RAMSEY THEORY OVER PARTITIONS III: STRONGLY LUZIN SETS AND PARTITION RELATIONS

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Abstract. The strongest type of coloring of pairs of countable ordinals, gotten by Todorčević from a strongly Luzin set, is shown to be equivalent to the existence of a nonmeager set of reals of size $\aleph_1$. In the other direction, it is shown that the existence of both a strongly Luzin set and a coherent Souslin tree is compatible with the existence of a countable partition of pairs of countable ordinals such that no coloring is strong over it.

This clarifies the interaction between a gallery of coloring assertions going back to Luzin and Sierpinski a hundred years ago.

1. Introduction

Each of the following propositions is a consequence of Cantor's Continuum Hypothesis (CH):

(M) There is a nonmeager set of reals of size $\aleph_1$;

(L) There is an uncountable set of reals whose intersection with every meager set is countable;

(L*) There is an uncountable set of reals X such that, for every positive integer d, and every meager subset Y of $\mathbb{R}^d$, the intersection $Y \cap X^d$ contains no uncountable pairwise disjoint subfamily;[1]

(S) There is a sequence $\langle f_n \mid n \in \mathbb{N}\rangle$ of functions from from $\aleph_1$ to $\aleph_1$ such that for every uncountable $I \subseteq \aleph_1$, for all but finitely many n's, $f_n[I] = \aleph_1$;

(EHM) There is a coloring $c : [\aleph_1]^2 \rightarrow \aleph_1$ such that for every infinite $A \subseteq \aleph_1$ and every uncountable $B \subseteq \aleph_1$ there is $\alpha \in A$ such that $c[\{\alpha\} \times B] = \aleph_1$;

(G) There is a coloring $c : [\aleph_1]^2 \rightarrow 2$ such that for every uncountable pairwise disjoint family $\mathcal{B} \subseteq [\aleph_1]^{<\aleph_0}$ and every $\delta < 2$, there are $a, b \in \mathcal{B}$ with $\max(a) < \min(b)$ such that $c[a \times b] = \{\delta\}$;

(T) There is a coloring $c : [\aleph_1]^2 \rightarrow \aleph_1$ such that for all $k, l < \omega$, for every infinite pairwise disjoint family $\mathcal{A} \subseteq [\aleph_1]^k$ and every uncountable pairwise disjoint family $\mathcal{B} \subseteq [\aleph_1]^l$ there is $a \in \mathcal{A}$ such that for every function $f : k \times l \rightarrow \aleph_1$, there is $b \in \mathcal{B}$ such that $c(a(i), b(j)) = f(i, j)$ for all $(i, j) \in k \times l$.

(L) was derived from CH by Mahlo and independently by Luzin around 1913; such a set of reals is called a *Luzin set*. (L*) was derived by Todorčević [27, p. 51], and such a set was named *strongly Luzin*. (S) was derived by Sierpiński in 1932, and may be found in his monograph [24]. (EHM) was derived by Erdős, Hajnal and Milner in 1966 [7] and (G) was derived by Galvin in 1980 [8]. A special case of (T)

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[1] Two d-tuples $(p_1, \ldots, p_d)$ and $(q_1, \ldots, q_d)$ are understood to be *disjoint* iff $\{p_1, \ldots, p_d\} \cap \{q_1, \ldots, q_d\} \neq \emptyset$.

in which either k or l is 1 and the number of colors is just 2 was gotten from CH by Hajnal and Juhász [10] in their work on HFC and HFD spaces from the 1970's .

Evidently, (L*) $\implies$ (L) $\implies$ (M), (T) $\implies$ (G) and (T) $\implies$ (EHM) $\implies$ (S). In 1980, Shelah [22, 12] established that (M) $\not\Longrightarrow$ (L). In 1987, Todorčević [25, pp. 290–291] proved that (L) $\implies$ (S) $\iff$ (EHM). In 1989, Todorčević [27, Proposition 6.4] proved that (L*) $\implies$ (T). Recently, Miller [18] proved (S) $\implies$ (M) and Guzmán [9] proved (M) $\implies$ (S), establishing (M) $\iff$ (S).

The first main result of this paper expands this circle of equivalences:

Theorem A. *(M)* $\iff$ *(T).*

Proposition (T), which in the language of Definition 2.1 below is denoted

$$\mathrm{Pr}_0(\aleph_1, {}^{\aleph_0 \circledast \aleph_1}\!/_{1 \circledast \aleph_1}, \aleph_1, \aleph_0)$$

asserts the existence of an extremely strong coloring, yet one can ask for even more. The notion of a strong coloring *over a partition* p was introduced recently in [4], where it was shown that for every strong coloring $c : [\aleph_1]^2 \to \aleph_1$ there is a partition $p : [\aleph_1]^2 \to 2$ such that c is not strong over p. Nevertheless, by [4, Lemma 9], if the space of strong colorings which witness $\mathrm{Pr}_0(\aleph_1, {}^{\aleph_0 \circledast \aleph_1}\!/_{1 \circledast \aleph_1}, \aleph_1, \aleph_0)$ is non-empty, then for every partition $p : [\aleph_1]^2 \to \mu$ with $(\aleph_1)^\mu = \aleph_1$ there is a strong coloring which witnesses it over p; the existence of such a coloring is denoted by

$$\mathrm{Pr}_0(\aleph_1, {}^{\aleph_0 \circledast \aleph_1}\!/_{1 \circledast \aleph_1}, \aleph_1, \aleph_0)_p.$$

Altogether, by Theorem A, (M) implies $\mathrm{Pr}_0(\aleph_1, {}^{\aleph_0 \circledast \aleph_1}\!/_{1 \circledast \aleph_1}, \aleph_1, \aleph_0)_p$ for all partitions $p : [\aleph_1]^2 \to \theta$ with a finite θ, and CH implies $\mathrm{Pr}_0(\aleph_1, {}^{\aleph_0 \circledast \aleph_1}\!/_{1 \circledast \aleph_1}, \aleph_1, \aleph_0)_p$ also for $\theta = \aleph_0$. By Theorem 3.4 below, $\mathrm{Pr}_0(\aleph_1, {}^{\aleph_0 \circledast \aleph_1}\!/_{1 \circledast \aleph_1}, \aleph_1, \aleph_0)_p$ can hold for all $p : [\aleph_1]^2 \to \aleph_0$ also in models with an arbitrarily large continuum.

It is natural to ask, then, whether (M) implies $\mathrm{Pr}_0(\aleph_1, {}^{\aleph_0 \circledast \aleph_1}\!/_{1 \circledast \aleph_1}, \aleph_1, \aleph_0)_p$ for all countable p. We prove in Section 3 below:

Theorem A'. *(M) is equivalent to* $\mathrm{Pr}_0(\aleph_1, {}^{\aleph_0 \circledast \aleph_1}\!/_{1 \circledast \aleph_1}, \aleph_1, \aleph_0)_p$ *for all* ℓ_∞*-coherent partitions* $p : [\aleph_1]^2 \to \aleph_0$.

This leaves open the question whether Theorem A' can be extended to cover all countable partitions. The second main result of this paper shows that this is not the case. In fact, (L*) does not even imply $\aleph_1 \to_p [\aleph_1]^2_{\aleph_0}$ for all countable p, whereas without the p this relation holds in ZFC by Todorčević's celebrated theorem [25].

Theorem B. *It is consistent that (L*) holds and there is a partition* $p : [\aleph_1]^2 \to \aleph_0$ *such that the positive Ramsey relation* $\aleph_1 \to_p (\aleph_1)^2_{\aleph_0}$ *holds.*

Theorem B is proved in Section 4. The model witnessing the theorem is obtained by forcing over a ground model of CH to which the partition p, whose existence is equivalent to the statement $\mathfrak{d} = \aleph_1$, belongs. In the forcing extension there exist a strongly Luzin set and a coherent Souslin tree, but every coloring $c : [\aleph_1]^2 \to \aleph_0$ is *p-special*: there is a decomposition $\langle X_i \mid i < \omega \rangle$ of $\aleph_1$ into (p, c)-homogeneous sets, that is, for each $i < \omega$, $p(\alpha, \beta)$ determines $c(\alpha, \beta)$ for all $(\alpha, \beta) \in [X_i]^2$.

2. Strong colorings and partitions

Surveys of the rich theory of strong colorings that was developed since Sierpiński's time to the present may be found in the introductions to [21, 4]. For the scope of this paper, we just need the following.

Definition 2.1 ([14])**.** Let $p : [\omega_1]^2 \to \omega$ be a partition. For cardinals $\chi \le \omega$ and $\theta \le \omega_1$, a coloring $c : [\omega_1]^2 \to \theta$ is said to witness

- $\mathrm{Pr}_1(\omega_1, \omega_1, \theta, \chi)_p$ iff for every uncountable pairwise disjoint subfamily $\mathcal{A} \subseteq [\omega_1]^n$, every $n < \chi$, and every function $\tau : \omega \to \theta$ there are $a, b \in \mathcal{A}$ with $a < b$ such that
$$c(\alpha, \beta) = \tau(p(\alpha, \beta)) \text{ for all } \alpha \in a \text{ and } \beta \in b;$$
- $\mathrm{Pr}_1(\omega_1, {}^{\omega\circledast\omega_1}\!/\!_{1\circledast\omega_1}, \theta, \chi)_p$ iff for every pairwise disjoint subfamilies $\mathcal{A}, \mathcal{B}$ of $[\omega_1]^n$ with $|\mathcal{A}| = \omega$, $|\mathcal{B}| = \omega_1$ and $n < \chi$ there is $a \in \mathcal{A}$ such that for every function $\tau : \omega \to \theta$, there is $b \in \mathcal{B}$ with $a < b$ such that
$$c(\alpha, \beta) = \tau(p(\alpha, \beta)) \text{ for all } \alpha \in a \text{ and } \beta \in b;$$
- $\mathrm{Pr}_0(\omega_1, {}^{\omega\circledast\omega_1}\!/\!_{1\circledast\omega_1}, \theta, \chi)_p$ iff for every pairwise disjoint subfamilies $\mathcal{A}, \mathcal{B}$ of $[\omega_1]^n$ with $|\mathcal{A}| = \omega$, $|\mathcal{B}| = \omega_1$ and $n < \chi$, there is $a \in \mathcal{A}$ such that for every matrix $(\tau_{i,j})_{i,j<n}$ of functions from ω to θ, there is $b \in \mathcal{B}$ with $a < b$ such that
$$c(a(i), b(j)) = \tau_{i,j}(p(a(i), b(j))) \text{ for all } i, j < n.$$

Remark 2.2. Here $a(i)$ stands for the i^{th} element of a.

