tau} \pitchfork_{\gamma_{\tau'},1}(p'_{\tau'})(p'_\tau \restriction 1 + t) = p'_\tau + t = \pitchfork_{\gamma_\tau,1}(p'_\tau)(p'_\tau \restriction 1 + t),$$

where the last equality follows from Lemma 5.6 for $(\pitchfork_{\gamma_\tau,1}, \pi_{\gamma_\tau,1})$.

Altogether, we have shown that $p'_\tau \leq^{\varpi_n^{\gamma_\tau}}_{\gamma_\tau} p \restriction \gamma_\tau$ and $q \restriction \gamma_\tau = p'_\tau + t$. Additionally, $p'_\tau \restriction \gamma_{\tau'} = p'_{\tau'}$ for all $\tau' < \tau$.

[60] Note that for the right-most equality we have used that $p'_\tau \restriction 1 = p'_{\tau'} \restriction 1$, for all $\tau' < \tau$.

Finally, putting $p' := p'_\theta$ we obtain a condition in $(\mathbb{P}_\alpha)_{\geq n}$ such that

$$p' \leq_\alpha^{\varpi_n^\alpha} p \text{ and } q = p' + \varpi_n^\alpha(q).$$

This completes the argument. $\square$

This completes the proof of the lemma. $\square$

Recalling Definition 3.3(2), for all nonzero $\alpha \leq \mu^+$ and $n < \omega$, we need to identify a candidate for a dense subposet $(\mathring{\mathbb{P}}_\alpha)_n = ((\mathring{P}_\alpha)_n, \leq_\alpha)$ of $(\mathbb{P}_\alpha)_n$.

Definition 7.5. For each nonzero $\gamma < \mu^+$, we let $\mathrm{tp}_{\gamma+1}$ be a type witnessing that $(\pitchfork_{\gamma+1,\gamma}, \pi_{\gamma+1,\gamma})$ has the weak mixing property.

Definition 7.6. Let $n < \omega$. Set $\mathring{P}_{1n} := {}^1(\mathring{Q}_n)$.[61] Then, for each $\alpha \in [2, \mu^+]$, define $\mathring{P}_{\alpha n}$ by recursion:

$$\mathring{P}_{\alpha n} := \begin{cases} \{p \in P_\alpha \mid \pi_{\alpha,\beta}(p) \in \mathring{P}_{\beta n} \ \&\ \mathrm{mtp}_{\beta+1}(p) = 0\}, & \text{if } \alpha = \beta + 1; \\ \{p \in P_\alpha \mid \pi_{\alpha,1}(p) \in \mathring{P}_{1n} \ \&\ \forall \gamma \in B_p \ \mathrm{mtp}_\gamma(\pi_{\alpha,\gamma}(p)) = 0\}, & \text{otherwise.} \end{cases}$$

Lemma 7.7. *Let $n < \omega$ and $1 \leq \beta < \alpha \leq \mu^+$. Then:*

(1) $\pi_{\alpha,\beta}$“$\mathring{P}_{\alpha n} \subseteq \mathring{P}_{\beta n}$;
(2) For every $p \in \mathring{P}_{\beta n}$, $p * \emptyset_\delta \in \mathring{P}_{\alpha n}$.

Proof. By induction, relying on Clause (4) of Definition 5.7. $\square$

We now move to establish Clause (iv) of Goal 7.2.

Lemma 7.8. *Let $\alpha \in [2, \mu^+]$. Then, for all $n < \omega$ and every directed set of conditions D in $(\mathring{\mathbb{P}}_\alpha)_n$ (resp. $(\mathring{\mathbb{P}}_\alpha)_n^{\varpi_n^\alpha}$) of size $<\aleph_1$ (resp. $<\sigma_n^*$) there is $q \in (\mathring{P}_\alpha)_n$ such that q is a $\leq_\alpha$ (resp. $\leq_n^{\varpi_n^\alpha}$) lower bound for D.*

Moreover, $B_q = \bigcup_{p \in D} B_p$.

Proof. The argument is similar to that of [PRS22, Lemma 3.13]. $\square$

Remark 7.9. A straightforward modification of the lemma shows that for all $\alpha \in [2, \mu^+]$ and $n < \omega$, $(\mathbb{P}_\alpha)_n^{\pi_{\alpha,1}}$ is σ_n^*-directed-closed.

Theorem 7.10. *For all nonzero $\alpha \leq \mu^+$, $(\mathbb{P}_\alpha, \ell_\alpha, c_\alpha, \vec{\varpi}_\alpha)$ satisfies all the requirements to be a $(\Sigma, \vec{\mathbb{S}})$-Prikry quadruple, with the possible exceptions of Clause (7) and the density requirement in Clauses (2) and (9).*

Additionally, $\emptyset_\alpha$ is the greatest condition in $\mathbb{P}_\alpha$, $\ell_\alpha = \ell_1 \circ \pi_{\alpha,1}$, $\emptyset_\alpha \Vdash_{\mathbb{P}_\alpha} \check{\mu} = \kappa^+$ and $\vec{\varpi}_\alpha$ is a coherent sequence of nice projections such that

$$\vec{\varpi}_\alpha = \vec{\varpi}_\gamma \bullet \pi_{\alpha,\gamma} \text{ for every } \gamma \leq \alpha.$$

Under the extra hypothesis that for each $\alpha \in \mathrm{acc}(\mu^+ + 1)$ and every $n < \omega$, $(\mathring{\mathbb{P}}_\alpha^{\varpi_n^\alpha})_n$ is a dense subposet of $(\mathbb{P}_\alpha^{\varpi_n^\alpha})_n$, we have that for all nonzero $\alpha \leq \mu^+$, $(\mathbb{P}_\alpha, \ell_\alpha, c_\alpha, \vec{\varpi}_\alpha)$ is a $(\Sigma, \vec{\mathbb{S}})$-Prikry quadruple having property $\mathcal{D}$.

[61] Here, $\mathring{Q}_n$ is obtained from Clause (2) of Definition 3.3 with respect to the triple $(\mathbb{Q}, \ell, c)$ given by Building Block I.

Proof. We argue by induction on $\alpha \leq \mu^+$. The base case $\alpha = 1$ follows from the fact that $\mathbb{P}_1$ is isomorphic to $\mathbb{Q}$ given by Building Block I. The successor step $\alpha = \beta + 1$ follows from the fact that $\mathbb{P}_{\beta+1}$ was obtained by invoking Building Block II.

Next, suppose that $\alpha \in \mathrm{acc}(\mu^+ + 1)$ is such that the conclusion of the lemma holds below α. In particular, the hypothesis of Lemma 7.4 are satisfied, so that, for all nonzero $\beta \leq \gamma \leq \alpha$, $(\pitchfork_{\gamma,\beta}, \pi_{\gamma,\beta})$ is a nice forking projection from $(\mathbb{P}_\gamma, \ell_\gamma, \vec{\varpi}_\gamma)$ to $(\mathbb{P}_\beta, \ell_\beta, \vec{\varpi}_\beta)$. By the very same proof of [PRS22, Lemma 3.14], we have that Clauses (1) and (3)–(6) of Definition 3.3 hold for $(\mathbb{P}_\alpha, \ell_\alpha, c_\alpha, \vec{\varpi}_\alpha)$, and that $\ell_\alpha = \ell_1 \circ \pi_{\alpha,1}$. Also, Clauses (2) and (9) –without the density requirement– follow from Lemma 7.8.

On the other hand, the equality $\vec{\varpi}_\alpha = \vec{\varpi}_\gamma \bullet \pi_{\alpha,\gamma}$ follows from Lemma 7.4. Arguing as in [PRS22, Claim 3.14.2], we also have that $\mathbb{1}_{\mathbb{P}_\alpha} \Vdash_{\mathbb{P}_\alpha} \check{\mu} = \check{\kappa}^+$. Finally, since $\vec{\varpi}_1$ is coherent (see Building Block I) and $(\pitchfork_{\alpha,1}, \pi_{\alpha,1})$ is a nice forking projection, Lemma 5.17 implies that $\vec{\varpi}_\alpha$ is coherent.

