SUMS OF TRIPLES IN ABELIAN GROUPS

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Abstract. Motivated by a problem in additive Ramsey theory, we extend Todorčević's partitions of three-dimensional combinatorial cubes to handle additional three-dimensional objects. As a corollary, we get that if the continuum hypothesis fails, then for every Abelian group G of size $\aleph_2$, there exists a coloring $c : G \to \mathbb{Z}$ such that for every uncountable $X \subseteq G$ and every integer k, there are three distinct elements x, y, z of X such that $c(x + y + z) = k$.

1. Introduction

By Hindman's celebrated theorem (see [HS12, Corollary 5.9]), for every partition of an infinite commutative cancellative semigroup $(G, +)$ into two cells A and B, there exists an infinite subset $X \subseteq G$ such that the set of its finite sums

$$\mathrm{FS}(X) := \{x_1 + \cdots + x_n \mid x_1, \ldots, x_n \text{ are distinct elements of } X \ \& \ n \in \mathbb{N} \setminus 2\}$$

is completely contained in A or completely contained in B. Equivalently, for every coloring $c : G \to 2$, there exists an infinite $X \subseteq G$ such that $c \restriction \mathrm{FS}(X)$ is constant.

Hindman's theorem does not generalize to the uncountable, as it follows from a theorem of Milliken (see [Mil78, Theorem 9]) that the following assertion is consistent with the usual axioms of set theory: for every (not necessarily Abelian) group $(G, *)$ whose size is a regular uncountable cardinal, there is a coloring $c : G \to G$ such that $c \restriction \mathrm{FS}_2(X)$ is *onto* G for every $X \subseteq G$ of size $|G|$, where this time

$$\mathrm{FS}_n(X) := \{x_1 * \cdots * x_n \mid x_1, \ldots, x_n \text{ are distinct elements of } X\}.$$

A few years ago, starting with a paper by Hindman, Leader and Strauss [HLS17], the study of higher analogs of Hindman's theorem regained interest. We mention only a few results that are relevant to this paper:

(1) Improving upon a theorem from [HLS17], Komjáth [Kom16], and independently Soukup and Weiss [SW16], proved that there exists a coloring $c : \mathbb{R} \to 2$ such that for every uncountable $X \subseteq \mathbb{R}$ and every $i \in \{0, 1\}$, there are $x \neq y$ in X such that $c(x + y) = i$.

(2) Solving a problem of Weiss, Komjáth [Kom20] proved that there exists a coloring $c : \mathbb{R} \to 2$ such that for every uncountable $X \subseteq \mathbb{R}$ and every $i \in \{0, 1\}$, there are $x \neq y$ in X such that $c(|x - y|) = i$. As for dimension $d > 1$, assuming the continuum hypothesis, there exists a coloring $c : \mathbb{R} \to 2$ such that for every uncountable $X \subseteq \mathbb{R}^d$ and every $i \in \{0, 1\}$, there are $x \neq y$ in X such that $c(\|x - y\|) = i$.

(3) In [FBR17], Fernández-Bretón and Rinot proved that there exists a coloring $c : \mathbb{R} \to \mathbb{N}$ such that for every $X \subseteq \mathbb{R}$ of size $|\mathbb{R}|$ and every $i \in \mathbb{N}$, there are $x \neq y$ in X such that $c(x + y) = i$.

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(4) By [FBR17], for class many cardinals κ (including $\kappa=\aleph_n$ for every positive integer n), for every commutative cancellative semigroup $(G,+)$ of size κ, there exists a coloring $c:G\rightarrow G$ such that for all $X,Y\subseteq G$ of size κ and every $g\in G$, there are $x\in X$ and $y\in Y$ such that $c(x+y)=g$.[1]
(5) By [FBR17], for every regular uncountable cardinal κ that is not Jónsson, for every commutative cancellative semigroup $(G,+)$ of size κ, there exists a coloring $c:G\rightarrow G$ such that for every $X\subseteq G$ of size κ and every $g\in G$, there are finitely many distinct elements $x_1,\ldots,x_n\in X$ such that $c(x_1+\cdots+x_n)=g$.
(6) In [Pro97], Protasov proved that for every commutative cancellative semigroup $(G,+)$, there exists a coloring $c:G\rightarrow\mathbb{N}$ such that $c\restriction\mathrm{FS}(X)$ is onto $\mathbb{N}$ for every uncountable $X\subseteq G$. By [FBR17], it is also consistent that the same holds after replacing $\mathbb{N}$ by $\mathbb{R}$.

Note that in the results listed in (1), (2) and (6), the triggering set X may have cardinality smaller than that of G, whereas in (3)–(5), $|X|$ coincides with $|G|$. Another important difference is that unlike the results of (1)–(4), in (5) and (6), no bound is asserted on the length of the sums needed to generate all the infinite colors. This raises a natural question whose simplest instance reads as follows.

Question. Suppose that $(G,+)$ is an Abelian group of size $\aleph_2$.
Must there exist a positive integer n and a coloring $c:G\rightarrow\mathbb{N}$ such that $c\restriction\mathrm{FS}_n(X)$ is onto $\mathbb{N}$ for every uncountable $X\subseteq G$?

A moment's reflection makes it clear that an affirmative answer (even for just one particular group) immediately implies the relation $\aleph_2\nrightarrow[\aleph_1]^n_{\aleph_0}$ from the classical study of partition relations for cardinal numbers [EHR65]. By a theorem of Erdős and Rado, the above relation may consistently fail for $n=2$, and it is a remarkable theorem of Todorčević [Tod94] that it does hold for $n=3$. The first main result of this paper gives a consistent extension of Todorčević's theorem.

Theorem A. *If the continuum hypothesis fails, then for every Abelian group $(G,+)$ of size $\aleph_2$, there exists a coloring $c:G\rightarrow\mathbb{N}$ such that for every uncountable $X\subseteq G$ and every $i\in\mathbb{N}$, there are three distinct elements x,y,z of X such that $c(x+y+z)=i$.*

Theorem A is not limited to Abelian groups. In fact, it works for all so-called *well-behaved magmas*, as follows.

Definition. A *magma* is a structure $(G,*)$, where $*$ is a binary operation. We say that it is *well-behaved* iff there exists a map $\varphi:G\rightarrow[G]^{<\omega}$ such that:[2]

- G is countable-to-one;
- for all $x\neq y$ in G, $\varphi(x)\mathbin{\triangle}\varphi(y)\subseteq\varphi(x*y)\subseteq\varphi(x)\cup\varphi(y)$.

Every infinite commutative cancellative semigroup $(G,+)$ is well-behaved (see, e.g., [FBR17, Lemma 2.2]). Also, every free group $(G,*)$ is well-behaved, as witnessed by the map that sends a word to the set of its letters. As a third example, consider the magma appearing in result (2) above, namely, $(\mathbb{R},d)$ where $d(x,y):=|x-y|$. Indeed, viewing $\mathbb{R}$ as a $\mathbb{Q}$-vector space over some Hamel basis B, any $x\in\mathbb{R}\setminus\{0\}$ is the unique linear combination $\sum_{i\leq n}q_iv_i$ of nonzero rational

[1] More is true, see [FBR17, Corollary 4.5].
[2] Here, $[G]^{<\omega}$ denotes the collection of all finite subsets of G.

numbers $q_0, \ldots, q_n$, and an injective sequence $\langle v_i \mid i \leq n\rangle$ of elements of B. So $\varphi : \mathbb{R} \to [\mathbb{R}]^{<\omega}$ sending x to the unique $\{v_i \mid i \leq n\}$ (and sending 0 to the emptyset) is countable-to-one, and for all $x \neq y$, $\varphi(x) \triangle \varphi(y) \subseteq \varphi(|x-y|) \subseteq \varphi(x) \cup \varphi(y)$.

The full statement of Theorem A reads as follows.

Theorem A′. *For every infinite cardinal μ such that $\mu^{<\mu} < \mu^+ < 2^\mu$, for every well-behaved magma $(G, *)$ of size μ^{++}, there is a coloring $c : G \to \mathbb{N}$ such that for every $X \subseteq G$ of size μ^+ and every $i \in \mathbb{N}$, there are three distinct elements x, y, z of X such that $c(x * y * z) = i$.*[3]

While not so explicit, the approach of going through well-behaved magmas is already present in [FBR17]. In particular, the coloring of result (4) attains all possible colors not only over evaluations of the form $x + y$, but also over any nontrivial $\mathbb{Q}$-combination of x and y, such as $|x - y|$. This suggests that it is possible to obtain a coloring simultaneously witnessing result (1) together with the first half of (2). Indeed, Komjáth's theorems follow from the following finding (using $\theta := \aleph_0$):

Theorem B. *For every infinite cardinal θ such that $2^{<\theta} = \theta$, for every set G with $\theta < |G| \leq 2^\theta$, and every map $\varphi : G \to [G]^{<\omega}$, there exists a corresponding coloring $c : G \to 2$ satisfying the following.*

For every binary operation $*$ on G, if φ witnesses that $(G, *)$ is well-behaved, then for every $X \subseteq G$ of size θ^+ and every $i \in \{0, 1\}$, there are $x \neq y$ in X such that $c(x * y) = i$.

The proofs of Theorems A′ and B are obtained in a few steps. As a first step, we consider a coloring principle $S_n(\kappa, \lambda, \theta)$ that is sufficient to imply that any well-behaved magma $(G, *)$ of size κ admits a coloring $c : G \to \theta$ that takes on every possible color on $\mathrm{FS}_n(X)$ for every set $X \subseteq G$ of size λ. The next step is the introduction of an extraction principle $\mathrm{Extract}_n(\kappa, \lambda, \ldots)$ that is sufficient for the reduction of $S_n(\kappa, \lambda, \theta)$ into a rectangular-type strengthening $\kappa \overset{\text{sup}}{\not\to} [\lambda, \lambda]^n_\theta$ of the classical partition relation $\kappa \not\to [\lambda]^n_\theta$. This leaves us with two independent tasks: proving instances of $\mathrm{Extract}_n(\kappa, \lambda, \ldots)$, and proving instances of $\kappa \overset{\text{sup}}{\not\to} [\lambda, \lambda]^n_\theta$. The harder task is the latter, and the second main result of this paper is an extension of Todorčević's theorem [Tod94] that Chang's conjecture fails iff $\omega_2 \not\to [\omega_1]^3_{\omega_1}$ holds. Here $\omega_2 \not\to [\omega_1]^3_{\omega_1}$ is improved to $\omega_2 \overset{\text{sup}}{\not\to} [\omega_1, \omega_1]^3_{\omega_1}$. Specifically:

Theorem C. *The following are equivalent:*

1. *$(\aleph_2, \aleph_1) \twoheadrightarrow (\aleph_1, \aleph_0)$ fails;*
2. *There exists a coloring $c : [\omega_2]^3 \to \omega_1$ with the property that for all disjoint $A, B \subseteq \omega_2$ of order-type ω_1 such that $\sup(A) = \sup(B)$, for every color $\tau < \omega_1$, there is $(\alpha, \beta, \gamma) \in [A \cup B]^3 \setminus ([A]^3 \cup [B]^3)$ such that $c(\alpha, \beta, \gamma) = \tau$.*

1.1. **Organization of this paper.** In Section 2, we provide some necessary preliminaries.

In Section 3, we recall the definition of a weak Kurepa tree and study related objects such as the branch spectrum $T(\mu, \theta)$. This will play a role in both getting instances of $\mathrm{Extract}_n(\kappa, \lambda, \ldots)$ and of $\kappa \overset{\text{sup}}{\not\to} [\lambda, \lambda]^n_\theta$.

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[3] As $*$ is not assumed to be associative, the claim is that we get $c(x * y * z) = i$ for both implementations of $x * y * z$.

In Section 4, we prove that $S_n(\kappa,\lambda,\theta)$ implies that any well-behaved magma $(G,*)$ of size κ admits a coloring with the strong properties mentioned earlier. It is proved that in the special case of $\lambda=\kappa$, $S_2(\kappa,\lambda,\theta)$ already follows from $\kappa\nrightarrow[\lambda;\lambda]^2_\theta$, and that, in general, $S_n(\kappa,\lambda,\theta)$ follows from $\kappa\overset{\sup}{\nrightarrow}[\lambda,\lambda]^n_\theta$ together with $\mathrm{Extract}_n(\kappa,\lambda,\omega,\omega)$. We then use tree combinatorics to obtain sufficient conditions for $\mathrm{Extract}_n(\kappa,\lambda,\dots)$ to hold. The definitions of $\mathrm{Extract}_n(\kappa,\lambda,\theta,\chi)$ and $\kappa\overset{\sup}{\nrightarrow}[\lambda,\lambda]^n_\theta$ will be found in this section as Definitions 4.17 and 4.20.

In Section 5, we prove the general case of Theorem C in which $\aleph_2$ is substituted by the double successor of a cardinal μ satisfying $\mu^{<\mu}=\mu$. The proof is a bit long, since the analysis goes through a division into a total of six cases and subcases.

In Section 6, we verify that Todorčević's theorems on the correspondence between unstable sets and oscillation remain valid in the rectangular context. We then combine it with the results of Section 5 and get that $\lambda^+\overset{\sup}{\nrightarrow}[\lambda,\lambda]^3_\omega$ holds for every successor $\lambda=\mu^+$ of an infinite cardinal $\mu=\mu^{<\mu}$.

In Section 7, we obtain the intended applications in additive Ramsey theory. Theorem A′ is gotten as a corollary of the results of Sections 4 and 6, and Theorem B is gotten as a corollary of a theorem asserting that $S_2(\kappa,\mu^+,2)$ holds whenever there exists a weak μ-Kurepa tree with κ-many branches.

2. Preliminaries

In this section, $\kappa,\lambda,\mu,\theta,\chi$ stand for nonzero cardinals, and n stand for a positive integer. We let H_κ denote the collection of all sets of hereditary cardinality less than κ. We write $[\kappa]^\lambda:=\{A\subseteq\kappa\mid |A|=\lambda\}$ and $[\kappa]^{<\lambda}:=\{A\subseteq\kappa\mid |A|<\lambda\}$. Let $E^\kappa_\chi:=\{\alpha<\kappa\mid \mathrm{cf}(\alpha)=\chi\}$, and define $E^\kappa_{\le\chi}$, $E^\kappa_{<\chi}$, $E^\kappa_{\ge\chi}$, $E^\kappa_{>\chi}$, $E^\kappa_{\neq\chi}$ analogously. For two distinct functions $f,g\in{}^\theta\mu$, write $f<_{\mathrm{lex}}g$ to mean that $f(\delta)<g(\delta)$ for the least $\delta<\lambda$ such that $f(\delta)\neq g(\delta)$. For functions $f,g\in{}^{\le\theta}\mu$, we write $f\sqsubseteq g$ to mean that $\mathrm{dom}(f)\le\mathrm{dom}(g)$ and $g\restriction\mathrm{dom}(f)=f$.

For sets of ordinals $A_1,\dots,A_n$, we define

$$A_1\circledast\cdots\circledast A_n:=\{(\alpha_1,\dots,\alpha_n)\in A_1\times\cdots\times A_n\mid \alpha_1<\cdots<\alpha_n\}.$$

By convention, whenever we write $(\alpha_1,\dots,\alpha_n)\in[A]^n$ (as opposed to $\{\alpha_1,\dots,\alpha_n\}\in[A]^n$), we mean that $(\alpha_1,\dots,\alpha_n)\in A\circledast\cdots\circledast A$.