Definition 2.3. For a partition $p : [\omega_1]^2 \to \omega$:

- p has *injective fibers* iff $p(\alpha, \beta) \neq p(\alpha', \beta)$ for all $\alpha < \alpha' < \beta$;
- p has *finite-to-one fibers* iff $\{\alpha < \beta \mid p(\alpha, \beta) = j\}$ is finite for all $\beta < \omega_1$ and $j < \omega$;
- p has *almost-disjoint fibers* iff $\{p(\alpha, \beta) \mid \alpha < \beta\} \cap \{p(\alpha, \beta') \mid \alpha < \beta\}$ is finite for all $\beta < \beta' < \omega_1$;
- p has *Cohen fibers* iff for every injection $g : a \to \omega$ with $a \in [\omega_1]^{<\omega}$, there are cofinally many $\beta < \omega_1$ such that $g(\alpha) = p(\alpha, \beta)$ for all $\alpha \in a$;
- p is *coherent* iff $\{\alpha < \beta \mid p(\alpha, \beta) \neq p(\alpha, \beta')\}$ is finite for all $\beta < \beta' < \omega_1$;
- p is *ℓ_∞-coherent* iff $\{p(\alpha, \beta) - p(\alpha, \beta') \mid \alpha < \beta\}$ is finite for all $\beta < \beta' < \omega_1$.

Remark 2.4. An example of an ℓ_∞-coherent partition which is not coherent is the map $\rho_2 : [\omega_1]^2 \to \omega$ from the theory of *walks on ordinals* [25, p. 269].

For many cardinal characteristics $\mathfrak{x}$ of the continuum, the assertion "$\mathfrak{x} = \aleph_1$" may be reformulated as a statement about the existence of a partition $p : [\omega_1]^2 \to \omega$ with certain properties. In Section 4 we shall need following reformulation of "$\mathfrak{d} = \aleph_1$".

Lemma 2.5. *$\mathfrak{d} = \aleph_1$ iff there exists a partition $p : [\omega_1]^2 \to \omega$ with injective, almost-disjoint and Cohen fibers which satisfies the following:*

For every function $h : \epsilon \to \omega$ with $\epsilon < \omega_1$ there exists $\gamma < \omega_1$ such that for every $b \in [\omega_1 \setminus \gamma]^{<\aleph_0}$ there exists $\Delta \in [\epsilon]^{<\aleph_0}$ such that:

- *for all $\alpha \in \epsilon \setminus \Delta$ and $\beta \in b$, $h(\alpha) < p(\alpha, \beta)$;*
- *$p \restriction ((\epsilon \setminus \Delta) \times b)$ is injective.*

Proof. For the backwards implication, derive an ω_1-sized cofinal family $\{r_\beta \mid \omega \le \beta < \omega_1\}$ in $({}^\omega\omega, <^*)$ by letting $r_\beta(n) := p(n, \beta)$.

We turn now to the forward implication. Fix a coherent $q : [\omega_1]^2 \to \omega$ having injective fibers (see, e.g., [15, Theorem 5.9]). Fix an enumeration $\langle g_\beta \mid \beta < \omega_1 \rangle$ of all injections g with $\mathrm{dom}(g) \in [\omega_1]^{<\aleph_0}$ and $\mathrm{Im}(g) \subseteq \omega$ in which each such injection occurs cofinally often. For each $\beta<\omega_1$, let $m_\beta := \sup(\mathrm{Im}(g_\beta))+1$. Fix a bijection $\pi:\omega\times\omega\leftrightarrow\omega$. Derive a function $\psi:\omega\rightarrow\omega$ via

$$\psi(m) := \max\{i<\omega \mid \exists j<\omega\,(\pi(i,j)\le m)\}.$$

Using $\mathfrak{d}=\aleph_1$, it is easy to construct recursively a sequence $\vec{d}=\langle d_\beta \mid \beta<\omega_1\rangle$ such that $\vec{d}$ is increasing and cofinal in $({}^\omega\omega,<^*)$, and, for every $\beta<\omega_1$, $\min(\mathrm{Im}(d_\beta))>\psi(m_\beta)$. Finally, define a partition $p:[\omega_1]^2\rightarrow\omega$ via:

$$p(\alpha,\beta) := \begin{cases} g_\beta(\alpha) & \text{if } \alpha\in\mathrm{dom}(g_\beta);\\ \pi(d_\beta(q(\alpha,\beta)),q(\alpha,\beta)) & \text{otherwise.}\end{cases}$$

Claim 2.5.1. *Let $\alpha<\beta<\omega_1$. Then $\alpha\in\mathrm{dom}(g_\beta)$ iff $p(\alpha,\beta)\in\mathrm{Im}(g_\beta)$.*

Proof. The forward implication is clear, so suppose that $\alpha\notin\mathrm{dom}(g_\beta)$, and set $j:=q(\alpha,\beta)$. By the choice of d_β, $i:=d_\beta(j)$ is greater than $\psi(m_\beta)$, and hence $\pi(i,j)>m_\beta>\sup(\mathrm{Im}(g_\beta))$. Altogether, $p(\alpha,\beta)=\pi(i,j)>\sup(\mathrm{Im}(g_\beta))$. □

As π is injective, q has injective fibers and each g_β is injective, it follows that p has injective fibers. It is also clear that p has Cohen fibers.

Claim 2.5.2. *p has almost-disjoint fibers.*

Proof. Fix an arbitrary pair $(\beta,\beta')\in[\omega_1]^2$ and consider the set

$$A := \{p(\alpha,\beta) \mid \alpha<\beta\}\cap\{p(\alpha,\beta') \mid \alpha<\beta\}.$$

Evidently, $|A|\le m_\beta+m_{\beta'}+|\{n<\omega \mid d_\beta(n)=d_{\beta'}(n)\}|<\omega$. □

To see that p is as sought, fix arbitrary ordinal $\epsilon<\omega_1$ and function $h:\epsilon\rightarrow\omega$. As q has coherent fibers, for every $\beta<\omega_1$ above ϵ, the following set is finite

$$A^0_\beta := \{\alpha<\epsilon \mid q(\alpha,\epsilon)\neq q(\alpha,\beta)\}.$$

Define a real $r:\omega\rightarrow\omega$ via

$$r(n) := \begin{cases} 0 & \text{if } \forall\alpha<\epsilon\,(q(\alpha,\epsilon)\neq n);\\ \psi(h(\alpha)) & \text{if } q(\alpha,\epsilon)=n.\end{cases}$$

Find a large enough ordinal $\gamma<\omega_1$ such that $\epsilon<\gamma$ and $r<^* d_\beta$ for every $\beta\in[\gamma,\omega_1)$. Now, let $b\in[\omega_1\setminus\gamma]^{<\aleph_0}$ be arbitrary. As q has injective fibers, for every $\beta\in[\gamma,\omega_1)$, the following set is finite

$$A^1_\beta := \{\alpha<\epsilon \mid r(q(\alpha,\beta))\ge d_\beta(q(\alpha,\beta))\}.$$

As $\vec{d}$ is $<^*$-increasing, we may find some $m^*<\omega$ such that, for all $n\in[m^*,\omega)$ and $(\beta,\beta')\in[b]^2$, $d_\beta(n)<d_{\beta'}(n)$. Now, as q has injective fibers, it follows that the following set is finite:

$$\Delta := \bigcup_{\beta\in b}(A^0_\beta\cup A^1_\beta\cup\mathrm{dom}(g_\beta)\cup\{\alpha<\epsilon \mid q(\alpha,\beta)<m^*\}).$$

Claim 2.5.3. (1) *for all $\alpha\in\epsilon\setminus\Delta$ and $\beta\in b$, $h(\alpha)<p(\alpha,\beta)$;*
(2) *$p\restriction((\epsilon\setminus\Delta)\times b)$ is injective.*

Proof. (1) Let $\alpha \in \epsilon \setminus \Delta$ and $\beta \in b$. Set $n := q(\alpha, \epsilon)$. As $\alpha \in \epsilon \setminus A^0_\beta$, $q(\alpha, \beta) = n$, so that $\psi(h(\alpha)) = r(n) = r(q(\alpha, \beta))$. As $\alpha \in \epsilon \setminus A^1_\beta$, $r(q(\alpha, \beta)) < d_\beta(q(\alpha, \beta))$. Altogether, $\psi(h(\alpha)) < d_\beta(q(\alpha, \beta))$ and hence $\pi(d_\beta(q(\alpha, \beta)), j) > h(\alpha)$ for all $j < \omega$. In particular, since $\alpha \notin \mathrm{dom}(g_\beta)$, $p(\alpha, \beta) = \pi(d_\beta(q(\alpha, \beta)), q(\alpha, \beta)) > h(\alpha)$.