To complete the proof let us additionally assume that for every n, $(\mathring{\mathbb{P}}_\alpha^{\varpi_n^\alpha})_n$ is a dense subposet of $(\mathbb{P}_\alpha^{\varpi_n^\alpha})_n$. Then, in particular, $(\mathring{\mathbb{P}}_\alpha)_n$ is a dense subposet of $(\mathbb{P}_\alpha)_n$. In effect, the density requirement in Clauses (2) and (9) is automatically fulfilled. About Clause (7), we take advantage of this extra assumption to invoke [PRS22, Corollary 3.12] and conclude that $(\mathbb{P}_\alpha, \ell_\alpha)$ has property $\mathcal{D}$. Consequently, Clause (7) for $(\mathbb{P}_\alpha, \ell_\alpha, c_\alpha, \vec{\varpi}_\alpha)$ follows by combining this latter fact with Lemma 5.12. $\square$

8. A proof of the Main Theorem

In this section, we arrive at the primary application of the framework developed thus far. We will be constructing a model where GCH holds below $\aleph_\omega$, $2^{\aleph_\omega} = \aleph_{\omega+2}$ and every stationary subset of $\aleph_{\omega+1}$ reflects.

8.1. Setting up the ground model.

We want to obtain a ground model with GCH and ω-many supercompact cardinals, which are Laver indestructible under GCH-preserving forcing. The first lemma must be well-known, but we could not find it in the literature, so we give an outline of the proof.

Lemma 8.1. *Suppose $\vec{\kappa} = \langle \kappa_n \mid n < \omega \rangle$ is an increasing sequence of supercompact cardinals. Then there is a generic extension where GCH holds and $\vec{\kappa}$ remains an increasing sequence of supercompact cardinals.*

Proof. Let $\mathbb{J}$ be Jensen's iteration to force the GCH. Namely, $\mathbb{J}$ is the inverse limit of the Easton-support iteration $\langle \mathbb{J}_\alpha; \dot{\mathbb{Q}}_\beta \mid \beta \leq \alpha \in \mathrm{Ord} \rangle$ such that, if $\mathbb{1}_{\mathbb{J}_\alpha} \Vdash_{\mathbb{J}_\alpha}$ "α is a cardinal", then $\mathbb{1}_{\mathbb{J}_\alpha} \Vdash_{\mathbb{J}_\alpha}$ "$\dot{\mathbb{Q}}_\alpha = \dot{\mathrm{Add}}(\alpha^+, 1)$" and $\mathbb{1}_{\mathbb{J}_\alpha} \Vdash_{\mathbb{J}_\alpha}$ "$\dot{\mathbb{Q}}_\alpha$ trivial", otherwise. A standard argument shows that $\mathbb{J}$ preserves the supercompactness of κ_n. $\square$

Note that in the model of the conclusion of the above lemma, the κ_n's are highly-destructible by κ_n-directed-closed forcing. Our next task is to remedy that, while maintaining GCH. For this, we need the following slight variation of the usual Laver preparation [Lav78].

Lemma 8.2. *Suppose that GCH holds, $\chi < \kappa$ are infinite regular cardinals, and κ is supercompact. Then there exists a χ-directed-closed notion of forcing $\mathbb{L}^{\kappa}_{\chi}$ that preserves GCH and makes the supercompactness of κ indestructible under κ-directed-closed forcings that preserve GCH.*

Proof. Let f be a Laver function on κ, as in [Cum10, Theorem 24.1]. Let $\mathbb{L}^{\kappa}_{\chi}$ be the direct limit of the forcing iteration $\langle \mathbb{R}_\alpha; \dot{\mathbb{Q}}_\beta \mid \chi \leq \beta < \alpha < \kappa\rangle$ where, if α is inaccessible, $\mathbb{1}_{\mathbb{R}_\alpha} \Vdash_{\mathbb{R}_\alpha} \mathsf{GCH}$, and $f(\alpha)$ encodes an $\mathbb{R}_\alpha$-name $\tau \in H_{\alpha^+}$ for some α-directed-closed forcing that preserves the GCH of $V^{\mathbb{R}_\alpha}$, then $\dot{\mathbb{Q}}_\alpha$ is chosen to be such $\mathbb{R}_\alpha$-name. Otherwise, $\dot{\mathbb{Q}}_\alpha$ is chosen to be the trivial forcing.

As in the proof of [Cum10, Theorem 24.12], we have that after forcing with $\mathbb{L}^{\kappa}_{\chi}$, the supercompactness of κ becomes indestructible under κ-directed-closed forcings that preserve GCH.

We claim that GCH holds in $V^{\mathbb{L}^{\kappa}_{\chi}}$. This is clear for cardinals $\geq \kappa$, since the iteration has size κ. Now, let $\lambda < \kappa$ and inductively assume $\mathsf{GCH}_{<\lambda}$. Observe that $\mathbb{L}^{\kappa}_{\chi} \cong \mathbb{R}_{\lambda+1} * \dot{\mathbb{Q}}$, where $\dot{\mathbb{Q}}$ is an $\mathbb{R}_{\lambda+1}$-name for a λ^+-directed-closed forcing. In particular, $\mathcal{P}(\lambda)^{V^{\mathbb{L}^{\kappa}_{\chi}}} = \mathcal{P}(\lambda)^{V^{\mathbb{R}_{\lambda+1}}}$, and so it is enough to show that $V^{\mathbb{R}_{\lambda+1}} \models \mathsf{CH}_\lambda$. There are two cases.

If λ is singular, then $|\mathbb{R}_\lambda| = \lambda^+$, and $\dot{\mathbb{Q}}_\lambda$ is trivial, so $V^{\mathbb{R}_{\lambda+1}} \models \mathsf{CH}_\lambda$.

Otherwise, let α be the largest inaccessible, such that $\alpha \leq \lambda$. Then $\mathbb{R}_{\lambda+1}$ is just $\mathbb{R}_\alpha * \dot{\mathbb{Q}}_\alpha$ followed by trivial forcing. Since $|\mathbb{R}_\alpha| = \alpha$ and by construction $\dot{\mathbb{Q}}_\alpha$ preserves GCH, the result follows. □

Corollary 8.3. *Suppose that $\vec{\kappa} = \langle \kappa_n \mid n < \omega\rangle$ is an increasing sequence of supercompact cardinals. Then, in some forcing extension, all of the following hold:*

(1) GCH*;*
(2) $\vec{\kappa}$ is an increasing sequence of supercompact cardinals;
(3) For every $n < \omega$, the supercompactness of κ_n is indestructible under notions of forcing that are κ_n-directed-closed and preserves the GCH.

Proof. By Lemma 8.1, we may assume that we are working in a model in which Clauses (1) and (2) already hold. Next, let $\mathbb{L}$ be the direct limit of the iteration $\langle \mathbb{L}_n * \dot{\mathbb{Q}}_n \mid n < \omega\rangle$, where $\mathbb{L}_0$ is the trivial forcing and, for each n, if $\mathbb{1} \Vdash_{\mathbb{L}_n}$ "κ_n is supercompact", then $\mathbb{1} \Vdash_{\mathbb{L}_n} \dot{\mathbb{Q}}_n$ is the (κ_{n-1})-directed-closed, GCH-preserving forcing making the supercompactness of κ_n indestructible under GCH-preserving κ_n-directed-closed notions of forcing. (More precisely, in the notation of the previous lemma, $\dot{\mathbb{Q}}_n$ is $\dot{\mathbb{L}}^{\kappa_n}_{\kappa_{n-1}}$, where, by convention, κ_{-1} is $\aleph_0$).

Note that, by induction on $n < \omega$, and Lemma 8.2, we maintain that $\mathbb{1} \Vdash_{\mathbb{L}_n}$ "κ_n is supercompact and GCH holds". And then when we force with

$\dot{\mathbb{Q}}_n$ over that model, we make this supercompacness indestructible under GCH -preserving forcing.

Then, after forcing with $\mathbb{L}$, GCH holds, and each κ_n remains supercompact, indestructible under κ_n-directed-closed forcings that preserve GCH.

□

8.2. **Connecting the dots.**

Setup 8. For the rest of this section, we make the following assumptions:

- $\vec{\kappa} = \langle \kappa_n \mid n < \omega \rangle$ is an increasing sequence of supercompact cardinals. By convention, we set $\kappa_{-1} := \aleph_0$;
- For every $n < \omega$, the supercompactness of κ_n is indestructible under notions of forcing that are κ_n-directed-closed and preserve the GCH;
- $\kappa := \sup_{n<\omega} \kappa_n$, $\mu := \kappa^+$ and $\lambda := \kappa^{++}$;
- GCH holds below λ. In particular, $2^\kappa = \kappa^+$ and $2^\mu = \mu^+$;
- $\Sigma := \langle \sigma_n \mid n < \omega \rangle$, where $\sigma_0 := \aleph_1$ and $\sigma_{n+1} := (\kappa_n)^+$ for all $n < \omega$;[62]
- $\vec{\mathbb{S}}$ is as in Definition 4.12.