For a set of ordinals A, we write $\mathrm{ssup}(A):=\sup\{\alpha+1\mid\alpha\in A\}$, $\mathrm{acc}^+(A):=\{\alpha<\mathrm{ssup}(A)\mid \sup(a\cap\alpha)=\alpha>0\}$, and $\mathrm{acc}(A):=A\cap\mathrm{acc}^+(a)$. For two sets of ordinals A and B, we write $A<B$ to mean that $A\times B$ coincides with $A\circledast B$.

Definition 2.1 (Positive round-bracket relations, [EHR65, §3])**.** $\kappa\rightarrow(\lambda)^n_\theta$ asserts that for every coloring $c:[\kappa]^n\rightarrow\theta$, there exists $A\subseteq\kappa$ of order-type λ such that c is constant over $[A]^n$.

Definition 2.2 (Negative square-bracket relations, [EHR65, §18])**.** A coloring $c:[\kappa]^n\rightarrow\theta$ is said to witness:

- $\kappa\nrightarrow[\lambda]^n_\theta$ iff $c[[A]^n]=\theta$ for every $A\in[\kappa]^\lambda$;
- $\kappa\nrightarrow[\lambda_1,\dots,\lambda_n]^n_\theta$ iff $c[A_1\times\cdots\times A_n]=\theta$ for every $\langle A_i\mid 1\le i\le n\rangle\in\prod_{i=1}^n[\kappa]^{\lambda_i}$;
- $\kappa\nrightarrow[\lambda_1;\dots;\lambda_n]^n_\theta$ iff $c[A_1\circledast\cdots\circledast A_n]=\theta$ for every $\langle A_i\mid 1\le i\le n\rangle\in\prod_{i=1}^n[\kappa]^{\lambda_i}$.

Note that $(\kappa\nrightarrow[\lambda;\dots;\lambda]^n_\theta)\implies(\kappa\nrightarrow[\lambda,\dots,\lambda]^n_\theta)\implies(\kappa\nrightarrow[\lambda]^n_\theta)$.

Definition 2.3 (Fiber maps)**.** Given a coloring of pairs $c : [\kappa]^2 \to \theta$ and some $\beta < \kappa$, we sometimes write c_β for the β^{th}-fiber map of c, that is, for the unique map $c_\beta : \beta \to \theta$ to satisfy $c_\beta(\alpha) = c(\alpha, \beta)$ for every $\alpha < \beta$.

We say that c has *injective fibers* iff c_β is injective of every $\beta < \kappa$.

Definition 2.4 ([LHR18])**.** $\mathrm{U}(\kappa, \mu, \theta, \chi)$ asserts the existence of a coloring $c : [\kappa]^2 \to \theta$ such that for every $\sigma < \chi$, every pairwise disjoint subfamily $\mathcal{A} \subseteq [\kappa]^\sigma$ of size κ, for every $\tau < \theta$, there exists $\mathcal{B} \subseteq \mathcal{A}$ of size μ such that $\min(c[a \times b]) > \tau$ for all $a \neq b$ from $\mathcal{B}$.

Remark 2.5. Of special interest are witnesses $c : [\kappa]^2 \to \theta$ to $\mathrm{U}(\kappa, \mu, \theta, \chi)$ that are moreover *subadditive*, i.e., satisfying that for all $\alpha < \beta < \gamma < \kappa$, the following hold:

- $c(\alpha, \gamma) \leq \max\{c(\alpha, \beta), c(\beta, \gamma)\}$;
- $c(\alpha, \beta) \leq \max\{c(\alpha, \gamma), c(\beta, \gamma)\}$.

These colorings are studied in [LHR23], and they will show up here in Section 5.

Given a coloring $c : [\kappa]^2 \to \theta$ and a subset $X \subseteq \kappa$ of order-type λ, we say that "$c \restriction [X]^2$ witnesses $\mathrm{U}(\lambda, \mu, \theta, \chi)$" if for the order-preserving bijection $\pi : \lambda \leftrightarrow X$, the coloring $d : [\lambda]^2 \to \theta$ defined via $d(\alpha, \beta) := c(\pi(\alpha), \pi(\beta))$ is a witness for $\mathrm{U}(\lambda, \mu, \theta, \chi)$. To be able to express that this happens globally, we introduce the following 5-cardinal extension of the principle of Definition 2.4.

Definition 2.6. $\mathrm{U}(\kappa, \lambda, \mu, \theta, \chi)$ asserts the existence of a coloring $c : [\kappa]^2 \to \theta$ such that for every $\sigma < \chi$, every pairwise disjoint subfamily $\mathcal{A} \subseteq [\kappa]^\sigma$ of size λ, for every $\tau < \theta$, there exists $\mathcal{B} \subseteq \mathcal{A}$ of size μ such that $\min(c[a \times b]) > \tau$ for all $a \neq b$ from $\mathcal{B}$.

Fact 2.7 ([Tod07, Lemma 9.2.3])**.** *For every regular uncountable cardinal λ, if $\mathrm{U}(\lambda^+, \lambda, 2, \lambda, 2)$ holds, then there exists a subadditive witness to $\mathrm{U}(\lambda^+, \lambda, \lambda, \lambda, \omega)$.*

Finally, we arrive at the notion motivating this paper.

Definition 2.8. For a magma $(G, *)$, we write $G \nrightarrow [\lambda]^{\mathrm{FS}_n}_\theta$ to assert that there exists a coloring $c : G \to \theta$ with the property that for every subset $A \subseteq G$ of size λ and every prescribed color $\tau < \theta$, there is an injective sequence $\langle a_i \mid 1 \leq i \leq n\rangle$ of elements of A such that $c(a_1 * \cdots * a_n) = \tau$ for *all* implementations of $a_1 * \cdots * a_n$.[4]

In the special case of $n = 2$, [FBR17, Corollary 4.5] and [RZ21, Corollary 2.20] provide sufficient conditions for $G \nrightarrow [\lambda]^{\mathrm{FS}_n}_\theta$ to follow from $|G| \nrightarrow [\lambda]^n_\theta$ for all values of θ. Higher dimensional reductions are out of reach at present.

2.1. **Walks on ordinals.** In this subsection, we provide a minimal background on walks on ordinals. This background is only necessary for Section 6, hence the exposition here is quite succinct. A thorough treatment may be found in [Tod07].

For the rest of this subsection, κ denotes a regular uncountable cardinal, and we fix some *C-sequence over* κ, that is, a sequence $\vec{C} = \langle C_\beta \mid \beta < \kappa\rangle$ such that, for every $\beta < \kappa$, C_β is closed subset of β with $\sup(C_\beta) = \sup(\beta)$.

Definition 2.9 (Todorčević)**.** From $\vec{C}$, derive maps $\mathrm{Tr} : [\kappa]^2 \to {}^\omega\kappa$, $\rho_2 : [\kappa]^2 \to \omega$, and $\mathrm{tr} : [\kappa]^2 \to {}^{<\omega}\kappa$, by letting for all $\alpha < \beta < \kappa$:

[4] The issue of implementation arises from the fact that we do not assume $*$ to be associative, e.g., it is possible that $(a_1 * a_2) * a_3 \neq a_1 * (a_2 * a_3)$.

- $\mathrm{Tr}(\alpha,\beta):\omega\to\kappa$ is defined by recursion on $n<\omega$:

$$\mathrm{Tr}(\alpha,\beta)(n):=\begin{cases}\beta, & n=0\\ \min(C_{\mathrm{Tr}(\alpha,\beta)(n-1)}\setminus\alpha), & n>0\ \&\ \mathrm{Tr}(\alpha,\beta)(n-1)>\alpha\\ \alpha, & \text{otherwise}\end{cases}$$

- $\rho_2(\alpha,\beta):=\min\{l<\omega\mid \mathrm{Tr}(\alpha,\beta)(l)=\alpha\}$;
- $\mathrm{tr}(\alpha,\beta):=\mathrm{Tr}(\alpha,\beta)\restriction\rho_2(\alpha,\beta)$.

To explain: Given a pair of ordinals $\alpha<\beta$ below κ, one would like to *walk* from β down to α. This is done by recursion, letting $\beta_0:=\beta$, and $\beta_{n+1}:=\min(C_{\beta_n}\setminus\alpha)$, thus, obtaining an ordinal β_{n+1} such that $\alpha\leq\beta_{n+1}\leq\beta_n$. Since the ordinals are well-founded, there must exist some integer k such that $\beta_{k+1}=\alpha$, so that, the walk is $\beta=\beta_0>\beta_1>\cdots>\beta_{k+1}=\alpha$. This walk is recorded by $\mathrm{Tr}(\alpha,\beta)$, since, for every $n\leq k$, we have that $\mathrm{Tr}(\alpha,\beta)=\beta_n$, and for every $n>k$, we have that $\mathrm{Tr}(\alpha,\beta)=\alpha$. The length of the walk is recorded by the positive integer $\rho_2(\alpha,\beta)$. Now, since $\mathrm{Tr}(\alpha,\beta)$ is eventually constant with value α, its nontrivial part is those ordinals greater than α, i.e., $\beta_0>\beta_1>\cdots>\beta_k$; this is recorded by $\mathrm{tr}(\alpha,\beta)$.

Definition 2.10 ([Rin14, Definition 2.8])**.** Define a function $\lambda_2:[\kappa]^2\to\kappa$ via

$$\lambda_2(\alpha,\beta):=\sup(\alpha\cap\{\sup(C_\eta\cap\alpha)\mid\eta\in\mathrm{Im}(\mathrm{tr}(\alpha,\beta))\}).$$

Note that $\lambda_2(\alpha,\beta)<\alpha$ whenever $0<\alpha<\beta<\kappa$, since $\mathrm{tr}(\alpha,\beta)$ is a finite sequence.

Fact 2.11 ([LHR18, Lemma 4.7])**.** *Suppose that $\lambda_2(\alpha,\beta)<\epsilon<\alpha<\beta<\kappa$. Then $\mathrm{tr}(\epsilon,\beta)$ end-extends $\mathrm{tr}(\alpha,\beta)$, and one of the following cases holds:*

(1) $\alpha\in\mathrm{Im}(\mathrm{tr}(\epsilon,\beta))$; or
(2) $\alpha\in\mathrm{acc}(C_{\eth})$ for $\eth:=\min(\mathrm{Im}(\mathrm{tr}(\alpha,\beta)))$.

Definition 2.12 ([RZ21, Definition 2.10])**.** For every $(\alpha,\beta)\in[\kappa]^2$, we define an ordinal $\eth_{\alpha,\beta}\in[\alpha,\beta]$ via:

$$\eth_{\alpha,\beta}:=\begin{cases}\min(\mathrm{Im}(\mathrm{tr}(\alpha,\beta))), & \alpha\in\mathrm{acc}(C_{\min(\mathrm{Im}(\mathrm{tr}(\alpha,\beta)))});\\ \alpha, & \text{otherwise;}\end{cases}$$

Remark 2.13. It is easy to see that $\sup(C_{\eth_{\alpha,\beta}})=\sup(\alpha)$ for all $\alpha<\beta<\kappa$, and it follows from Fact 2.11 that

$$\mathrm{tr}(\epsilon,\beta)=\mathrm{tr}(\eth_{\alpha,\beta},\beta)^\frown\mathrm{tr}(\epsilon,\eth_{\alpha,\beta}),$$

whenever $\lambda_2(\alpha,\beta)<\epsilon<\alpha<\beta<\kappa$.

Fact 2.14 (Todorčević, [Tod07, §9])**.** *If $\kappa=\lambda^+$ for a regular cardinal λ and $\mathrm{otp}(C_\beta)\leq\lambda$ for all $\beta<\kappa$, then there exists a subadditive coloring $\rho:[\kappa]^2\to\lambda$ with the property that $\rho(\alpha,\beta)\geq\mathrm{otp}(C_\eta\cap\alpha)$ for all $\alpha<\beta<\kappa$ and $\eta\in\mathrm{Im}(\mathrm{tr}(\alpha,\beta))$.*

3. Weak Kurepa trees and the branch spectrum

In this section, μ denotes a cardinal and θ denotes an ordinal.

Definition 3.1. $\mathcal{T}(\mu,\theta)$ denotes the collection of all subsets $T\subseteq{}^{<\theta}\mu$ such that the following two hold:

(1) T is downward-closed, i.e, for every $t\in T$, $\{t\restriction\alpha\mid\alpha<\theta\}\subseteq T$;
(2) for every $\alpha<\theta$, the set $T_\alpha:=T\cap{}^{\alpha}\mu$ is nonempty and has size $<\mu$.

We say that T is a *tree of height* θ if there exists a cardinal μ such that $T \in \mathcal{T}(\mu,\theta)$.[5] Note that θ is uniquely determined. For such a tree T, we shall refer to T_α as the α^{th}-*level* of T, and the set $\{b \in {}^\theta\mu \mid \forall\alpha<\theta\,(b\restriction\alpha \in T_\alpha)\}$ of all branches through T is denoted by $\mathcal{B}(T)$. Also, for all $f,g \in {}^{\le\mu}\theta$, we let

$$\Delta(f,g) := \begin{cases} \min\{\delta \in \mathrm{dom}(f)\cap\mathrm{dom}(g) \mid f(\delta)\neq g(\delta)\}, & \text{if } f \not\subseteq g \ \&\ g \not\subseteq f;\\ \min\{\mathrm{dom}(f),\mathrm{dom}(g)\}, & \text{otherwise.}\end{cases}$$

Definition 3.2. $T \in \mathcal{T}(\mu,\theta)$ is said to be *normal* iff for all $\alpha<\beta<\theta$ and $t \in T_\alpha$, there exists $t' \in T_\beta$ with $t \sqsubseteq t'$.

Definition 3.3. Given a tree T and a subset $B \subseteq \mathcal{B}(T)$, we consider the subtree:

$$T^{\rightsquigarrow B} := \{t \in T \mid |\{b \in B \mid t \sqsubseteq b\}| = |B|\}.$$

Lemma 3.4. *Suppose that $T \in \mathcal{T}(\mu,\theta)$, and λ is an infinite regular cardinal.*

(1) If $\lambda \ge \mu$, then for every $B \in [\mathcal{B}(T)]^\lambda$, $T^{\rightsquigarrow B}$ is in $\mathcal{T}(\mu,\theta)$ and is normal;
(2) If $\lambda \ge \max\{\mu,|\theta|^+\}$, then for all $A,B \in [\mathcal{B}(T)]^\lambda$, there are $s \in T$ and $i \neq i'$ such that $s^\frown\langle i\rangle \in T^{\rightsquigarrow A}$ and $s^\frown\langle i'\rangle \in T^{\rightsquigarrow B}$.

Proof. (1) Suppose that $B \in [\mathcal{B}(T)]^\lambda$ and $\lambda \ge \mu$. It is clear that $\emptyset \in T^{\rightsquigarrow B}$. Thus, to prove that $T^{\rightsquigarrow B}$ has height θ and is normal, let $\alpha<\beta<\theta$ and $t \in (T^{\rightsquigarrow B})_\alpha$, and we shall show that there exists $t' \in (T^{\rightsquigarrow B})_\beta$ extending t.

By the choice of t, $B' := \{b \in B \mid t \sqsubseteq b\}|$ has size λ. Since $T \in \mathcal{T}(\mu,\theta)$, it is the case that $0 < |T_\beta| < |B'| = \mathrm{cf}(|B'|)$, and then the pigeonhole principle provides $t' \in T_\beta$ such that $\{b \in B' \mid t' \sqsubseteq b\}$ has size λ. Evidently, t' is as sought.