(2) Fix $(\alpha, \beta), (\alpha', \beta') \in (\epsilon \setminus \Delta) \times b$ with $p(\alpha, \beta) = p(\alpha', \beta')$. If $\beta = \beta'$, then since p has injective injective fibers, $\alpha = \alpha'$ and we are done. So, suppose that $\beta \neq \beta'$, say, $\beta < \beta'$. Denote $(k, n) := (d_\beta(q(\alpha, \beta)), q(\alpha, \beta))$. As $p(\alpha, \beta) = p(\alpha', \beta')$, $\alpha \notin \mathrm{dom}(g_\beta)$ and $\alpha' \notin \mathrm{dom}(g_{\beta'})$, it follows that $(d_{\beta'}(q(\alpha', \beta')), q(\alpha', \beta')) = (k, n)$. In particular, $d_\beta(n) = d_{\beta'}(n)$. As $\alpha \in \epsilon \setminus \Delta$, we infer that $n \geq m^*$, so $d_\beta(n) < d_{\beta'}(n)$. This is a contradiction. □

This completes the proof. □

3. Strong colorings from a nonmeager set

In the next Theorem, which proves Theorems A and A': Clause (1) is proposition (M). Clause (2) is a syntactic weakening of proposition (S), but addressing a concern raised by Bagemihl and Sprinkle [2], it was shown by Miller [18] to be equivalent to it. Clause (3) is a high-dimensional version of Clause (2). Clause (4) asserts the existence of a coloring $c : [\omega_1]^2 \to \omega_1 \times \omega$ for which the map $(\alpha, \beta) \mapsto \delta$ iff $\exists \iota[c(\alpha, \beta) = (\delta, \iota)]$ witnesses $\mathrm{Pr}_1(\aleph_1, {}^{\aleph_0 \circledast \aleph_1}/_{1 \circledast \aleph_1}, \aleph_1, \aleph_0)$ of Definition 2.1 and c itself has finite-to-one fibers. Clause (5) is proposition (T) over ℓ_∞-coherent partitions. Clause (6) is slightly weaker than proposition (EHM). The implication (7) $\implies$ (1) is due to Miller [17] and the implication (1) $\implies$ (2) is due to Guzmán [9].

Theorem 3.1. *All of the following are equivalent:*

(1) $\mathrm{non}(\mathcal{M}) = \aleph_1$;

(2) *There exists a sequence* $\vec{f} = \langle f_m \mid m < \omega \rangle$ *of functions from* ω_1 *to* ω_1 *satisfying that for every cofinal subset* $B \subseteq \omega_1$ *there exists* $m < \omega$ *such that* $f_m[B] = \omega_1$;

(3) *There exists a sequence* $\vec{g} = \langle g_n \mid n < \omega \rangle$ *of functions from* ω_1 *to* ω_1 *satisfying that for every uncountable pairwise disjoint subfamily* $\mathcal{B} \subseteq [\omega_1]^{<\aleph_0}$ *there are infinitely many* $n < \omega$ *such that for every* $\gamma < \omega_1$, *for some* $b \in \mathcal{B}$, $g_n[b] = \{\gamma\}$;

(4) *There exists a coloring* $c : [\omega_1]^2 \to \omega_1 \times \omega$ *with finite-to-one fibers, such that for every*
   - $k < \omega$ *and an infinite pairwise disjoint subfamily* $\mathcal{A} \subseteq [\omega_1]^k$
   - $l < \omega$ *and an uncountable pairwise disjoint subfamily* $\mathcal{B} \subseteq [\omega_1]^l$

   there exists $a \in \mathcal{A}$ *such that for every* $\delta < \omega_1$ *there are* $\iota < \omega$ *and* $b \in \mathcal{B}$ *such that*
$$\{\alpha < \beta \mid c(\alpha, \beta) = (\delta, \iota)\} = a \text{ for every } \beta \in b;$$

(5) *For every* ℓ_∞*-coherent partition* $p : [\omega_1]^2 \to \omega$, *there exists a corresponding coloring* $d : [\omega_1]^2 \to \omega_1$ *satisfying that for every*
   - $k < \omega$ *and an infinite pairwise disjoint subfamily* $\mathcal{A} \subseteq [\omega_1]^k$
   - $l < \omega$ *and an uncountable pairwise disjoint subfamily* $\mathcal{B} \subseteq [\omega_1]^l$

   there exists $a \in \mathcal{A}$ *such that for every matrix* $\langle \tau_{n,m} \mid n < k, m < l \rangle$ *of functions from* ω *to* ω_1 *there exists* $b \in \mathcal{B}$ *such that*
$$d(a(n), b(m)) = \tau_{n,m}(p(a(n), b(m))) \text{ for all } n < k \text{ and } m < l;$$

(6) *There exists a coloring* $e : [\omega_1]^2 \to \omega$ *such that for every infinite* $A \subseteq \omega_1$ *and uncountable* $B \subseteq \omega_1$, *there is* $\alpha \in A$ *such that* $\{e(\alpha,\beta) \mid \beta \in B \setminus (\alpha+1)\} = \omega$;
(7) *There exists a subset* $X \subseteq {}^\omega\omega$ *of size* $\aleph_1$ *with the property that for every real* $y : \omega \to \omega$, *for some* $x \in X$, $x \cap y$ *is infinite.*

Proof. For the rest of the proof we fix a bijection $\pi : \omega \leftrightarrow \omega \times \omega$ and, by the Engelking-Karlowicz theorem [5], we fix a sequence $\langle h_j \mid j < \omega \rangle$ of functions from ω_1 to ω such that for every set $x \in [\omega_1]^{<\aleph_0}$ and a function $h : x \to \omega$ there exists $j < \omega$ such that $h \subseteq h_j$.

(1) $\implies$ (2): This is Proposition 2.2 of [9].

(2) $\implies$ (3): Let $\vec{f}$ witness Clause (2). For every $\beta < \omega_1$ fix a surjection $e_\beta : \omega \to \beta + 1$. Define a sequence $\vec{g} = \langle g_n \mid n < \omega \rangle$ of functions from ω_1 to ω_1, as follows. Given $n < \omega$, let $(m, j) := \pi(n)$ and for every $\beta < \omega_1$ set

$$g_n(\beta) := f_m(e_\beta(h_j(\beta))).$$

To see that $\vec{g}$ witnesses Clause (3), fix an arbitrary uncountable pairwise disjoint subfamily $\mathcal{B} \subseteq [\omega_1]^{<\aleph_0}$ and some $k < \omega$. We shall find an integer $n > k$ such that, for every $\gamma < \omega_1$ there is some $b \in \mathcal{B}$ such that $g_n[b] = \{\gamma\}$.

For every $b \in \mathcal{B}$, define a function $h^b : b \to \omega$ via:

$$h^b(\beta) := \min\{i < \omega \mid e_\beta(i) = \min(b)\}.$$

Fix $j' < \omega$ for which $\mathcal{B}' := \{b \in \mathcal{B} \mid h^b \subseteq h_{j'}\}$ is uncountable, and then let

$$m' := \max(\{0\} \cup \{m < \omega \mid \pi^{-1}(m, j') \leq k\}).$$

Evidently, $B_0 := \{\min(b) \mid b \in \mathcal{B}'\}$ is uncountable. Next, for every $i \leq m'$ such that B_i has already been defined, proceed as follows:

▶ If $Z_i := \{\beta \in B_i \mid f_i(\beta) \neq 0\}$ is uncountable, then let $B_{i+1} := Z_i$;

▶ Otherwise, let $B_{i+1} := B_i \setminus Z_i$.

In either case, $B_{i+1} \subseteq B_i$ is uncountable with $f_i[B_{i+1}] \neq \omega_1$.

Finally, as $B_{m'+1}$ is uncountable, let us pick, by the choice of $\vec{f}$, an integer $m < \omega$ such that $f_m[B_{m'+1}] = \omega_1$.

For all $i \leq m'$, $f_i[B_{m'+1}] \subseteq f_i[B_{i+1}] \subsetneq \omega_1$, so $m > m'$. In particular, $n := \pi(m, j')$ is larger than k. To see that n is as sought, let $\gamma \in \omega_1 = f_m[B_{m'+1}]$ be arbitrary. Pick $\beta' \in B_{m'+1}$ with $f_m(\beta') = \gamma$. As $\beta' \in B_{m'+1} \subseteq B_0$, let us pick $b \in \mathcal{B}$ such that $h^b \subseteq h_{j'}$ and $\min(b) = \beta'$. Let $\beta \in b$ be arbitrary. Then

$$g_n(\beta) = f_m(e_\beta(h_{j'}(\beta))) = f_m(e_\beta(h^b(\beta))) = f_m(\beta') = \gamma.$$

So $g_n[b] = \{\gamma\}$, as required.

(3) $\implies$ (4): The proof here is inspired by Miller's proof of [18, Proposition 4]. Define an eventually increasing sequence of integers $\langle m_n \mid n < \omega \rangle$ by recursion, setting $m_0 := 1$, and $m_{n+1} := n! \cdot (\sum_{i \leq n} m_i)$ for every $n < \omega$. For every $n < \omega$, let $\Phi_n := \bigcup\{{}^x(\omega_1 \times \omega) \mid x \subseteq \omega_1, |x| = m_n\}$. Evidently, $|\Phi_n| = \omega_1$, so we may fix an injective enumeration $\langle \phi_n^\gamma \mid \gamma < \omega_1 \rangle$ of Φ_n.