We now want to appeal to the iteration scheme of the previous section. First, observe that μ, $\langle (\sigma_n, \mu) \mid n < \omega \rangle$, $\vec{\mathbb{S}}$ and Σ respectively fulfill all the blanket assumptions of Setup 7.

We now introduce our three building blocks of choice:

Building Block I. We let $(\mathbb{Q}, \ell, c, \vec{\varpi})$ be EBPFC as defined in Section 4. By Corollary 4.26, this is a $(\Sigma, \vec{\mathbb{S}})$-Prikry that has property $\mathcal{D}$, and $\vec{\varpi}$ is a coherent sequence of nice projection. Also, $\mathbb{Q}$ is a subset of H_{μ^+} and, by Lemma 4.25, $\mathbb{1}_{\mathbb{Q}} \Vdash_{\mathbb{Q}} \check{\mu} = \check{\kappa}^+$. In addition, κ is singular, so that we have $\mathbb{1}_{\mathbb{Q}} \Vdash_{\mathbb{Q}}$ "κ is singular". Finally, for all $n < \omega$, $\mathring{\mathbb{Q}}_n = \mathbb{Q}_n$ (see Lemmas 4.19 and 4.24).

Building Block II. Suppose that $(\mathbb{P}, \ell, c, \vec{\varpi})$ is a $(\Sigma, \vec{\mathbb{S}})$-Prikry quadruple having property $\mathcal{D}$ such that $\mathbb{P} = (P, \leq)$ is a subset of H_{μ^+}, $\vec{\varpi}$ is a coherent sequence of nice projections, $\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}} \check{\mu} = \check{\kappa}^+$ and $\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}}$ "κ is singular".

For every $r^\star \in P$ and a $\mathbb{P}$-name z for an $r^\star$-fragile stationary subset of μ, there are a $(\Sigma, \vec{\mathbb{S}})$-Prikry quadruple $(\mathbb{A}, \ell_{\mathbb{A}}, c_{\mathbb{A}}, \vec{\varsigma})$ having property $\mathcal{D}$, and a pair of maps $(\pitchfork, \pi)$ such that all the following hold:

(a) $(\pitchfork, \pi)$ is a super nice forking projection from $(\mathbb{A}, \ell_{\mathbb{A}}, c_{\mathbb{A}}, \vec{\varsigma})$ to $(\mathbb{P}, \ell, c, \vec{\varpi})$ that has the weak mixing property;
(b) $\vec{\varsigma}$ is a coherent sequence of nice projections;
(c) $\mathbb{1}_{\mathbb{A}} \Vdash_{\mathbb{A}} \check{\mu} = \check{\kappa}^+$;
(d) $\mathbb{A} = (A, \trianglelefteq)$ is a subset of H_{μ^+};
(e) For every $n < \omega$, $\mathring{\mathbb{A}}^\pi_n$ is μ-directed-closed;
(f) if $r^\star \in P$ and z is a $\mathbb{P}$-name for an $r^\star$-fragile stationary subset of μ then
$$\lceil r^\star \rceil^{\mathbb{A}} \Vdash_{\mathbb{A}} \text{“}z \text{ is nonstationary”}.$$

[62]By convention, we set σ_{-2} and σ_{-1} to be $\aleph_0$.

Remark 8.4.

- ▶ If $r^\star \in P$ forces that z is a $\mathbb{P}$-name for an $r^\star$-fragile subset of μ, we first find some $\mathbb{P}$-name $\tilde{z}$ such that $\mathbb{1}_{\mathbb{P}}$ forces that $\tilde{z}$ is a stationary subset of μ, $r^\star \Vdash_{\mathbb{P}} z = \tilde{z}$ and $\tilde{z}$ is $\mathbb{1}_{\mathbb{P}}$-fragile. Subsequently, we obtain $(\mathbb{A}, \ell_{\mathbb{A}}, c_{\mathbb{A}}, \vec{\varsigma})$ and $(\pitchfork, \pi)$ by appealing to Corollary 6.20 with the $(\Sigma, \vec{\mathbb{S}})$-Prikry triple $(\mathbb{P}, \ell_{\mathbb{P}}, c_{\mathbb{P}}, \vec{\varpi})$, the condition $\mathbb{1}_{\mathbb{P}}$ and the $\mathbb{P}$-name $\tilde{z}$. In effect, $\lceil \mathbb{1}_{\mathbb{P}} \rceil^{\mathbb{A}}$ forces that $\tilde{z}$ is nonstationary, so that $\lceil r^\star \rceil^{\mathbb{A}}$ forces that z is nonstationary.
- ▶ Otherwise, we invoke Corollary 6.20 with the $(\Sigma, \vec{\mathbb{S}})$-Prikry forcing $(\mathbb{P}, \ell_{\mathbb{P}}, c_{\mathbb{P}}, \vec{\varpi})$, the condition $\mathbb{1}_{\mathbb{P}}$ and the name $z := \emptyset$.

Building Block III. As $2^\mu = \mu^+$, we fix a surjection $\psi : \mu^+ \to H_{\mu^+}$ such that the preimage of any singleton is cofinal in μ^+.

The next lemma deals with the extra assumption in Theorem 7.10:

Lemma 8.5 (Density of the rings)**.** *For each $\alpha \in \operatorname{acc}(\mu^+ + 1)$ and every integer $n < \omega$, $(\mathring{\mathbb{P}}_\alpha^{\varpi_n^\alpha})_n$ is a dense subposet of $(\mathbb{P}_\alpha^{\varpi_n^\alpha})_n$.*[63]

Proof. This follows along the same lines of [PRS22, Lemma 4.28], with the only difference that now we use the following:

1. (1) At successor stages we can get into the ring $(\mathring{\mathbb{P}}_\alpha^{\varpi_n^\alpha})_n$ by $\le_\alpha^{\varpi_n^\alpha}$-extending. This is granted by Lemma 6.3.
2. (2) For all $\gamma < \alpha$, $(\mathring{\mathbb{P}}_\gamma^{\varpi_n^\gamma})_n$ is σ_n-directed-closed. With this property we take care of the limit stages. (In [PRS22], the full ring $(\mathring{\mathbb{P}}_\gamma)_n$ was σ_n-closed). We can make this replacement, because of the first item above. □

Now, we can appeal to the iteration scheme of Section 7 with these building blocks, and obtain, in return, a sequence $\langle (\mathbb{P}_\alpha, \ell_\alpha, c_\alpha, \vec{\varpi}_\alpha) \mid 1 \le \alpha \le \mu^+ \rangle$ of $(\Sigma, \vec{\mathbb{S}})$-Prikry quadruples. By Lemma 7.8 and Theorem 7.10 (see also Remark 7.9), for all nonzero $\alpha \le \mu^+$, $(\mathring{\mathbb{P}}_\alpha)_n^{\pi_{\alpha,1}}$ is μ-directed-closed and $\mathbb{1}_{\mathbb{P}_\alpha} \Vdash_{\mathbb{P}_\alpha} \check{\mu} = \check{\kappa}^+$. Note that by the first clause of Goal 7.2, $|P_\alpha| \le \mu^+$ for every $\alpha \le \mu^+$.

Lemma 8.6. *Let $n \in \omega \setminus 2$ and $\alpha \in [2, \mu^+)$. Then $((\mathbb{P}_\alpha)_n, \mathbb{S}_n, \mathbb{R}_n, \varpi_n^\alpha)$ is suitable for reflection with respect to $\langle \sigma_{n-2}, \kappa_{n-1}, \kappa_n, \mu \rangle$.*

Proof. Let $(\mathbb{P}, \ell, c, \vec{\varpi})$ denote the first $(\Sigma, \vec{\mathbb{S}})$-Prikry forcing of our iteration; namely, Gitik's forcing from Section 4. In Corollary 4.32 we show that $(\mathbb{P}_n, \mathbb{S}_n, \mathbb{R}_n, \varpi_n)$ is suitable for reflection with respect to $\langle \sigma_{n-2}, \kappa_{n-1}, \kappa_n, \mu \rangle$. We are going to use this, along with the properties of the iteration, to show that the same property holds for $((\mathbb{P}_\alpha)_n, \mathbb{S}_n, \mathbb{R}_n, \varpi_n^\alpha)$.

Let us go over the clauses of Definition 2.10. Clause (1) holds simply because $(\mathbb{P}_\alpha, \ell_\alpha, c_\alpha, \vec{\varpi}_\alpha)$ is $(\Sigma, \vec{\mathbb{S}})$-Prikry.[64] Clause (2) holds by Lemma 8.5.

[63] $(\mathring{\mathbb{P}}_\alpha)_n$ is as in Definition 7.6.