(2) Suppose that $A,B \in [\mathcal{B}(T)]^\lambda$ and $\lambda \ge \max\{\mu,|\theta|^+\}$. By possibly passing to λ-sized subsets of A and B, we may assume that $A \cap B = \emptyset$. Let $\langle a_j \mid j<\lambda\rangle$ be some injective enumeration of A, and likewise let $\langle b_j \mid j<\lambda\rangle$ be some injective enumeration of B. For each $j<\lambda$, as $a_j \neq b_j$, we may let $\delta_j := \Delta(a_j,b_j)+1$. As λ is a regular cardinal greater than $|\theta|$, we may fix some $J \in [\lambda]^\lambda$ on which the map $j \mapsto \delta_j$ is constant with value, say, δ. As $T_{\delta+1}$ has size $<\mu\le\lambda$, we may moreover assume that the map $j \mapsto ((a_j\restriction\delta+1),(b_j\restriction\delta+1))$ is constant over J, with value, say, $(s^\frown\langle i\rangle, s^\frown\langle i'\rangle)$. Then, we are done. □

Corollary 3.5. *Suppose that $T \in \mathcal{T}(\lambda,\theta)\cap\mathcal{P}({}^{<\theta}2)$, where $\lambda = \mathrm{cf}(\lambda) > \mathrm{cf}(\theta) \ge \omega$. Suppose that we are given $i<2$ and $X \in [\mathcal{B}(T)]^\lambda$. Then, for λ-many $x \in X$, there are cofinally many $\delta<\theta$ such that the following two hold:*

(1) $x(\delta) = i$;
(2) $\{y \in X \mid \Delta(x,y)=\delta\}$ has size λ.

Proof. Suppose not. In particular, the set Y of all $x \in X$ for which there are boundedly many $\delta<\theta$ satisfying Clauses (1) and (2) has size λ. So, for each $x \in Y$, the following ordinal is smaller than θ:

$$\epsilon_x := \sup\{\delta<\theta \mid x(\delta)=i \ \&\ |\{y \in X \mid \Delta(x,y)=\delta\}| = \lambda\}.$$

As $|Y| = \mathrm{cf}(\lambda) > \mathrm{cf}(\theta)$, we may find some $\epsilon<\theta$ such that $Z := \{x \in Y \mid \epsilon_x = \epsilon\}$ has size λ. As $|T_{\epsilon+1}| < \lambda$, we may also find some $t \in T_{\epsilon+1}$ such that $Z_t := \{x \in Z \mid t \sqsubseteq x\}$ has size λ. Now, by appealing to Lemma 3.4(2) with $\mu := \lambda$, $A := Z_t$ and $B := Z_t$, we may find $s \in T$ and $j<2$ such that $\hat{A} := \{a \in A \mid s^\frown\langle j\rangle \sqsubseteq a\}$ and $\hat{B} := \{b \in B \mid s^\frown\langle 1-j\rangle \sqsubseteq b\}$ are both of size λ. As $A = B$ and by possibly

[5] There is no loss of generality here, see [BR21, Lemma 2.5(2)].

switching the roles of $\hat{A}$ and $\hat{B}$, we may assume that $j = i$. Denote $\delta := \mathrm{dom}(s)$. For all $a \in \hat{A}$ and $b \in \hat{B}$, since $a, b \in Z_t$, both $s^\frown\langle i\rangle$ and $s^\frown\langle 1-i\rangle$ are compatible with t, so that $\delta = \mathrm{dom}(s) \geq \mathrm{dom}(t) > \epsilon$. Now, for every $x \in \hat{A}$, it is the case that $x(\delta) = i$ and $\{y \in X \mid \Delta(x, y) = \delta\}$ covers $\hat{B}$, but $|\hat{B}| = \lambda$, so we got a contradiction to the fact that $\delta > \epsilon = \epsilon_x$. $\square$

Lemma 3.6. *Suppose that $T \in \mathcal{T}(\mu, \mu)$, and μ is a regular uncountable cardinal. Suppose also that $\langle b_\xi \mid \xi < \mu\rangle$ is an injective enumeration of some $B \in [\mathcal{B}(T)]^\mu$. For every $\langle t_\alpha \mid \alpha < \mu\rangle \in \prod_{\alpha<\mu}(T^{\rightsquigarrow B} \cap {}^\alpha\mu)$, for club many $\alpha < \mu$,*

$$\sup(\{\Delta(b_\beta, t_\alpha) \mid \beta < \alpha\} \cap \alpha) = \alpha.$$

In particular, for club many $\alpha < \mu$,

$$\sup\{\gamma < \mu \mid \alpha \in \mathrm{acc}^+(\{\Delta(b_\beta, b_\gamma) \mid \beta < \alpha\})\} = \mu.$$

Proof. The ‘In particular’ part follows the main claim together with Lemma 3.4(1), using $\lambda := \mu$. Next, to prove the main claim, let $\langle t_\alpha \mid \alpha < \mu\rangle \in \prod_{\alpha<\mu}(T^{\rightsquigarrow B} \cap {}^\alpha\mu)$. Denote $\Gamma_\alpha := \{\gamma < \mu \mid t_\alpha \sqsubseteq b_\gamma\}$. Consider the club

$$C := \{\alpha \in \mathrm{acc}(\mu) \mid \forall\bar{\alpha} < \alpha\,[\min(\Gamma_{\bar{\alpha}} \setminus \bar{\alpha}) < \alpha]\}.$$

Note that for every $\alpha < \mu$, $D^\alpha := \{\Delta(b_\beta, t_\alpha) \mid \beta \in \alpha \setminus \Gamma_\alpha\}$ is a subset of α.

Claim 3.6.1. *The following set covers a club in μ:*

$$A := \{\alpha < \mu \mid \sup(D^\alpha) = \alpha\}$$

Proof. Suppose not. Fix an ordinal $\epsilon < \mu$ for which the following set is stationary:

$$S := \{\alpha \in C \mid \sup(D^\alpha) = \epsilon\}.$$

There are two cases to consider:

▶ Suppose that there exists a function $b : \mu \to \mu$ such that $S_b := \{\alpha \in S \mid b \restriction \alpha = t_\alpha\}$ is cofinal in μ. Pick $\bar{\alpha} \in S_b \setminus (\epsilon + 1)$. Since $\Gamma_{\bar{\alpha}}$ has more than one element, we may now find $\beta \in \Gamma_{\bar{\alpha}}$ such $b \neq b_\beta$. Then $\epsilon < \bar{\alpha} \leq \Delta(b_\beta, b) < \mu$. Pick $\alpha \in S_b$ above $\max\{\Delta(b_\beta, b), \beta\}$. Then $\epsilon < \Delta(b_\beta, t_\alpha) < \alpha$, contradicting the fact that $\sup(D^\alpha) = \epsilon$.

▶ Suppose the first case fails. First, since $|T_{\epsilon+1}| < \mu$, pick a node $t \in T_{\epsilon+1}$ such that $S' := \{\alpha \in S \mid t_\alpha \restriction (\epsilon+1) = t\}$ is stationary. Since for every function $b : \mu \to \mu$, the set $S_b := \{\alpha \in S \mid b \restriction \alpha = t_\alpha\}$ is bounded in μ, we may now pick a pair $(\bar{\alpha}, \alpha) \in S'$ such that $t_{\bar{\alpha}} \not\sqsubseteq t_\alpha$, so that $\epsilon < \Delta(t_{\bar{\alpha}}, t_\alpha) < \bar{\alpha}$. Let $\beta := \min(\Gamma_{\bar{\alpha}} \setminus \bar{\alpha})$. Then $b_\beta \restriction \bar{\alpha} = t_{\bar{\alpha}}$ and hence $\Delta(b_\beta, t_\alpha) = \Delta(t_{\bar{\alpha}}, t_\alpha)$. That is, $\epsilon < \Delta(b_\beta, t_\alpha) < \bar{\alpha} \leq \beta < \alpha$, contradicting the fact that $\sup(D^\alpha) = \epsilon$. $\square$

Let $\alpha \in A$. Recall that Γ_α has size μ, and note that, for every $\gamma \in \Gamma_\alpha$,

$$\{\Delta(b_\beta, b_\gamma) \mid \beta < \alpha\} \cap \alpha = \{\Delta(b_\beta, t_\alpha) \mid \beta < \alpha\} \cap \alpha = D^\alpha,$$

so we are done. $\square$

Lemma 3.7. *Suppose that $T \in \mathcal{T}(2^\mu, \mu) \cap \mathcal{P}({}^{<\mu}\mu)$, where μ is an infinite regular cardinal. For every $B \in [\mathcal{B}(T)]^\mu$, there exist $B' \in [B]^\mu$ and $\theta \leq \mu$ such that:*

1. *For every $B'' \in [B']^\mu$, $T^{\rightsquigarrow B''}$ is in $\mathcal{T}(\mu, \theta)$ and is normal;*
2. *If $\theta < \mu$ or if T contains no μ-Aronszajn subtrees, then $|\mathcal{B}(T^{\rightsquigarrow B''})| = \mu$ for every $B'' \in [B']^\mu$;*
3. *If $\theta < \mu$, then $|\{b \in B' \mid \mathrm{ssup}\{\Delta(t, b \restriction \theta) \mid t \in T^{\rightsquigarrow B'}\ \&\ t \not\sqsubseteq b\} < \theta\}| < \mu$.*

Proof. Let $B \in [\mathcal{B}(T)]^\mu$.

Claim 3.7.1. *If $T^{\rightsquigarrow B} \in \mathcal{T}(\mu,\mu)$, then the pair $(B',\theta) := (B,\mu)$ is as sought.*

Proof. Suppose that $T^{\rightsquigarrow B}$ is in $\mathcal{T}(\mu,\mu)$. For every $B'' \in [B]^\mu$, $T^{\rightsquigarrow B''} = (T^{\rightsquigarrow B})^{\rightsquigarrow B''}$, and hence Lemma 3.4(1) implies that $T^{\rightsquigarrow B''} \in \mathcal{T}(\mu,\mu)$ and is normal. In addition, if there exists some $B'' \in [B]^\mu$ such that $|\mathcal{B}(T^{\rightsquigarrow B''})| < \mu$, then looking at $B''' := B'' \setminus \mathcal{B}(T^{\rightsquigarrow B''})$, we get that $T^{\rightsquigarrow B'''}$ is a μ-Aronszajn subtree of T. □

From now on, suppose that $T^{\rightsquigarrow B} \in \mathcal{T}(2^\mu,\mu) \setminus \mathcal{T}(\mu,\mu)$. Let $\theta < \mu$ be the least such that $(T^{\rightsquigarrow B})_\theta$ has size $\geq \mu$. Let $\langle t_i \mid i < \mu\rangle$ be an injective sequence of elements of $(T^{\rightsquigarrow B})_\theta$. For each $i < \mu$, pick $b_i \in B$ such that $t_i \sqsubseteq b_i$, and set $B' := \{b_i \mid i < \mu\}$. To see that the pair (B',θ) is as sought, let $B'' \in [B']^\mu$.

Claim 3.7.2. *$T^{\rightsquigarrow B''}$ is in $\mathcal{T}(\mu,\theta)$ and is normal.*

Proof. For every $\alpha \in [\theta,\mu)$, it is the case that for every $t \in (T^{\rightsquigarrow B''})_\alpha$, there exists a unique $i < \mu$ such that $t_i \sqsubseteq t$, and hence $\{b \in B'' \mid t \sqsubseteq b\} \subseteq \{b_i\}$ is finite. Therefore, $(T^{\rightsquigarrow B''})_\alpha$ is empty. In addition, as $B'' \subseteq B' \subseteq B$, it is the case that $|(T^{\rightsquigarrow B''})_\alpha| \leq |(T^{\rightsquigarrow B})_\alpha| < \mu$ for all $\alpha < \theta$.

Clearly, $\emptyset \in T^{\rightsquigarrow B''}$. Finally, let $\alpha < \beta < \mu$ with $t \in (T^{\rightsquigarrow B''})_\alpha$, and we shall find $t' \in (T^{\rightsquigarrow B''})_\beta$ extending t. By the choice of t, $B^* := \{b \in B'' \mid t \sqsubseteq b\}|$ has size μ. By the minimality of θ, the map $b \mapsto b \restriction \beta$ from B^* to T_β cannot have an image of size μ, and hence there exists $B^{**} \in [B^*]^\mu$ on which the said map is constant, with some value, say t'. Clearly, t' is as sought. □

Claim 3.7.3. $|\mathcal{B}(T^{\rightsquigarrow B''})| = \mu$.

Proof. Suppose not. In particular, $I := \{i < \mu \mid b_i \in B'' \;\&\; t_i \notin \mathcal{B}(T^{\rightsquigarrow B''})\}$ has size μ. It follows that there exists an $\alpha < \theta$ such that $I_\alpha := \{i \in I \mid (t_i \restriction \alpha) \notin T^{\rightsquigarrow B''}\}$ has size μ. However, μ is a regular cardinal greater than $|T_\alpha| \geq |(T^{\rightsquigarrow B''})_\alpha|$, and hence there must exist some $s \in (T^{\rightsquigarrow B''})_\alpha$ such that $\{i \in I_\alpha \mid (t_i \restriction \alpha) = s\}$ has size μ. This is a contradiction. □

Claim 3.7.4. $|\{i < \mu \mid \operatorname{ssup}\{\Delta(t,t_i) \mid t \in T^{\rightsquigarrow B'} \;\&\; t \not\sqsubseteq t_i\} < \theta\}| < \mu$.

Proof. Suppose not, and pick $\epsilon < \theta$ such that the following set has size μ:

$$I := \{i < \mu \mid \operatorname{ssup}\{\Delta(t,t_i) \mid t \in T^{\rightsquigarrow B'} \;\&\; t \not\sqsubseteq t_i\} = \epsilon\}.$$

Then pick $s \in T_\epsilon$ such that $\{i \in I \mid t_i \restriction \epsilon = s\}$ has size μ. Finally, as in the proof of Lemma 3.4(2), we may find some $s' \in T^{\rightsquigarrow B'}$ extending s and $j \neq j'$ such that $\{i \in I \mid s'{}^\frown\langle j\rangle \sqsubseteq t_i\}$ and $\{i \in I \mid s'{}^\frown\langle j'\rangle \sqsubseteq t_i\}$ are both of size μ. In particular, there exist $i \neq i'$ in I such that $t_i \cap t_{i'} = s'$. So $\Delta(s'{}^\frown\langle i'\rangle, t_i) \geq \epsilon$, contradicting the fact that $i \in I$. □

This completes the proof. □

Definition 3.8. Let μ denote an infinite cardinal.

1. A *weak μ-Kurepa tree* is a tree T of height μ, of size μ, satisfying $|\mathcal{B}(T)| > \mu$;
2. A *μ-Kurepa tree* is a tree T of height μ for which $\{\alpha < \mu \mid |T_\alpha| > |\alpha|\}$ is nonstationary, and $|\mathcal{B}(T)| > \mu$.

Remark 3.9. As in Exercise 34 of [Kun80, §II], if there exists a μ-Kurepa tree (resp. weak μ-Kurepa tree), then there exists one which is a subset of ${}^{<\mu}2$.

Definition 3.10 (Branch spectrum)**.** $T(\mu,\theta)$ stands for the collection of all cardinals κ for which there exists $T \in \mathcal{T}(\mu,\theta)$ with $\kappa \leq |\mathcal{B}(T)|$.