Let $\vec{g}$ witness Clause (3). Define a coloring $d : [\omega_1]^2 \to (\omega_1 \times \omega) \times \omega$ by letting for all $\alpha < \beta < \omega_1$:

$$d(\alpha, \beta) := \begin{cases} ((\alpha, 0), 0) & \text{if } \alpha \notin \bigcup_{i<\omega} \operatorname{dom}(\phi_i^{g_i(\beta)}); \\ (\phi_n^{g_n(\beta)}(\alpha), n+1) & \text{if } n = \min\{i < \omega \mid \alpha \in \operatorname{dom}(\phi_i^{g_i(\beta)})\}. \end{cases}$$

Finally, define a coloring $c : [\omega_1]^2 \to \omega_1 \times \omega$ by letting $c(\alpha, \beta) := (\gamma, \pi(\iota, n))$ iff $d(\alpha, \beta) = ((\gamma, \iota), n)$. It is clear that d has finite-to-one fibers, and hence so does c.

To see that c witnesses Clause (4), fix positive integers k, l along with $\mathcal{A}, \mathcal{B}$ such that:

- $\mathcal{A}$ is an infinite pairwise disjoint subfamily of $[\omega_1]^k$,
- $\mathcal{B}$ is an uncountable pairwise disjoint subfamily of $[\omega_1]^l$, and
- $a < b$ for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$.

By the choice of $\vec{g}$, let us fix an integer $n > \max\{k, l\}$ such that, for every $\gamma < \omega_1$, for some $b \in \mathcal{B}$, $g_{n+1}[b] = \{\gamma\}$. As m_{n+1} is divisible by k, we now fix an injective sequence $\langle a_\iota \mid \iota < \frac{m_{n+1}}{k}\rangle$ consisting of elements of $\mathcal{A}$.

Claim 3.1.1. *There exists $\iota < \frac{m_{n+1}}{k}$ such that, for every $\delta < \omega_1$, there is $b \in \mathcal{B}$, such that, for every $\beta \in b$:*

$$\{\alpha < \beta \mid c(\alpha, \beta) = (\delta, \pi(\iota, n+2))\} = a_\iota.$$

Proof. Suppose not. Then, for every $\iota < \frac{m_{n+1}}{k}$, we may find some $\delta_\iota < \omega_1$ such that for all $b \in \mathcal{B}$, for some $\beta \in b$,

$$\{\alpha < \beta \mid d(\alpha, \beta) = ((\delta_\iota, \iota), n+2)\} \neq a_\iota.$$

Define a function $\phi : \biguplus\{a_\iota \mid \iota < \frac{m_{n+1}}{k}\} \to \omega_1 \times \omega$ by letting $\phi(\alpha) := (\delta_\iota, \iota)$ iff $\alpha \in a_\iota$. As $|\biguplus\{a_\iota \mid \iota < \frac{m_{n+1}}{k}\}| = \frac{m_{n+1}}{k} \cdot k = m_{n+1}$, we infer that $\phi \in \Phi_{n+1}$, so we may fix $\gamma < \omega_1$ such that $\phi = \phi^\gamma_{n+1}$. Now, pick $b \in \mathcal{B}$ with $g_{n+1}[b] = \{\gamma\}$.

For every $i \leq n$ and $\beta \in b$, let $x^\beta_i := \mathrm{dom}(\phi^{g_i(\beta)}_i)$, so that $|x^\beta_i| = m_i$. Next, set $x := \bigcup\{x^\beta_i \mid i \leq n, \beta \in b\}$. As $|b| = l$, we infer that $|x| \leq l \cdot \sum_{i \leq n} m_i$. Thus

$$k \cdot |x| \leq k \cdot l \cdot \sum_{i \leq n} m_i < n! \cdot \sum_{i \leq n} m_i = m_{n+1}.$$

In particular, $|x| < \frac{m_{n+1}}{k}$, so we may fix $\iota < \frac{m_{n+1}}{k}$ such that $a_\iota \cap x = \emptyset$.

Let $\beta \in b$ be arbitrary. Consider the set

$$A := \{\alpha < \beta \mid d(\alpha, \beta) = ((\delta_\iota, \iota), n+2)\}.$$

As $g_{n+1}(\beta) = \gamma$, we infer that $\phi^{g_{n+1}(\beta)}_{n+1} = \phi$, so, by the definition of d:

$$A \subseteq \{\alpha < \beta \mid \phi^{g_{n+1}(\beta)}_{n+1}(\alpha) = (\delta_\iota, \iota)\} \subseteq \{\alpha < \beta \mid \phi(\beta)(\alpha) = (\delta_\iota, \iota)\} = a_\iota.$$

On the other hand, for every $\alpha \in a_\iota \subseteq \mathrm{dom}(\phi^{g_{n+1}(\beta)}_{n+1})$, as $\alpha \notin x$, it follows that $\min\{i < \omega \mid \alpha \in \mathrm{dom}(\phi^{g_i(\beta)}_i)\} = n+1$, and hence

$$d(\alpha, \beta) = (\phi^{g_{n+1}(\beta)}_{n+1}(\alpha), n+2) = (\phi(\alpha), n+2) = ((\delta_\iota, \iota), n+2),$$

so that $\alpha \in A_i$. Altogether, $A = a_\iota$, contradicting the choice of δ_ι. $\square$

(4) $\implies$ (5): Fix c witnessing Clause (4). Let $\langle \eta_\gamma \mid \gamma < \omega_1\rangle$ be some injective enumeration of $\bigcup\{{}^{k \times l \times t}\omega_1 \mid k, l, t < \omega\}$ and let $\langle (i_\delta, j_\delta, \gamma_\delta) \mid \delta < \omega_1\rangle$ be some injective enumeration of $\omega \times \omega \times \omega_1$,

Now, given any ℓ_∞-coherent partition $p : [\omega_1]^2 \to \omega$, define a coloring $d : [\omega_1]^2 \to \omega_1$ as follows. Given $(\alpha, \beta) \in [\omega_1]^2$, let $(\delta, \iota) := c(\alpha, \beta)$ and then set

$$d(\alpha, \beta) := \begin{cases} \eta_{\gamma_\delta}(h_{i_\delta}(\alpha), h_{j_\delta}(\beta), p(\alpha, \beta)) & \text{if } (h_{i_\delta}(\alpha), h_{j_\delta}(\beta), p(\alpha, \beta)) \in \mathrm{dom}(\eta_{\gamma_\delta}) \\ 0 & \text{otherwise.} \end{cases}$$

To see that d witnesses Clause (5), fix $k, l, \mathcal{A}, \mathcal{B}$ and $\epsilon < \omega_1$ such that:

- $\mathcal{A}$ is an infinite pairwise disjoint subfamily of $[\omega_1]^k$,

- $\mathcal{B}$ is an uncountable pairwise disjoint subfamily of $[\omega_1]^l$ and
- $\max(a) < \epsilon \leq \min(b)$ for all $a \in \mathcal{A}$ and all $b \in \mathcal{B}$.

For every $x \in \mathcal{A} \cup \mathcal{B}$, define a function $h^x : x \to \omega$ via:

$$h^x(\beta) := \mathrm{otp}(x \cap \beta).$$

Now pick $j' < \omega$ for which $\mathcal{B}' := \{b \in \mathcal{B} \mid h^b \subseteq h_{j'}\}$ is uncountable. As p is ℓ_∞-coherent, we may shrink $\mathcal{B}'$ further and assume the existence of some $q < \omega$ such that, for all $b \in \mathcal{B}'$:

$$\{|p(\alpha, \epsilon) - p(\alpha, \beta)| \mid \beta \in b\} \subseteq q.$$

Now, as $|\mathcal{A}| = \aleph_0$ and $|\mathcal{B}'| = \aleph_1$, by the choice of c, we may fix $a \in \mathcal{A}$ such that, for every $\delta < \omega_1$, there are $b \in \mathcal{B}'$ and $\iota < \omega$ such that $c[a \times b] = \{(\delta, \iota)\}$.

Claim 3.1.2. *Let $\langle \tau_{n,m} \mid n < k, m < l\rangle$ be a matrix of functions from ω to ω_1. Then there exists $b \in \mathcal{B}'$ satisfying that, for all $n < k$ and $m < l$,*

$$d(a(n), b(m)) = \tau_{n,m}(p(a(n), b(m))).$$

Proof. Fix $i' < \omega$ such that $h^a \subseteq h_{i'}$. Let $t := \max\{p(\alpha, \epsilon) + q \mid \alpha \in a\}$. Define a function $\eta : k \times l \times t \to \omega_1$ via:

$$\eta(n, m, s) := \tau_{n,m}(s).$$

Let $\delta < \omega_1$ be such that $(i_\delta, j_\delta, \eta_{\gamma_\delta}) = (i', j', \eta)$. Pick $b \in \mathcal{B}'$ and $\iota < \omega$ such that $c[a \times b] = \{(\delta, \iota)\}$. Now, given $n < k$ and $m < l$, we have $c(a(n), b(m)) = (\delta, \iota)$, $p(a(n), b(m)) < p(a(n), \epsilon) + q \leq t$, so that

$$\begin{aligned} d(a(n), b(m)) &= \eta_{\gamma_\delta}(h_{i_\delta}(a(n)), h_{j_\delta}(b(m)), p(a(n), b(m))) \\ &= \eta(h^a(a(n)), h^b(b(m)), p(a(n), b(m))) \\ &= \eta(n, m, p(a(n), b(m))) \\ &= \tau_{n,m}(p(a(n), b(m))), \end{aligned}$$

as sought. □

(5) $\implies$ (6): Let d witness Clause (5) with respect to the constant partition $p : [\omega_1]^2 \to 1$. Define a function $e : [\omega_1]^2 \to \omega$ by letting $e(\alpha, \beta) := d(\alpha, \beta)$ whenever $d(\alpha, \beta) < \omega$, and $e(\alpha, \beta) := 0$, otherwise. Clearly, e witnesses (6).