[64] Here the map inducing $\varpi_n^\alpha := \pi_{\mathbb{P}_n} \circ \pi_{\alpha,1}$ is $\pi_{\mathbb{P}_n} : \mathbb{P}_n \to \mathbb{S}_n \times \mathbb{R}_n$.

To address Clause (3), let us fix $a \in (P_\alpha)_n$ with $\varpi^\alpha_n(a) \neq \mathbb{1}_{\mathbb{S}_n}$. Let $p := \pi_{\alpha,1}(a)$ and note that $\varpi_{1,n}(p) \neq \mathbb{1}_{\mathbb{S}_n}$. Because of suitability for reflection of $(\mathbb{P}_n, \mathbb{S}_n, \mathbb{R}_n, \varpi_n)$ we infer the existence of a cardinal δ with $\tau^+ < \delta < \sigma$ and forcing $\mathbb{Q}$ of size $< \delta$ such that

$$\mathbb{S}_n \downarrow \varpi^\alpha_n(a) \simeq \mathbb{Q} \times \mathrm{Col}(\delta, {<}\sigma).$$

We move on to subclause (3a).

Claim 8.6.1. *"$|\mu| = \mathrm{cf}(\mu) = \kappa_n = (\kappa_{n-1})^{++}$" holds in $V^{\mathbb{S}_n \times \mathbb{R}_n}$ and in $V^{\mathbb{S}_n \times (\mathbb{P}_\alpha)^{\varpi^\alpha_n}_n}$.*

Proof. This configuration holds in the first generic extension by suitability for reflection of $(\mathbb{P}_n, \mathbb{S}_n, \mathbb{R}_n, \varpi_n)$. We now turn to address the α^{th}-stage. The upcoming discussion assumes the notation of Section 4.

By Lemma 4.34, we have:

(1) $\mathbb{T}_n$ has the κ_n-cc and has size κ_n;
(2) ψ_n defines a nice projection;
(3) $\mathbb{P}^{\psi_n}_n$ is κ_n-directed-closed;
(4) for each $p \in P_n$, $\mathbb{P}_n \downarrow p$ and $(\mathbb{T}_n \downarrow \psi_n(p)) \times ((\mathbb{P}^{\psi_n})_n \downarrow p)$ are forcing equivalent.

By Lemma 4.31, $\mathbb{P}_n$ forces $|\mu| = \mathrm{cf}(\mu) = \kappa_n = (\sigma_n)^+ = (\kappa_{n-1})^{++}$, and by our remark before the statement of this lemma, $(\mathbb{P}_\alpha)^{\pi_{\alpha,1}}_n$ is μ-directed-closed, in particular, κ_n-directed-closed. Combining Clauses (1), (3) and (4) above with Easton's Lemma, $(\mathbb{P}^{\psi_n})_n \times (\mathbb{P}_\alpha)^{\pi_{\alpha,1}}_n$ is κ_n-distributive over $V^{\mathbb{T}_n}$, and so $\mathbb{P}_n \times (\mathbb{P}_\alpha)^{\pi_{\alpha,1}}_n$ forces $\kappa_n = (\kappa_{n-1})^{++}$. Moreover, as $\mathbb{P}_n \times (\mathbb{P}_\alpha)^{\pi_{\alpha,1}}_n$ projects to $\mathbb{P}_n$ and the former preserves κ_n, it also forces $|\mu| = \mathrm{cf}(\mu)$. Altogether, $\mathbb{P}_n \times (\mathbb{P}_\alpha)^{\pi_{\alpha,1}}_n$ forces $|\mu| = \mathrm{cf}(\mu) = \kappa_n = (\kappa_{n-1})^{++}$. To establish that the same configuration is being forced by $\mathbb{S}_n \times (\mathbb{P}_\alpha)^{\varpi^\alpha_n}_n$, we give a sandwich argument:

- $\mathbb{P}_n \times (\mathbb{P}_\alpha)^{\pi_{\alpha,1}}_n$ projects to $\mathbb{S}_n \times (\mathbb{P}_\alpha)^{\varpi^\alpha_n}_n$, as witnessed by $(p, q) \mapsto (\varpi_n(p), q)$;
- For any condition p in $(\mathbb{P}_\alpha)_n$, $(\mathbb{S}_n \downarrow \varpi^\alpha_n(p)) \times (\mathbb{P}_\alpha)^{\varpi^\alpha_n}_n \downarrow p)$ projects to $(\mathbb{P}_\alpha)_n \downarrow p$, by Definition 2.2(4).
- $(\mathbb{P}_\alpha)_n$ projects to $\mathbb{P}_n$ via $\pi_{\alpha,1}$. $\square$

Recall that $\varpi_{n,\aleph}\colon \mathbb{P}_n \to \mathbb{Q}$ and $\varpi_{n,\beth}\colon \mathbb{P}_n \to \mathrm{Col}(\delta, {<}\kappa_{n-1}) \times \mathbb{R}_n$ are the projections induced by $\varpi_n\colon \mathbb{P}_n \to \mathbb{S}_n$ (see Definition 4.14). Denote

$$\varpi^\alpha_{n,\aleph} := \varpi_{n,\aleph} \circ \pi_{\alpha,1} \text{ and } \varpi^\alpha_{n,\beth} := \varpi_{n,\beth} \circ \pi_{\alpha,1}.$$

For a condition $a \in (P_\alpha)_n$, $\varpi^\alpha_{n,\aleph}(a)$ is just the collapsing component of the EBPFC-part of a (except for the right-most collapse in $\mathrm{Col}(\delta, {<}\kappa_{n-1})$). Similarly, $\varpi^\alpha_{n,\beth}(a)$ gives the other components (i.e., Cohens, right-most collapse and 4-tuples) of the EBPFC-part of a.

Postponing the verification of (3b) and (3c) to a bit later, we turn now to the verification of Clause (3d).

Claim 8.6.2. *Clause* (3d) *holds.*

Proof. Suppose $\langle b_i \mid i < m\rangle$ is a finite sequence of conditions in $(P_\alpha)_n$ that are $\le_\alpha$-below a, that the elements of $\langle \varpi^\alpha_{n,\aleph}(b_i) \mid i < m\rangle$ are pairwise $\mathbb{Q}$-incompatible, and that r is a condition such that $r \le_{\mathrm{Col}(\delta,<\kappa_{n-1})\times\mathbb{R}_n} \varpi^\alpha_{n,\beth}(b_i)$ for all $i < m$. We want a condition $b \le^{\varpi^\alpha_{n,\aleph}} a$ with $\varpi^\alpha_{n,\beth}(b) = r$ such that $b + \varpi^\alpha_{n,\aleph}(b_i) \le b_i$.

For each $i < m$, let p and p_i denote the EBPFC-parts of a and b_i, respectively. In our scenario the above hypotheses amount to saying the following:

- the collapses s_i of the p_i's (except for the right-most one) are pairwise incompatible;
- the rest of entries of p_i (i.e., the Cohens, 4-tuples and right-most collapse) are compatible as witnessed by a lower bound r.

Let q be the condition in EBPFC whose $\mathbb{Q}$-component is s (the one of p) and the other are determined by r. Clearly, $q \le^{\varpi_{n,\aleph}} p$, $q + s_i \le^{\varpi_{n,\aleph}} p_i$ and $\varpi_{n,\beth}(q) = r$.

Let us first describe how to build the target condition b in the more simpler scenario where $b_i = (p_i, \vec{S}^i)$ and $a = (p, \vec{S})$ (i.e., when $\alpha = 2$).

