RAMSEY THEORY OVER PARTITIONS II: NEGATIVE RAMSEY RELATIONS AND PUMP-UP THEOREMS

MENACHEM KOJMAN, ASSAF RINOT, AND JURIS STEPRĀNS

Abstract. In this series of papers we advance Ramsey theory over partitions. In this part, we concentrate on anti-Ramsey relations, or, as they are better known, strong colorings, and in particular solve two problems from [CKS21].

It is shown that for every infinite cardinal λ, a strong coloring on λ^+ by λ colors over a partition can be stretched to one with λ^+ colors over the same partition. Also, a sufficient condition is given for when a strong coloring witnessing $\mathrm{Pr}_1(\ldots)$ over a partition may be improved to witness $\mathrm{Pr}_0(\ldots)$.

Since the classical theory corresponds to the special case of a partition with just one cell, the two results generalize pump-up theorems due to Eisworth and Shelah, respectively.

1. Introduction

1.1. **Strong colorings.** Shortly after Ramsey [Ram30] proved his groundbreaking result that every infinite graph contains an infinite clique or an infinite anti-clique, Sierpiński [Sie33] defined a graph over the reals with neither an uncountable clique nor an uncountable anti-clique. As a graph may be identified with a 2-coloring, Sierpiński's counterexample suggested that there was a class of *strong colorings* waiting to be discovered on the uncountable cardinals. A function $c : [\kappa]^2 \to \theta$ of unordered pairs from a cardinal κ by θ colors is a *strong coloring* iff for every $A \subseteq \kappa$ of full cardinality κ the coloring c hits all colors from θ on the pairs from A, that is $c[[A]^2] = \theta$. Such colorings, whose existence is asserted by the symbol $\kappa \nrightarrow [\kappa]^2_\theta$, witness powerful failures of analogs of Ramsey's theorem.

Surveys of the rich theory of strong colorings that was developed since Sierpiński's time to the present may be found in the introductions to [Rin14a, CKS21]. We mention here, therefore, only the milestones which are most relevant to the present work: the various ways in which strong colorings can become stronger.

Sierpiński's example in particular verified that $\aleph_1 \nrightarrow [\aleph_1]^2_2$ holds. Improving it to handle a larger number of colors was very challenging. After a few decades and considerable effort by many, Todorčević extended in [Tod87] Sierpiński's result to one with the maximal number of colors, which witnessed $\aleph_1 \nrightarrow [\aleph_1]^2_{\aleph_1}$. Furthermore, $\kappa \nrightarrow [\kappa]^2_\kappa$ holds for every uncountable cardinal κ that is the successor of a regular cardinal. Whether Todorčević's theorem extends to successors of singulars is still open.

A second way of making a strong coloring stronger was to require that it attains all possible colors on additional graphs beyond *squares*, i.e., sets of the form $[A]^2 =$

Date: Preprint as of May 12, 2022. For the latest version, visit http://assafrinot.com/paper/50.
2010 *Mathematics Subject Classification.* Primary 03E02; Secondary 03E35.
Key words and phrases. Partition relations, Strong colorings, Walks on ordinals, Singular cardinals combinatorics.

$\{(\alpha,\beta)\in A\times A \mid \alpha<\beta\}$. Work by Shelah [She90, She91, She97] and by Moore [Moo06] has established that for every κ which is a successor of a regular cardinal carries a coloring $c:[\kappa]^2\rightarrow\kappa$ with the property that $c[A\circledast B]=\kappa$ for all $A,B\subseteq\kappa$ of full cardinality, where $A\circledast B$ stands for the *rectangle* $\{(\alpha,\beta)\in A\times B \mid \alpha<\beta\}$. We denote this by $\kappa\nrightarrow[\kappa\circledast\kappa]^2_\kappa$.

Assuming the Generalized Continuum Hypothesis (GCH), Erdős, Hajnal and Rado [EHR65, Theorem 17A] constructed for every infinite cardinal λ a coloring $c:[\lambda^+]^2\rightarrow\lambda^+$ with the property that $c[A\circledast B]=\lambda^+$ for all $A\subseteq\lambda^+$ of (small) size λ and $B\subseteq\lambda^+$ of size λ^+, and then Erdős, Hajnal and Milner [EHM66, Lemma 14.1] used GCH to construct a coloring $c:[\lambda^+]^2\rightarrow\lambda^+$ with the property that for all $A\subseteq\lambda^+$ of size λ and $B\subseteq\lambda^+$ of size λ^+, there is $\alpha\in A$ such that $c[\{\alpha\}\circledast B]=\lambda^+$. We denote the former by $\lambda^+\nrightarrow[\lambda\circledast\lambda^+]^2_{\lambda^+}$, and the latter by $\lambda^+\nrightarrow[{}^{\lambda\circledast\lambda^+}\!/_{1\circledast\lambda^+}]^2_{\lambda^+}$. Here, the class of graphs is enlarged and yet all colors are attained on a subgraph of a prescribed form.

A third aspect in which some strong coloring were shown to be stronger than others is a coloring's ability to handle patterns of higher dimension. In [Gal80], Galvin constructed from the Continuum Hypothesis (CH) a coloring $c:[\aleph_1]^2\rightarrow 2$ with the property that for every finite dimension k, every uncountable pairwise disjoint subfamily $\mathcal{A}\subseteq[\aleph_1]^k$, and every color $\gamma<2$, there are $a,b\in\mathcal{A}$ with $\max(a)<\min(b)$ such that $c[a\times b]=\{\gamma\}$. In Shelah's notation [She88], this is denoted by $\mathrm{Pr}_1(\aleph_1,\aleph_1,2,\aleph_0)$. An extension of Galvin's theorem due to Todorčević is studied in Part III of this series [KRS22b].

Shelah's principle $\mathrm{Pr}_1(\kappa,\kappa,\theta,\chi)$ asserts the existence of a coloring $c:[\kappa]^2\rightarrow\theta$ with the property that for every $\sigma<\chi$, for every pairwise disjoint family $\mathcal{A}\subseteq[\kappa]^\sigma$ of size κ, and every color $\gamma<\theta$, there are $a,b\in\mathcal{A}$ with $\sup(a)<\min(b)$ such that $c[a\times b]=\{\gamma\}$. So, $\mathrm{Pr}_1(\kappa,\kappa,\theta,2)$ coincides with $\kappa\nrightarrow[\kappa]^2_\theta$, and $\mathrm{Pr}_1(\kappa,\kappa,\theta,3)$ implies the rectangular relation $\kappa\nrightarrow[\kappa\circledast\kappa]^2_\theta$. A survey of key results in the study of $\mathrm{Pr}_1(\kappa,\kappa,\theta,\chi)$ is given in the introduction to [RZ22].

Motivated by work of Hajnal and Juhász [HJ74] and by Roitman [Roi78] that connected strong colorings and topology, Shelah [She88] identified a fourth aspect of strengthening a coloring: instead of requiring $c\restriction(a\times b)$ to be a constant function with some prescribed value γ, one requires $c\restriction(a\times b)$ to realize some arbitrary prescribed finite pattern g. Specifically, the principle $\mathrm{Pr}_0(\kappa,\kappa,\theta,\chi)$ asserts the existence of a coloring $c:[\kappa]^2\rightarrow\theta$ with the property that for every $\sigma<\chi$, for every pairwise disjoint family $\mathcal{A}\subseteq[\kappa]^\sigma$ of size κ, and every pattern $g:\sigma\times\sigma\rightarrow\theta$, there are $a,b\in\mathcal{A}$ with $\sup(a)<\min(b)$ such that $c(a(i),b(j))=g(i,j)$ for all $i,j<\sigma$.[1]

1.2. **Strong colorings over partitions.** In order to motivate the definition of strong colorings *over partitions* [CKS21], we first explain how to adapt classical Ramsey relations to this context. The standard positive Ramsey relation $\kappa\rightarrow(\lambda)^2_\theta$ asserts that for every coloring $c:[\kappa]^2\rightarrow\theta$ there exists a set $A\subseteq\kappa$ of cardinality λ such that $c\restriction[A]^2$ is constant. Such a set A for which $\{c(\alpha,\beta)\mid(\alpha,\beta)\in[A]^2\}$ has size no more than 1 is called *c-homogeneous*.

Given a partition $p:[\kappa]^2\rightarrow\mu$ of the unordered pairs from κ into μ cells, it is possible to relax the notion of c-homogeneity to *relative* c-homogeneity *over* p. A set $A\subseteq\kappa$ is *c-homogeneous over* p, or (p,c)*-homogeneous* for short, if all pairs from A which lie in the same p-cell are colored by one color which depends on the cell.

[1] Here $a(i)$ stands the for the i^{th} element of a, that is, the unique $\alpha\in a$ to satisfy $\mathrm{otp}(a\cap\alpha)=i$.

More formally, for every cell $\epsilon < \mu$, the set $\{c(\alpha,\beta) \mid (\alpha,\beta) \in [A]^2 \ \&\ p(\alpha,\beta) = \epsilon\}$ has size no more than 1. Put differently, when restricted to $[A]^2$, p refines c, namely the p-cell of a pair in A determines its c-color. Thus, A is (p,c)-homogeneous iff there is a function $\tau : \mu \to \theta$ such that $c(\alpha,\beta) = \tau(p(\alpha,\beta))$ for every $(\alpha,\beta) \in [A]^2$.

The Ramsey relation $\kappa \to (\lambda)^2_\theta$ can now be relaxed, for a partition p, to its "over p" version, $\kappa \to_p (\lambda)^2_\theta$, to mean that for every coloring $c : [\kappa]^2 \to \theta$ there is a set $A \subseteq \kappa$ of size λ which is *c-homogeneous over p*. Note that if A is c-homogeneous, it is also (p,c)-homogeneous for every partition p of the pairs from κ, so $\kappa \to_p (\lambda)^2_\theta$ follows from $\kappa \to (\lambda)^2_\theta$ for any partition p.

Part I of this series [KRS22a] is devoted to the consistency of positive Ramsey relations over partitions. For instance, it is shown that Martin's Axiom for $\aleph_1$ implies that $\aleph_1 \to_p (\aleph_1)^2_{\aleph_0}$ holds for many partitions $p : [\aleph_1]^2 \to \aleph_0$. This means that also the Sierpiński coloring can be relatively homogenizied over suitable countable partitions.