(6) $\implies$ (7): Let e witness Clause (6). Define $X = \{x_\beta \mid \beta < \omega_1\}$, as follows. For every $\beta < \omega_1$, define a function $x_\beta : \omega \to \omega$ via $x_\beta(n) := e(n, \beta)$. Towards a contradiction, suppose that $y : \omega \to \omega$ is a counterexample. It follows that there exists a large enough $n < \omega$ for which $B := \{\beta < \omega_1 \mid \mathrm{dom}(x_\beta \cap y) \subseteq n\}$ is uncountable. By the choice of e, we may now fix an integer $\alpha > n$ such that $\{e(\alpha, \beta) \mid \beta \in B \setminus (\alpha + 1)\} = \omega$. In particular, we may find $\beta \in B$ such that $e(\alpha, \beta) = y(\alpha)$. Altogether, $x_\beta(\alpha) = e(\alpha, \beta) = y(\alpha)$ contradicting the fact that $\beta \in B$ and $\alpha > n$.

(7) $\implies$ (1): By Theorem 1.3 of [17]. □

Corollary 3.2 (Theorem A'). *$non(\mathcal{M}) = \aleph_1$ iff $\mathrm{Pr}_0(\aleph_1, {}^{\aleph_0 \circledast \aleph_1}\!/\!_{1 \circledast \aleph_1}, \aleph_1, \aleph_0)_p$ holds for all ℓ_∞-coherent partitions $p : [\omega_1]^2 \to \omega$.* □

Corollary 3.3. *In the following,* (1) $\implies$ (2) $\implies$ (3) *and none of the implications is revertible.*

1. $\mathrm{Pr}_0(\aleph_1, {}^{\aleph_0 \circledast \aleph_1}\!/\!_{1 \circledast \aleph_1}, \aleph_1, \aleph_0)$;
2. $\mathrm{Pr}_0(\aleph_1, \aleph_1, \aleph_1, \aleph_0)$;

(3) $\mathrm{Pr}_0(\aleph_1, \aleph_1, \aleph_1, n)$ *for all* $n < \omega$.

Proof. To see that (2) does not imply (1), recall that $\mathrm{non}(\mathcal{M}) > \aleph_1 = \mathfrak{b}$ is consistent (e.g., after adding $\aleph_2$ random reals to a model of CH) and that Todorčević [26] proved that Clause (2) is a consequence of $\mathfrak{b} = \aleph_1$.

To see that (3) does not imply (2) recall that Clause (2) is refuted by $\mathsf{MA}_{\aleph_1}$, and that Peng and Wu [19] proved Clause (3) in ZFC. $\square$

We conclude this section by pointing out that by a proof similar to that of [4, Theorem 27], $\mathrm{Pr}_0(\aleph_1, {}^{\aleph_0 \circledast \aleph_1}\!/_{1 \circledast \aleph_1}, \aleph_1, \aleph_0)_p$ for all countable p is compatible with the failure of CH:

Theorem 3.4. *In the forcing extension after adding adding $\aleph_2$ many Cohen reals, for every partition $p : [\omega_1]^2 \to \omega$, $\mathrm{Pr}_0(\aleph_1, {}^{\aleph_0 \circledast \aleph_1}\!/_{1 \circledast \aleph_1}, \aleph_1, \aleph_0)_p$ holds.* $\square$

4. Strongly Luzin sets and strong colorings over partitions

Luzin sets are tightly connected with strong colorings. In addition to Todorčević's theorems [25, pp. 291],[27, Proposition 6.4] that were improved by the main result of the previous section, an earlier result connecting Luzin sets with strong colorings may be found in [6, Theorem 5.3]. Likewise, Souslin trees give rise to strong colorings (see [11, Lemma 6.6], [23, Lemma 1], [27, §5], and [20, §3]), and *coherent* Souslin trees have further strong coloring applications (see [16, §3.3]). By [1, §6], the existence of a Souslin tree does not imply the existence of a coherent one.

Now we show that a strongly Luzin set together with a coherent Souslin tree do not suffice to entail $\aleph_1 \nrightarrow_p [\aleph_1]^2_{\aleph_0}$ for all countable partitions p.

Theorem 4.1. *It is consistent that all of the following hold simultaneously:*

- *There exists a strongly Luzin set;*
- *There exists a coherent Souslin tree;*
- *There exists a partition $p : [\omega_1]^2 \to \omega$ such that all colorings $c : [\omega_1]^2 \to \omega$ are p-special, that is, for every coloring $c : [\omega_1]^2 \to \omega$, there is a decomposition $\omega_1 = \biguplus_{i<\omega} X_i$ such that for all $i, j < \omega$, c is constant over $\{(\alpha, \beta) \in [X_i]^2 \mid p(\alpha, \beta) = j\}$.*

The model of Theorem 4.1 will be the outcome of a finite support iteration of posets $\mathbb{Q}(p, c)$ of the following form.

Definition 4.2. $\mathbb{Q}(p, c)$ consists of all triples $q = (a_q, f_q, w_q)$ satisfying all of the following:

(1) $a_q \in [\omega_1]^{<\aleph_0}$;
(2) $f_q : a_q \to \omega$ is a function;
(3) w_q is a function from a finite subset of $\omega \times \omega$ to ω;
(4) for all $(\alpha, \beta) \in [a_q]^2$, if $f_q(\alpha) = f_q(\beta)$, then $(f_q(\alpha), p(\alpha, \beta)) \in \mathrm{dom}(w_q)$ and $c(\alpha, \beta) = w_q(f_q(\alpha), p(\alpha, \beta))$.

For a generic $G \subseteq \mathbb{Q}(p, c)$, let $X_{i,G} = \{\alpha < \omega_1 \mid \exists q \in G\, (f_q(\alpha) = i)\}$. It is not hard to see that for every partition $p : [\omega_1]^2 \to \omega$ with injective and almost-disjoint fibers, $\mathbb{Q}(p, c)$ has Property K,[2] and for all $i, j < \omega$,

$$\mathbb{1} \Vdash_{\mathbb{Q}(p,c)} \text{“}|\{c(\alpha, \beta) \mid (\alpha, \beta) \in [X_{i,\dot{G}}]^2 \text{ and } p(\alpha, \beta) = j\}| \leq 1\text{”}.$$

[2] In fact, by a result from [13, §3], $\mathbb{Q}(p, c)$ satisfies the stationary-cc.

Definition 4.3. For all $q \in \mathbb{Q}(p,c)$, $k < \omega$ and $z \in [\omega_1]^{<\aleph_0}$, define $q^\wedge(k,z)$ to be the triple (a,f,w) satisfying:

- $a := a_q \cup z$;
- $f : a \to \omega$ is a function extending f_q and satisfying $f(\alpha) = k + \mathrm{otp}(z \cap \alpha)$ for all $\alpha \in a \setminus a_q$;
- $w_q := w$.

Note that $q^\wedge(k,z)$ may not be in $\mathbb{Q}(p,c)$, but it will be, provided that $k \supseteq \mathrm{Im}(f_q)$.

Corollary 4.4. *For every $\beta < \omega_1$, $D_\beta := \{q \in \mathbb{Q}(p,c) \mid \beta \in a_q\}$ is dense, so that*

$$\mathbb{1} \Vdash_{\mathbb{Q}(p,c)} \text{“}\biguplus_{i<\omega} X_{i,\dot{G}} = \omega_1\text{”}.$$

Proof. Given arbitrary $q \in \mathbb{Q}(p,c)$ and $\beta < \omega_1$, for all sufficiently large k, $q^\wedge(k,\{\beta\})$ is a condition in D_β, extending q. □

Corollary 4.5. $\mathbb{1} \Vdash_{\mathbb{Q}(p,c)}$ *“c is p-special”.* □

Definition 4.6. Let $p : [\omega_1]^2 \to \omega$ be a partition. For any ordinal η, a finite-support iteration $\{\mathbb{Q}_\xi\}_{\xi\in\eta}$ will be called a *p-iteration* iff $\mathbb{Q}_0$ is the trivial forcing, and, for each ordinal ξ with $\xi + 1 < \eta$ there is a $\mathbb{Q}_\xi$-name $\dot{c}_\xi$ such that

(1) $\mathbb{1} \Vdash_{\mathbb{Q}_\xi}$ “$\dot{c}_\xi : [\omega_1]^2 \to \omega$ is a coloring”,
(2) $\mathbb{Q}_{\xi+1} = \mathbb{Q}_\xi * \mathbb{Q}(p,\dot{c}_\xi)$.

Convention 4.7. If $\{\mathbb{Q}_\xi\}_{\xi\in\eta}$ is a p-iteration, with $\eta > 0$ a limit ordinal, then we denote its direct limit by $\mathbb{Q}_\eta$.

From now on, we fix a p-iteration $\{\mathbb{Q}_\xi\}_{\xi\in\eta}$ for some partition $p : [\omega_1]^2 \to \omega$ with injective and almost-disjoint fibers, hence each of the iterands has Property K, and so does the whole iteration.

Definition 4.8. A structure $\mathfrak{M}$ is said to be *good* for the p-iteration $\{\mathbb{Q}_\xi\}_{\xi\in\eta}$ iff there is a large enough regular cardinal $\kappa > \eta$ such that all of the following hold:

- $\mathfrak{M}$ is a countable elementary submodel of $(\mathcal{H}_\kappa, \in, \lhd_\kappa)$, where $\lhd_\kappa$ is a well-ordering of $\mathcal{H}_\kappa$;
- $p, \{\mathbb{Q}_\xi\}_{\xi\in\eta}$ and $\{\dot{c}_\xi \mid \xi + 1 < \eta\}$ are in $\mathfrak{M}$.