Denote $\beta := \mathrm{dom}(\vec{S})$ and $\beta_i := \mathrm{dom}(\vec{S}^i)$, and put $\beta^* := \max_{i<m}\beta_i$. Define a sequence $\vec{Q} := \langle Q_\gamma \mid \gamma \le \beta^*\rangle$ as follows. Given $u \in W(q)$ (i.e., u is of the form $q^\frown\vec{\nu}$) and a vector of collapses $c \preceq \varpi_{\ell(u)}(u)$,

$$Q_\gamma(u,c) := \begin{cases} S^i_{\min\{\gamma,\beta_i\}}(w(p_i, u+c), c), & \text{if } c \text{ is stronger than } s_i;\\ S_{\min\{\gamma,\beta\}}(w(p, u+c), c), & \text{otherwise.}\end{cases}$$

Note that there is at most one i witnessing the first clause as the s_i's are pairwise $\mathbb{Q}$-incompatible. One can check that $(q, \vec{Q})$ is a well-defined condition in the forcing of Definition 6.7. Also, $(q, \vec{Q}) \le^{\varpi^2_{n,\aleph}} (p, \vec{S})$ and

$$(q,\vec{Q}) + s_i = \pitchfork(q,\vec{Q})(q+s_i) \le^{\varpi^2_{n,\beth}} (p_i, \vec{S}_i).$$

Let us now describe how to handle the general case. Let $\langle \gamma_\xi \mid \xi < \theta\rangle$ be an injective enumeration of the support of the conditions $\bigcup_{i<m} B_{b_i}$. As usual, we denote by β_ξ the ordinal predecessor of γ_ξ. Suppose by induction that we have succeeded in defining $b \restriction \gamma_\xi$ for all $\xi < \xi^*$. Define

$$b \restriction \beta_{\xi^*} := \left[\bigcup\nolimits_{\xi<\xi^*} b \restriction \gamma_\xi\right]^{\mathbb{P}_{\beta_{\xi^*}}}.$$

We claim that, for each $i < m$, $(b \restriction \beta_{\xi^*}) + s_i \le_{\beta_{\xi^*}} b_i \restriction \beta_{\xi^*}$. Indeed,

$$(b \restriction \beta_{\xi^*}) + s_i = \pitchfork_{\beta_{\xi^*},1}(b \restriction \beta_{\xi^*})(q+s_i)$$

and the restriction to $\mathbb{P}_{\gamma_\xi}$ of this latter condition is

$$\pitchfork_{\gamma_\xi,1}(b \restriction \gamma_\xi)(q+s_i) = (b \restriction \gamma_\xi) + s_i \le_{\gamma_\xi} b_i \restriction \gamma_\xi.$$

Here the right-most inequality is justified by the induction hypothesis.

Being mindful of $(b \restriction \beta_{\xi^*}) + s_i \le_{\beta_\xi} b_i \restriction \beta_{\xi^*}$ we can produce (as in the preceding case) a sequence $\vec{Q}$ such that $b \restriction \gamma_{\xi^*} := (b \restriction \beta_{\xi^*}, \vec{Q})$ is as wished.

A crucial observation is that there is an univocal correspondence between $W(b\restriction\beta_{\xi^*})$ and $W(q)$ as granted by property (4) of Definition 5.1 with respect to $\pitchfork_{\beta_{\xi^*},1}$. In the end, one takes b be the union of its restrictions. $\square$

Hereafter, to streamline the presentation we omit $\downarrow a$ in our notations. Namely, whenever we write $(\mathbb{P}_\alpha^{\varpi^\alpha_{n,\aleph}})_n$ we have in mind the more cumbersome $(\mathbb{P}_\alpha^{\varpi^\alpha_{n,\aleph}})_n\downarrow a$. We now turn to verify Clause (3b).

Claim 8.6.3. *$(\mathbb{P}_\alpha^{\varpi^\alpha_{n,\aleph}})_n/\operatorname{Col}(\delta,{<}\kappa_{n-1})\times\mathbb{R}_n$ has a δ-closed dense set.*

Proof. Note that the map $\varpi^\alpha_{n,\beth}$ is a projection from $(\mathbb{P}_\alpha)_n^{\varpi^\alpha_{n,\aleph}}$ to $\operatorname{Col}(\delta,{<}\kappa_{n-1})\times\mathbb{R}_n$ by Claim 8.6.2 above (see also Remark 2.11).

Let $F\subseteq\operatorname{Col}(\delta,{<}\kappa_{n-1})\times\mathbb{R}_n$ be generic over V. Working over $V[F]$ define

$$D:=\{b\in(P_\alpha)_n/F\mid b\in(\mathring{P}_\alpha)_n\}.$$

Density: Let $b\leq^{\varpi^\alpha_{n,\aleph}} a$ be in $(P_\alpha)_n/F$. Since ϖ^α_n is a nice projection there is $a^*\leq^{\varpi^\alpha_n} a$ such that $b=a^*+\varpi^\alpha_n(b)$. Since $(\mathring{\mathbb{P}}_\alpha^{\varpi^\alpha_n})_n$ is dense in $(\mathbb{P}_\alpha^{\varpi^\alpha_n})_n$ (Lemma 8.5) there is $b^*\in(\mathring{P}_\alpha)_n$ such that $b^*\leq^{\varpi^\alpha_n} a$. Denote

$$c:=b^*+\varpi^\alpha_{n,\beth}(b).$$

Note that $c\in(P_\alpha)_n/F$ because $\varpi^\alpha_{n,\beth}(c)=\varpi^\alpha_{n,\beth}(b)\in F$. Also, $c\in(\mathring{P}_\alpha)_n$ because

$$b^*+\varpi^\alpha_{n,\beth}(b)=\pitchfork_{\alpha,1}(b^*)(\pi_{\alpha,1}(b^*)+\varpi^\alpha_{n,\beth}(b))$$

and $b^*\in(\mathring{P}_\alpha)_n$ (see Remark 5.8). Finally, $c\leq^{\varpi^\alpha_{n,\aleph}} b$, because

$$c=(b^*+\varpi^\alpha_{n,\aleph}(b^*))+\varpi^\alpha_{n,\beth}(b)=(b^*+\varpi^\alpha_{n,\aleph}(b))+\varpi^\alpha_{n,\beth}(b)=b^*+\varpi^\alpha_n(b)\leq b$$

and

$$\varpi^\alpha_{n,\aleph}(c)=\varpi^\alpha_{n,\aleph}(b^*+\varpi^\alpha_{n,\beth}(b))=\varpi^\alpha_{n,\aleph}(b^*)=\varpi^\alpha_{n,\aleph}(a).$$

Closure: Let $\langle b_i\mid i<\chi\rangle$ be a $\leq^{\varpi^\alpha_{n,\aleph}}$-decreasing sequence in $V[F]$ for some $\chi<\delta$. Clearly, $\langle\varpi^\alpha_{n,\beth}(b_i)\mid i<\chi\rangle\in V[F]$ is $\leq^{\varpi_{n,\aleph}}$-decreasing in $\operatorname{Col}(\delta,{<}\kappa_{n-1})\times\mathbb{R}_n$. Since the latter is δ-closed (over V), the sequence belongs to V and thus there is a lower bound for it $(c,r)\in F$. Let s be the common $\varpi^\alpha_{n,\aleph}$-value for the b_i's. Since $\mathbb{P}_n$ is isomorphic to $\mathbb{S}_n\times(\operatorname{Col}(\delta,{<}\kappa_{n-1})\times\mathbb{R}_n)$, there is a unique condition $p_\chi\in P_n$ coded by (s,c,r). By the properties of Definition 5.1, $\langle\pitchfork_{\alpha,1}(b_i)(p_\chi)\mid i<\chi\rangle$ is a $\leq^{\pi_{\alpha,1}}$-decreasing sequence of conditions in $(\mathring{P}_\alpha)_n$. Thus, there is $b_\chi\in(\mathring{P}_\alpha)_n$ a $\leq^{\pi_{\alpha,1}}$-lower bound for it. Note that $b_\chi\in(\mathring{P}_\alpha)_n/F$ because $\varpi^\alpha_{n,\beth}(b)=\varpi_{n,\beth}(q_\chi)=(c,r)\in F$. $\square$

Our last verification is that of Clause (3c).

Claim 8.6.4. *$(\mathbb{P}_\alpha^{\varpi^\alpha_n})_n\times\operatorname{Col}(\delta,{<}\kappa_{n-1})/(\mathbb{P}_\alpha^{\varpi^\alpha_{n,\aleph}})_n$ has a δ-closed dense set.*

Proof. Let $D := ((\mathring{\mathbb{P}}_\alpha^{\varpi_n^\alpha})_n \times \mathrm{Col}(\delta, {<}\kappa_{n-1}))$. This is a dense set in the product $(\mathbb{P}_\alpha^{\varpi_n^\alpha})_n \times \mathrm{Col}(\delta, {<}\kappa_{n-1})$ by virtue of Lemma 8.5.

Let $G \subseteq (\mathbb{P}_\alpha)_n^{\varpi_{n,\aleph}^\alpha}$ be a V-generic filter. Since there is a projection

$$D^* := \{(b,c) \in D \mid (b,c) \in ((P_\alpha)_n \times \mathrm{Col}(\delta, {<}\kappa_{n-1})/G)\}.$$

is a dense set in the quotient $(\mathbb{P}_\alpha^{\varpi_n^\alpha})_n \times \mathrm{Col}(\delta, {<}\kappa_{n-1})/G$.