In this paper, which constitutes Part II of this series, we deal with strong colorings over partitions, that is, with the failure of the "over p" version of the weak Ramsey relation $\kappa \to [\lambda]^2_\theta$.

We recall that the relation $\kappa \to [\lambda]^2_\theta$ means that for every coloring $c : [\kappa]^2 \to \theta$ there exists a set $A \subseteq \kappa$ of cardinality λ such that $c \restriction [A]^2$ is not surjective, i.e., one of the colors $\gamma < \theta$ is *omitted* in the sense that $c(\alpha,\beta) \neq \gamma$ for every pair $(\alpha,\beta) \in [A]^2$. Now, given a partition $p : [\kappa]^2 \to \mu$, the relation $\kappa \to_p [\lambda]^2_\theta$ is defined to mean that for every coloring $c : [\kappa]^2 \to \theta$ there is a set $A \subseteq \kappa$ of cardinality λ such that at least one color from θ is omitted by c in every p-cell. That is, for some function $\tau : \mu \to \theta$, $c(\alpha,\beta) \neq \tau(p(\alpha,\beta))$ for every $(\alpha,\beta) \in [A]^2$.

The strong coloring symbol over p, $\kappa \nrightarrow_p [\lambda]^2_\theta$, which is simply the negation of the positive relation above, means, then, that for every $A \subseteq \kappa$ of cardinality λ, for every function $\tau : \mu \to \theta$ there is a pair $(\alpha,\beta) \in [A]^2$ such that $c(\alpha,\beta) = \tau(p(\alpha,\beta))$. By [CKS21, Fact 5], this is the same as asserting that for every $A \subseteq \kappa$ of cardinality λ, there is an $\epsilon < \mu$ such that $\{c(\alpha,\beta) \mid (\alpha,\beta) \in [A]^2 \ \&\ p(\alpha,\beta) = \epsilon\} = \theta$.

The referee of this paper kindly shared with us the way they visualize the latter definition. We reproduce it now. Think of the product coloring $p \times c$ as a map that takes a pair (α,β) from $[\kappa]^2$ and sends it to some (ϵ,τ) in $\mu \times \theta$. Then, interpreting $\mu \times \theta$ as a matrix with μ rows and θ columns, $\kappa \nrightarrow_p [\lambda]^2_\theta$ means that given any subset $A \subseteq \kappa$ of cardinality λ, the image of $[A]^2$ under the product map contains a row of $\mu \times \theta$.

To sum up, in Ramsey theory over partitions the role of a color γ is taken by a function τ from p-cells to colors. Restriction to a single color becomes restriction to a single function, omitting a color becomes omitting a function and attaining all colors becomes attaining all functions τ via $c(\alpha,\beta) = \tau(p(\alpha,\beta))$.

Therefore, the "over p" versions of the strong coloring principles Pr_1 and Pr_0 are defined as follows:

Definition 1.1 ([CKS21])**.** Let $p : [\kappa]^2 \to \mu$ be a partition. A coloring $c : [\kappa]^2 \to \theta$ is said to witness

- $\mathrm{Pr}_1(\kappa,\kappa,\theta,\chi)_p$ iff for every $\sigma < \chi$, every pairwise disjoint family $\mathcal{A} \subseteq [\kappa]^\sigma$ of size κ, and every function $\tau : \mu \to \theta$ there are $a, b \in \mathcal{A}$ with $\sup(a) < \min(b)$ such that

$$c(\alpha,\beta) = \tau(p(\alpha,\beta)) \text{ for all } \alpha \in a \text{ and } \beta \in b;$$

- $\mathrm{Pr}_0(\kappa,\kappa,\theta,\chi)_p$ iff for for every $\sigma<\chi$, every pairwise disjoint family $\mathcal{A}\subseteq[\kappa]^\sigma$ of size κ, and every matrix $(\tau_{i,j})_{i,j<\sigma}$ of functions from μ to θ there are $a,b\in\mathcal{A}$ with $\sup(a)<\min(b)$ such that

$$c(a(i),b(j))=\tau_{i,j}(p(a(i),b(j)))\text{ for all }i,j<\sigma.$$

Every coloring which is strong over p is also strong over any partition p' coarser than p. However, no coloring is strong in any of the ways defined above over any non-trivial coarsening of itself (see Proposition 2.2 below). In particular, for every coloring c by more than one color there are partitions of pairs to just two cells over which the coloring is no longer strong.

The study of strong colorings over partitions is, then, concentrated on the strcuture of the space of all colorings witnessing any of the relations defined above over small partitions: for which colorings c and small partitions p, does c witness a negative Ramsey relation over p.

1.3. **The results.** The findings of Part I show that unlike classical Ramsey theory of the uncountable, which steers towards the negative side, suitable forcing axioms for a prescribed uncountable cardinal λ imply that certain small partitions p satisfy $\lambda^+\rightarrow_p(\lambda^+)^2_\lambda$. Thus, arbitrary large successor cardinals κ may be "p-weakly compact" in the sense that $\kappa\rightarrow_p(\kappa)^2_2$.

On the other hand, strong negative Ramsey relations over partitions are also available: Under GCH-type assumptions, results from the classical theory prevail to the new context, for instance, by [CKS21, Lemma 9], for any partition $p:[\kappa]^2\rightarrow\mu$ and $i<2$, $\mathrm{Pr}_i(\kappa,\kappa,\theta^\mu,\chi)$ outright implies $\mathrm{Pr}_i(\kappa,\kappa,\theta,\chi)_p$.

In the present paper we aim for absolute results rather than independence results. When the space of strong colorings over a prescribed partition $p:[\kappa]^2\rightarrow\mu$ is not empty, we explore which colorings this space must contain.

A standard fact from classical theory is that for every successor cardinal $\kappa=\lambda^+$, $\kappa\nrightarrow[\kappa]^2_\lambda$ implies $\kappa\nrightarrow[\kappa]^2_\kappa$. What happens in the new context? Is it the case that $\kappa\nrightarrow_p[\kappa]^2_\lambda$ implies $\kappa\nrightarrow_p[\kappa]^2_\kappa$ for every partition p? And what happens with the stronger relations in Definition 1.1? For instance, in [Eis13], Eisworth solved a longstanding open problem by proving that for every singular cardinal λ, $\mathrm{Pr}_1(\lambda^+,\lambda^+,\lambda,\mathrm{cf}(\lambda))$ implies $\mathrm{Pr}_1(\lambda^+,\lambda^+,\lambda^+,\mathrm{cf}(\lambda))$. Is it the case that $\mathrm{Pr}_1(\lambda^+,\lambda^+,\lambda,\mathrm{cf}(\lambda))_p$ implies $\mathrm{Pr}_1(\lambda^+,\lambda^+,\lambda^+,\mathrm{cf}(\lambda))_p$ for every partition p? Whether the same implication holds with $\lambda=\aleph_0$ was asked in [CKS21, Question 47]. The first result of this paper answers this question in the affirmative.

Since the classical theory corresponds to the special case of a partition p with just one cell, the following result generalizes (and also provides a new proof of) Eisworth's pump-up theorem:

Theorem A. *For every infinite cardinal λ, for every partition $p:[\lambda^+]^2\rightarrow\lambda$, and for every cardinal $\chi\leq\mathrm{cf}(\lambda)$,*

$$\mathrm{Pr}_1(\lambda^+,\lambda^+,\lambda,\chi)_p\iff\mathrm{Pr}_1(\lambda^+,\lambda^+,\lambda^+,\chi)_p.$$

Our proof of the preceding brings the method of walks on ordinals into the study of strong colorings over partitions. The proof actually provides an operator from colorings c by λ colors to colorings c^+ by λ^+ colors with the property that for every partition p to $\leq\lambda$ cells, if c is strong over p then c^+ is also strong over p.

In [She94, Lemma 4.5], Shelah presented sufficient cardinal arithmetic conditions for $\mathrm{Pr}_1(\kappa,\kappa,\theta,\chi)$ to imply the stronger $\mathrm{Pr}_0(\kappa,\kappa,\theta,\chi)$. [CKS21, Question 46] asks whether it is possible to obtain the same pump-up over a partition p for $\kappa=\aleph_1$. The following theorem provides a general affirmative answer.

Theorem B. *For a regular uncountable cardinal κ and cardinals $\mu,\lambda,\chi,\theta\leq\kappa$ satisfying $\lambda^{<\chi}<\kappa\leq 2^\lambda$ and $\lambda^{<\chi}\leq\theta^{<\chi}=\theta$, for every partition $p:[\kappa]^2\to\mu$,*

$$\mathrm{Pr}_1(\kappa,\kappa,\theta,\chi)_p \iff \mathrm{Pr}_0(\kappa,\kappa,\theta,\chi)_p.$$

2. Preliminaries

For cardinals $\chi<\kappa$, $E^\kappa_{\geq\chi}$ denotes the set $\{\alpha<\kappa\mid \mathrm{cf}(\alpha)\geq\chi\}$. For an ordinal σ and a set of ordinals A, we write $[A]^\sigma$ for $\{B\subseteq A\mid \mathrm{otp}(B)=\sigma\}$. For a cardinal χ and a set $\mathcal{A}$, we write $[\mathcal{A}]^\chi:=\{\mathcal{B}\subseteq\mathcal{A}\mid |\mathcal{B}|=\chi\}$ and $[\mathcal{A}]^{<\chi}:=\{\mathcal{B}\subseteq\mathcal{A}\mid |\mathcal{B}|<\chi\}$. For a set of ordinals a and b, we let $\mathrm{acc}(a):=\{\alpha\in a\mid \sup(a\cap\alpha)=\alpha>0\}$, and we write $a<b$ if $\alpha<\beta$ for all $\alpha\in a$ and $\beta\in b$. For a set $\mathcal{A}$ which is either an ordinal or a collection of sets of ordinals, we identify $[\mathcal{A}]^2$ with $\{(a,b)\in\mathcal{A}\times\mathcal{A}\mid a<b\}$.

Definition 2.1. For $q:X\to\nu$ and $p:X\to\mu$, q is a *coarsening* of p, or p is a *refinement* of q, iff $p(x)=p(y)$ implies $q(x)=q(y)$ for all $x,y\in X$.

Proposition 2.2. *For cardinals κ,ν,μ,θ, suppose a partition $p:[\kappa]^2\to\mu$ and a coloring $c:[\kappa]^2\to\theta$ have a common coarsening $q:[\kappa]^2\to\nu$ that is not constant. Then c does not witness $\kappa\nrightarrow_p[\kappa]^2_\theta$.*

Proof. Let X be an arbitrary p-cell, and we shall show that $c\restriction X$ is not surjective.