Definition 4.9. Define $q \in \mathbb{Q}_\xi$ to be *determined* by recursion on $\xi \in \eta$:

▶ For $\xi = 0$, all the conditions are determined.
▶ For any ξ, a condition $q \in \mathbb{Q}_{\xi+1}$ is determined if:

(1) $q \restriction \xi$ is determined;
(2) $q \restriction \xi \Vdash_{\mathbb{Q}_\xi}$ “$q(\xi) = (a_{q,\xi}, f_{q,\xi}, w_{q,\xi})$” for an actual triple of finite sets;
(3) for all $(\alpha,\beta) \in [a_{q,\xi}]^2$ there is $n < \omega$ such that $q \restriction \xi \Vdash_{\mathbb{Q}_\xi}$ “$\dot{c}_\xi(\alpha,\beta) = n$”.

▶ For any $\xi \in \mathrm{acc}(\eta)$, $q \in \mathbb{Q}_\xi$ is determined if $q \restriction \zeta$ is determined for all $\zeta < \xi$.

By a standard argument, the determined conditions are dense in $\mathbb{Q}_\eta$.

Definition 4.10. For a determined condition q in the p-iteration, we say that k is *sufficiently large for* q iff $k \supseteq \mathrm{Im}(f_{q,\xi})$ for all ξ in the support of q.

Definition 4.11. For a condition q in the p-iteration, $k < \omega$ and $z \in [\omega_1]^{<\aleph_0}$, define $q^\wedge(k,z)$ by letting $q^\wedge(k,z)(\xi) := q(\xi)^\wedge(k,z)$ for each ξ in the support of q.

Note that if q is determined and k is sufficiently large for q, then for each ξ in the support of q, $q \restriction \xi \Vdash_{\mathbb{Q}_\xi}$ "$q^\wedge(k,z) \in \mathbb{Q}(p, \dot{c}_\xi)$". In effect, if k is sufficiently large for q, then $q^\wedge(k,z)$ is a legitimate condition.

Definition 4.12. For any structure $\mathfrak{M}$ good for the p-iteration $\{\mathbb{Q}_\xi\}_{\xi\in\eta}$, for all $\xi \in \eta$ and a determined condition $q \in \mathbb{Q}_\xi$, we define $q^{\mathfrak{M}}$, as follows. The definition is by recursion on $\xi \in \eta$:

► For $\xi = 0$ there is nothing to do.

► For any ξ such that $q^{\mathfrak{M}}$ has been defined for all determined q in $\mathbb{Q}_\xi$, given a determined condition $q \in \mathbb{Q}_{\xi+1}$, we consider two cases:

►► If $\xi \in \mathfrak{M}$, then let $q^{\mathfrak{M}} := (q \restriction \xi)^{\mathfrak{M}} * (a_{q,\xi} \cap \mathfrak{M}, f_{q,\xi} \cap \mathfrak{M}, w_{q,\xi})$;

►► Otherwise, just let $q^{\mathfrak{M}} := (q \restriction \xi)^{\mathfrak{M}} * (\emptyset, \emptyset, \emptyset)$.

► For any $\xi \in \operatorname{acc}(\eta)$, since this is a finite-support iteration, there is nothing new to define.

If q is determined, then, for every coordinate ξ in the support of q, $q^{\mathfrak{M}}(\xi)$ is a triple consisting of finite sets lying in $\mathfrak{M}$. It is important to note that $q^{\mathfrak{M}}$ may not, in general, be a condition in $\mathbb{Q}_\xi$, because the last clause of Definition 4.2 may fail. Nevertheless, $(q^{\mathfrak{M}})^\wedge(k,z)$ is a well-defined object, since its definition does not depend on the $\dot{c}_\xi$'s.

Notation 4.13. For any determined condition $q \in \mathbb{Q}_\xi$, we denote by A_q the union of $a_{q,\xi}$ over all ξ in the support of q.

We now arrive at the main technical lemma of this section.

Lemma 4.14. *Suppose $p : [\omega_1]^2 \to \omega$ satisfies the conclusion of Lemma 2.5, and $\mathfrak{M}$ is a structure which is good for the p-iteration $\{\mathbb{Q}_\xi\}_{\xi\in\eta}$.*

For all $\zeta \leq \sup(\eta)$ and a determined condition $r \in \mathbb{Q}_\zeta$, there is a finite set $\bar{z} \subseteq \mathfrak{M} \cap \omega_1$ such that:

- **A:** *For every $z \in [\mathfrak{M}\cap\omega_1]^{<\aleph_0}$ covering $\bar{z}$ and every integer k that is sufficiently large for r, $(r^{\mathfrak{M}})^\wedge(k,z)$ is in $\mathfrak{M} \cap \mathbb{Q}_\zeta$ and is determined;*
- **B:** *For every $z \in [\mathfrak{M}\cap\omega_1]^{<\aleph_0}$ covering $\bar{z}$ and every integer k that is sufficiently large for r, for the condition $\bar{r} := (r^{\mathfrak{M}})^\wedge(k,z)$ and a condition $q \in \mathfrak{M} \cap \mathbb{Q}_\zeta$, if the following three requirements hold:*
  1. *$\mathfrak{M} \models q \leq \bar{r}$ and q is determined;*
  2. *the mapping $(\alpha,\beta) \mapsto p(\alpha,\beta)$ is injective over $(A_q \setminus A_{\bar{r}}) \times (A_r \setminus A_{\bar{r}})$;*
  3. *$p(\alpha,\beta) > p(\alpha',\beta')$ for all $(\alpha,\beta) \in (A_q \setminus A_{\bar{r}}) \times (A_r \setminus A_{\bar{r}})$ and $(\alpha',\beta') \in [A_r]^2 \cup [A_q]^2$,*

  then $q \not\perp r$.

Proof. Proceed by induction on $\zeta \leq \sup(\eta)$ proving **A** and **B** simultaneously. The case $\zeta = 0$ is immediate. The case $\zeta = 1$ is simple as well, but it may be instructive to consider it in detail. So c_0 is a coloring in the ground model and all conditions are determined. In effect, given $r \in \mathbb{Q}_1$, $r^{\mathfrak{M}}$ is a condition, as well. It will be shown that $\bar{z} = \emptyset$ satisfies the conclusion.

Let k be sufficiently large for r. We know that $(r^{\mathfrak{M}})^\wedge(k,z) \in \mathfrak{M} \cap \mathbb{Q}_1$ for any $z \in [\mathfrak{M}\cap\omega_1]^{<\aleph_0}$. Hence **A** is immediate. To see that **B** holds, suppose that we are given $z \in [\mathfrak{M}\cap\omega_1]^{<\aleph_0}$, we let $\bar{r} := (r^{\mathfrak{M}})^\wedge(k,z)$, and we are also given a condition $q \in \mathfrak{M} \cap \mathbb{Q}_1$ satisfying requirements (1)–(3) above.

To see that $q \not\perp r$, let $a := a_{q,0} \cup a_{r,0}$, $f := f_{q,0} \cup f_{r,0}$ and $w := w_{q,0} \cup w_{r,0}$. It is immediate to see that f and w are functions, $A_r = a_{r,0}$, $A_q = a_{q,0}$ and $A_q \cap A_r = A_{\bar{r}}$.

We need to show that there exists a function w^* extending w for which (a, f, w^*) is a legitimate condition. For this, suppose that we are given $i, j < \omega$, $(\alpha, \beta), (\alpha', \beta') \in [a]^2$, with $f(\alpha) = f(\beta) = i = f(\alpha') = f(\beta')$ and $p(\alpha, \beta) = j = p(\alpha', \beta')$. It must be shown that $c_0(\alpha, \beta) = c_0(\alpha', \beta')$. There are two cases to consider:

Case I: If $(\alpha, \beta), (\alpha', \beta') \in [A_q]^2 \cup [A_r]^2$, then since w extends $w_{q,0}$ and $w_{r,0}$, $c_0(\alpha, \beta) = w(i, j) = c_0(\alpha', \beta')$.

Case II: If $(\alpha, \beta) \in [a]^2 \setminus ([A_q]^2 \cup [A_r]^2)$, then since $A_q \cap A_r = A_{\bar{r}}$ and $\alpha < \beta$, we infer that $(\alpha, \beta) \in (A_q \setminus A_{\bar{r}}) \times (A_r \setminus A_{\bar{r}})$. So, by Clause (3), $(\alpha', \beta') \in [a]^2 \setminus ([A_q]^2 \cup [A_r]^2)$, as well. Then, likewise $(\alpha', \beta') \in (A_q \setminus A_{\bar{r}}) \times (A_r \setminus A_{\bar{r}})$. Altogether, by Clause (2), $(\alpha, \beta) = (\alpha', \beta')$. In particular, $c_0(\alpha, \beta) = c_0(\alpha', \beta')$.

Next, assume that $\zeta \leq \sup(\eta)$ and that **A** and **B** have been established for all $\xi < \zeta$. If ζ is a limit, then the finite-support nature of the iteration also establishes both **A** and **B** hold, so suppose that $\zeta = \xi + 1$. The successor case in which $\xi \notin \mathfrak{M}$ also follows directly from the induction hypothesis by the definition of $q^{\mathfrak{M}}$, so assume that $\xi \in \mathfrak{M}$.

Let $r \in \mathbb{Q}_\zeta$ be determined. Let $\bar{z} \in [\mathfrak{M} \cap \omega_1]^{<\aleph_0}$ be given by the induction hypothesis with respect to $r \restriction \xi$. In particular, for every $z \in [\mathfrak{M} \cap \omega_1]^{<\aleph_0}$ covering $\bar{z}$, and k sufficiently large for $r \restriction \xi$, $((r \restriction \xi)^{\mathfrak{M}})^\wedge(k, z)$ is in $\mathfrak{M} \cap \mathbb{Q}_\xi$ and is determined.