Proposition 3.11. *Let μ and θ denote infinite cardinals. Then:*

1. $\sup(T(\mu,\theta)) \leq \mu^\theta$;
2. *If θ is the least cardinal to satisfy $\mu^\theta > \mu$, then* $\max(T(\mu^+,\theta)) = (\mu^+)^\theta$;
3. *If there exists a weak μ-Kurepa tree, then $\mu^+ \in T(\mu^+,\mu)$;*
4. *If there exists a μ-Kurepa tree, then $\mu^+ \in T(\mu,\mu)$;*
5. *If μ is a strong limit, then $2^\mu \in T(\mu,\mathrm{cf}(\mu))$.*

Proof. Clear. $\square$

Remark 3.12. $T(\mu,\theta)$ need not have a maximal element. By [Poó21], it is consistent for $T(\omega_1,\omega_1)$ to have $\aleph_{\omega_2}$ has a supremum that is not attained. Note, however, that $\mathcal{T}(\mu,\theta)$ is closed under unions of length $< \mathrm{cf}(\mu)$, and hence $T(\mu,\theta)$ is $<\mathrm{cf}(\mu)$-closed.

Proposition 3.13. *For every $\kappa \in T(\mu,\theta)$, there exists a coloring $c : [\kappa]^2 \to \theta$ witnessing $\mathrm{U}(\kappa,\lambda,\lambda,\theta,2)$ for every regular cardinal $\lambda \in [\mu,\kappa]$.*

Proof. Given $\kappa \in T(\mu,\theta)$, let us fix $T \in \mathcal{T}(\mu,\theta)$ admitting an injective sequence $\langle b_\xi \mid \xi < \kappa\rangle$ consisting of elements of $\mathcal{B}(T)$. Define $c : [\kappa]^2 \to \theta$ via $c(\alpha,\beta) := \Delta(b_\alpha,b_\beta)$. Now, given $\tau < \theta$ and $A \in [\kappa]^\lambda$ for a regular cardinal $\lambda \in [\mu,\kappa]$, since $|T_{\tau+1}| < \mu$, it is possible to find $x \in T_{\tau+1}$ for which $B := \{\alpha \in A \mid x \sqsubseteq b_\alpha\}$ has size λ. Evidently, $c(\alpha,\beta) > \tau$ for all $\alpha \neq \beta$ from B. $\square$

4. Coloring well-behaved magmas

In this section, we obtain sufficient conditions for $G \nrightarrow [\lambda]^{\mathrm{FS}_n}_\theta$ to hold. To ease on the reader, we start with the special case of $n = 2$. The upcoming Lemma 4.2 reduces this case to the following simple combinatorial principle.

Definition 4.1. $S_2(\kappa,\lambda,\theta)$ asserts the existence of a coloring $d : [\kappa]^{<\omega} \to \theta$ such that, for every $\mathcal{X} \subseteq [\kappa]^{<\omega}$ of size λ and every prescribed color $\tau < \theta$, there exist two distinct $x, y \in \mathcal{X}$ such that $d(z) = \tau$ whenever $(x \mathbin{\triangle} y) \subseteq z \subseteq (x \cup y)$.

Lemma 4.2 ([FBR17, Theorem 4.7])**.** *Suppose that $S_2(\kappa,\lambda,\theta)$ holds, for given cardinals $\theta \leq \lambda \leq \kappa$ with λ regular and uncountable.*

Then $G \nrightarrow [\lambda]^{\mathrm{FS}_2}_\theta$ holds for every well-behaved magma $(G,*)$ with $|G| = \kappa$.

Proof. Let d be coloring witnessing $S_2(\kappa,\lambda,\theta)$. Suppose that $(G,*)$ is a well-behaved magma with $|G| = \kappa$. By identifying $[G]^{<\omega}$ with $[\kappa]^{<\omega}$, we may thus fix a map $\varphi : G \to [\kappa]^{<\omega}$ such that:

- φ is $<\lambda$-to-one;
- for all $x \neq y$ in G, $\varphi(x) \mathbin{\triangle} \varphi(y) \subseteq \varphi(x * y) \subseteq \varphi(x) \cup \varphi(y)$.

Define a coloring $c : G \to \theta$ by letting $c := d \circ \varphi$. To see that c is as sought, let $X \in [G]^\lambda$. As φ is $<\lambda$-to-one, $\mathcal{X} := \{\varphi(x) \mid x \in X\}$ has size λ. Thus, given a prescribed color $\tau < \theta$, we may find $x, y \in X$ with $\varphi(x) \neq \varphi(y)$ such that $d(z) = \tau$ whenever $(\varphi(x) \mathbin{\triangle} \varphi(y)) \subseteq z \subseteq (\varphi(x) \cup \varphi(y))$. In particular, $x \neq y$ and $c(x * y) = d(\varphi(x * y)) = \tau$. $\square$

The question arises: How do one obtain instances of $S_2(\ldots)$? The proof of [FBR17, Lemma 3.4] makes it clear that the following holds:

Fact 4.3. *Suppose that λ is a regular uncountable cardinal and that θ is an infinite cardinal. Then $\mathrm{Pr}_1(\kappa,\lambda,\theta,\omega)$ implies $S_2(\kappa,\lambda,\theta)$.*

Remark 4.4. The principle $\mathrm{Pr}_1(\kappa,\lambda,\theta,\omega)$ is a particular strengthening of $\kappa \nrightarrow [\lambda;\lambda]^2_\theta$. Since it will not play a role in this paper, we omit its definition, and settle for pointing out the following corollary. By a theorem of Fleissner [Fle78, §5], for every regular uncountable cardinal κ, in the forcing extension for adding κ-many Cohen reals, $\mathrm{Pr}_1(\kappa,\omega_1,\omega,\omega)$ holds. It thus follows that if κ is a regular cardinal $\geq \mathfrak{c}$, then after adding κ-many Cohen reals, $S_2(2^{\aleph_0},\aleph_1,\aleph_0)$ holds.

In case that $\lambda=\kappa$, we can now improve Fact 4.3, as follows.

Theorem 4.5. *Suppose that κ is a regular uncountable cardinal and that θ is an infinite cardinal. Then $\kappa \nrightarrow [\kappa;\kappa]^2_\theta$ implies $S_2(\kappa,\kappa,\theta)$.*

Proof. Suppose that $c:[\kappa]^2\rightarrow\theta$ is a coloring witnessing $\kappa \nrightarrow [\kappa;\kappa]^2_\theta$. Fix a bijection $\pi:\theta\leftrightarrow\theta\times\omega$, and then find $c_0:[\kappa]^2\rightarrow\theta$ and $c_1:[\kappa]^2\rightarrow\omega$ such that $\pi(c(\alpha,\beta))=(c_0(\alpha,\beta),c_1(\alpha,\beta))$ for every $(\alpha,\beta)\in[\kappa]^2$.

Define a coloring $d:[\kappa]^{<\omega}\rightarrow\theta$, as follows. For $z\in[\kappa]^{<2}$, just let $d(z):=0$. Next, for $z\in[\kappa]^{<\omega}$ of size ≥ 2, first let $\langle \alpha_i \mid i<|z|\rangle$ denote the increasing enumeration of z, and then let $d(z):=c_0(\alpha_{j_z},\alpha_{j_z+1})$, for

$$j_z := \min\{j<|z|-1 \mid c_1(\alpha_j,\alpha_{j+1})=\max\{c_1(\alpha_i,\alpha_{i+1}) \mid i<|z|-1\}\}.$$

To see this works, suppose that we are given a κ-sized family $\mathcal{X}\subseteq[\kappa]^{<\omega}$, and a prescribed color $\tau<\theta$, By thinning out, we may assume that $\mathcal{X}$ forms an head-tail-tail Δ-system with some root r, i.e., for all $x\neq y$ from $\mathcal{X}$, r is an initial segment of both x and y, and either $x<(y\setminus r)$ or $y<(x\setminus r)$. By further thinning out, we may assume the existence of some $n<\omega$ such that $c_1``[x]^2\subseteq n$ for all $x\in\mathcal{X}$. Split $\mathcal{X}$ into two κ-sized sets $\mathcal{X}=\mathcal{X}_0\cup\mathcal{X}_1$. Set $A:=\{\max(x)\mid x\in\mathcal{X}_0\}$ and $B:=\{\min(x\setminus r)\mid x\in\mathcal{X}_1\}$. As c witnesses $\kappa \nrightarrow [\kappa;\kappa]^2_\theta$, fix $(\alpha,\beta)\in A\circledast B$ such that $c(\alpha,\beta)=\pi^{-1}(\tau,n)$. Pick the unique $x,y\in\mathcal{X}$ such that $\alpha=\max(x)$ and $\beta=\min(y\setminus r)$. As $\mathcal{X}_0\cap\mathcal{X}_1=\emptyset$, $x\neq y$. Consequently, $x<(y\setminus r)$. Now fix an arbitrary set z such that $(x\mathbin{\triangle} y)\subseteq z\subseteq(x\cup y)$. Clearly $|z|\geq 2$. Let $\langle\alpha_i \mid i<|z|\rangle$ denote the increasing enumeration of z. For every $i<|z|$, if $\{\alpha_i,\alpha_{i+1}\}\subseteq x$ then $c_1(\alpha_i,\alpha_{i+1})<n$, and likewise, if $\{\alpha_i,\alpha_{i+1}\}\subseteq y$ then $c_1(\alpha_i,\alpha_{i+1})<n$. As $x<(y\setminus r)$ and $\{\alpha,\beta\}\subseteq z$, it follows that there exists $j<|z|$ such that $\alpha_j=\alpha$ and $\alpha_{j+1}=\beta$. For this j, we would have $c_1(\alpha_j,\alpha_{j+1})=c_1(\alpha,\beta)=n$. Altogether, $d(z)=c_0(\alpha,\beta)=\tau$, as sought. $\square$

As for the general case (i.e., $\lambda\leq\kappa$), we now present an extraction principle that is sufficient to derive $S_2(\kappa,\lambda,\theta)$ from $\kappa \nrightarrow [\lambda;\lambda]^2_\theta$. Roughly speaking, the upcoming principle asserts the existence of a map $e:[\kappa]^{<\omega}\rightarrow[\kappa]^2$ that, in some scenarios, manages to extract two distinguished points $e(z)$ from any given set $z\in[\kappa]^{<\omega}$. When reading the next definition for the first time, the readers may want to ease on themselves and assume that $\theta=\chi=\omega$.

Definition 4.6. $\mathrm{Extract}_2(\kappa,\lambda,\theta,\chi)$ asserts the existence of a map $e:[\kappa]^{<\omega}\rightarrow[\kappa]^2$ satisfying that for every sequence $\langle x_\gamma \mid \gamma<\lambda\rangle$ of subsets of κ, every $r\in[\kappa]^{<\theta}$, and every nonzero $\sigma<\chi$ such that:

(1) for every $(\gamma,\gamma')\in[\lambda]^2$, $x_\gamma\cap x_{\gamma'}\subseteq r$;
(2) for every $\gamma<\lambda$, $y_\gamma:=x_\gamma\setminus r$ has order-type σ,

there exist $j<\sigma$ and disjoint cofinal subsets Γ_0,Γ_1 of λ satisfying the following:

(a) For every $(\gamma,\gamma')\in[\Gamma_0\cup\Gamma_1]^2$, $y_\gamma(j)<y_{\gamma'}(j)$;
(b) For every $(\gamma,\gamma')\in(\Gamma_0\circledast\Gamma_1)\cup(\Gamma_1\circledast\Gamma_0)$, for every $z\in[x_\gamma\cup x_{\gamma'}]^{<\omega}$ covering $\{y_\gamma(j),y_{\gamma'}(j)\}$, we have
$$e(z)=(y_\gamma(j),y_{\gamma'}(j)).$$

Remark 4.7. Without loss of generality, we may assume that $e(z)\in[z]^2$ for every $z\in[\kappa]^{<\omega}$ of size ≥ 2. Also note that $\mathrm{Extract}_2(\kappa,\kappa,\mathrm{cf}(\kappa),2)$ is a theorem of ZFC.

Lemma 4.8. *Suppose that λ is a regular uncountable cardinal and that θ is an arbitrary cardinal. If $\kappa\nrightarrow[\lambda;\lambda]^2_\theta$ and $\mathrm{Extract}_2(\kappa,\lambda,\omega,\omega)$ both hold, then so does $S_2(\kappa,\lambda,\theta)$.*

Proof. Suppose that $c:[\kappa]^2\rightarrow\theta$ is a witness for $\kappa\nrightarrow[\lambda;\lambda]^2_\theta$, and that $e:[\kappa]^{<\omega}\rightarrow[\kappa]^2$ is a witness for $\mathrm{Extract}_2(\kappa,\lambda,\omega,\omega)$. We claim that $d:=c\circ e$ is a witness for $S_2(\kappa,\lambda,\theta)$. To this end, suppose that we are given a subfamily $\mathcal{X}\subseteq[\kappa]^{<\omega}$ of size λ, and a prescribed color $\tau<\theta$. As λ is regular and uncountable, by the Δ-system lemma, we may find a sequence $\langle x_\gamma\mid\gamma<\lambda\rangle$ consisting of elements of $\mathcal{X}$, some $r\in[\kappa]^{<\omega}$, and a nonzero $\sigma<\chi$ such that:

(1) for every $(\gamma,\gamma')\in[\lambda]^2$, $x_\gamma\cap x_{\gamma'}=r$;
(2) for every $\gamma<\lambda$, $y_\gamma:=x_\gamma\setminus r$ has order-type σ.

It thus follows from the choice of e that we may pick some integer $j<\sigma$ and cofinal subsets Γ_0,Γ_1 of λ satisfying the following:

(a) For every $(\gamma,\gamma')\in[\Gamma_0\cup\Gamma_1]^2$, $y_\gamma(j)<y_{\gamma'}(j)$;
(b) For every $(\gamma,\gamma')\in(\Gamma_0\circledast\Gamma_1)\cup(\Gamma_1\circledast\Gamma_0)$, for every $z\in[x_\gamma\cup x_{\gamma'}]^{<\omega}$ covering $\{y_\gamma(j),y_{\gamma'}(j)\}$, we have
$$e(z)=(y_\gamma(j),y_{\gamma'}(j)).$$

Put $A:=\{y_\gamma(j)\mid\gamma\in\Gamma_0\}$ and $B:=\{y_\gamma(j)\mid\gamma\in\Gamma_1\}$. By the choice of c, we may find $(\alpha,\beta)\in A\circledast B$ such that $c(\alpha,\beta)=\tau$. Pick $\gamma\in\Gamma_0$ such that $y_\gamma(j)=\alpha$, and pick $\gamma'\in\Gamma_1$ such that $y_{\gamma'}(j)=\beta$. As $\alpha<\beta$, Clause (a) implies that $(\gamma,\gamma')\in(\Gamma_0\circledast\Gamma_1)$. As $x_\gamma\cap x_{\gamma'}=r$, we infer that $\{\alpha,\beta\}\subseteq x_\gamma\bigtriangleup x_{\gamma'}$. So, for every set z such that $(x_\gamma\bigtriangleup x_{\gamma'})\subseteq z\subseteq(x_\gamma\cup x_{\gamma'})$, we get that
$$d(z)=c(e(z))=c(\alpha,\beta)=\tau,$$
as sought. $\square$

Motivated by the preceding reduction, one would like to see how to get $\mathrm{Extract}_2(\kappa,\lambda,\omega,\omega)$. The next lemma provides a sufficient condition.