As p refines q, we may fix a q-cell $\hat{X}$ that covers X. As q is not constant there is some q-cell $\hat{Y}$ disjoint from $\hat{X}$. Fix $y\in\hat{Y}$ and let $\gamma:=c(y)$. Since c refines q, $c^{-1}[\{\gamma\}]\subseteq\hat{Y}$. As $X\subseteq\hat{X}$ and $\hat{X}\cap\hat{Y}=\emptyset$, it holds that $\gamma\neq c(x)$ for all $x\in X$. □

Here and also in Part III of this series, we study *unbalanced* versions of the principles of Definition 1.1.

Definition 2.3. Let $p:[\kappa]^2\to\mu$ be a partition. A coloring $c:[\kappa]^2\to\theta$ is said to witness

- $\mathrm{Pr}_1(\kappa,\nicefrac{\nu\circledast\kappa}{\nu'\circledast\kappa},\theta,\chi)_p$ iff for every pairwise disjoint subfamilies $\mathcal{A},\mathcal{B}$ of $[\kappa]^\sigma$ with $|\mathcal{A}|=\nu$, $|\mathcal{B}|=\kappa$ and $\sigma<\chi$ there is $\mathcal{A}'\in[\mathcal{A}]^{\nu'}$ such that for every function $\tau:\mu\to\theta$, there are $a\in\mathcal{A}'$ and $b\in\mathcal{B}$ with $a<b$ such that
$$c(\alpha,\beta)=\tau(p(\alpha,\beta))\text{ for all }\alpha\in a\text{ and }\beta\in b;$$
- $\mathrm{Pr}_0(\kappa,\nicefrac{\nu\circledast\kappa}{\nu'\circledast\kappa},\theta,\chi)_p$ iff for every pairwise disjoint subfamilies $\mathcal{A},\mathcal{B}$ of $[\kappa]^\sigma$ with $|\mathcal{A}|=\nu$, $|\mathcal{B}|=\kappa$ and $\sigma<\chi$, there is $\mathcal{A}'\in[\mathcal{A}]^{\nu'}$ such that for every matrix $(\tau_{i,j})_{i,j<\sigma}$ of functions from μ to θ, there are $a\in\mathcal{A}'$ and $b\in\mathcal{B}$ with $a<b$ such that
$$c(a(i),b(j))=\tau_{i,j}(p(a(i),b(j)))\text{ for all }i,j<\sigma.$$

Remark 2.4. We write $\mathrm{Pr}_i(\kappa,\nu\circledast\kappa,\theta,\chi)_p$ for $\mathrm{Pr}_i(\kappa,\nicefrac{\nu\circledast\kappa}{\nu\circledast\kappa},\theta,\chi)_p$.

Definition 2.5. A partition $p:[\kappa]^2\to\mu$ is said to have *injective fibers* iff for all $\alpha<\alpha'<\beta$, $p(\alpha,\beta)\neq p(\alpha',\beta)$.

The next proposition shows that in order to obtain strong colorings over all partitions, it suffices to focus on partitions with injective fibers.

Proposition 2.6. *For every infinite cardinal λ and every partition $p : [\lambda^+]^2 \to \lambda$, there exists a corresponding partition $\bar{p} : [\lambda^+]^2 \to \lambda$ with injective fibers such that if any strong coloring relation from Definitions 1.1 and 2.3 holds for $\bar{p}$, then it also holds for p.*

Proof. Given $p : [\lambda^+]^2 \to \lambda$, we define $q : [\lambda^+]^2 \to \lambda \times \lambda$ as follows. Fix an arbitrary nonzero $\beta < \lambda^+$. Fix a bijection $i_\beta : |\beta| \leftrightarrow \beta$. Then, for every $\epsilon < |\beta|$, let

$$q(i_\beta(\epsilon), \beta) := (p(i_\beta(\epsilon), \beta), \mathrm{otp}\{\varepsilon < \epsilon \mid p(i_\beta(\varepsilon), \beta) = p(i_\beta(\epsilon), \beta)\}).$$

It is easy to check that, for all $\alpha < \beta < \lambda^+$:

- $q(\alpha, \beta) = (p(\alpha, \beta), \zeta)$ for some $\zeta < \lambda$;
- $q(\alpha', \beta) \neq q(\alpha, \beta)$ for all $\alpha' < \alpha$.

Finally, fix a bijection $\pi : \lambda \leftrightarrow \lambda \times \lambda$ and set $\bar{p} := \pi^{-1} \circ q$.

Then, to any function $\tau \in {}^\lambda\theta$, we define the corresponding function $\bar{\tau} \in {}^\lambda\theta$ such that, for all $\eta < \lambda$, if $\pi(\eta) = (\xi, \zeta)$, then $\bar{\tau}(\eta) = \tau(\xi)$. □

3. Theorem A: increasing the number of colors

This section deals with the problem of pumping up a strong coloring by λ colors into one by λ^+ colors. The special case of a singular cardinal λ with no partition involved is a result that was first obtained by Eisworth as a corollary to his transformation theorem of [Eis13]. While the theory of transformations has advanced considerably [RZ21b, RZ21a], at the moment it is unclear whether such transformations can overcome partitions. Thus, the proof given below is different.

When taking partitions into account, as the phrasing of Question 46 from [CKS21] hints, one ought to expect that different stretchings of the same coloring might be needed for different partitions. It is surprising, then, that a coloring can be stretched once in a way which uniformly works for all partitions it is strong over. Indeed, the main corollary of this section reads as follows.

Corollary 3.1. *Let λ be an infinite cardinal.*

For every coloring $c : [\lambda^+]^2 \to \lambda$ there exists a corresponding coloring $c^+ : [\lambda^+]^2 \to \lambda^+$ such that for every partition $p : [\lambda^+]^2 \to \lambda$ and every cardinal $\chi \leq \mathrm{cf}(\lambda)$:

1. *if c witnesses $\mathrm{Pr}_1(\lambda^+, \lambda^+, \lambda, \chi)_p$ then c^+ witnesses $\mathrm{Pr}_1(\lambda^+, \lambda^+, \lambda^+, \chi)_p$;*
2. *if c witnesses $\mathrm{Pr}_1(\lambda^+, \lambda^+ \circledast \lambda^+, \lambda, \chi)_p$ then c^+ witnesses $\mathrm{Pr}_1(\lambda^+, \lambda^+ \circledast \lambda^+, \lambda^+, \chi)_p$.*

Proof. The proof is split into three cases:

▶ If λ is regular then the stationary subset of ordinals of cofinality λ below λ^+ is non-reflecting, so Theorem 3.5 below applies.

▶ If λ is singular of countable cofinality, then sets of cardinality below $\mathrm{cf}(\lambda)$ are finite, so Theorem 3.4 below applies.

▶ If λ is a singular of uncountable cofinality, then appeal to Theorem 3.6 below. □

So how do one pump up a strong coloring by λ colors into one by λ^+ colors? The classical stretching argument (cf. [Tod87, p. 277]) employs a sequence of surjections $\langle e_\beta : \lambda \to \beta + 1 \mid \beta < \lambda^+\rangle$. Specially, defining $c^+(\alpha, \beta) := e_\beta(c(\alpha, \beta))$ stretches a strong coloring $c : [\lambda^+]^2 \to \lambda$ to a strong coloring $c^+ : [\lambda^+]^2 \to \lambda^+$ via a pigeonhole consideration for stabilizing the stretch: for a prescribed color $\delta < \lambda^+$ many e_β

will map the same $i < \lambda$ to δ, and the original coloring c will indeed produce any possible $i < \lambda$.

The above one-dimensional stretching (that depends only on β) is incompatible with a two-dimensional partition. What we do here, then, is instead of letting $c^+(\alpha,\beta) := e_\beta(c(\alpha,\beta))$, we let $c^+(\alpha,\beta) := e_\gamma(i)$, where i is again computed from $c(\alpha,\beta)$, but γ is computed from the *triple* $(\alpha,\beta,c(\alpha,\beta))$.

Based on a feedback from the referee, we commence by illustrating the basic idea through a warm-up proof of the simplest pump up theorem, though this is covered by later theorems.

Proposition 3.2. *Let λ be an infinite cardinal.*

For every coloring $c : [\lambda^+]^2 \to \lambda$, there exists a corresponding coloring $c^+ : [\lambda^+]^2 \to \lambda^+$ such that for every $\mu \leq \lambda$ and every $p : [\lambda^+]^2 \to \mu$,

1. *if c witnesses $\lambda^+ \nrightarrow_p [\lambda^+]^2_\lambda$ then c^+ witnesses $\lambda^+ \nrightarrow_p [\lambda^+]^2_{\lambda^+}$;*
2. *if c witnesses $\lambda^+ \nrightarrow_p [\varkappa \circledast \lambda^+]^2_\lambda$ then c^+ witnesses $\lambda^+ \nrightarrow_p [\varkappa \circledast \lambda^+]^2_{\lambda^+}$.*

Proof. Fix a bijection $\pi : \lambda \to \lambda \times \lambda$. For every $\beta < \lambda^+$, fix a surjection $e_\beta : \lambda \to \beta + 1$ and let e_γ^{-1} be a right inverse of e_γ, that is, satisfy that $e_\gamma(e_\gamma^{-1}(\delta)) = \delta$ for all $\delta \leq \gamma$.

Now, given $c : [\lambda^+]^2 \to \lambda$, define $c^+ : [\lambda^+]^2 \to \lambda^+$, as follows. For $\alpha < \beta < \lambda^+$, let $(i,j) := \pi(c(\alpha,\beta))$ and then let:

$$c^+(\alpha,\beta) := e_{e_\beta(j)}(i).$$

Thus, rather than apply the bijection e_β to $c(\alpha,\beta)$ itself, we split $c(\alpha,\beta)$ to two terms via π and apply e_β to one of them, to find some γ whose e_γ is applied to the other. This allows the choice of the stretching to be done by the coloring c, and enables the coloring c^+ to overcome every partition p which c overcomes.