To establish **A**, note that, since $\xi \in \mathfrak{M}$, it follows that for any $z \in [\mathfrak{M} \cap \omega_1]^{<\aleph_0}$ covering $\bar{z}$, and k sufficiently large for r (in particular, sufficiently large for $r \restriction \xi$), $s_{k,z} := ((r \restriction \xi)^{\mathfrak{M}})^\wedge(k, z) * (a_{r,\xi} \cap \mathfrak{M}, f_{r,\xi} \cap \mathfrak{M}, w_{r,\xi})$ is in $\mathfrak{M}$. It must also be shown that $s_{k,z}$ belongs to $\mathbb{Q}_\zeta$. For this, it suffices to show that for all $i, j < \omega$,

$$((r \restriction \xi)^{\mathfrak{M}})^\wedge(k, z) \Vdash_{\mathbb{Q}_\xi} \text{“}\forall(\alpha, \beta) \in [f_{r,\xi}^{-1}[\{i\}] \cap p^{-1}[\{j\}] \cap \mathfrak{M}]^2 \ \dot{c}_\xi(\alpha, \beta) = w_{r,\xi}(i, j)\text{”}.$$

As $\dot{c}_\xi$ belongs to $\mathfrak{M}$, for each $(\alpha, \beta) \in [a_{r,\xi} \cap \mathfrak{M}]^2$ there is a countable, maximal antichain deciding $\dot{c}_\xi(\alpha, \beta)$ and belonging to $\mathfrak{M}$ because; in other words, all possible decisions about the value of $\dot{c}_\xi(\alpha, \beta)$ can be forced without leaving $\mathfrak{M}$. So, if the above displayed assertion fails, then there must be some $q^* \leq ((r \restriction \xi)^{\mathfrak{M}})^\wedge(k, z)$ in $\mathfrak{M}$ and (α, β) in $[f_{r,\xi}^{-1}[\{i\}] \cap \mathfrak{M}]^2$ such that $p(\alpha, \beta) = j$, but $q^* \Vdash_{\mathbb{Q}_\xi}$ “$\dot{c}_\xi(\alpha, \beta) \neq w_{r,\xi}(i, j)$”. Fix k sufficiently large for r. Then, under the assumption that for any $z \in [\mathfrak{M} \cap \omega_1]^{<\aleph_0}$ covering $\bar{z}$, $s_{k,z}$ does not belong to $\mathbb{Q}_\zeta$, it is possible to construct recursively a sequence $\{(z_n, q_n, i_n, j_n, (\alpha_n, \beta_n)\}_{n\in\omega}$ such that:

- $z_0 = \bar{z}$;
- $q_n \leq ((r \restriction \xi)^{\mathfrak{M}})^\wedge(k, z_n)$ and q_n is determined;
- $(\alpha_n, \beta_n) \in [f_{r,\xi}^{-1}[\{i_n\}] \cap p^{-1}[\{j_n\}] \cap \mathfrak{M}]^2$;
- $q_n \Vdash_{\mathbb{Q}_\xi}$ “$\dot{c}_\xi(\alpha_n, \beta_n) \neq w_{r,\xi}(i_n, j_n)$”;
- $z_{n+1} \supsetneq A_{q_n}$.

By making canonical choices (e.g., by consulting with $\lhd_\kappa$), this construction can be carried out in $\mathfrak{M}$. Let $\epsilon := \sup(\bigcup_{n\in\omega} A_{q_n}) + 1$. Define a function $h : \epsilon \to \omega$ via

$$h(\alpha) := \max\{k, p(\alpha', \beta') \mid (\alpha', \beta') \in [A_{q_{n+1}}]^2 \text{ and } \alpha \in A_{q_{n+1}} \setminus A_{q_n}\},$$

and note that h is in $\mathfrak{M}$.

Let γ satisfy the conclusion of Lemma 2.5 for h. As $h \in \mathfrak{M}$, $\gamma \in \mathfrak{M}$, so, since $b := A_r \setminus \mathfrak{M}$ is an element of $[\omega_1 \setminus \gamma]^{<\aleph_0}$, there exists $\Delta \in [\epsilon]^{<\aleph_0}$ such that:

- $p \restriction ((\epsilon \setminus \Delta) \times b)$ is injective;
- for all $\alpha \in \epsilon \setminus \Delta$ and $\beta \in b$, $h(\alpha) < p(\alpha, \beta)$.

Fix a large enough $n < \omega$ such that $A_{q_{n+1}} \setminus A_{q_n}$ is disjoint from Δ. Denote $\bar{r} := ((r \restriction \xi)^{\mathfrak{M}})^\frown(k, z_{n+1})$. As $z_{n+1} \supseteq A_{q_n}$, $(A_{q_{n+1}} \setminus A_{\bar{r}}) \subseteq (A_{q_{n+1}} \setminus A_{q_n}) \subseteq (\epsilon \setminus \Delta)$, $(A_{r\restriction\xi} \setminus A_{\bar{r}}) \subseteq b$, and all of the following hold:

(1) $\mathfrak{M} \models q_{n+1} \leq \bar{r}$ and q is determined;
(2) the mapping $(\alpha, \beta) \mapsto p(\alpha, \beta)$ is injective over $(A_{q_{n+1}} \setminus A_{\bar{r}}) \times (A_{r\restriction\xi} \setminus A_{\bar{r}})$;
(3) $p(\alpha, \beta) > p(\alpha', \beta')$ for all $(\alpha, \beta) \in (A_{q_{n+1}} \setminus A_{\bar{r}}) \times (A_{r\restriction\xi} \setminus A_{\bar{r}})$ and $(\alpha', \beta') \in [A_{r\restriction\xi}]^2 \cup [A_{q_{n+1}}]^2$.

Then applying the induction hypothesis for **B** yields that $q_{n+1} \not\perp (r \restriction \xi)$. Pick a determined condition q^* in $\mathbb{Q}_\xi$ simultaneously extending q_{n+1} and $(r \restriction \xi)$. As $q^* \leq q_{n+1}$, we infer that

$$q^* \Vdash_{\mathbb{Q}_\xi} \text{“}\dot{c}_\xi(\alpha_{n+1}, \beta_{n+1}) \neq w_{r,\xi}(i_{n+1}, j_{n+1})\text{”}.$$

As $(\alpha_n, \beta_n) \in [f_{r,\xi}^{-1}[\{i_n\}] \cap \mathfrak{M}]^2$, $p(\alpha_n, \beta_n) = j_n$ and $q^* \leq r \restriction \xi$, we infer that

$$q^* \Vdash_{\mathbb{Q}_\xi} \text{“}\dot{c}_\xi(\alpha_{n+1}, \beta_{n+1}) = w_{r,\xi}(i_{n+1}, j_{n+1})\text{”}.$$

This is a contradiction. So **A** does hold.

Next, let us establish **B**. Recall that we have a determined condition $r \in \mathbb{Q}_\zeta$ and $\bar{z} \in [\mathfrak{M} \cap \omega_1]^{<\aleph_0}$ satisfying that for every $z \in [\mathfrak{M} \cap \omega_1]^{<\aleph_0}$ covering $\bar{z}$, and k sufficiently large for $r \restriction \xi$, $((r \restriction \xi)^{\mathfrak{M}})^\frown(k, z)$ is in $\mathfrak{M} \cap \mathbb{Q}_\xi$ and is determined. We have just established **A**, proving that we may fix a finite z^* with $\bar{z} \subseteq z^* \subseteq \mathfrak{M} \cap \omega_1$, satisfying that for every $z \in [\mathfrak{M} \cap \omega_1]^{<\aleph_0}$ covering z^*, and every integer k that is sufficiently large for r, $(r^{\mathfrak{M}})^\frown(k, z)$ is in $\mathfrak{M} \cap \mathbb{Q}_\zeta$ and is determined.

Now, fix arbitrary $z \in [\mathfrak{M} \cap \omega_1]^{<\aleph_0}$ covering z^*, an integer k that is sufficiently large for r, and a condition $q \in \mathfrak{M} \cap \mathbb{Q}_\zeta$. Set $\bar{r} := (r^{\mathfrak{M}})^\frown(k, z)$ and suppose that the requirements (1)–(3) of **B** for $\bar{r}$ and q hold. In particular, they hold for $\bar{r} \restriction \xi$ and $q \restriction \xi$. That is:

- $\mathfrak{M} \models q \restriction \xi \leq \bar{r} \restriction \xi$ and $q \restriction \xi$ is determined;
- the mapping $(\alpha, \beta) \mapsto p(\alpha, \beta)$ is injective over $(A_{q\restriction\xi} \setminus A_{\bar{r}\restriction\xi}) \times (A_{r\restriction\xi} \setminus A_{\bar{r}\restriction\xi})$;
- $p(\alpha, \beta) > p(\alpha', \beta')$ for all $(\alpha, \beta) \in (A_{q\restriction\xi} \setminus A_{\bar{r}\restriction\xi}) \times (A_{r\restriction\xi} \setminus A_{\bar{r}\restriction\xi})$ and $(\alpha', \beta') \in [A_{r\restriction\xi}]^2 \cup [A_{q\restriction\xi}]^2$.

Now, as $z^* \supseteq \bar{z}$, we get from **B** of the previous stage that $(q \restriction \xi) \not\perp (r \restriction \xi)$. Pick a determined condition q^* in $\mathbb{Q}_\xi$ simultaneously extending $(q \restriction \xi)$ and $(r \restriction \xi)$. Let $a := a_{q,\xi} \cup a_{r,\xi}$, $f := f_{q,\xi} \cup f_{r,\xi}$ and $w := w_{q,\xi} \cup w_{r,\xi}$. It is immediate to see that f and w are functions, $A_r \supseteq a_{r,\xi}$, $A_q \supseteq a_{q,\xi}$ and $A_q \cap A_r = A_{\bar{r}}$. To see that $q \not\perp r$, it suffices to prove that there exists $w^* \supseteq w$ such that $q^* * (a, f, w^*) \in \mathbb{Q}_\zeta$.