Lemma 4.9. *Suppose that $\kappa\in T(\mu,\theta)$. Then there exists a map $e:[\kappa]^{<\omega}\rightarrow[\kappa]^2$ witnessing $\mathrm{Extract}_2(\kappa,\lambda,\mathrm{cf}(\theta),\omega)$ for every regular cardinal λ with $\max\{\mu,\theta^+\}\leq\lambda\leq\kappa$.*

Proof. As $\kappa\in T(\mu,\theta)$, let us fix $T\in\mathcal{T}(\mu,\theta)$ admitting an injective sequence $\langle b_\xi\mid\xi<\kappa\rangle$ consisting of elements of $\mathcal{B}(T)$. For notational simplicity, we shall write $\Delta(\alpha,\beta)$ for $\Delta(b_\alpha,b_\beta)$. First, for every $z\in[\kappa]^{<\omega}$, let:

- $M_z:=\{(\alpha,\beta)\in[z]^2\mid\Delta(\alpha,\beta)=\max(\Delta``[z]^2)\}$, and
- $M^*_z:=\{(\alpha,\beta)\in M_z\mid\alpha=\min\{\alpha'\mid(\alpha',\beta')\in M_z\}\}$.

Then, pick any function $e:[\kappa]^{<\omega}\rightarrow[\kappa]^2$ satisfying that for every $z\in[\kappa]^{<\omega}$:

- for every $z\in[\kappa]^{<\omega}$ of size ≥ 2, $e(z)\in[z]^2$;
- if M^*_z is a singleton, then $e(z)$ is its unique element.

To see that e is as sought, suppose that λ is a regular cardinal satisfying $\max\{\mu,\theta^+\}\le\lambda\le\kappa$, and that we are given $\langle x_\gamma\mid\gamma<\lambda\rangle$, $r\in[\kappa]^{<\mathrm{cf}(\theta)}$ and $\sigma<\omega$ as in Definition 4.6. By the pigeonhole principle and the Dushnik-Miller theorem, we may find a cofinal subset $\Gamma\subseteq\lambda$, an ordinal $\delta<\theta$, and a sequence $\langle t_j\mid j<\sigma\rangle$ of nodes in $T_{\delta+1}$ such that for every $(\gamma,\gamma')\in[\Gamma]^2$:

(I) $\sup(\Delta``[r\cup y_\gamma]^2)=\delta$;
(II) for every $j<\sigma$, $b_{y_\gamma(j)}\restriction(\delta+1)=t_j$;
(III) for every $j<\sigma$, $y_\gamma(j)<y_{\gamma'}(j)$.

Claim 4.9.1. *There exist $\Gamma_0,\Gamma_1\in[\Gamma]^\lambda$ and a sequence $\langle(s_j,i_j,i'_j)\mid j<\sigma\rangle$ of triples in $T\times\mu\times\mu$ such that, for every $j<\sigma$:*

- *for every $\gamma\in\Gamma_0$, $s_j{}^\frown\langle i_j\rangle\sqsubseteq b_{y_\gamma(j)}$,*
- *for every $\gamma\in\Gamma_1$, $s_j{}^\frown\langle i'_j\rangle\sqsubseteq b_{y_\gamma(j)}$, and*
- *$i_j\neq i'_j$.*

Proof. We shall define by recursion a sequence of pairs $\langle(A_j,B_j)\mid j\le\sigma\rangle$ such that, for all $j<\sigma$, $A_{j+1}\in[A_j\cap\Gamma]^\lambda$ and $B_{j+1}\in[B_j\cap\Gamma]^\lambda$.

We commence by letting both A_0 and B_0 be Γ. Now, for every $j<\sigma$ for which the pair (A_j,B_j) has already been defined, we do the following. Set $A^j:=\{y_\gamma(j)\mid\gamma\in A_j\}$ and $B^j:=\{y_\gamma(j)\mid\gamma\in B_j\}$. By Clause (III), A^j and B^j have size λ. So, by Lemma 3.4(2), we may find $s_j\in T$ and $i_j\neq i'_j$ such that $\{\alpha\in A^j\mid s_j{}^\frown\langle i_j\rangle\sqsubseteq b_\alpha\}$ and $\{\beta\in B^j\mid s_j{}^\frown\langle i'_j\rangle\sqsubseteq b_\beta\}$ are both of size λ. Then, let $A_{j+1}:=\{\gamma\in A_j\mid s_j{}^\frown\langle i_j\rangle\subseteq b_{y_\gamma(j)}\}$ and $B_{j+1}:=\{\gamma\in B_j\mid s_j{}^\frown\langle 1-i_j\rangle\subseteq b_{y_\gamma(j)}\}$.

Clearly, $\Gamma_0:=A_\sigma$ and $\Gamma_1:=B_\sigma$ are as sought. □

Let Γ_0,Γ_1 and $\langle(s_j,i_j,i'_j)\mid j<\sigma\rangle$ be given by the preceding claim. Note that Γ_0 is disjoint from Γ_1. Set $\delta^*:=\max\{\mathrm{dom}(s_j)\mid j<\sigma\}$ and $j^*:=\min\{j<\sigma\mid\mathrm{dom}(s_j)=\delta^*\}$.

Claim 4.9.2. *Let $(\gamma,\gamma')\in(\Gamma_0\circledast\Gamma_1)\cup(\Gamma_1\circledast\Gamma_0)$. Let $z\in[x_\gamma\cup x_{\gamma'}]^{<\omega}$ be such that $\{y_\gamma(j^*),y_{\gamma'}(j^*)\}\subseteq z$. Then $e(z)=(y_\gamma(j^*),y_{\gamma'}(j^*))$.*

Proof. It is clear that $2\le\mathrm{otp}(z)<\omega$, so that M_z is nonempty. Let $(\alpha,\beta)\in[z]^2$. By the choice of z, we must analyze the following cases:

(1) Suppose that $\alpha\in r$.
As $\beta\in r\cup y_\gamma\cup y_{\gamma'}$, it follows from Clause (I) that $\Delta(\alpha,\beta)\le\delta$.
(2) Suppose that $\alpha\in y_\gamma$.
  (a) If $\beta\in r\cup y_\gamma$, then it follows from Clause (I) that $\Delta(\alpha,\beta)\le\delta$;
  (b) If $\beta\in y_{\gamma'}$, then let $j_\alpha,j_\beta<\sigma$ be such that, $\alpha=y_\gamma(j_\alpha)$ and $\beta=y_{\gamma'}(j_\beta)$. There are two possible options:
    (i) If $j_\alpha=j_\beta=j$, then by Clause (II), $f_\alpha\restriction(\delta+1)=t_j=f_\beta\restriction(\delta+1)$. So $\Delta(\alpha,\beta)>\delta$.
    (ii) If $j_\alpha\neq j_\beta$, then by Clauses (I) and (II),
$$\Delta(y_{\gamma'}(j_\alpha),\beta)\le\delta<\Delta(\alpha,y_{\gamma'}(j_\alpha)),$$
and hence $\Delta(\alpha,\beta)=\Delta(y_{\gamma'}(j_\alpha),\beta)\le\delta$.
(3) If $\alpha\in y_{\gamma'}$, then the analysis is analogous to that of (2).

Altogether, so far we have shown that
$$\emptyset\subsetneq M_z\subseteq\{(y_\gamma(j),y_{\gamma'}(j))\mid j<\sigma\}.$$

Recalling that $(\gamma,\gamma')\in(\Gamma_0\circledast\Gamma_1)\cup(\Gamma_1\circledast\Gamma_0)$, we infer from the choice of δ^* that

$$\emptyset\subsetneq M_z\subseteq\{(y_\gamma(j),y_{\gamma'}(j))\mid j<\sigma,\operatorname{dom}(s_j)=\delta^*\}.$$

So, since $\{y_\gamma(j^*),y_{\gamma'}(j^*)\}\subseteq z$, it is the case that $M_z^*=\{(y_\gamma(j^*),y_{\gamma'}(j^*))\}$. In particular, $e(z)=(y_\gamma(j^*),y_{\gamma'}(j^*))$, as sought. □

This completes the proof. □

Corollary 4.10. *Suppose that λ is an infinite regular cardinal, and $\nu<\lambda$.*

If there exists a cardinal $\theta<\lambda$ such that $\nu^\theta\geq\lambda$, then $\operatorname{Extract}_2(\nu^\theta,\lambda,\operatorname{cf}(\theta),\omega)$ *holds for the least such θ.*

Proof. Let θ denote the least cardinal such that $\nu^\theta\geq\lambda$. Then $T:={}^{<\theta}\nu$ belongs to $\mathcal{T}(\theta,\lambda)$, so that $\nu^\theta=|\mathcal{B}(T)|$ is in $T(\theta,\lambda)$. Now, appeal to Lemma 4.9 with $(\kappa,\mu):=(\nu^\theta,\lambda)$. □

Corollary 4.11. *For every regular uncountable cardinal κ that is not a strong limit,* $\operatorname{Extract}_2(\kappa,\kappa,\omega,\omega)$ *holds.* □

Proposition 4.12. *Suppose that λ is a regular uncountable cardinal.*

For every cardinal $\kappa>2^{<\lambda}$, $\operatorname{Extract}_2(\kappa,\lambda,2,2)$ *fails.*

Proof. Set $\nu:=2^{<\lambda}$, and note that $\nu^\theta=\nu$ for every $\theta<\lambda$. Towards a contradiction, suppose that $e:[\kappa]^{<\omega}\rightarrow[\kappa]^2$ is a map witnessing $\operatorname{Extract}_2(\kappa,\lambda,2,2)$, and yet $\kappa>\nu$. Without loss of generality, we may assume that $e(z)\in[z]^2$ for every $z\in[\kappa]^3$.

Claim 4.12.1. *For every $\delta<\kappa$, there exist no subset $A\subseteq\delta$ of order-type λ such that $\delta\in e(\{\alpha,\beta,\delta\})$ for every $(\alpha,\beta)\in[A]^2$.*

Proof. Otherwise, fix a counterexample δ and a witnessing $A\subseteq\delta$. Let $\langle\alpha_\gamma\mid\gamma<\lambda\rangle$ be the increasing enumeration of A. Now let $r:=\{\delta\}$ and, for every $\gamma<\lambda$, put $x_\gamma:=y_\gamma\uplus r$ where $y_\gamma:=\{\alpha_\gamma\}$. Then, for every $(\gamma,\gamma')\in[\lambda]^2$, setting $z:=x_\gamma\cup x_{\gamma'}$, we get that $r\cap e(z)\neq\emptyset$. This is a contradiction. □

Denote $\varkappa:=\nu^+$. It follows from the claim that for every $\delta\in E^\varkappa_\lambda$, we may fix some $A_\delta\in[\delta]^{<\lambda}$ with the property that for every ordinal β such that $\sup(A_\delta)<\beta<\delta$, there exists $\alpha\in A_\delta$ such that $e(\{\alpha,\beta,\delta\})=(\alpha,\beta)$. Now, using Fodor's lemma, we may find $\varepsilon<\varkappa$ and $\theta<\lambda$ such that $\{\delta\in E^\varkappa_\lambda\mid\operatorname{ssup}(A_\delta)=\varepsilon\ \&\ |A_\delta|=\theta\}$ is stationary. Recalling that $\nu^\theta=\nu<\varkappa$, we may then find some $A\in[\varepsilon]^\theta$ for which $S:=\{\delta\in E^\varkappa_\lambda\mid A_\delta=A\}$ is stationary. Define a coloring $c:[S]^2\rightarrow A$ by letting for every $(\beta,\delta)\in[S]^2$:

$$c(\beta,\delta):=\min\{\alpha\in A\mid e(\{\alpha,\beta,\delta\})=(\alpha,\beta)\}.$$

By the Erdős-Rado theorem, $\varkappa\rightarrow(\lambda)^2_\theta$ holds, so we may pick $B\subseteq S$ of order-type λ that is c-homogeneous, with value, say, α. Let $\langle\beta_\gamma\mid\gamma<\lambda\rangle$ be the increasing enumeration of B. Finally, let $r:=\{\alpha\}$ and, for every $\gamma<\lambda$, put $x_\gamma:=y_\gamma\uplus r$ where $y_\gamma:=\{\beta_\gamma\}$. Then, for every $(\gamma,\gamma')\in[\lambda]^2$, setting $z:=x_\gamma\cup x_{\gamma'}$, we get that $r\cap e(z)\neq\emptyset$. This is a contradiction. □

Corollary 4.13. *If κ is a strong limit cardinal, then* $\operatorname{Extract}_2(\kappa,\lambda,2,2)$ *fails for every infinite cardinal $\lambda<\kappa$.* □

Corollary 4.14. $\operatorname{Extract}_2(\aleph_2,\aleph_1,\aleph_0,\aleph_0)$ *holds iff* CH *fails.*

Proof. By Corollary 4.10 and Proposition 4.12. □

Moving on from the case $n=2$ to the general case, we consider the following two definitions.

Definition 4.15. $S_n(\kappa,\lambda,\theta)$ asserts the existence of a coloring $d:[\kappa]^{<\omega}\rightarrow\theta$ such that, for every $\mathcal{X}\subseteq[\kappa]^{<\omega}$ of size λ and every prescribed color $\tau<\theta$, there exist $\{a_j\mid j<n\}\in[\mathcal{X}]^n$ such that $d(z)=\tau$ for every z satisfying

$$a_0\vartriangle(\bigcup\nolimits_{0<j<n}a_j)\subseteq z\subseteq\bigcup\nolimits_{j<n}a_j.$$

Proposition 4.16 (monotonicity)**.** *Suppose that:*

(1) $2\le n\le n'<\omega$*;*
(2) $\omega\le\lambda\le\lambda'$*;*
(3) $\theta\le\theta'$*.*

Then $S_n(\kappa,\lambda,\theta')$ *implies* $S_{n'}(\kappa,\lambda',\theta)$*.*

Proof. This is mostly trivial, so we settle for proving that if $d:[\kappa]^{<\omega}\rightarrow\theta$ witnesses $S_n(\kappa,\lambda,\theta)$ for some integer $n\geq 2$, then it also witnesses $S_{n+1}(\kappa,\lambda,\theta)$.

To this end, let $\mathcal{X}\subseteq[\kappa]^{<\omega}$ be a given family of size λ. Pick $x\in\mathcal{X}$, and note that $\mathcal{X}':=\{a\cup x\mid a\in\mathcal{X}\setminus\{x\}\}$ is a λ-sized subset of $[\kappa]^{<\omega}$. Now, given a prescribed color $\tau<\theta$, pick $\{a'_j\mid j<n\}\in[\mathcal{X}']^n$ such that $d(z)=\tau$ for every z satisfying

$$a'_0\vartriangle(\bigcup\nolimits_{0<j<n}a'_j)\subseteq z\subseteq\bigcup\nolimits_{j<n}a'_j.$$

For each $j<n$, pick $a_j\in\mathcal{X}\setminus\{x\}$ such that $a'_j=a_j\cup x$. As $\langle a'_j\mid j<n\rangle$ is an injective sequence, so is $\langle a_j\mid j<n\rangle$. Set $a_n:=x$. Altogether, $\{a_j\mid j<n+1\}\in[\mathcal{X}]^{n+1}$. It is clear that $\bigcup_{j<n}a'_j=\bigcup_{j<n+1}a_j$. In addition,

$$\begin{array}{rcl} a'_0\vartriangle(\bigcup_{0<j<n}a'_j) & = & (a'_0\setminus(\bigcup_{0<j<n}a'_j))\quad\cup\quad((\bigcup_{0<j<n}a'_j)\setminus a'_0)\\ & \subseteq & (a_0\setminus(\bigcup_{0<j<n}a_j))\quad\cup\quad((\bigcup_{0<j<n}a_j)\setminus a_0)\\ & \subseteq & (a_0\setminus(\bigcup_{0<j<n+1}a_j))\quad\cup\quad((\bigcup_{0<j<n+1}a_j)\setminus a_0)\\ & = & a_0\vartriangle(\bigcup_{0<j<n+1}a_j). \end{array}$$

Therefore, $d(z)=\tau$ for every set z with $a_0\vartriangle(\bigcup_{0<j<n+1}a_j)\subseteq z\subseteq\bigcup_{j<n+1}a_j$. $\square$

Definition 4.17. $\mathrm{Extract}_n(\kappa,\lambda,\theta,\chi)$ asserts the existence of a map $e:[\kappa]^{<\omega}\rightarrow{}^n\kappa$ such that for every sequence $\langle x_\gamma\mid\gamma<\lambda\rangle$ of subsets of κ, every $r\in[\kappa]^{<\theta}$, and every nonzero $\sigma<\chi$ such that:

(1) for every $(\gamma,\gamma')\in[\lambda]^2$, $x_\gamma\cap x_{\gamma'}\subseteq r$;
(2) for every $\gamma<\lambda$, $y_\gamma:=x_\gamma\setminus r$ has order-type σ,

there exist $j<\sigma$ and disjoint cofinal subsets Γ_0,Γ_1 of λ satisfying the following:

(a) For every $(\gamma,\gamma')\in[\Gamma_0\cup\Gamma_1]^2$, $y_\gamma(j)<y_{\gamma'}(j)$;
(b) For every strictly increasing sequence $\langle\gamma_i\mid i<n\rangle$ of ordinals from $\Gamma_0\cup\Gamma_1$ such that $\{\gamma_i\mid i<n\}\notin([\Gamma_0]^n\cup[\Gamma_1]^n)$, for every $z\in[\bigcup_{i<n}x_{\gamma_i}]^{<\omega}$ that covers $\{y_{\gamma_i}(j)\mid i<n\}$, we have

$$e(z)=\langle y_{\gamma_i}(j)\mid i<n\rangle.$$

Remark 4.18. Without loss of generality, we may assume that for every $z\in[\kappa]^{<\omega}$ of size $\geq n$, $e(z)$ consists of ordinals from z.