Claim 3.2.1. *For every cofinal $B \subseteq \lambda^+$ there exists $j < \lambda$ such that*

$$\sup\left\{\gamma < \lambda^+ \mid \sup\{\beta \in B \mid e_\beta(j) = \gamma\} = \lambda^+\right\} = \lambda^+.$$

Proof. Let $B \subseteq \lambda^+$ be cofinal. For every $\gamma < \lambda^+$ and $\beta \geq \gamma$ in B there is some $j_{\gamma,\beta} < \lambda$ such that $e_\beta(j_{\gamma,\beta}) = \gamma$. As λ^+ is regular, there is some j_γ and $B_\gamma \subseteq B$ with $\sup(B_\gamma) = \lambda^+$ such that $e_\beta(i_\gamma) = \gamma$ for all $\beta \in B_\gamma$. Finally, by regularity of λ^+, there is some $j < \lambda$ such that $j = j_\gamma$ for an unbounded set of $\gamma < \lambda^+$, as required. □

We shall only verify Clause (1), and encourage the reader to see they know how to adapt the verification to the context of Clause (2).

Suppose that $p : [\lambda^+]^2 \to \mu$ with $\mu \leq \lambda$ is some partition for which c witnesses $\lambda^+ \nrightarrow_p [\lambda^+]_\lambda$. To see that c^+ witnesses $\lambda^+ \nrightarrow_p [\lambda^+]^2_{\lambda^+}$ suppose that $B \subseteq \lambda^+$ is cofinal and that some function $\tau : \mu \to \lambda^+$ is given. We need to find a pair $\alpha < \beta$ of ordinal in B such that $c^+(\alpha,\beta) = \tau(p(\alpha,\beta))$.

Let $j < \lambda$ be given by the claim with respect to B. For every $\gamma < \lambda^+$, denote $B_\gamma := \{\beta \in B \mid e_\beta(j) = \gamma\}$. By the choice of j, the set $\Gamma = \{\gamma < \lambda^+ \mid \sup(B_\gamma) = \lambda^+\}$ is cofinal in λ^+. So, by the regularity of λ^+, we may fix some $\gamma \in \Gamma$ above $\sup(\mathrm{Im}(\tau))$.

Define a function $\tau^* : \mu \to \lambda$ as follows. For every $\epsilon < \mu$ let

$$\tau^*(\epsilon) = \pi^{-1}(e_\gamma^{-1}(\tau(\epsilon)), j).$$

As c witnesses $\lambda^+ \not\to_p [\lambda^+]^2_\lambda$ and B_γ is in particular a cofinal subset of λ^+, we may fix $(\alpha,\beta)\in[B_\gamma]^2$ such that $c(\alpha,\beta)=\tau^*(p(\alpha,\beta))$. Denote $(i',j'):=\pi(c(\alpha,\beta))$ and $\epsilon:=p(\alpha,\beta)$. By the definition of τ^*, necessarily $j'=j$ and $i'=e_\gamma^{-1}(\tau(\epsilon))$. In particular, $e_\beta(j')=\gamma$. By the definition of c^+, then,

$$c^+(\alpha,\beta)=e_{e_\beta(j')}(i')=e_\gamma(e_\gamma^{-1}(\tau(\epsilon)))=\tau(\epsilon)=\tau(p(\alpha,\beta)),$$

as required. □

To motivate the statement of our next theorem, notice that if $\lambda^+ \not\to_p [\lambda^+]^2_{\lambda^+}$ holds, then so do $\lambda^+ \not\to_p [\lambda^+]^2_\lambda$ and $\lambda^+ \not\to [\lambda^+]^2_{\lambda^+}$. The theorem shows that it is possible to combine these two consequences — a witness for λ many colors over a partition with a witness for λ^+ many colors but not over a partition — into a single strong coloring. Note that in the next theorem there is no restriction on the value of χ.

Theorem 3.3. *Suppose $\nu,\mu\le\lambda$ are cardinals with λ infinite, and:*

- *$p:[\lambda^+]^2\to\mu$ is a partition;*
- *$\nu=1$ or $\nu=\lambda$. More generally, $\mathrm{cf}([\lambda]^\nu,\subseteq)\le\lambda$ suffices.*

If $\mathrm{Pr}_1(\lambda^+,{}^{\lambda\circledast\lambda^+}\!/_{\nu\circledast\lambda^+},\lambda,\chi)_p$ *and* $\mathrm{Pr}_1(\lambda^+,\lambda\circledast\lambda^+,\lambda^+,\chi)$ *both hold, then so does* $\mathrm{Pr}_1(\lambda^+,{}^{\lambda\circledast\lambda^+}\!/_{\nu\circledast\lambda^+},\lambda^+,\chi)_p$.

Proof. Fix a coloring $c:[\lambda^+]^2\to\lambda$ which witnesses $\mathrm{Pr}_1(\lambda^+,{}^{\lambda\circledast\lambda^+}\!/_{\nu\circledast\lambda^+},\lambda,\chi)_p$ and a coloring $d:[\lambda^+]^2\to\lambda^+$ which witnesses $\mathrm{Pr}_1(\lambda^+,\lambda\circledast\lambda^+,\lambda^+,\chi)$. For every $\beta<\lambda^+$ fix a surjection $e_\beta:\lambda\to\beta+1$. Fix a bijection $\pi:\lambda\leftrightarrow\lambda\times\lambda$. Define a coloring $c^+:[\lambda^+]^2\to\lambda^+$, as follows: For all $\alpha<\beta<\lambda$, let $c^+(\alpha,\beta):=0$; for $\alpha<\beta<\lambda^+$ with $\beta\ge\lambda$ denote $(i,j):=\pi(c(\alpha,\beta))$ and let:

$$c^+(\alpha,\beta):=e_{d(j,\beta)}(i).$$

To verify that c^+ witnesses $\mathrm{Pr}_1(\lambda^+,{}^{\lambda\circledast\lambda^+}\!/_{\nu\circledast\lambda^+},\lambda^+,\chi)_p$ fix pairwise disjoint subfamilies $\mathcal{A},\mathcal{B}\subseteq[\lambda^+]^{<\chi}$ with $|\mathcal{A}|=\lambda$ and $|\mathcal{B}|=\lambda^+$. Denote $\mathcal{B}^\gamma_j:=\{b\in\mathcal{B}\mid \min(b)\ge\lambda\ \&\ d[\{j\}\times b]=\{\gamma\}\}$.

Claim 3.3.1. *There exists $j<\lambda$ for which $\{\gamma<\lambda^+\mid |\mathcal{B}^\gamma_j|=\lambda^+\}$ is cofinal in λ^+.*

Proof. Suppose not. Then, for every $j<\lambda$, $\delta_j:=\sup\{\gamma<\lambda^+\mid |\mathcal{B}^\gamma_j|=\lambda^+\}$ is $<\lambda^+$. Consider $\delta:=(\sup_{j<\lambda}\delta_j)+1$. Then, for every $j<\lambda$, $|\mathcal{B}^\delta_j|<\lambda^+$. Consequently, $\mathcal{B}':=\{b\in\mathcal{B}\mid\min(b)\ge\lambda\ \&\ \forall j<\lambda\,(d[\{j\}\times b]\neq\{\delta\})\}$ has size λ^+. Appealing to d with $\mathcal{A}':=[\lambda]^1$ and $\mathcal{B}'$, there must exist $a\in\mathcal{A}'$ and $b\in\mathcal{B}'$ such that $d[a\times b]=\{\delta\}$. But $a=\{j\}$ for some $j<\lambda$, contradicting the fact that $b\in\mathcal{B}'$. □

Let $j<\lambda$ be given by the claim. By the choice of c, for every $\gamma<\lambda^+$ such that $|\mathcal{B}^\gamma_j|=\lambda^+$ there exists $\mathcal{A}^\gamma\in[\mathcal{A}]^\nu$ such that for every function $\tau:\mu\to\lambda$, there are $a\in\mathcal{A}^\gamma$ and $b\in\mathcal{B}^\gamma_j$ with $a<b$ such that $c(\alpha,\beta)=\tau(p(\alpha,\beta))$ for all $(\alpha,\beta)\in a\times b$. As $\{\gamma<\lambda^+\mid|\mathcal{B}^\gamma_j|=\lambda^+\}$ is cofinal in λ^+ and $\mathrm{cf}([|\mathcal{A}|]^\nu,\subseteq)<\lambda^+$, we may find some $\mathcal{A}'\in[\mathcal{A}]^\nu$ for which $\Gamma:=\{\gamma<\lambda^+\mid|\mathcal{B}^\gamma_j|=\lambda^+\ \&\ \mathcal{A}^\gamma\subseteq\mathcal{A}'\}$ is cofinal in λ^+. We claim that $\mathcal{A}'$ is as sought.

Claim 3.3.2. *Let $\tau:\mu\to\lambda^+$. There are $a\in\mathcal{A}'$ and $b\in\mathcal{B}$ with $a<b$ such that $c(\alpha,\beta)=\tau(p(\alpha,\beta))$ for all $(\alpha,\beta)\in a\times b$.*

Proof. As $\mu \leq \lambda$, we may fix a large enough $\gamma \in \Gamma$ such that $\mathrm{Im}(\tau) \subseteq \gamma$. For every $\epsilon < \mu$, fix $i_\epsilon < \lambda$ such that $e_\gamma(i_\epsilon) = \tau(\epsilon)$. Define a function $\tau' : \mu \to \lambda$ via $\tau'(\epsilon) := \pi^{-1}(i_\epsilon, j)$. Pick $a \in \mathcal{A}^\gamma$ and $b \in \mathcal{B}^\gamma_j$ with $a < b$ such that $c(\alpha,\beta) = \tau'(p(\alpha,\beta))$ for all $(\alpha,\beta) \in a \times b$. Clearly, $a \in \mathcal{A}'$ and $b \in \mathcal{B}$. Set $\epsilon := p(\alpha,\beta)$. Then $c(\alpha,\beta) = \tau'(\epsilon) = \pi^{-1}(i_\epsilon, j)$, so that $c^+(\alpha,\beta) = e_{d(j,\beta)}(i_\epsilon) = e_\gamma(i_\epsilon) = \tau(\epsilon)$. □

This completes the proof. □

Another approach for stretching strong colorings for high-dimensional relations, is to try to encode sequences of ordinals in a single value. In the next theorem, this is done by appealing to the Engelking-Karlowicz theorem, which arranges λ^+ many patterns in a matrix with just λ rows.