For this, suppose that we are given $i, j < \omega$, $(\alpha, \beta), (\alpha', \beta') \in [a]^2$, with $f(\alpha) = f(\beta) = i = f(\alpha') = f(\beta')$ and $p(\alpha, \beta) = j = p(\alpha', \beta')$. It must be shown that $q^* \Vdash_{\mathbb{Q}_\xi}$ “$\dot{c}_\xi(\alpha, \beta) = \dot{c}_\xi(\alpha', \beta')$”. There are two cases to consider:

Case I: If $(\alpha, \beta), (\alpha', \beta') \in [a_{q,\xi}]^2 \cup [a_{r,\xi}]^2$, then since w extends $w_{q,\xi}$ and $w_{r,\xi}$, the conclusion follows from the fact that q^* extends $q \restriction \xi$ and $r \restriction \xi$.

Case II: If $(\alpha, \beta) \in [a]^2 \setminus ([a_{q,\xi}]^2 \cup [a_{r,\xi}]^2)$, then, as seen earlier, requirements (2) and (3) imply that $(\alpha, \beta) = (\alpha', \beta')$.

So, we are done. □

Lemma 4.15. *Suppose:*

- $p : [\omega_1]^2 \to \omega$ *satisfies the conclusion of Lemma 2.5;*
- $L = \{l_\delta\}_{\delta\in\omega_1}$ *is a strongly Luzin subset of* 2^ω*;*
- $\{\mathbb{Q}_\xi\}_{\xi\in\eta}$ *is a p-iteration with* $\eta > 0$ *a limit ordinal.*

Then $\mathbb{1} \Vdash_{\mathbb{Q}_\eta}$ *"L is strongly Luzin".*

Proof. Suppose not. Then it can be assumed that there is a $\mathbb{Q}_\eta$-name $\dot{T}$ and a positive integer d such that for the set $\mathcal{Q}_d := \bigcup_{n<\omega}(2^n)^d$ of the 'rationals' of $(2^\omega)^d$:

- $\mathbb{1} \Vdash_{\mathbb{Q}_\eta}$ "$\dot{T} \subseteq \mathcal{Q}_d$ is a closed nowhere dense tree", and
- $\mathbb{1} \Vdash_{\mathbb{Q}_\eta}$ "$[\dot{T}] \cap L^d$ contains an uncountable pairwise disjoint subfamily".

It follows that for each $\gamma < \omega_1$, we may fix a determined condition $r_\gamma \in \mathbb{Q}_\eta$ and a sequence $\langle \delta_i^\gamma \mid i < d\rangle$ of ordinals in $\omega_1 \setminus \gamma$ such that $r_\gamma \Vdash_{\mathbb{Q}_\eta}$ "$\vec{l_\gamma} := \langle l_{\delta_i^\gamma} \mid i < d\rangle$ is a branch through $\dot{T}$". Pick an uncountable $\Gamma \subseteq \omega_1$ along with $k < \omega$ which is sufficiently large for r_γ for all $\gamma \in \Gamma$. By possibly shrinking Γ further, we may also assume that $\{A_{r_\gamma} \mid \gamma \in \Gamma\}$ forms a Δ-system with root ρ, and that $\langle \{\delta_i^\gamma \mid i < d\} \mid \gamma \in \Gamma\rangle$ consists of pairwise disjoint sets.

Let $\mathfrak{M}$ be a structure good for the p-iteration $\{\mathbb{Q}_\xi\}_{\xi\in\eta}$, with $\rho, \dot{T}, \mathbb{Q}_\eta \in \mathfrak{M}$.

For each $\gamma \in \Gamma$, let $\bar{z}_\gamma$ be given by Lemma 4.14 with respect to r_γ. Fix an uncountable $\Gamma' \subseteq \Gamma$ and some $\bar{z} \in [\omega_1 \cap \mathfrak{M}]^{<\omega}$ such that $\bar{z}_\gamma = \bar{z}$ for all $\gamma \in \Gamma'$. By possibly shrinking further, we may assume the existence of q such that $(r_\gamma)^{\mathfrak{M}} = q$ for all $\gamma \in \Gamma'$. In particular, for every $z \in [\mathfrak{M}\cap\omega_1]^{<\aleph_0}$ covering $\bar{z}$, $q^\wedge(k, z) \in \mathfrak{M}\cap\mathbb{Q}_\eta$ is determined. Let $\{\tau_n\}_{n\in\omega}$ enumerate the set $\mathcal{Q}_d$. Construct recursively a sequence $\{(z_n, q_n, t_n)\}_{n\in\omega}$ such that:

- $z_0 = \bar{z} \cup \rho$;
- $q_n \leq q^\wedge(k, z_n)$ and q_n is a determined condition lying in $\mathfrak{M}$;
- $\tau_n \subseteq t_n \in \mathcal{Q}_d$ with $q_n \Vdash_{\mathbb{Q}_\eta}$ "$t_n \notin \dot{T}$";
- $z_{n+1} \supsetneq A_{q_n}$.

Let $\epsilon := \sup(\bigcup_{n\in\omega} A_{q_n}) + 1$. Define a function $h : \epsilon \to \omega$ via

$$h(\alpha) := \max\{k, p(\alpha', \beta') \mid (\alpha', \beta') \in [A_{q_{n+1}}]^2 \text{ and } \alpha \in A_{q_{n+1}} \setminus A_{q_n}\}.$$

Recalling that p was given by Lemma 2.5, we now fix $\gamma^* < \omega_1$ satisfying that for every $b \in [\omega_1 \setminus \gamma^*]^{<\aleph_0}$, there exists $\Delta \in [\epsilon]^{<\aleph_0}$ such that:

(I) $p \restriction ((\epsilon \setminus \Delta) \times b)$ is injective;
(II) for all $\alpha \in \epsilon \setminus \Delta$ and $\beta \in b$, $h(\alpha) < p(\alpha, \beta)$.

Clearly, $\Gamma^* := \{\gamma \in \Gamma' \mid \min(A_{r_\gamma} \setminus \rho) > \gamma^*\}$ is uncountable. For each $n < \omega$, consider the open set $U_n := \{\vec{l} \in (2^\omega)^d \mid \bigwedge_{i<d}(t_n(i) \subseteq \vec{l}(i))\}$. Then $W := \bigcap_{j=0}^\infty \bigcup_{j=n}^\infty U_{n+1}$ is a dense G_δ set, and hence $L^d \setminus W$ contains no uncountable pairwise disjoint subfamily. Consequently, we may find some $\gamma \in \Gamma^*$ such that $\vec{l_\gamma}$ is in W. Set $b := A_{r_\gamma} \setminus \rho$ and then find $\Delta \in [\epsilon]^{<\aleph_0}$ satisfying (I) and (II). Fix a large enough $j < \omega$ such that $A_{q_{n+1}} \setminus A_{q_n}$ is disjoint from Δ for all $n \geq j$. As $\vec{l_\gamma} \in W$, we may now fix some $n \geq j$ such that $\vec{l_\gamma} \in U_{n+1}$. Denote $\bar{r} := (q^{\mathfrak{M}})^\wedge(k, z_{n+1})$. Then $(A_{q_{n+1}} \setminus A_{\bar{r}}) \subseteq (A_{q_{n+1}} \setminus A_{q_n}) \subseteq (\epsilon \setminus \Delta)$, $(A_{r_\gamma} \setminus A_{\bar{r}}) \subseteq b$, and all of the following hold:

(1) $\mathfrak{M} \models q_{n+1} \leq \bar{r}$ and q is determined;
(2) the mapping $(\alpha, \beta) \mapsto p(\alpha, \beta)$ is injective over $(A_{q_{n+1}} \setminus A_{\bar{r}}) \times (A_{r_\gamma} \setminus A_{\bar{r}})$;
(3) $p(\alpha, \beta) > p(\alpha', \beta')$ for all $(\alpha, \beta) \in (A_{q_{n+1}} \setminus A_{\bar{r}}) \times (A_{r_\gamma} \setminus A_{\bar{r}})$ and $(\alpha', \beta') \in [A_{r_\gamma}]^2 \cup [A_{q_{n+1}}]^2$.

Since $z_{n+1} \supseteq \bar{z}$ and $\bar{z}$ was given by Lemma 4.14, we may apply **B** and infer that $q_{n+1} \not\perp r_\gamma$. However, $q_{n+1} \Vdash_{\mathbb{Q}_\eta}$ "$t_{n+1} \notin \dot{T}$" and $r_\gamma \Vdash_{\mathbb{Q}_\eta}$ "$\vec{l_\gamma}$ is a branch through $\dot{T}$", contradicting the fact that $t_{n+1} \subseteq \vec{l_\gamma}$. $\square$

Proof of Theorem 4.1. Start with a model V of GCH in which there exists a coherent Souslin tree (see [3, Proposition 2.5 and Theorem 3.6]). Using CH, fix a strongly Luzin set L and a partition p as in Lemma 2.5. Let $\mathbb{Q}_{\omega_2}$ be the corresponding p-iteration, using $\mathcal{H}_{\aleph_2}$ as our bookkeeping device of names of colorings $\dot{c}_\xi$. The iteration satisfies Property K, being a finite-support iteration of Property K posets, hence the coherent Souslin tree survives. In addition, the *ccc* of the iteration implies that for every coloring $c : [\omega_1]^2 \rightarrow \omega$ in the extension, there is a tail of $\xi \in \omega_2$ such that c admits a $\mathbb{Q}_\xi$-name in $\mathcal{H}_{\aleph_2}$ of V. So, in $V^{\mathbb{Q}_{\omega_2}}$, all colorings $c : [\omega_1]^2 \rightarrow \omega$ are p-special. Finally, by Lemma 4.15, the strongly Luzin set L survives. □

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