Let us check that D^* is δ-closed. Let $\langle (b_i, c_i) \mid i < \chi \rangle$ be decreasing, with $\chi < \delta$. Let F be the generic for $\mathrm{Col}(\delta, {<}\kappa_{n-1})$ induced by G and the projection between $(\mathbb{P}_\alpha^{\varpi_{n,\aleph}^\alpha})$ and $\mathrm{Col}(\delta, {<}\kappa_{n-1})$. Then, G is generic (over $V[F]$) for $(\mathbb{P}_\alpha^{\varpi_{n,\aleph}^\alpha})/F$ and $V[G] = V[F][G]$. Since the latter contains a δ-closed dense set it follows that $\langle (b_i, c_i) \mid i < \chi \rangle$ is in $V[F]$. Similarly, $\mathrm{Col}(\delta, {<}\kappa_{n-1})$ is δ-closed (over V) so $\langle (b_i, c_i) \mid i < \chi \rangle \in V$. We could now take lower bounds for the b_i's and c_i's but there is the additional caveat of finding a lower bound in $(\mathbb{P}_\alpha^{\varpi_n^\alpha})_n \times \mathrm{Col}(\delta, {<}\kappa_{n-1})/G$. To ensure this we do a density argument. Look at the set (in V) consisting of all $u \in (P_\alpha)_n$ such that:

- Either there is $i < \chi$ such that u_i is $\leq^{\varpi_{n,\aleph}^\alpha}$-incompatible with $b_i + c_i$, or
- there is a lower bound (b_χ, c_χ) for the (b_i, c_i)'s such that
$$u \leq^{\varpi_{n,\aleph}^\alpha} b_\chi + c_\chi.$$

If we show that this is $\leq^{\varpi_{n,\aleph}^\alpha}$-dense, then there would be a lower bound (b_χ, c_χ) for the (b_i, c_i) in the quotient $(\mathbb{P}_\alpha^{\varpi_n^\alpha})_n \times \mathrm{Col}(\delta, {<}\kappa_{n-1})/G$.

Let $u \in (P_\alpha)_n$ that is $\leq^{\varpi_{n,\aleph}^\alpha}$-compatible with all the $b_i + c_i$'s. Without loss of generality, $u \leq^{\varpi_{n,\aleph}^\alpha} b_i + c_i$ for all $i < \chi$.[65] Look at the EBPFC-part of u (in symbols, $\pi_{\alpha,1}(u)$). We can decouple this as $p_\chi + c_\chi$ where

- $c_\chi \in \mathrm{Col}(\delta, {<}\kappa_{n-1})$ is the last collapse of the EBPFC-part of u;
- p_χ is equal to be the EBPFC-part of u except for its last collapse, which is the common one to the b_i's.

Clearly, $p_\chi \leq^{\varpi_n} \pi_{\alpha,1}(b_i)$ so $\langle \pitchfork_{\alpha,1}(b_i)(p_\chi) \mid i < \chi \rangle$ is a $\leq^{\pi_{\alpha,1}}$-decreasing sequence in $(\mathring{P}_\alpha)_n$. Let b_χ be the $\leq^{\pi_{\alpha,1}}$-greatest lower bound for it. In particular, $u \leq^{\varpi_{n,\aleph}^\alpha} b_\chi$. Also, it is not hard to check that $u \leq^{\varpi_{n,\aleph}^\alpha} b_\chi + c_\chi$. $\square$

This completes the proof of Lemma 8.6. $\square$

Lemma 8.7. *Let $n < \omega$ and $0 < \alpha < \mu^+$. Then $(\mathbb{P}_\alpha)_n^{\varpi_n^\alpha}$ preserves* GCH.

Proof. The case $\alpha = 1$ is taken care of by Lemma 4.35.

Now, let $\alpha \geq 2$. Since $(\mathbb{P}_\alpha)_n^{\varpi_n^\alpha}$ contains a σ_n-directed-closed dense subset, it preserves GCH below σ_n. By the sandwich analysis from the proof of Lemma 8.6, in any generic extension by $(\mathbb{P}_\alpha)_n^{\varpi_n^\alpha}$, $|\mu| = \mathrm{cf}(\mu) = \kappa_n = (\sigma_n)^+$.

[65] Here we use that $(\mathring{P}_\alpha)_n$ is a δ-closed $\leq^{\varpi_{n,\aleph}^\alpha}$-dense set in $(P_\alpha)_n$.

So, as $(\mathbb{P}_\alpha)_n^{\varpi_n^\alpha}$ is a notion of forcing of size $\leq \mu^+$, collapsing μ to κ_n, it preserves GCH_θ for any cardinal $\theta > \kappa_n$.

It thus left to verify that $(\mathbb{P}_\alpha)_n^{\varpi_n^\alpha}$ forces $2^\theta = \theta^+$ for $\theta \in \{\sigma_n, \kappa_n\}$.

► Arguing as in Lemma 8.6, for any condition p in $(\mathbb{P}_\alpha)_n$, $(\mathbb{T}_n \downarrow \psi_n(p)) \times \big(((\mathbb{P}^{\psi_n})_n \downarrow p) \times (\mathbb{P}_\alpha)_n^{\pi_{\alpha,1}}\big)$ projects onto $(\mathbb{P}_\alpha)_n^{\varpi_n^\alpha}$. Recall that the first factor of the product is a κ_n-cc forcing of size $\leq \kappa_n$. By Lemma 7.8, the second factor is forcing equivalent to a κ_n-directed-closed forcing. Thus, by Easton's lemma, this product preserves CH_{σ_n} if and only if $\mathbb{T}_n \downarrow \psi_n(p)$ does. And this is indeed the case, as the number of $\mathbb{T}_n$-nice names for subsets of σ_n is at most $\kappa_n^{<\kappa_n} = \kappa_n = (\sigma_n)^+$.

► Again, arguing as in Lemma 8.6, $(\mathbb{P}_1)_n \times (\mathbb{P}_\alpha)_n^{\pi_{\alpha,1}}$ projects onto $\mathbb{S}_n \times (\mathbb{P}_\alpha)_n^{\varpi_n^\alpha}$, which projects onto $(\mathbb{P}_\alpha)_n^{\varpi_n^\alpha}$. Since $(\mathbb{P}_\alpha)_n^{\pi_{\alpha,1}}$ is forcing equivalent to a μ-directed-closed, it preserves CH_{σ_n}. Also, it preserves μ and so, by Lemma 4.35(1) and the absolutness of the μ^+-Linked property, $(\mathbb{P}_1)_n$ is also μ^+-Linked in $V^{(\mathbb{P}_\alpha)_n^{\pi_{\alpha,1}}}$. Once again, counting-of-nice-names arguments implies that this latter forcing forces $2^{\kappa_n} \leq \mu^+ = (\kappa_n)^+$. Thus, $(\mathbb{P}_1)_n \times (\mathbb{P}_\alpha)_n^{\pi_{\alpha,1}}$ preserves CH_{κ_n} and so does $(\mathbb{P}_\alpha)_n^{\varpi_n^\alpha}$. $\square$

Theorem 8.8. *In $V^{\mathbb{P}_{\mu^+}}$, all of the following hold true:*

1. *All cardinals $\geq \kappa$ are preserved;*
2. *$\kappa = \aleph_\omega$, $\mu = \aleph_{\omega+1}$ and $\lambda = \aleph_{\omega+2}$;*
3. *$2^{\aleph_n} = \aleph_{n+1}$ for all $n < \omega$;*
4. *$2^{\aleph_\omega} = \aleph_{\omega+2}$;*
5. *Every stationary subset of $\aleph_{\omega+1}$ reflects.*

Proof. (1) We already know that $\mathbb{1}_{\mathbb{P}_{\mu^+}} \Vdash_{\mathbb{P}_\alpha} \check{\mu} = \check{\kappa}^+$. By Lemma 3.15(2), κ remains strong limit cardinal in $V^{\mathbb{P}_{\mu^+}}$. Finally, as Clause (3) of Definition 3.3 holds for $(\mathbb{P}_{\mu^+}, \ell_{\mu^+}, c_{\mu^+}, \vec{\varpi}_{\mu^+})$, $\mathbb{P}_{\mu^+}$ has the μ^+-chain-condition, so that all cardinals $\geq \kappa^{++}$ are preserved.

(2) Let $G \subseteq \mathbb{P}_{\mu^+}$ be an arbitrary generic over V. By virtue of Clause (1) and Setup 8, it suffices to prove that $V[G] \models \kappa = \aleph_\omega$. Let G_1 the $\mathbb{P}_1$-generic filter generated by G and $\pi_{\mu^+,1}$. By Theorem 4.1, $V[G_1] \models \kappa = \aleph_\omega$. Thus, let us prove that $V[G]$ and $V[G_1]$ have the same cardinals $\leq \kappa$.