Theorem 3.4. *Let λ be an infinite cardinal.*

For every coloring $c : [\lambda^+]^2 \to \lambda$ there exists a corresponding coloring $c^+ : [\lambda^+]^2 \to \lambda^+$ such that for every partition $p : [\lambda^+]^2 \to \mu$ with $\mu \leq \lambda$ and every cardinal χ such that $\lambda^{<\chi} = \lambda$:

1. *if c witnesses $\mathrm{Pr}_1(\lambda^+, \lambda^+, \lambda, \chi)_p$ then c^+ witnesses $\mathrm{Pr}_1(\lambda^+, \lambda^+, \lambda^+, \chi)_p$;*
2. *if c witnesses $\mathrm{Pr}_1(\lambda^+, \lambda^+ \circledast \lambda^+, \lambda, \chi)_p$ then c^+ witnesses $\mathrm{Pr}_1(\lambda^+, \lambda^+ \circledast \lambda^+, \lambda^+, \chi)_p$;*
3. *if c witnesses $\mathrm{Pr}_1(\lambda^+, {}^{\lambda \circledast \lambda^+}/_{\nu \circledast \lambda^+}, \lambda, \chi)_p$ with $\mathrm{cf}([\lambda]^\nu, \subseteq) \leq \lambda$, then c^+ witnesses $\mathrm{Pr}_1(\lambda^+, {}^{\lambda \circledast \lambda^+}/_{\nu \circledast \lambda^+}, \lambda^+, \chi)_p$.*

Proof. Using the Engelking-Karlowicz theorem, fix a sequence $\langle h_j \mid j < \lambda \rangle$ of functions from λ^+ to λ with the property that for every $a \subseteq \lambda^+$ with $\lambda^{|a|} = \lambda$ and a function $h : a \to \lambda$, there exists $j < \lambda$ with $h \subseteq h_j$. Define a function $d : \lambda \times \lambda^+ \to \lambda^+$ via

$$d(j,\beta) := e_\beta(h_j(\beta)).$$

For each $\mathcal{B} \subseteq \mathcal{P}(\lambda^+)$, denote $\mathcal{B}^\gamma_j := \{b \in \mathcal{B} \mid \min(b) \geq \lambda \ \&\ d[\{j\} \times b] = \{\gamma\}\}$. The following is clear.

Claim 3.4.1. *Assuming $\lambda^{<\chi} = \lambda$, for every $\mathcal{B} \subseteq [\lambda^+]^{<\chi}$ of size λ^+, there exists $j < \lambda$ for which $\{\gamma < \lambda^+ \mid |\mathcal{B}^\gamma_j| = \lambda^+\}$ is cofinal in λ^+.* □

The rest of the proof is now very similar to that of Theorem 3.3. We fix a bijection $\pi : \lambda \leftrightarrow \lambda \times \lambda$ and, for every $\beta < \lambda^+$, we fix a surjection $e_\beta : \lambda \to \beta + 1$. Given a coloring $c : [\lambda^+]^2 \to \lambda$, we define the corresponding coloring $c^+ : [\lambda^+]^2 \to \lambda^+$ by letting $c^+(\alpha,\beta) := 0$ for all $\alpha < \beta < \lambda$ and, given $\alpha < \beta < \lambda^+$ with $\beta \geq \lambda$, we denote $(i,j) := \pi(c(\alpha,\beta))$ and let:

$$c^+(\alpha,\beta) := e_{d(j,\beta)}(i).$$

The verification of the three clauses of this theorem is now similar to the verification in the proof of Theorem 3.3. □

The proofs of the next theorem and the one following it employ walks on ordinals in order to pick the γ in the template formula "$c^+(\alpha,\beta) := e_\gamma(i)$".

Theorem 3.5. *Let λ be an infinite cardinal.*

Suppose that $\chi \leq \mathrm{cf}(\lambda)$ and that $E^{\lambda^+}_{\geq\chi}$ admits a non-reflecting stationary set. For every coloring $c : [\lambda^+]^2 \to \lambda$ there exists a corresponding coloring $c^+ : [\lambda^+]^2 \to \lambda^+$ which satisfies that for every partition $p : [\lambda^+]^2 \to \mu$ with $\mu \leq \lambda$:

1. *if c witnesses $\mathrm{Pr}_1(\lambda^+, \lambda^+, \lambda, \chi)_p$ then c^+ witnesses $\mathrm{Pr}_1(\lambda^+, \lambda^+, \lambda^+, \chi)_p$;*

(2) *if* c *witnesses* $\mathrm{Pr}_1(\lambda^+,\lambda^+\circledast\lambda^+,\lambda,\chi)_p$ *then* c^+ *witnesses* $\mathrm{Pr}_1(\lambda^+,\lambda^+\circledast\lambda^+,\lambda^+,\chi)_p$.

Proof. Fix a bijection $\pi:\lambda\leftrightarrow\lambda\times\lambda$. For every $\beta<\lambda^+$, fix a surjection $e_\beta:\lambda\to\beta+1$. Fix a non-reflecting stationary set $\Gamma\subseteq E^{\lambda^+}_{\ge\chi}$ and a surjection $h:\lambda^+\to\lambda^+$ with the property that $H_\gamma:=\{\alpha\in\Gamma\mid h(\alpha)=\gamma\}$ is stationary for all $\gamma<\lambda^+$. Fix a sequence $\vec Z=\langle Z_\gamma\mid\gamma<\lambda^+\rangle$ of elements of $[\lambda]^{\mathrm{cf}(\lambda)}$ such that, for all $\gamma<\delta<\lambda^+$, $|Z_\gamma\cap Z_\delta|<\mathrm{cf}(\lambda)$.

Let $\vec C=\langle C_\alpha\mid\alpha<\lambda^+\rangle$ be a sequence such that C_α is a closed subset of α with $\sup(C_\alpha)=\sup(\alpha)$ and $\mathrm{acc}(C_\alpha)\cap\Gamma=\emptyset$, for every $\alpha<\lambda^+$. We shall be conducting walks on ordinals along $\vec C$ (see [Tod07] for a comprehensive treatment). First, for all $\alpha<\beta<\lambda^+$, define a function $\mathrm{Tr}(\alpha,\beta):\omega\to\beta+1$, by recursion on $n<\omega$, as follows:

$$\mathrm{Tr}(\alpha,\beta)(n):=\begin{cases}\beta, & n=0\\ \min(C_{\mathrm{Tr}(\alpha,\beta)(n-1)}\setminus\alpha), & n>0\ \&\ \mathrm{Tr}(\alpha,\beta)(n-1)>\alpha\\ \alpha, & \text{otherwise}\end{cases}$$

Then, derive a function $\rho_2:[\lambda^+]^2\to\omega$ via

$$\rho_2(\alpha,\beta):=\min\{n<\omega\mid\mathrm{Tr}(\alpha,\beta)(n)=\alpha\}.$$

Now, given a coloring $c:[\lambda^+]^2\to\lambda$, we define a corresponding coloring $c^+:[\lambda^+]^2\to\lambda^+$, as follows. For every pair $(\alpha,\beta)\in[\lambda^+]^2$, first let $(i,\zeta):=\pi(c(\alpha,\beta))$; then, if there exists $n<\omega$ such that $\zeta\in Z_{\mathrm{Tr}(\alpha,\beta)(n)}$, let

$$c^+(\alpha,\beta):=e_{h(\mathrm{Tr}(\alpha,\beta)(n))}(i)$$

for the least such n. Otherwise, let $c^+(\alpha,\beta):=0$.

Observe that the color $c(\alpha,\beta)$ is again split to two terms, but this time the term ζ is used as a halting condition in the walk, and then h translates the halting point δ into the ordinal γ corresponding to the surjection e_γ.

To see that c^+ is as sought, let $p:[\lambda^+]^2\to\mu$ be an arbitrary partition with $\mu\le\lambda$. Assume one of the following:

(1) c witnesses $\mathrm{Pr}_1(\lambda^+,\lambda^+,\lambda,\chi)_p$ and we are given a pairwise disjoint subfamily $\mathcal A$ of $[\lambda^+]^{<\chi}$ of size λ^+, and a prescribed function $\tau:\mu\to\lambda^+$;

(2) c witnesses $\mathrm{Pr}_1(\lambda^+,\lambda^+\circledast\lambda^+,\lambda,\chi)_p$ and we are given two pairwise disjoint subfamilies $\mathcal A,\mathcal B$ of $[\lambda^+]^{<\chi}$ of size λ^+, and a prescribed function $\tau:\mu\to\lambda^+$.

The proofs from either of the assumptions above are very similar. We will present them simultaneously, indicating by "Case (1)" and "Case (2)" the different parts.

In case (2), for every $\alpha<\lambda^+$, pick $a_\alpha\in\mathcal A$ and $b_\alpha\in\mathcal B$ with $\min(x_\alpha)>\alpha$, where $x_\alpha:=a_\alpha\cup b_\alpha$. In case (1), for every $\alpha<\lambda^+$, pick $a_\alpha\in\mathcal A$ with $\min(x_\alpha)>\alpha$, where $x_\alpha:=a_\alpha$.

Let D be some club in λ^+ such that, for every $\delta\in D$ and $\alpha<\delta$, $\sup(x_\alpha)<\delta$. This ensures that for every $(\alpha,\delta)\in[D]^2$, $\sup(x_\alpha)<\delta<\min(x_\delta)$, so that $\langle x_\delta\mid\delta\in D\rangle$ is $<$-increasing.

Set $\gamma:=\sup(\mathrm{Im}(\tau))$. As $\mu\le\lambda$, it is the case that $\gamma<\lambda^+$, and hence $\Delta:=\{\delta\in D\cap\Gamma\mid h(\delta)=\gamma\}$ is stationary. Next, define two functions $f:\Delta\to\lambda^+$ and $g:\Delta\to\lambda$ via:

- $f(\delta):=\sup\{\sup(C_{\mathrm{Tr}(\delta,\beta)(i)}\cap\delta)\mid\beta\in x_\delta, i<\rho_2(\delta,\beta)\}$ and
- $g(\delta):=\min(Z_\delta\setminus\bigcup\{Z_{\mathrm{Tr}(\delta,\beta)(i)}\mid\beta\in x_\delta, i<\rho_2(\delta,\beta)\}$.