The proof of [FBR17, Theorem 4.7] makes it clear that the following holds.

Proposition 4.19. *Suppose that $S_n(\kappa,\lambda,\theta)$ holds for given cardinals $\theta\le\lambda\le\kappa$ with λ regular and uncountable. For every map $\varphi : G\rightarrow [G]^{<\omega}$ that is $<\lambda$-to-one, there exists a corresponding coloring $c : G\rightarrow\theta$ satisfying the following.*

For every binary operation $*$ on G such that, for all $x\neq y$ in G,

$$\varphi(x)\mathbin{\triangle}\varphi(y)\subseteq\varphi(x*y)\subseteq\varphi(x)\cup\varphi(y),$$

for every $X\in[G]^{\lambda}$ and every $\tau<\theta$, there is an injective sequence $\langle x_j\mid 1\le j\le n\rangle$ of elements of X such that $c(x_1*\cdots*x_n)=\tau$.[6] □

In order to generalize Lemma 4.8, we now introduce the relation $\kappa\overset{\sup}{\nrightarrow}[\lambda,\lambda]^n_\theta$. It is a strengthening of $\kappa\nrightarrow[\lambda]^n_\theta$, and a weakening of $\kappa\nrightarrow[\lambda,\dots,\lambda]^n_\theta$.

Definition 4.20. $\kappa\overset{\sup}{\nrightarrow}[\lambda,\lambda]^n_\theta$ asserts the existence of a coloring $c:[\kappa]^n\rightarrow\theta$ such that for all $\tau<\theta$ and disjoint $A,B\in\mathcal{P}(\kappa)$ satisfying the two:

- (i) $\mathrm{otp}(A)=\mathrm{otp}(B)=\lambda$,
- (ii) $\sup(A)=\sup(B)$,

there is $\vec{x}\in[A\cup B]^n\setminus([A]^n\cup[B]^n)$ with $c(\vec{x})=\tau$.

Remark 4.21. In the special case of $\lambda=\kappa$, $\kappa\overset{\sup}{\nrightarrow}[\lambda,\lambda]^2_\theta$ coincides with the classical relation $\kappa\nrightarrow[\lambda,\lambda]^2_\theta$.

Lemma 4.22. *Suppose that:*

- $2\le n<\omega$*;*
- $\theta\le\lambda\le\kappa$ *are cardinals with λ regular and uncountable;*
- $\kappa\overset{\sup}{\nrightarrow}[\lambda,\lambda]^n_\theta$ *holds;*
- $\mathrm{Extract}_n(\kappa,\lambda,\omega,\omega)$ *holds.*

Then $S_n(\kappa,\lambda,\theta)$ holds.

Proof. The proof is similar to that of Lemma 4.8, so we settle for a sketch. Fix a map $c:[\kappa]^n\rightarrow\theta$ witnessing $\kappa\overset{\sup}{\nrightarrow}[\lambda,\lambda]^n_\theta$ and a map $e:[\kappa]^{<\omega}\rightarrow{}^n\kappa$ witnessing $\mathrm{Extract}_n(\kappa,\lambda,\omega,\omega)$. We claim that $d:=c\circ e$ is a witness for $S_n(\kappa,\lambda,\theta)$. To this end, suppose that we are given a subfamily $\mathcal{X}\subseteq[\kappa]^{<\omega}$ of size λ, and a prescribed color $\tau<\theta$. By the Δ-system lemma, find a sequence $\langle x_\gamma\mid\gamma<\lambda\rangle$ consisting of elements of $\mathcal{X}$, some $r\in[\kappa]^{<\omega}$, and a nonzero $\sigma<\chi$ such that:

1. (1) for every $(\gamma,\gamma')\in[\lambda]^2$, $x_\gamma\cap x_{\gamma'}=r$;
2. (2) for every $\gamma<\lambda$, $y_\gamma:=x_\gamma\setminus r$ has order-type σ.

Now, let $j<\sigma$ and disjoint cofinal subsets Γ_0,Γ_1 of λ be given, as in Definition 4.17. Put $A:=\{y_\gamma(j)\mid\gamma\in\Gamma_0\}$ and $B:=\{y_\gamma(j)\mid\gamma\in\Gamma_1\}$. By the choice of c, pick $\vec{x}\in[A\cup B]^n\setminus([A]^n\cup[B]^n)$ such that $c(\vec{x})=\tau$. Find a sequence $\vec{\gamma}=\langle\gamma_i\mid i<n\rangle$ of ordinals from $\Gamma_0\cup\Gamma_1$ such that $\vec{x}=\langle y_{\gamma_i}(j)\mid i<n\rangle$. Clearly, $\vec{\gamma}$ is strictly increasing, $\{\gamma_i\mid i<n\}\nsubseteq\Gamma_0$ and $\{\gamma_i\mid i<n\}\nsubseteq\Gamma_1$. So, for every set z of interest,

$$d(z)=c(e(z))=c(\vec{x})=\tau,$$

as sought. □

Lemma 4.23. *Suppose that $\kappa\in T(\mu,\theta)$. Then there exists a map $e:[\kappa]^{<\omega}\rightarrow{}^3\kappa$ witnessing* $\mathrm{Extract}_3(\kappa,\lambda,\mathrm{cf}(\theta),\omega)$ *for every regular cardinal λ with* $\max\{\mu,\theta^+\}\le\lambda\le\kappa$.

[6] As always, we mean that this holds true for all implementations of $x_1*\cdots*x_n$.

Proof. As $\kappa \in T(\mu,\theta)$, let us fix $T \in \mathcal{T}(\mu,\theta)$ admitting an injective sequence $\langle b_\xi \mid \xi < \kappa\rangle$ consisting of elements of $\mathcal{B}(T)$. For notational simplicity, we shall write $\Delta(\alpha,\beta)$ for $\Delta(b_\alpha,b_\beta)$. For any triplet $w \in [\kappa]^3$, let $\Delta_2(w) := \min\{\Delta(\alpha,\beta) \mid (\alpha,\beta) \in [w]^2\}$. First, given $z \in [\kappa]^{<\omega}$, let:

- $M_z := \{(\alpha,\beta,\gamma) \in [z]^3 \mid \Delta_2(\{\alpha,\beta,\gamma\}) = \max\{\Delta_2(w) \mid w \in [z]^3\}\}$, and
- $M^*_z := \{(\alpha,\beta,\gamma) \in M_z \mid \alpha = \min\{\alpha' \mid (\alpha',\beta',\gamma') \in M_z\}\}$.

Then, pick any function $e : [\kappa]^{<\omega} \to {}^3\kappa$ satisfying that for every $z \in [\kappa]^{<\omega}$:

- for every $z \in [\kappa]^{<\omega}$, $e(z)$ is a strictly increasing sequence of ordinals in κ. If $|z| \geq 3$, then $e(z)$ consists of ordinals from z;
- if M^*_z is a singleton, then $e(z)$ is its unique element.

To see that e is as sought, suppose that λ is a regular cardinal satisfying $\max\{\mu,\theta^+\} \leq \lambda \leq \kappa$, and that we are given $\langle x_\gamma \mid \gamma < \lambda\rangle$, $r \in [\kappa]^{<\mathrm{cf}(\theta)}$ and $\sigma < \omega$ as in Definition 4.17. As in the proof of Lemma 4.9, we may find a cofinal subset $\Gamma \subseteq \lambda$, an ordinal $\delta < \theta$, and a sequence $\langle t_j \mid j < \sigma\rangle$ of nodes in $T_{\delta+1}$ such that for every $(\gamma,\gamma') \in [\Gamma]^2$:

(I) $\sup(\Delta``[r \cup y_\gamma]^2) = \delta$;
(II) for every $j < \sigma$, $b_{y_\gamma(j)} \restriction (\delta+1) = t_j$;
(III) for every $j < \sigma$, $y_\gamma(j) < y_{\gamma'}(j)$.

In addition, we may fix $\Gamma_0,\Gamma_1 \in [\Gamma]^{\lambda^+}$ and a sequence $\langle (s_j,i_j,i'_j) \mid j < m\rangle$ of triples in $T \times \mu \times \mu$ such that, for every $j < \sigma$:

- for every $\gamma \in \Gamma_0$, $s_j{}^\frown\langle i_j\rangle \sqsubseteq b_{y_\gamma(j)}$,
- for every $\gamma \in \Gamma_1$, $s_j{}^\frown\langle i'_j\rangle \sqsubseteq b_{y_\gamma(j)}$, and
- $i_j \neq i'_j$.

By possibly passing to cofinal subsets, we may assume that $\Gamma_0 \cap \Gamma_1 = \emptyset$. Let $\delta^* := \max\{\mathrm{dom}(s_j) \mid j < \sigma\}$ and $j^* := \min\{j < \sigma \mid \mathrm{dom}(s_j) = \delta^*\}$.

Claim 4.23.1. *Let $(\gamma,\gamma',\gamma'') \in [\Gamma_0\cup\Gamma_1]^3 \setminus ([\Gamma_0]^3 \cup [\Gamma_1]^3)$. Let $z \in [x_\gamma \cup x_{\gamma'} \cup x_{\gamma''}]^{<\omega}$ be such that $\{y_\gamma(j^*), y_{\gamma'}(j^*), y_{\gamma''}(j^*)\} \subseteq z$. Then $e(z) = (y_\gamma(j^*), y_{\gamma'}(j^*), y_{\gamma''}(j^*))$.*

Proof. As made clear by the proof of Claim 4.9.2, for every $(\xi,\zeta) \in [\{\gamma,\gamma',\gamma''\}]^2$, for every $\alpha \in r \cup a_\xi$ and $\beta \in r \cup y_\zeta$, $\Delta(\alpha,\beta) > \delta$ iff there is a $j < \sigma$ such that $\alpha = y_\xi(j)$ and $\beta = y_\zeta(j)$. In addition, if $\{\xi,\zeta\} \nsubseteq \Gamma_0$ and $\{\xi,\zeta\} \nsubseteq \Gamma_1$, then for every $j < m$, $\Delta(y_\xi(j), y_\zeta(j)) = \mathrm{dom}(s_j)$. So, in this case,

$$\begin{aligned}&\{(\alpha,\beta) \in [r \cup y_\xi \cup y_\zeta]^2 \mid \Delta(\alpha,\beta) = \max(\Delta``[r \cup y_\xi \cup y_\zeta]^2)\}\\ =\ &\{(y_\xi(j), y_\zeta(j)) \mid \mathrm{dom}(s_j) = \delta^*\}.\end{aligned}$$

Now, since $\{\gamma,\gamma',\gamma''\} \nsubseteq \Gamma_0$ and $\{\gamma,\gamma',\gamma''\} \nsubseteq \Gamma_1$, it follows that

$$\Delta_2(y_\gamma(j^*), y_{\gamma'}(j^*), y_{\gamma''}(j^*)) = \delta^*.$$

Consequently,

$$\emptyset \subsetneq M_z = \{(y_\gamma(j), y_{\gamma'}(j), y_{\gamma''}(j)) \mid \mathrm{dom}(s_j) = \delta^*\}.$$

So, by Clause (III),

$$e(z) = (y_\gamma(j^*), y_{\gamma'}(j^*), y_{\gamma''}(j^*)),$$

as sought. □

This completes the proof. □

Proposition 4.24. *Suppose that $\lambda \leq \kappa$ is a pair of infinite cardinals.*
If $\mathrm{Extract}_2(\kappa, \lambda, 3, 3)$ *holds, then so does* $\kappa \overset{\sup}{\not\to} [\lambda, \lambda]^4_2$.

Proof. Suppose that $\kappa \overset{\sup}{\not\to} [\lambda, \lambda]^4_2$ fails, and we shall prove that $\mathrm{Extract}_2(\kappa, \lambda, 3, 3)$ fails, as well. To this end, let $e : [\kappa]^{<\omega} \to [\kappa]^2$ be given. Define a coloring $c : [\kappa]^4 \to 2$ by letting for all $\alpha_0 < \alpha_1 < \alpha_2 < \alpha_3 < \kappa$:

$$c(\alpha_0, \alpha_1, \alpha_2, \alpha_3) := 1 \text{ iff } e(\alpha_0, \alpha_1, \alpha_2, \alpha_3) = (\alpha_2, \alpha_3).$$

Now, since $\kappa \overset{\sup}{\longrightarrow} [\lambda, \lambda]^4_2$ holds, we may find $\tau < 2$ and disjoint $A, B \in \mathcal{P}(\kappa)$ satisfying all of the following:

(i) $\mathrm{otp}(A) = \mathrm{otp}(B) = \lambda$,
(ii) $\sup(A) = \sup(B)$,
(iii) for every $(\alpha_0, \alpha_1, \alpha_2, \alpha_3) \in [A \cup B]^4 \setminus ([A]^4 \cup [B]^4)$,

$$c(\alpha_0, \alpha_1, \alpha_2, \alpha_3) \neq \tau.$$

Using Clauses (i) and (ii), fix a sequence $\langle (\alpha_i, \beta_i) \mid i < \lambda \rangle$ of pairs in $A \times B$ such that, for all $i < j < \lambda$, $\alpha_i < \beta_i < \alpha_j$.

▶ If $\tau = 1$, then let $r := \{\alpha_0, \alpha_1\}$, and for every $\gamma < \lambda$, let $x_\gamma := r \uplus y_\gamma$, where $y_\gamma := \{\beta_{\gamma+1}\}$. Now, for every $(\gamma, \gamma') \in [\lambda]^2$, as $z := x_\gamma \cup x_{\gamma'}$ is in $[A \cup B]^4 \setminus ([A]^4 \cup [B]^4)$, $c(z) = 0$, and then $e(z)$ is not disjoint from r.