Of course, $V[G_1] \subseteq V[G]$, and so any $V[G]$-cardinal is also a $V[G_1]$-cardinal. Towards a contradiction, suppose that there is a $V[G_1]$-cardinal $\theta < \kappa$ that ceases to be so in $V[G]$. Any surjection witnessing this can be encoded as a bounded subset of κ, hence as a bounded subset of some σ_n for some $n < \omega$. Thus, Lemma 3.15(1) implies that θ is not a cardinal in $V[H_n]$, where H_n is the $\mathbb{S}_n$-generic filter generated by G_1 and ϖ_n^1. As $V[H_n] \subseteq V[G_1]$, θ is not a cardinal in $V[G_1]$, which is a contradiction.

(3) On one hand, by Lemma 3.15(1), $\mathcal{P}(\aleph_n)^{V^{\mathbb{P}_{\mu^+}}} = \mathcal{P}(\aleph_n)^{V^{\mathbb{S}_m}}$ for some $m < \omega$. On the other hand, as $\mathsf{GCH}_{<\lambda}$ holds (cf. Setup 8), Remark 4.13 shows that $\mathbb{S}_m$ preserves $\mathsf{CH}_{\aleph_n}$. Altogether, $V^{\mathbb{P}_{\mu^+}} \models \mathsf{CH}_{\aleph_n}$.

(4) By Setup 8, $V \models 2^\kappa = \kappa^+$. In addition, $\mathbb{P}_{\mu^+}$ is isomorphic to a notion of forcing lying in H_{μ^+} (see [PRS22, Remark 3.3(1)]) and $|H_{\mu^+}| = \lambda$. Thus, $V^{\mathbb{P}_{\mu^+}} \models 2^\kappa \leq \lambda$. In addition, $\mathbb{P}_{\mu^+}$ projects to $\mathbb{P}_1$, which is isomorphic to $\mathbb{Q}$, being a poset blowing up 2^κ to λ, as seen in Theorem 4.1, so that $V^{\mathbb{P}_{\mu^+}} \models 2^\kappa \geq \lambda$. So, $V^{\mathbb{P}_{\mu^+}} \models 2^\kappa = \lambda$. Thus, together with Clause (2), $V^{\mathbb{P}_{\mu^+}} \models 2^{\aleph_\omega} = \aleph_{\omega+2}$.

(5) Let G be $\mathbb{P}_{\mu^+}$-generic over V and hereafter work in $V[G]$. Towards a contradiction, suppose that there exists a stationary set $T \subseteq \mu$ that does not reflect. By shrinking, we may assume the existence of some regular cardinal $\theta < \mu$ such that $T \subseteq E^\mu_\theta$. Fix $r^* \in G$ and a $\mathbb{P}_{\mu^+}$-name τ such that τ_G is equal to such a T and such that r^* forces τ to be a stationary subset of μ that does not reflect. Since $\mu = \kappa^+$ and κ is singular in V, by possibly enlarging r^*, we may assume that r^* forces τ to be a subset of $\Gamma_{\ell(r^*)}$ (see the opening of Subsection 6.2). Furthermore, we may require that τ be a *nice name*, i.e., each element of τ is a pair $(\check{\xi}, p)$ where $(\xi, p) \in \Gamma_{\ell(r^*)} \times P_{\mu^+}$, and, for each ordinal $\xi \in \Gamma_{\ell(r^*)}$, the set $\{p \in P_{\mu^+} \mid (\check{\xi}, p) \in \tau\}$ is a maximal antichain.

As $\mathbb{P}_{\mu^+}$ satisfies Clause (3) of Definition 3.3, $\mathbb{P}_{\mu^+}$ has in particular the μ^+-cc. Consequently, there exists a large enough $\beta < \mu^+$ such that

$$B_{r^*} \cup \bigcup \{B_p \mid (\check{\xi}, p) \in \tau\} \subseteq \beta.$$

Let $r := r^* \restriction \beta$ and set

$$\sigma := \{(\check{\xi}, p \restriction \beta) \mid (\check{\xi}, p) \in \tau\}.$$

From the choice of Building Block III, we may find a large enough $\alpha < \mu^+$ with $\alpha > \beta$ such that $\psi(\alpha) = (\beta, r, \sigma)$. As $\beta < \alpha$, $r \in P_\beta$ and σ is a $\mathbb{P}_\beta$-name, the definition of our iteration at step $\alpha + 1$ involves appealing to Building Block II with $(\mathbb{P}_\alpha, \ell_\alpha, c_\alpha, \vec{\varpi}_\alpha)$, $r^\star := r * \emptyset_\alpha$ and $z := i^\alpha_\beta(\sigma)$.[66] For each ordinal $\eta < \mu^+$, denote $G_\eta := \pi_{\mu^+,\eta}[G]$. By our choice of β and since $\alpha > \beta$, we have

$$\tau = \{(\check{\xi}, p * \emptyset_{\mu^+}) \mid (\check{\xi}, p) \in \sigma\} = \{(\check{\xi}, p * \emptyset_{\mu^+}) \mid (\check{\xi}, p) \in z\},$$

so that, in $V[G]$,

$$T = \tau_G = \sigma_{G_\beta} = z_{G_\alpha}.$$

In addition, $r^* = r^\star * \emptyset_{\mu^+}$ and so $\ell(r^*) = \ell(r^\star)$.

As r^* forces that τ is a non-reflecting stationary subset of $\Gamma_{\ell(r^\star)}$, it follows that $r^\star$ $\mathbb{P}_\alpha$-forces the same about z.

Claim 8.8.1. *z is $r^\star$-fragile.*

Proof. Recalling Lemma 6.21, it suffices to prove that for every $n < \omega$,

$$V^{(\mathbb{P}_\alpha)_n} \models \mathrm{Refl}(E^\mu_{<\sigma_{n-2}}, E^\mu_{<\sigma_n}).$$

This is trivially the case for $n \leq 1$. So, let us fix an arbitrary $n \geq 2$. By Lemma 8.6, $((\mathbb{P}_\alpha)_n, \mathbb{S}_n, \varpi^\alpha_n)$ is suitable for reflection with respect to

[66] Recall Convention 7.1.

$\langle\sigma_{n-2},\kappa_{n-1},\kappa_n,\mu\rangle$. Since $(\mathbb{P}_\alpha)_n^{\varpi_n^\alpha}$ is forcing equivalent to a σ_n-directed-closed forcing and (by Lemma 8.7) it preserves GCH, κ_{n-1} is a supercompact cardinal indestructible under forcing with $(\mathbb{P}_\alpha)_n^{\varpi_n^\alpha}$. So, recalling Setup 8, $(\mathbb{P}_\alpha)_n^{\varpi_n^\alpha}$ preserves the supercompactness of κ_{n-1}. Thus, by Lemma **??**, $V^{(\mathbb{P}_\alpha)_n}\models \mathrm{Refl}(E^\mu_{<\sigma_{n-2}},E^\mu_{<\sigma_n})$. □

As z is $r^\star$-fragile and $\pi_{\mu^+,\alpha+1}(r^*)=r^\star*\emptyset_{\alpha+1}=\lceil r^\star\rceil^{\mathbb{P}_{\alpha+1}}\in G_{\alpha+1}$, Clause (f) of Building Block II implies that there exists (in $V[G_{\alpha+1}]$) a club subset of μ disjoint from T. In particular, T is nonstationary in $V[G_{\alpha+1}]$ and thus nonstationary in $V[G]$. This contradicts the very choice of T. The result follows from the above discussion and the previous claim. □

We are now ready to derive the Main Theorem.

Theorem 8.9. *Suppose that there exist infinitely many supercompact cardinals. Then there exists a forcing extension where all of the following hold:*

(1) $2^{\aleph_n}=\aleph_{n+1}$ *for all* $n<\omega$*;*
(2) $2^{\aleph_\omega}=\aleph_{\omega+2}$*;*
(3) every stationary subset of $\aleph_{\omega+1}$ *reflects.*

Proof. Using Corollary 8.3, we may assume that all the blanket assumptions of Setup 8 are met. Specifically:

- $\vec\kappa=\langle\kappa_n\mid n<\omega\rangle$ is an increasing sequence of supercompact cardinals that are indestructible under κ_n-directed-closed notions of forcing that preserve the GCH;
- $\kappa:=\sup_{n<\omega}\kappa_n$, $\mu:=\kappa^+$ and $\lambda:=\kappa^{++}$;
- GCH holds.

Now, appeal to Theorem 8.8. □

9. Acknowledgments

Poveda was supported by postdoctoral fellowships from the Einstein Institute of Mathematics (Hebrew University of Jerusalem) and from the Center of Mathematical Sciences and Applications (Harvard University). Rinot was partially supported by the European Research Council (grant agreement ERC-2018-StG 802756) and by the Israel Science Foundation (grant agreement 203/22). Sinapova was partially supported by the National Science Foundation, Career-1454945 and DMS-1954117.