For all $\delta \in \Delta$, $\beta \in x_\delta$ and $i < \rho_2(\delta,\beta)$, $\operatorname{acc}(C_{\operatorname{Tr}(\delta,\beta)(i)}) \cap \Gamma = \emptyset$, so $\sup(C_{\operatorname{Tr}(\delta,\beta)(i)} \cap \delta) < \delta$. It thus follows from $|x_\delta| < 2 \cdot \chi \leq \operatorname{cf}(\delta)$ that $f(\delta) < \delta$. Also, since $|x_\delta| < \operatorname{cf}(\lambda)$, $g(\delta)$ is well-defined. Fix $(\xi', \zeta') \in \lambda^+ \times \lambda$ for which $\Delta' := \{\delta \in \Delta \mid f(\delta) = \xi' \ \&\ g(\delta) = \zeta'\}$ is stationary.

As the prescribed τ is a function from μ to $\gamma+1$, we may fix, for every $\epsilon < \mu$, an $i_\epsilon < \lambda$ such that $e_\gamma(i_\epsilon) = \tau(\epsilon)$. Define a function $\tau' : \mu \to \lambda$ via $\tau'(\epsilon) := \pi^{-1}(i_\epsilon, \zeta')$. As $\Delta' \subseteq D$ and $|\Delta'| = \lambda^+$, we infer that $\mathcal{A}' := \{a_\delta \mid \delta \in \Delta'\}$ is a subfamily of $\mathcal{A}$ of size λ^+. Likewise, in Case (2), we also have that $\mathcal{B}' := \{b_\delta \mid \delta \in \Delta'\}$ is a subfamily of $\mathcal{B}$ of size λ^+. So, in Case (1) (resp. Case (2)), we may fix $a, b \in \mathcal{A}'$ (resp. $a \in \mathcal{A}'$ and $b \in \mathcal{B}'$) with $a < b$ such that $c(\alpha,\beta) = \tau'(p(\alpha,\beta))$ for all $\alpha \in a$ and $\beta \in b$,

Claim 3.5.1. *Let $(\alpha,\beta) \in a \times b$. Then $c^+(\alpha,\beta) = \tau(p(\alpha,\beta))$.*

Proof. Denote $\epsilon := p(\alpha,\beta)$. By the definition of τ', $c(\alpha,\beta) = \tau'(\epsilon) = \pi^{-1}(i_\epsilon, \zeta')$.

By the choice of b, let us fix $\delta \in \Delta'$ such that $b \subseteq x_\delta$. As $\beta \in x_\delta$, $\xi' = f(\delta) < \delta < \beta$. Likewise, since $\alpha \in a \in \mathcal{A}'$, $\xi' < \alpha$. Altogether,

$$\max\{\sup(C_{\operatorname{Tr}(\delta,\beta)(i)} \cap \delta) \mid i < \rho_2(\delta,\beta)\} \leq f(\delta) = \xi' < \alpha < \delta < \beta.$$

Now, by a standard fact from the theory of walks on ordinals (see [Rin14b, Claim 3.1.2]), $\operatorname{Tr}(\alpha,\beta)(i) = \operatorname{Tr}(\delta,\beta)(i)$ for all $i < \rho_2(\delta,\beta)$, and $\operatorname{Tr}(\alpha,\beta)(\rho_2(\delta,\beta)) = \delta$. Recalling that $g(\delta) = \zeta'$, this means that $n := \rho_2(\delta,\beta)$ is the least integer for which $\zeta' \in Z_{\operatorname{Tr}(\alpha,\beta)(n)}$. Therefore, by the definition of c^+,

$$c^+(\alpha,\beta) = e_{h(\operatorname{Tr}(\alpha,\beta)(n))}(i_\epsilon) = e_{h(\delta)}(i_\epsilon) = e_\gamma(i_\epsilon) = \tau(\epsilon),$$

as sought. □

This completes the proof. □

Unlike the preceding theorem, the proof of the next does not employ a surjection $h : \lambda^+ \to \lambda^+$, since it is still open whether for every singular cardinal λ there is a $\vec{C}$-sequence that gives rise to a decomposition of λ^+ into λ^+ many walk-wise-large sets. In the template formula "$c^+(\alpha,\beta) := e_\gamma(i)$", instead of letting $\gamma := h(\delta)$ for some well-chosen δ in the walk from β down to α, we shall let γ be the ξ^{th} element of C_δ, for well-chosen δ in the walk *and* $\xi < \lambda$.

Theorem 3.6. *Suppose that λ is a singular cardinal of uncountable cofinality and $\chi \leq \operatorname{cf}(\lambda)$. For every coloring $c : [\lambda^+]^2 \to \lambda$, there exists a corresponding coloring $c^+ : [\lambda^+]^2 \to \lambda^+$ satisfying that for every partition $p : [\lambda^+]^2 \to \mu$ with $\mu \leq \lambda$:*

1. *if c witnesses $\operatorname{Pr}_1(\lambda^+, \lambda^+, \lambda, \chi)_p$ then c^+ witnesses $\operatorname{Pr}_1(\lambda^+, \lambda^+, \lambda^+, \chi)_p$;*
2. *if c witnesses $\operatorname{Pr}_1(\lambda^+, \lambda^+ \circledast \lambda^+, \lambda, \chi)_p$ then c^+ witnesses $\operatorname{Pr}_1(\lambda^+, \lambda^+ \circledast \lambda^+, \lambda^+, \chi)_p$.*

Proof. By the proof of Case 1 of Theorem 4.21 from [LHR18], we may fix a C-sequence $\vec{C} = \langle C_\alpha \mid \alpha < \lambda^+ \rangle$ such that $\operatorname{otp}(C_\alpha) < \lambda$ for all $\alpha < \lambda^+$ and such that the functions Tr and ρ_2 derived from walking along $\vec{C}$ (as defined in the proof of Theorem 3.5) satisfy the following.

Claim 3.6.1. *Let $\mathcal{X}$ be a pairwise disjoint subfamily of $[\lambda^+]^{<\operatorname{cf}(\lambda)}$ of size λ^+. Then there exists a stationary set $\Delta \subseteq \lambda^+$, a sequence $\langle x_\gamma \mid \gamma \in \Delta \rangle$, and an ordinal $\varepsilon < \lambda^+$, such that, for every $\gamma \in \Delta$:*

- *$x_\gamma \in \mathcal{X}$ with $\min(x_\gamma) > \gamma > \varepsilon$;*
- *for all $\alpha \in (\varepsilon, \gamma)$ and $\beta \in x_\gamma$, $\gamma \in \operatorname{Im}(\operatorname{Tr}(\alpha,\beta))$;*

- $\mathrm{cf}(\gamma) > \sup\{\mathrm{otp}(C_{\mathrm{Tr}(\gamma,\beta)(n)}) \mid \beta \in x_\gamma, n < \rho_2(\gamma,\beta)\}$.

Proof. It suffices to prove that for every club $D \subseteq \lambda^+$ there are $\gamma \in D$, $x_\gamma \in \mathcal{X}$ and an ordinal $\varepsilon < \gamma$ such that the three bullets above hold. Now, given an arbitrary club $D \subseteq \lambda^+$, Claim 4.21.2 from [LHR18] provides $\gamma \in D$, $x_\gamma \in \mathcal{X}$ and an ordinal $\varepsilon < \gamma$ such that the first two bullets hold. The proof of that claim makes it clear that $\mathrm{cf}(\gamma) > |C|$ where $C := \bigcup\{C_{\mathrm{Tr}(\gamma,\beta)(n)} \mid \beta \in x_\gamma, n \leq \rho_2(\gamma,\beta)\}$, and goes through even if we require $\mathrm{cf}(\gamma) > |C|^+$. In particular, this will give $\mathrm{cf}(\gamma) > \sup\{\mathrm{otp}(C_{\mathrm{Tr}(\gamma,\beta)(n)}) \mid \beta \in x_\gamma, n < \rho_2(\gamma,\beta)\}$. □

Fix a bijection $\pi : \lambda \leftrightarrow \lambda \times \lambda$. For every $\beta < \lambda^+$, fix a surjection $e_\beta : \lambda \to \beta + 1$. Now, given a coloring $c : [\lambda^+]^2 \to \lambda$, we define a corresponding coloring $c^+ : [\lambda^+]^2 \to \lambda^+$ as follows. For every pair $(\alpha,\beta) \in [\lambda^+]^2$, first let $(i,\xi) := \pi(c(\alpha,\beta))$, and then, if there exists $n < \omega$ such that $\mathrm{otp}(C_{\mathrm{Tr}(\alpha,\beta)(n)}) > \xi$, let

$$c^+(\alpha,\beta) := e_{C_{\mathrm{Tr}(\alpha,\beta)(n)}(\xi)}(i)$$

for the least such n. Otherwise, just let $c^+(\alpha,\beta) := 0$.

So here the color $c(\alpha,\beta)$ is split to two terms. The term ξ is used both as a halting condition in the walk and as a pointer for a specific element $\gamma = C_\delta(\xi)$ (corresponding to the surjection e_γ) in the ladder C_δ of the halting point δ.

To see that c^+ is as sought, let $p : [\lambda^+]^2 \to \mu$ be an arbitrary partition with $\mu \leq \lambda$. There are two cases to consider:

(1) Assume c witnesses $\mathrm{Pr}_1(\lambda^+, \lambda^+, \lambda, \chi)_p$, and we are given a pairwise disjoint subfamily $\mathcal{A}$ of $[\lambda^+]^{<\chi}$ of size λ^+, and a prescribed function $\tau : \mu \to \lambda^+$.

 In this case, appeal to Claim 3.6.1 with $\mathcal{X} := \mathcal{A}$, and obtain a stationary set $\Delta \subseteq \lambda^+$, a sequence $\langle x_\gamma \mid \gamma \in \Delta\rangle$ and an ordinal $\varepsilon < \lambda^+$.

(2) Assume c witnesses $\mathrm{Pr}_1(\lambda^+, \lambda^+ \circledast \lambda^+, \lambda, \chi)_p$, and we are given two pairwise disjoint subfamilies $\mathcal{A},\mathcal{B}$ of $[\lambda^+]^{<\chi}$ of size λ^+, and a prescribed function $\tau : \mu \to \lambda^+$.