▶ If $\tau = 0$, then let $r := \emptyset$, and for every $\gamma < \lambda$, let $x_\gamma := r \uplus y_\gamma$, where $y_\gamma := \{\alpha_\gamma, \beta_\gamma\}$. Now, for every $(\gamma, \gamma') \in [\lambda]^2$, as $z := x_\gamma \cup x_{\gamma'}$ is in $[A \cup B]^4 \setminus ([A]^4 \cup [B]^4)$, $c(z) = 1$, and then $e(z) = y_{\gamma'}$ which is disjoint from y_γ. □

Corollary 4.25. *If $\lambda = \aleph_0$ or if λ is weakly compact, then* $\mathrm{Extract}_2(\kappa, \lambda, 3, 3)$ *fails for every cardinal $\kappa \geq \lambda$.* □

5. Maximal number of colors

This section is devoted to the proof of Theorem C. The main corollary of this section reads as follows:

Corollary 5.1. *Suppose that $\lambda = \mu^+$ for an infinite cardinal $\mu = \mu^{<\mu}$.*
Then the following are equivalent:

(1) $(\lambda^+, \lambda) \twoheadrightarrow (\mu^+, \mu)$ *fails;*
(2) $\lambda^+ \overset{\sup}{\not\to} [\lambda, \lambda]^3_\lambda$ *holds.*

Proof. We focus on the nontrivial (that is, forward) implication. As $\lambda = \mu^+$, by [Tod07, Lemma 9.2.3], the failure of $(\lambda^+, \lambda) \twoheadrightarrow (\mu^+, \mu)$ is equivalent to the existence of a subadditive coloring $\varrho : [\lambda^+]^2 \to \lambda$ witnessing $\mathrm{U}(\lambda^+, \lambda, \lambda, \lambda, \omega)$. In particular, $\varrho \restriction [X]^2$ witnesses $\mathrm{U}(\lambda, \lambda, \lambda, 3)$ for every $X \subseteq \lambda^+$ of order-type λ. Now, there are two cases to consider:

▶ If $2^\mu > \mu^+$, then since $\mu^{<\mu} = \mu$, $T := {}^{<\mu}\mu$ is a weak μ-Kurepa tree with λ^+-many branches. In addition, $\mu^{<\mu} = \mu$ implies that μ is regular, so that E^λ_μ is a nonreflecting stationary set. Now the result follows from the upcoming Theorem 5.2, using $\kappa := \lambda^+$.

▶ If $2^\mu = \mu^+$, then $\lambda \not\to [\mu; \lambda]^2_\lambda$ holds by a theorem of Sierpiński (see [IR23, Lemma 8.3]). Now the result follows from Theorem 5.3 below. □

When reading the hypotheses of the upcoming theorem, it may worth keeping in mind that if $\lambda = \mu^+$ for an infinite regular cardinal μ, then E^λ_μ is a nonreflecting stationary set, and if $\kappa = \lambda^+$, then Fact 2.14 provides a subadditive map $\rho : [\kappa]^2 \to \lambda$. The conclusion of the theorem is a conditional form of $\kappa \overset{\sup}{\not\to} [\lambda, \lambda]^3_\lambda$ in which a third clause is added to Definition 4.20.

Theorem 5.2. *Suppose that:*

- *$\mu < \lambda < \kappa$ are infinite regular cardinals;*
- *E^λ_μ admits a nonreflecting stationary set;*
- *$\varrho : [\kappa]^2 \to \lambda$ is a subadditive coloring of pairs;*
- *there exists a weak μ-Kurepa tree with at least κ-many branches.*

Then there exists a corresponding coloring of triples $c : [\kappa]^3 \to \lambda$ such that, for all $\tau < \lambda$ and disjoint $A, B \in \mathcal{P}(\kappa)$ satisfying the three:

(i) $\operatorname{otp}(A) = \operatorname{otp}(B) = \lambda$,
(ii) $\sup(A) = \sup(B)$,
(iii) $\varrho \restriction [A \cup B]^2$ *witnesses* $\mathrm{U}(\lambda, \lambda, \lambda, 3)$,

there exists $(\alpha, \beta, \gamma) \in [A \cup B]^3 \setminus ([A]^3 \cup [B]^3)$ such that $c(\alpha, \beta, \gamma) = \tau$.

Proof. Let $T \subseteq {}^{<\mu}2$ be a weak μ-Kurepa tree with at least κ-many branches, and let $\vec{b} = \langle b_\xi \mid \xi < \kappa \rangle$ be an injective sequence consisting of elements of $\mathcal{B}(T)$. For all $\alpha \neq \beta$ from κ, we write $\Delta(\alpha, \beta)$ for $\Delta(b_\alpha, b_\beta)$. For all $B \subseteq \kappa$ and $t \in T$, denote $B_t := \{\beta \in B \mid t \sqsubseteq b_\beta\}$. As E^λ_μ admits a nonreflecting stationary set, by [LHR23, Lemma 3.31], we may fix a coloring $e : [\lambda]^2 \to \mu$ for which the following set is stationary:

$$\partial(e) := \{\sigma \in E^\lambda_\mu \mid \forall \epsilon \in \lambda \setminus \sigma \, \forall \delta < \mu \, [\sup\{\zeta < \sigma \mid e(\zeta, \epsilon) \leq \delta\} < \sigma]\}.$$

Let $h : \lambda \to \lambda$ be a surjection such that $S_\tau := \{\sigma \in \partial(e) \mid h(\sigma) = \tau\}$ is stationary for every $\tau < \lambda$.

For every $(\alpha, \beta, \gamma) \in \kappa \times \kappa \times \kappa$, let

$$Z_{(\alpha,\beta,\gamma)} := \{\zeta \in \lambda \setminus \varrho(\{\alpha, \beta\}) \mid \max(\Delta``\{\alpha, \beta, \gamma\}^2) \geq e(\zeta, \varrho(\{\beta, \gamma\}))\}.$$

Now, derive a coloring $c : [\kappa]^3 \to \lambda$ by letting:

$$c(\{\alpha, \beta, \gamma\}) := \begin{cases} h(\min(Z_{(\beta,\alpha,\gamma)})), & \text{if } \gamma = \max\{\alpha, \beta, \gamma\} \ \&\ b_\alpha <_{\mathrm{lex}} b_\beta <_{\mathrm{lex}} b_\gamma; \\ h(\min(Z_{(\alpha,\beta,\gamma)})), & \text{if } \gamma = \max\{\alpha, \beta, \gamma\} \ \&\ b_\gamma <_{\mathrm{lex}} b_\alpha <_{\mathrm{lex}} b_\beta; \\ 0, & \text{otherwise.} \end{cases}$$

Suppose that $A, B \in \mathcal{P}(\kappa)$ are disjoint sets satisfying conditions (i)–(iii) above. Recalling Lemma 3.4(2), by possibly shrinking A and B, we may assume the existence of some $\chi < \mu$ such that $\Delta[A \times B] = \{\chi\}$. By possibly switching the roles of A and B, we may also assume the following:

(iv) for every $(\alpha, \beta) \in A \times B$, $b_\alpha <_{\mathrm{lex}} b_\beta$.

Note that $|T| = \mu < \lambda < \kappa \leq 2^\mu$. Now, given a prescribed color $\tau < \lambda$, let M be an elementary submodel of $\mathcal{H}_{(2^\mu)^+}$ containing $\{A, B, \varrho, \vec{b}, T\}$ such that $\sigma := M \cap \lambda$ is in S_τ. In particular, $|M| \geq \operatorname{cf}(\sigma) = \mu$. Denote $\upsilon := \sup(A)$ and $\upsilon_M := \sup(M \cap \upsilon)$. The proof is now divided into two cases:

Case 1: For every $\beta \in B$, $\sup(\varrho_\beta``A) < \lambda$. By Condition (iii), find $A' \in [A]^\lambda$ and $B' \in [B]^\lambda$ such that $\min(\varrho[A' \circledast B']) > \sigma$. Fix $\alpha \in A' \setminus \upsilon_M$ arbitrarily. Appeal to Corollary 3.5 with $X := B'$ and $i := 1$ to pick $\gamma \in B' \setminus (\alpha + 1)$ such that for cofinally many $\delta < \mu$, the two hold:

(1) $b_\gamma(\delta)=1$, and
(2) $\{\beta\in B'\mid \Delta(\beta,\gamma)=\delta\}$ has size λ.

Denote $D:=\{\delta<\mu\mid \text{Clauses (1) and (2) both hold}\}$.

As $(\alpha,\gamma)\in A'\circledast B'$, the ordinal $\epsilon:=\varrho(\alpha,\gamma)$ is bigger than σ. Pick $\delta\in D$ above $\max\{\chi,e(\sigma,\epsilon)\}$. Since $\sigma\in\partial(e)$, the following set is bounded below σ:

$$Z:=\{\zeta<\sigma\mid e(\zeta,\epsilon)\le\delta\}.$$

Set $t:=(b_\gamma\restriction\delta)^\frown\langle 0\rangle$. From $\delta\in D$ we infer that $(B')_t$ has size λ. As $t\in T\subseteq M$, in particular, B_t is a set of size λ lying in M. By Condition (iii) and elementarity of M one can find $\beta_0\neq\beta_1$ in $B_t\cap M$ such that $\varrho(\beta_0,\beta_1)>\sup(Z)$. As ϱ is subadditive, we may now find $\beta\in\{\beta_0,\beta_1\}$ such that $\varrho(\beta,\alpha)>\sup(Z)$. As $\beta\in B$, $\sup(\varrho_\beta``A)<\lambda$. As $\{\varrho,A\}\in M$, it follows that $\sup(\varrho_\beta``A)\in M$. In particular, $\varrho(\beta,\alpha)<\sigma$.

Claim 5.2.1. *All of the following hold:*

(1) $(\beta,\alpha,\gamma)\in B\circledast A\circledast B$;
(2) $b_\alpha<_{\mathrm{lex}}b_\beta<_{\mathrm{lex}}b_\gamma$;
(3) $\max(\Delta``\{\alpha,\beta,\gamma\}^2)=\delta$;
(4) $\min(Z_{(\beta,\alpha,\gamma)})=\sigma$.

Proof. (1) γ was chosen to be in $B\setminus(\alpha+1)$. In addition, $\alpha\in A\setminus\upsilon_M$, whereas $\beta\in M\cap B$.

(2) As $\alpha\in A$ and $\beta\in B$, Condition (iv) entails that $b_\alpha<_{\mathrm{lex}}b_\beta$. In addition, as $b_\beta\restriction(\delta+1)=t=(b_\gamma\restriction\delta)^\frown\langle0\rangle$ and $b_\gamma(\delta)=1$, we get that $b_\beta<_{\mathrm{lex}}b_\gamma$.

(3) By the previous analysis, $\Delta(\beta,\gamma)=\delta$. In addition, $\Delta(\alpha,\gamma)=\chi<\delta$. Recalling that $|\Delta``\{\alpha,\beta,\gamma\}^2|=2$, we are done.

(4) By Clause (3) and the fact that $\varrho(\alpha,\gamma)=\epsilon$, we infer that $Z_{(\beta,\alpha,\gamma)}=\{\zeta\in\lambda\setminus\varrho(\beta,\alpha)\mid\delta\ge e(\zeta,\epsilon)\}$. In particular, $\sigma\in Z_{(\beta,\alpha,\gamma)}$. Now, if $\zeta:=\min(Z_{(\beta,\alpha,\gamma)})$ is $<\sigma$, then $\zeta\in Z$, contradicting the fact that $\varrho(\beta,\alpha)>\sup(Z)$. $\square$

By the preceding claim and the definition of c,

$$c(\{\alpha,\beta,\gamma\})=h(\min(Z_{(\beta,\alpha,\gamma)}))=h(\sigma)=\tau,$$

as sought.

<u>**Case 2: There is** $\beta\in B$ **such that** $\sup(\varrho_\beta``A)=\lambda$**.**</u> As $\{\varrho,A\}\in M$, we may pick $\beta\in B\cap M$ such that $\sup(\varrho_\beta``A)=\lambda$. Clearly, $|\varrho_\beta``A|=\lambda$. Define a function $f:\varrho_\beta``A\to A$ via

$$f(\xi):=\min\{\alpha\in A\mid\varrho(\beta,\alpha)=\xi\}.$$

As $\{\beta,A,\varrho\}\in M$, we infer that $\varrho_\beta``A$, f and $\mathrm{Im}(f)$ are all in M. Note that f is injective, so that $|\mathrm{Im}(f)|=\lambda$. It also follows that

$$\Gamma:=\{\gamma\in\mathrm{Im}(f)\setminus(\beta+1)\mid\varrho(\beta,\gamma)\le\sigma\}$$

is bounded in $\mathrm{Im}(f)$.

Appeal to Corollary 3.5 with $X:=\mathrm{Im}(f)\setminus(\beta+1)$ and $i:=0$ to pick $\gamma\in X\setminus(\Gamma\cup\upsilon_M)$ such that for cofinally many $\delta<\mu$, the two hold:

(1) $b_\gamma(\delta)=0$, and
(2) $\{\alpha\in X\mid\Delta(\alpha,\gamma)=\delta\}$ has size λ.

Denote $D := \{\delta < \mu \mid \text{Clauses (1) and (2) both hold}\}$.

As $\gamma \notin \Gamma$, $\epsilon := \varrho(\beta,\gamma)$ is bigger than σ. Pick $\delta \in D$ above $\max\{\chi, e(\sigma,\epsilon)\}$. Since $\sigma \in \partial(e)$, the following set is bounded below σ:

$$Z := \{\zeta < \sigma \mid e(\zeta,\epsilon) \leq \delta\}.$$

Set $t := (b_\gamma \restriction \delta)^\frown\langle 1\rangle$. From $\delta \in D$ we infer that X_t is a set of size λ. As $t \in T \subseteq M$ and $X \in M$, X_t is in M. As $\alpha \mapsto \varrho(\beta,\alpha)$ is injective over X, we may find an $\alpha \in X_t \cap M$ such that $\varrho(\beta,\alpha) > \sup(Z)$. Because of the fact that $\{\beta,\alpha\} \in M$, we altogether get that $\sup(Z) < \varrho(\beta,\alpha) < \sigma$.

Claim 5.2.2. *All of the following hold:*

(1) $(\beta,\alpha,\gamma) \in B \circledast A \circledast A$*;*
(2) $b_\gamma <_{\text{lex}} b_\alpha <_{\text{lex}} b_\beta$*;*
(3) $\max(\Delta``\{\alpha,\beta,\gamma\}^2) = \delta$*;*
(4) $\min(Z_{(\alpha,\beta,\gamma)}) = \sigma$*.*

Proof. (1) β was chosen to be in $B \cap M$, γ was chosen to be in $A \setminus \upsilon_M$, whereas $\alpha \in M \cap A$ with $\alpha > \beta$.

(2) As $\alpha \in A$ and $\beta \in B$, Condition (iv) entails that $b_\alpha <_{\text{lex}} b_\beta$. In addition, $b_\alpha \restriction (\delta+1) = t = (b_\gamma \restriction \delta)^\frown\langle 1\rangle$ and $b_\gamma(\delta) = 0$, so $b_\gamma <_{\text{lex}} b_\alpha$.

(3) By the previous analysis, $\Delta(\alpha,\gamma) = \delta$. In addition, $\Delta(\alpha,\beta) = \chi < \delta$. Recalling that $|\Delta``\{\alpha,\beta,\gamma\}^2| = 2$, we are done.