We thank the referee for their thorough feedback, corrections and suggestions.

References

[Abr10] Uri Abraham. Proper forcing. In *Handbook of set theory*, pages 333–394. Springer, 2010.

[BHU19] Omer Ben-Neria, Yair Hayut, and Spencer Unger. Stationary reflection and the failure of SCH. *arXiv preprint arXiv:1908.11145*, 2019.

[Buk65] L. Bukovský. The continuum problem and powers of alephs. *Comment. Math. Univ. Carolinae*, 6:181–197, 1965.

[CFM01] James Cummings, Matthew Foreman, and Menachem Magidor. Squares, scales and stationary reflection. *J. Math. Log.*, 1(1):35–98, 2001.

[Coh63] Paul J. Cohen. The independence of the continuum hypothesis. *Proc. Nat. Acad. Sci. U.S.A.*, 50:1143–1148, 1963.

[Cum10] James Cummings. Iterated forcing and elementary embeddings. In *Handbook of set theory. Vols. 1, 2, 3*, pages 775–883. Springer, Dordrecht, 2010.

[DJ75] Keith I. Devlin and R. B. Jensen. Marginalia to a theorem of Silver. In $\vDash$*ISILC Logic Conference (Proc. Internat. Summer Inst. and Logic Colloq., Kiel, 1974)*, pages 115–142. Lecture Notes in Math., Vol. 499, 1975.

[Eas70] William B. Easton. Powers of regular cardinals. *Ann. Math. Logic*, 1:139–178, 1970.

[FMS88] M. Foreman, M. Magidor, and S. Shelah. Martin's maximum, saturated ideals, and nonregular ultrafilters. I. *Ann. of Math. (2)*, 127(1):1–47, 1988.

[FR11] Sakaé Fuchino and Assaf Rinot. Openly generated Boolean algebras and the Fodor-type reflection principle. *Fund. Math.*, 212(3):261–283, 2011.

[FT05] Matthew Foreman and Stevo Todorcevic. A new Löwenheim-Skolem theorem. *Trans. Amer. Math. Soc.*, 357(5):1693–1715, 2005.

[GH75] Fred Galvin and András Hajnal. Inequalities for cardinal powers. *Ann. of Math. (2)*, 101:491–498, 1975.

[Git02] Moti Gitik. The power set function. In *Proceedings of the International Congress of Mathematicians, Vol. I (Beijing, 2002)*, pages 507–513. Higher Ed. Press, Beijing, 2002.

[Git10] Moti Gitik. Prikry-type forcings. In *Handbook of set theory. Vols. 1, 2, 3*, pages 1351–1447. Springer, Dordrecht, 2010.

[Git19a] Moti Gitik. Blowing up the power of a singular cardinal of uncountable cofinality. *J. Symb. Log.*, 84(4):1722–1743, 2019.

[Git19b] Moti Gitik. Collapsing generators. *Preprint*, 2019.

[Git22] Moti Gitik. Reflection and not SCH with overlapping extenders. *Arch. Math. Logic*, 61(5-6):591–597, 2022.

[GM94] Moti Gitik and Menachem Magidor. Extender based forcings. *J. Symbolic Logic*, 59(2):445–460, 1994.

[Göd40] Kurt Gödel. *The Consistency of the Continuum Hypothesis*. Annals of Mathematics Studies, no. 3. Princeton University Press, Princeton, N. J., 1940.

[Lav78] Richard Laver. Making the supercompactness of κ indestructible under κ-directed closed forcing. *Israel Journal of Mathematics*, 29(4):385–388, 1978.

[Mag77a] Menachem Magidor. On the singular cardinals problem. I. *Israel J. Math.*, 28(1-2):1–31, 1977.

[Mag77b] Menachem Magidor. On the singular cardinals problem. II. *Ann. of Math. (2)*, 106(3):517–547, 1977.

[Mag82] Menachem Magidor. Reflecting stationary sets. *J. Symbolic Logic*, 47(4):755–771 (1983), 1982.

[Men76] Telis K. Menas. Consistency results concerning supercompactness. *Trans. Amer. Math. Soc.*, 223:61–91, 1976.

[Mit10] William J. Mitchell. The covering lemma. In *Handbook of set theory. Vols. 1, 2, 3*, pages 1497–1594. Springer, Dordrecht, 2010.

[Moo06] Justin Tatch Moore. The proper forcing axiom, Prikry forcing, and the singular cardinals hypothesis. *Ann. Pure Appl. Logic*, 140(1-3):128–132, 2006.

[Pov20] Alejandro Poveda. *Contributions to the theory of Large Cardinals through the method of Forcing*. 2020. Thesis (Ph.D.)– Universitat de Barcelona.

[Pri70] K. L. Prikry. Changing measurable into accessible cardinals. *Dissertationes Math. (Rozprawy Mat.)*, 68:55, 1970.

[PRS21] Alejandro Poveda, Assaf Rinot, and Dima Sinapova. Sigma-Prikry forcing I: The axioms. *Canad. J. Math.*, 73(5):1205–1238, 2021.

[PRS22] Alejandro Poveda, Assaf Rinot, and Dima Sinapova. Sigma-Prikry forcing II: Iteration scheme. *J. Math. Log.*, 22(3):Paper No. 2150019, 59pp, 2022.

[Rin08] Assaf Rinot. A topological reflection principle equivalent to Shelah's strong hypothesis. *Proc. Amer. Math. Soc.*, 136(12):4413–4416, 2008.

[Sak15] Hiroshi Sakai. Simple proofs of SCH from reflection principles without using better scales. *Arch. Math. Logic*, 54(5-6):639–647, 2015.

[Sha05] Assaf Sharon. *Weak squares, scales, stationary reflection and the failure of SCH.* 2005. Thesis (Ph.D.)–Tel Aviv University.

[She79] Saharon Shelah. On successors of singular cardinals. In *Logic Colloquium '78 (Mons, 1978)*, volume 97 of *Stud. Logic Foundations Math*, pages 357–380. North-Holland, Amsterdam-New York, 1979.

[She82] Saharon Shelah. *Proper forcing*, volume 940 of *Lecture Notes in Mathematics.* Springer-Verlag, Berlin-New York, 1982.

[She91] Saharon Shelah. Reflecting stationary sets and successors of singular cardinals. *Archive for Mathematical Logic*, 31:25–53, 1991.

[She92] Saharon Shelah. Cardinal arithmetic for skeptics. *Bull. Amer. Math. Soc. (N.S.)*, 26(2):197–210, 1992.

[She00] Saharon Shelah. The generalized continuum hypothesis revisited. *Israel J. Math.*, 116:285–321, 2000.

[She08] Saharon Shelah. Reflection implies the SCH. *Fund. Math.*, 198(2):95–111, 2008.

[Sil75] Jack Silver. On the singular cardinals problem. In *Proceedings of the International Congress of Mathematicians (Vancouver, B. C., 1974), Vol. 1*, pages 265–268, 1975.

[Sol74] Robert M. Solovay. Strongly compact cardinals and the GCH. In *Proceedings of the Tarski Symposium (Proc. Sympos. Pure Math., Vol. XXV, Univ. California, Berkeley, Calif., 1971)*, pages 365–372, 1974.

[Tod93] Stevo Todorčević. Conjectures of Rado and Chang and cardinal arithmetic. In *Finite and infinite combinatorics in sets and logic (Banff, AB, 1991)*, volume 411 of *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, pages 385–398. Kluwer Acad. Publ., Dordrecht, 1993.

[Vel92] Boban Veličković. Forcing axioms and stationary sets. *Adv. Math.*, 94(2):256–284, 1992.

[Via06] Matteo Viale. The proper forcing axiom and the singular cardinal hypothesis. *J. Symbolic Logic*, 71(2):473–479, 2006.

Einstein Institute of Mathematics, Hebrew University of Jerusalem, Givat-Ram, 91904, Israel.
Current address: Center of Mathematical Sciences and Applications, Harvard University, Cambridge, MA 02138, USA.

Department of Mathematics, Bar-Ilan University, Ramat-Gan 5290002, Israel.
URL: http://www.assafrinot.com

Department of Mathematics, Statistics, and Computer Science, University of Illinois at Chicago, Chicago, IL 60607-7045, USA.
Current address: Department of Mathematics, Rutgers University, Piscataway, NJ 08854-8019, USA.
URL: https://sites.math.rutgers.edu/~ds2005/