 In this case, appeal to Claim 3.6.1 with some pairwise disjoint subfamily $\mathcal{X}$ of $[\lambda^+]^{<\mathrm{cf}(\lambda)}$ of size λ^+ such that, for every $x \in \mathcal{X}$, there are $a \in \mathcal{A}$ and $b \in \mathcal{B}$ such that $x = a \uplus b$. In return, we obtain a stationary set $\Delta \subseteq \lambda^+$, a sequence $\langle x_\gamma \mid \gamma \in \Delta\rangle$ and an ordinal $\varepsilon < \lambda^+$. Then, for every $\gamma \in \Delta$, fix $a_\gamma \in \mathcal{A}$ and $b_\gamma \in \mathcal{B}$ such that $x_\gamma = a_\gamma \uplus b_\gamma$.

Next, let D be some club in λ^+ such that, for every $\delta \in D$ and $\alpha \in \Delta \cap \delta$, $\sup(x_\alpha) < \delta$. By shrinking D, we may also assume that $\min(D) > \sup(\mathrm{Im}(\tau))$. For every $\delta \in D \cap \Delta$, let ξ_δ denote the least ordinal $\xi < \lambda$ with

$$\sup\{\mathrm{otp}(C_{\mathrm{Tr}(\delta,\beta)(n)}) \mid \beta \in x_\delta, n < \rho_2(\delta,\beta)\} < \xi < \mathrm{otp}(C_\delta)$$

such that $C_\delta(\xi) > \sup(\mathrm{Im}(\tau))$. Fix $\xi < \lambda$ and $\gamma < \lambda^+$ for which $\Delta' := \{\delta \in \Delta \cap D \mid \xi_\delta = \xi \ \&\ C_\delta(\xi) = \gamma\}$ is stationary. As the prescribed τ is a function from μ to γ, for every $\epsilon < \mu$, we may fix $i_\epsilon < \lambda$ such that $e_\gamma(i_\epsilon) = \tau(\epsilon)$. Define a function $\tau' : \mu \to \lambda$ via $\tau'(\epsilon) := \pi^{-1}(i_\epsilon, \xi)$. As $\Delta' \subseteq D$ and $|\Delta'| = \lambda^+$, we infer that $\mathcal{A}' := \{a_\delta \mid \delta \in \Delta'\}$ is a subfamily of $\mathcal{A}$ of size λ^+. Likewise, in Case (2), we also have that $\mathcal{B}' := \{b_\delta \mid \delta \in \Delta'\}$ is a subfamily of $\mathcal{B}$ of size λ^+. So, in Case (1) (resp. Case (2)), we may fix $a, b \in \mathcal{A}'$ (resp. $a \in \mathcal{A}'$ and $b \in \mathcal{B}'$) with $a < b$ such that $c(\alpha,\beta) = \tau'(p(\alpha,\beta))$ for all $\alpha \in a$ and $\beta \in b$,

Claim 3.6.2. *Let* $(\alpha,\beta) \in a \times b$. *Then* $c^+(\alpha,\beta) = \tau(p(\alpha,\beta))$.

Proof. Denote $\epsilon := p(\alpha,\beta)$. By the definition of τ', $\pi(c(\alpha,\beta)) = \pi(\tau'(\epsilon)) = (i_\epsilon,\xi)$.

By the choice of b, let us fix $\delta \in \Delta'$ such that $b \subseteq x_\delta$. As $\beta \in x_\delta$ and $\alpha \in a \in \mathcal{A}'$,

$$\varepsilon < \alpha < \delta < \beta,$$

so that $\delta \in \mathrm{Im}(\mathrm{Tr}(\alpha,\beta))$. Now, by the same standard fact used in the proof of Claim 3.5.1, $\mathrm{Tr}(\alpha,\beta)(i) = \mathrm{Tr}(\delta,\beta)(i)$ for all $i < \rho_2(\delta,\beta)$, and $\mathrm{Tr}(\alpha,\beta)(\rho_2(\delta,\beta)) = \delta$. Recalling the choice of ξ, this means that $n := \rho_2(\delta,\beta)$ is the least integer to satisfy $\mathrm{otp}(C_{\mathrm{Tr}(\alpha,\beta)(n)}) > \xi$. So, by the definition of c^+, we infer that

$$c^+(\alpha,\beta) = e_{C_{\mathrm{Tr}(\alpha,\beta)(n)}(\xi)}(i) = e_{C_\delta(\xi)}(i_\epsilon) = e_\gamma(i_\epsilon) = \tau(\epsilon),$$

as sought. □

This completes the proof. □

4. Theorem B: Strengthening hi-dimensional colorings

In this short section, we prove Theorem B. Its proof follows Shelah's proof of the pump-up from Pr_1 to Pr_0 [She94, Lemma 4.5] and adds to it considerations to handle the partition. This answers [CKS21, Question 46] in the affirmative.

Theorem 4.1. *Suppose that κ is a regular uncountable cardinal and μ,λ,χ,θ are cardinals $\leq \kappa$. Assume $\lambda^{<\chi} < \kappa \leq 2^\lambda$ and $\lambda^{<\chi} \leq \theta^{<\chi} = \theta$.*

For every coloring $c_1 : [\kappa]^2 \to \theta$, there exists a corresponding coloring $c_0 : [\kappa]^2 \to \theta$ satisfying that for every partition $p : [\kappa]^2 \to \mu$:

1. *if c_1 witnesses $\mathrm{Pr}_1(\kappa,\kappa,\theta,\chi)_p$, then c_0 witnesses $\mathrm{Pr}_0(\kappa,\kappa,\theta,\chi)_p$;*
2. *if c_1 witnesses $\mathrm{Pr}_1(\kappa, {}^{\nu\circledast\kappa}\!/\!{}_{1\circledast\kappa},\theta,\chi)_p$, then c_0 witnesses $\mathrm{Pr}_0(\kappa, {}^{\nu\circledast\kappa}\!/\!{}_{1\circledast\kappa},\theta,\chi)_p$.*

Proof. As $\kappa \leq 2^\lambda$, we may fix an injective sequence $\langle X_\alpha \mid \alpha < \kappa\rangle$ of subsets of λ.

Claim 4.1.1. *For every $\sigma < \chi$ and $a \in [\kappa]^\sigma$, there are $y \in [\lambda]^{<\chi}$ and an injection $f : \sigma \to \mathcal{P}(y)$, such that, for all $\alpha \in a$, $X_\alpha \cap y = f(\mathrm{otp}(\alpha \cap a))$.*

Proof. For all $\alpha < \beta < \kappa$, let $\delta_{\alpha,\beta} := \min(X_\alpha \bigtriangleup X_\beta)$. Now, let $y := \{\delta(\alpha,\beta) \mid \alpha,\beta \in a, \alpha \neq \beta\}$ and then define a function $f : \sigma \to \mathcal{P}(y)$ via $f(i) := X_{a(i)} \cap y$. Evidently, y and f are as required. □

Consider the following set:

$$W := \left\{(y^0,y^1,\mathcal{Z},g) \mid y^0,y^1 \in [\lambda]^{<\chi},\ \mathcal{Z} \in [\mathcal{P}(y^0 \cup y^1)]^{<\chi} \text{ and } g : \mathcal{Z}\times\mathcal{Z} \to \theta\right\}.$$

It is clear that $|W| = \theta$, so let us fix an enumeration $\langle (y^0_j,y^1_j,\mathcal{Z}_j,g_j) \mid j < \theta\rangle$ of W.

For each $j < \theta$, define a function $h_j : [\kappa]^2 \to \theta$ via:

$$h_j(\alpha,\beta) := \begin{cases} g_j(X_\alpha \cap y^0_j, X_\beta \cap y^1_j) & \text{if } X_\alpha \cap y^0_j \in \mathcal{Z}_j \text{ and } X_\beta \cap y^1_j \in \mathcal{Z}_j;\\ 0 & \text{otherwise.}\end{cases}$$

Finally, given a coloring $c_1 : [\kappa]^2 \to \theta$, define the coloring $c_0 : [\kappa]^2 \to \theta$ via

$$c_0(\alpha,\beta) := h_{c_1(\alpha,\beta)}(\alpha,\beta).$$

(1) Suppose that c_1 witnesses $\mathrm{Pr}_1(\kappa,\kappa,\theta,\chi)_p$. To see that c_0 witnesses $\mathrm{Pr}_0(\kappa,\kappa,\theta,\chi)_p$ fix an arbitrary $\sigma < \chi$, a κ-sized pairwise disjoint family $\mathcal{A} \subseteq [\kappa]^\sigma$ and a matrix $(\tau_{\xi,\zeta})_{\xi,\zeta<\sigma}$ of functions from μ to θ. By Claim 4.1.1 and a pigeonhole argument, fix a set $y \in [\lambda]^{<\chi}$ and an injection $f : \sigma \to \mathcal{P}(y)$ such that, for all $a \in \mathcal{A}$ and $\alpha \in a$, $X_\alpha \cap y = f(\mathrm{otp}(\alpha \cap a))$.

Denote $\mathcal{Z} := \mathrm{Im}(f)$. For every $\epsilon < \mu$, define a function $g^\epsilon : \mathcal{Z} \times \mathcal{Z} \to \theta$ via

$$g^\epsilon(f(\xi), f(\zeta)) := \tau_{\xi,\zeta}(\epsilon).$$

Now pick $j_\epsilon < \theta$ such that $(y^0_{j_\epsilon}, y^1_{j_\epsilon}, \mathcal{Z}_{j_\epsilon}, g_{j_\epsilon}) = (y, y, \mathcal{Z}, g^\epsilon)$. Finally, define a function $\tau^* : \mu \to \theta$ via $\tau^*(\epsilon) := j_\epsilon$ and then pick $(a, b) \in [\mathcal{A}]^2$ such that $c_1(\alpha, \beta) = \tau^*(p(\alpha, \beta))$ for all $\alpha \in a$ and $\beta \in b$.

Claim 4.1.2. *Let $\xi, \zeta < \sigma$. Then $c_0(a(\xi), b(\zeta)) = \tau_{\xi,\zeta}(p(a(\xi), b(\zeta))$.*

Proof. Write $\epsilon := p(a(\xi), b(\zeta))$. Then

$$c_1(a(\xi), b(\zeta)) = \tau^*(p(a(\xi), b(\zeta))) = \tau^*(\epsilon) = j_\epsilon.$$

Altogether,

$$\begin{aligned} c_0(a(\xi), b(\zeta)) &= h_{j_\epsilon}(a(\xi), b(\zeta)) \\ &= g_{j_\epsilon}(X_{a(\xi)} \cap y^0_{j_\epsilon}, X_{b(\zeta)} \cap y^1_{j_\epsilon}) \\ &= g^\epsilon(f(\xi), f(\zeta)) = \tau_{\xi,\zeta}(\epsilon), \end{aligned}$$

as sought. □

(2) Suppose that c_1 witnesses $\mathrm{Pr}_1(\kappa, {}^{\nu\circledast\kappa}\!/\!{}_{1\circledast\kappa}, \theta, \chi)_p$. To see that c_0 witnesses $\mathrm{Pr}_0(\kappa, {}^{\nu\circledast\kappa}\!/\!{}_{1\circledast\kappa}, \theta, \chi)_p$, fix an arbitrary $\sigma < \chi$, a ν-sized pairwise disjoint family $\mathcal{A} \subseteq [\kappa]^\sigma$ and a κ-sized pairwise disjoint family $\mathcal{B} \subseteq [\kappa]^\sigma$. By Claim 4.1.1, we may assume the existence of a set $y^1 \in [\lambda]^{<\chi}$ and an injection $f^1 : \sigma \to \mathcal{P}(y^1)$, such that, for all $b \in \mathcal{B}$ and $\beta \in B$, $X_\beta \cap y^1 = f^1(\mathrm{otp}(\beta \cap b))$. Now, by the hypothesis on c_1, fix $a \in \mathcal{A}$ such that, for every $\tau : \mu \to \theta$, there exist $b \in \mathcal{B}$ with $a < b$ such that, for all $\alpha \in a$ and $\beta \in b$, $c_1(\alpha, \beta) = \tau(p(\alpha, \beta))$.

By Claim 4.1.1, fix $y^0 \in [\lambda]^{<\chi}$ and an injection $f^0 : \sigma \to \mathcal{P}(y^0)$ such that, for all $\alpha \in a$, $X_\alpha \cap y^0 = f^0(\mathrm{otp}(\alpha \cap a))$. Denote $\mathcal{Z} := \mathrm{Im}(f^0) \cup \mathrm{Im}(f^1)$.

Now, given a matrix $(\tau_{\xi,\zeta})_{\xi,\zeta<\sigma}$ of functions from μ to θ, for every $\epsilon < \mu$, pick any function $g^\epsilon : \mathcal{Z} \times \mathcal{Z} \to \theta$ satisfying that for all $\xi, \zeta < \sigma$:

$$g^\epsilon(f^0(\xi), f^1(\zeta)) = \tau_{\xi,\zeta}(\epsilon).$$

Then pick $j_\epsilon < \theta$ such that $(y^0_{j_\epsilon}, y^1_{j_\epsilon}, \mathcal{Z}_{j_\epsilon}, g_{j_\epsilon}) = (y^0, y^1, \mathcal{Z}, g^\epsilon)$. Finally, define a function $\tau^* : \mu \to \theta$ via $\tau^*(\epsilon) := j_\epsilon$ and then pick $b \in \mathcal{B}$ with $a < b$ such that $c_1(\alpha, \beta) = \tau^*(p(\alpha, \beta))$ for all $\alpha \in a$ and $\beta \in b$. The proof of Claim 4.1.2 makes clear that $c_0(a(\xi), b(\zeta)) = \tau_{\xi,\zeta}(p(a(\xi), b(\zeta)))$ for all $\xi, \zeta < \sigma$. □

5. Concluding remarks

The pump up theorems which are presented here furnish the toolbox for working in the theory of strong colorings over partitions with the basic tools which are available in the classical theory: stretching the number of colors and strengthening one high dimensional principle to a stronger one.

Also the spectrum of methods is expanded, with walks on ordinals making their first appearance here.

Together with the independence results of Parts I and III of this series and the absolute results of this part, Ramsey theory over partitions emerges now as a fruitful branch of Ramsey theory on uncountable cardinals.

Many more questions arise. We mention only a few of them.

Question 5.1. Can the restriction "$\chi \le \mathrm{cf}(\lambda)$" in Theorem A be waived?

Question 5.2. Is it consistent that some positive Ramsey relation holds over a small partition at a successor of a singular cardinal?

For a positive integer n and a partition $p : [\kappa]^n \to \mu$, define $\kappa \nrightarrow_p [\kappa]^n_\theta$ to assert that there exists a coloring $c : [\kappa]^n \to \theta$ such that for every $A \subseteq \kappa$ of full cardinality and every $\tau : \mu \to \theta$, there exists $x \in [A]^n$ such that $c(x) = \tau(p(x))$.

Question 5.3. Is it possible to extend Todorcevic's [Tod94] celebrated negative Ramsey relation $\aleph_2 \nrightarrow [\aleph_1]^3_{\aleph_0}$ to the new context? Dually, is it consistent that for some small partition p of $[\aleph_2]^3$, the positive Ramsey relation $\aleph_2 \to_p [\aleph_1]^3_{\aleph_0}$ holds?

Question 5.4. For which integers n and m and cardinals κ, θ, does $\kappa \to_p [\kappa]^n_\theta$ for some partition $p : [\kappa]^n \to \mu$ imply that $\kappa \to_{p^*} [\kappa]^m_\theta$ for some partition $p^* : [\kappa]^m \to \mu$?

For a successor cardinal $\kappa = \lambda^+$, we can show that the axiom $\text{⫯}(\lambda^+)$ entails a positive implication between all $n, m \geq 1$.

6. Acknowledgments

Kojman was partially supported by the Israel Science Foundation (grant agreement 665/20). Rinot was partially supported by the Israel Science Foundation (grant agreement 2066/18) and by the European Research Council (grant agreement ERC-2018-StG 802756). Steprāns was partially supported by NSERC of Canada.

The authors are grateful to an anonymous referee, whose thoughtful comments and suggestions helped to improve the presentation of this paper.

References

[CKS21] William Chen-Mertens, Menachem Kojman, and Juris Steprans. Strong colorings over partitions. *The Bulletin of Symbolic Logic*, 27(1):67–90, 2021.

[EHM66] P. Erdős, A. Hajnal, and E. C. Milner. On the complete subgraphs of graphs defined by systems of sets. *Acta Math. Acad. Sci. Hungar.*, 17:159–229, 1966.

[EHR65] P. Erdős, A. Hajnal, and R. Rado. Partition relations for cardinal numbers. *Acta Math. Acad. Sci. Hungar.*, 16:93–196, 1965.

[Eis13] Todd Eisworth. Getting more colors II. *J. Symbolic Logic*, 78(1):17–38, 2013.

[Gal80] Fred Galvin. Chain conditions and products. *Fund. Math.*, 108(1):33–48, 1980.

[HJ74] A. Hajnal and I. Juhász. On hereditarily α-Lindelöf and α-separable spaces. II. *Fund. Math.*, 81(2):147–158, 1973/74.

[KRS22a] Menachem Kojman, Assaf Rinot, and Juris Steprans. Advances in Ramsey theory over partitions I: Positive Ramsey relations from forcing axioms. *Israel J. Math.*, Accepted May 2022. `http://assafrinot.com/paper/49`.

[KRS22b] Menachem Kojman, Assaf Rinot, and Juris Steprans. Ramsey theory over partitions III: Strongly Luzin sets and partition relations. *Proc. Amer. Math. Soc.*, Accepted April 2022. `http://assafrinot.com/paper/55`.

[LHR18] Chris Lambie-Hanson and Assaf Rinot. Knaster and friends I: Closed colorings and precalibers. *Algebra Universalis*, 79(4):Art. 90, 39, 2018.

[Moo06] Justin Tatch Moore. A solution to the L space problem. *J. Amer. Math. Soc.*, 19(3):717–736 (electronic), 2006.

[Ram30] F.P. Ramsey. On a problem of formal logic. *Proc. London Math. Soc.*, pages 264–286, 1930.

[Rin14a] Assaf Rinot. Chain conditions of products, and weakly compact cardinals. *Bull. Symb. Log.*, 20(3):293–314, 2014.

[Rin14b] Assaf Rinot. Complicated colorings. *Math. Res. Lett.*, 21(6):1367–1388, 2014.

[Roi78] Judy Roitman. A reformulation of S and L. *Proc. Amer. Math. Soc.*, 69(2):344–348, 1978.

[RZ21a] Assaf Rinot and Jing Zhang. Strongest transformations. Submitted April 2021. http://assafrinot.com/paper/45.

[RZ21b] Assaf Rinot and Jing Zhang. Transformations of the transfinite plane. *Forum Math. Sigma*, 9, Paper No. e16, 2021.

[RZ22] Assaf Rinot and Jing Zhang. Complicated colorings, revisited. Submitted Februray 2022. http://assafrinot.com/paper/52.

[She88] Saharon Shelah. Successors of singulars, cofinalities of reduced products of cardinals and productivity of chain conditions. *Israel Journal of Mathematics*, 62:213–256, 1988.

[She90] Saharon Shelah. Strong negative partition above the continuum. *The Journal of Symbolic Logic*, 55:21–31, 1990.

[She91] Saharon Shelah. Strong negative partition relations below the continuum. *Acta Mathematica Hungarica*, 58:95–100, 1991.

[She94] Saharon Shelah. There are jonsson algebras in many inaccessible cardinals. In *Cardinal Arithmetic*, volume 29 of *Oxford Logic Guides*. Oxford University Press, 1994.

[She97] Saharon Shelah. Colouring and non-productivity of $\aleph_2$-cc. *Annals of Pure and Applied Logic*, 84:153–174, 1997.

[Sie33] Waclaw Sierpiński. Sur un problème de la théorie des relations. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (2)*, 2(3):285–287, 1933.

[Tod87] Stevo Todorčević. Partitioning pairs of countable ordinals. *Acta Math.*, 159(3-4):261–294, 1987.

[Tod94] Stevo Todorčević. Some partitions of three-dimensional combinatorial cubes. *J. Combin. Theory Ser. A*, 68(2):410–437, 1994.

[Tod07] Stevo Todorcevic. *Walks on ordinals and their characteristics*, volume 263 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 2007.

Department of Mathematics, Ben-Gurion University of the Negev, P.O.B. 653, Be'er Sheva, 84105 Israel

URL: https://www.math.bgu.ac.il/~kojman/

Department of Mathematics, Bar-Ilan University, Ramat-Gan 5290002, Israel.

URL: http://www.assafrinot.com

Department of Mathematics & Statistics, York University, 4700 Keele Street, Toronto, Ontario, Canada M3J 1P3

URL: http://www.math.yorku.ca/~steprans/