(4) By Clause (3) and the fact that $\varrho(\beta,\gamma) = \epsilon$, we infer that $Z_{(\alpha,\beta,\gamma)} = \{\zeta \in \lambda \setminus \varrho(\beta,\alpha) \mid \delta \geq e(\zeta,\epsilon)\}$. In particular, $\sigma \in Z_{(\alpha,\beta,\gamma)}$. Now, if $\zeta := \min(Z_{(\alpha,\beta,\gamma)})$ is $< \sigma$, then $\zeta \in Z$, contradicting the fact that $\varrho(\beta,\alpha) > \sup(Z)$. $\square$

By the preceding claim and the definition of c,

$$c(\{\alpha,\beta,\gamma\}) = h(\min(Z_{(\alpha,\beta,\gamma)})) = h(\sigma) = \tau,$$

as sought. $\square$

Theorem 5.3. *Suppose that:*

- $\mu = \mu^{<\mu}$ *is an infinite cardinal,* $\lambda = \mu^+$ *and* $\kappa = \lambda^+$*;*
- $\varrho : [\kappa]^2 \to \lambda$ *is a subadditive coloring of pairs;*
- $\lambda \nrightarrow [\mu;\lambda]^2_\lambda$ *holds.*

Then, there exists a corresponding coloring of triples $c : [\kappa]^3 \to \lambda$ *such that, for all* $\tau < \lambda$ *and disjoint* $A, B \in \mathcal{P}(\kappa)$ *satisfying the three:*

(i) $\text{otp}(A) = \text{otp}(B) = \lambda$*,*
(ii) $\sup(A) = \sup(B)$*,*
(iii) $\varrho \restriction [A \cup B]^2$ *witnesses* $\mathrm{U}(\lambda,\lambda,\lambda,3)$*,*

there exists $(\alpha,\beta,\gamma) \in [A \cup B]^3 \setminus ([A]^3 \cup [B]^3)$ *such that* $c(\alpha,\beta,\gamma) = \tau$*.*

Proof. Let $d : [\kappa]^2 \to \lambda$ be a coloring witnessing $\lambda \nrightarrow [\mu;\lambda]^2_\lambda$. Let $T := {}^{<\lambda}2$. Let $\vec{b} = \langle b_\xi \mid \xi < \kappa\rangle$ be an injective enumeration of elements of $\mathcal{B}(T)$. For $\alpha \neq \beta$ from κ, we write $\Delta(\alpha,\beta)$ for $\Delta(b_\alpha, b_\beta)$. Likewise, for $B \subseteq \kappa$, we write $T^{\rightsquigarrow B}$ for $T^{\rightsquigarrow \{b_\beta \mid \beta \in B\}}$.

For all $B \subseteq \kappa$ and $t \in T$, denote $B_t := \{\beta \in B \mid t \sqsubseteq b_\beta\}$. Let $e : [\lambda]^2 \to \mu$ be a map with injective fibers. Let $h : \lambda \to \lambda$ be a surjection such that $S_\tau := \{\sigma \in E^\lambda_\mu \mid h(\sigma) = \tau\}$ is stationary for every $\tau < \lambda$. For every $(\alpha,\beta,\gamma) \in \kappa \circledast \kappa \circledast \kappa$, define:

$$Z_{(\alpha,\beta,\gamma)} := \{\zeta \in \lambda \setminus \varrho(\alpha,\gamma) \mid e(\Delta(\alpha,\beta), \varrho(\beta,\gamma)) \geq e(\zeta, \varrho(\beta,\gamma))\}.$$

We define a coloring $c : [\kappa]^3 \to \lambda$ by letting for all $\alpha < \beta < \gamma < \kappa$:

$$c(\alpha,\beta,\gamma) := \begin{cases} d(\Delta(\alpha,\gamma),\Delta(\beta,\gamma)), & \text{if } \Delta(\alpha,\beta) < \Delta(\beta,\gamma);\\ d(\Delta(\alpha,\gamma),\varrho(\beta,\gamma)), & \text{if } \Delta(\alpha,\beta) = \Delta(\beta,\gamma);\\ d(\Delta(\alpha,\beta),\varrho(\alpha,\gamma)), & \text{if } \Delta(\alpha,\beta) > \Delta(\beta,\gamma) \ \&\ b_\alpha <_{\mathrm{lex}} b_\beta;\\ h(\min(Z_{(\alpha,\beta,\gamma)})), & \text{if } \Delta(\alpha,\beta) > \Delta(\beta,\gamma) \ \&\ b_\beta <_{\mathrm{lex}} b_\alpha. \end{cases}$$

Suppose that $A, B \in \mathcal{P}(\kappa)$ are disjoint sets satisfying conditions (i)–(iii) above and let $\tau < \lambda$ be a prescribed color. By possibly passing to a cofinal subset of A, we may assume that $A = A'$ in the sense of Lemma 3.7. In particular, we may assume the existence of $\theta_A \leq \lambda$ such that $T^{\rightsquigarrow A} \in \mathcal{T}(\lambda,\theta_A)$ and, in addition, if $\theta_A < \lambda$, then $|\mathcal{B}(T^{\rightsquigarrow A})| = \lambda$. As $\mu^{<\mu} < \mu^+ = \lambda$, this means that if $\theta_A < \lambda$, then $\theta_A \in E^\lambda_\mu$. Likewise, we may assume the existence of $\theta_B \in E^\lambda_\mu \cup \{\lambda\}$ such that $T^{\rightsquigarrow B} \in \mathcal{T}(\lambda,\theta_B)$. Without loss of generality, we may also assume that $\theta_A \leq \theta_B$. The proof is now divided into two main cases.

Case 1: $\theta_A = \theta_B = \lambda$. In this case, we shall need the following claim.

Claim 5.3.1. *There exists $t \in T^{\rightsquigarrow B}$ satisfying all of the following:*

- *If $t \notin T^{\rightsquigarrow A}$, then $D := \{\Delta(b_\beta, t) \mid \beta \in B\}$ has size μ, and there exists $t' \in T^{\rightsquigarrow A}$ incompatible with t such that $\sup(D) > \Delta(t,t')$;*
- *If $t \in T^{\rightsquigarrow A}$, then $D := \{\Delta(b_\alpha, t) \mid \alpha \in A\}$ has size μ.*

Proof. There are two cases to consider:

► Suppose that there exists $\epsilon < \lambda$ such that $T^{\rightsquigarrow A} \cap T^{\rightsquigarrow B} \cap {}^{\epsilon}2 = \emptyset$. Pick $t' \in T^{\rightsquigarrow A} \cap {}^{\epsilon}2$. For each $\alpha < \lambda$, pick $t_\alpha \in T^{\rightsquigarrow B} \cap {}^{\alpha}2$. Then, by Lemma 3.6, there exists $\alpha \in E^\lambda_\mu$ above ϵ such that $D := \{\Delta(b_\beta, t_\alpha) \mid \beta \in B\} \cap \alpha$ is cofinal in α. To see that $t := t_\alpha$ is as sought, notice that since $t \restriction \epsilon \notin T^{\rightsquigarrow A}$, it must be the case that $\Delta(t,t') < \epsilon$.

► Otherwise. Thus, for each $\alpha < \lambda$, we may pick $t_\alpha \in T^{\rightsquigarrow A} \cap T^{\rightsquigarrow B} \cap {}^{\alpha}2$. As $\langle t_\alpha \mid \alpha < \lambda\rangle \in \prod_{\alpha<\lambda} T^{\rightsquigarrow A} \cap {}^{\alpha}2$, Lemma 3.6 provides an $\alpha \in E^\lambda_\mu$ such that $D := \{\Delta(b_\beta, t_\alpha) \mid \beta \in A\} \cap \alpha$ is cofinal in α. So $t := t_\alpha$ is as sought. □

Let $t \in T^{\rightsquigarrow B}$ and the corresponding D be as in the claim. There are two subcases to consider:

Subcase 1.1: $t \in T^{\rightsquigarrow A}$. Pick $\bar{A} \in [A]^\mu$ such that $\bar{D} := \{\Delta(b_\alpha, t) \mid \alpha \in \bar{A}\}$ is a μ-sized subset of $D \cap \mathrm{dom}(t)$. Clearly, $B' := B_t \setminus \sup(\bar{A})$ is a set of size λ. Since $T^{\rightsquigarrow B'} \in \mathcal{T}(\lambda,\lambda)$, $E := \Delta[B' \circledast B']$ is of size λ, as well. By the choice of d, we may now find $(\delta,\epsilon) \in \bar{D} \circledast E$ such that $d(\delta,\epsilon) = \tau$. Pick $\alpha \in \bar{A}$ such that $\Delta(b_\alpha, t) = \delta$. Finally, find $(\beta,\gamma) \in B' \circledast B'$ such that $\Delta(\beta,\gamma) = \epsilon$. Then

$$\Delta(\beta,\gamma) = \epsilon > \delta = \Delta(b_\alpha, t) = \Delta(\alpha,\gamma),$$

and hence $\Delta(\alpha,\beta) = \Delta(\alpha,\gamma) < \Delta(\beta,\gamma)$. Altogether, $(\alpha,\beta,\gamma) \in A \circledast B \circledast B$, and

$$c(\alpha,\beta,\gamma) = d(\Delta(\alpha,\gamma),\Delta(\beta,\gamma)) = d(\delta,\epsilon) = \tau.$$

Subcase 1.2: $t \notin T^{\rightsquigarrow A}$. Pick $t' \in T^{\rightsquigarrow A}$ incompatible with t such that $\sup(D) > \Delta(t,t')$. As $\mathrm{cf}(\theta_B) = \mu$, we may now pick $\bar{B} \in [B]^\mu$ such that $\bar{D} := \{\Delta(b_\beta, t) \mid \beta \in \bar{B}\}$ is a μ-sized subset of D with $\min(\bar{D}) > \Delta(t,t')$.

As $t' \in T^{\rightsquigarrow A}$, $A_{t'}$ has size λ and so does $A' := A_{t'} \setminus \sup(\bar{B})$. As $t \in T^{\rightsquigarrow B}$, B_t has size λ, so since $\varrho \restriction [A \cup B]^2$ witnesses $\mathrm{U}(\lambda, 2, \lambda, 3)$, the set $E := \varrho[A' \circledast B_t]$ has size λ, as well. By the choice of d, find $(\delta,\epsilon) \in \bar{D} \circledast E$ such that $d(\delta,\epsilon) = \tau$. Find

$\alpha \in \bar{B}$ such that $\Delta(b_\alpha, t) = \delta$. Find $(\beta,\gamma) \in A' \circledast B_t$ such that $\varrho(\beta,\gamma) = \epsilon$. As $(\beta,\gamma) \in A_{t'} \circledast B_t$,

$$\Delta(\beta,\gamma) = \Delta(t',t) < \min(\bar{D}) \leq \delta = \Delta(b_\alpha, t) = \Delta(\alpha,\gamma),$$

and hence $\Delta(\alpha,\beta) = \Delta(\beta,\gamma)$. Altogether, $(\alpha,\beta,\gamma) \in B \circledast A \circledast B$, and

$$c(\alpha,\beta,\gamma) = d(\Delta(\alpha,\gamma), \varrho(\beta,\gamma)) = d(\delta,\epsilon) = \tau.$$

Case 2: $\theta_A < \lambda$. Set $\theta := \theta_A$. We shall need the following claim.

Claim 5.3.2. *There exist $\chi < \theta$, $A' \in [A]^\lambda$ and $B' \in [B]^\lambda$ such that $\Delta[A' \times B'] = \{\chi\}$.*

Proof. Denote $\bar{A} := \mathcal{B}(T^{\rightsquigarrow A})$ and $\bar{B} := \mathcal{B}(T^{\rightsquigarrow B})$. Recall that by our application of Lemma 3.7, $|\bar{A}| = \lambda$, and if $\theta_B < \lambda$, then $|\bar{B}| = \lambda$, as well. We shall prove the claim by showing that there exist $\chi < \theta$ and a pair $(t,t') \in (T^{\rightsquigarrow A})_{\chi+1} \times (T^{\rightsquigarrow B})_{\chi+1}$ such that $\Delta(t,t') = \chi$. Indeed, once we have such a pair (t,t'), the sets $A' := A_t$ and $B' := B_{t'}$ would be as sought.

There are two cases to consider:

▶ If $\theta_B = \theta$, then set $\bar{T} := T \cap {}^{<\theta}2$. In this case, $\bar{B}$ and $\bar{A}$ are λ-sized subsets of $\mathcal{B}(\bar{T})$. So Lemma 3.4(2) yields an $s \in \bar{T}$ together with $i \neq i'$ such that $s^\frown\langle i\rangle \in \bar{T}^{\rightsquigarrow \bar{A}} \subseteq T^{\rightsquigarrow A}$ and $s^\frown\langle i'\rangle \in \bar{T}^{\rightsquigarrow \bar{B}} \subseteq T^{\rightsquigarrow B}$. Evidently, $\chi := \mathrm{dom}(s)$, $t := s^\frown\langle i\rangle$ and $t' := s^\frown\langle i'\rangle$ are as sought.

▶ If $\theta_B > \theta$, then pick $r \in (T^{\rightsquigarrow B})_\theta$. For every $a \in \bar{A} \setminus \{r\}$, $\chi_a := \Delta(a,r)$ is smaller than θ. As $|\bar{A}| = \lambda$, we can find $\chi < \theta$ such that λ-many a's in $\bar{A}\setminus\{r\}$ satisfy $\chi_a = \chi$. As the χ^{th} level of $T^{\rightsquigarrow A}$ has size $< \lambda$, we may then find $t \in (T^{\rightsquigarrow A})_{\chi+1}$ such that that λ-many a's in $\bar{A} \setminus \{r\}$ satisfy $\chi_a = \chi$ and $a \restriction (\chi+1) = t$. Clearly, χ, t and $t' := r \restriction (\chi+1)$ are as sought. □

Let χ be given by the claim. For notational simplicity, we shall assume that $\Delta[A \times B] = \{\chi\}$. Let M be an elementary submodel of $\mathcal{H}_{(2^\lambda)^+}$ containing $\{\chi, A, B, T^{\rightsquigarrow A}, T^{\rightsquigarrow B}, \varrho, d\}$ such that $\sigma := M \cap \lambda$ is in S_τ. In particular, $|M| = \mu$. Denote $\upsilon := \sup(A)$ and $\upsilon_M := \sup(M \cap \upsilon)$. Note that since $T^{\rightsquigarrow A} \in \mathcal{T}(\lambda,\theta)$ and as $\lambda = \mu^+ > |\theta|$, it follows that $T^{\rightsquigarrow A}$ has size $\leq \mu$. So, $T^{\rightsquigarrow A} \subseteq M$.

Consider the following sets:

- $A^0 := \{\alpha \in A \mid |\varrho_\alpha[B]| = \lambda\}$,
- $A^1 := \{\alpha \in A \mid |\varrho_\alpha[B]| \leq \mu\}$.

Observe that $A^0, A^1 \in M$. We examine two subcases.

Subcase 2.1: A^0 has size λ. Appeal to Corollary 3.5 with $X := A^0$ and $i := 0$ to pick $\alpha \in A^0$ such that for cofinally many $\delta < \theta$, the two hold:

(1) $b_\alpha(\delta) = 0$, and
(2) $\{\beta \in A \mid \Delta(\alpha,\beta) = \delta\}$ has size λ.

Since $\theta \in E^\lambda_\mu$, $D := \{\delta < \theta \mid \text{Clauses (1) and (2) both hold}\}$ has size μ. For each $\delta \in D$, use Clause (2) to fix $\beta_\delta \in A$ above α such that $\Delta(\alpha,\beta_\delta) = \delta$.

Consider $\varsigma := \sup\{\beta_\delta \mid \delta < \mu\}$. As $|B \cap \varsigma| \leq \mu$, the fact that $\alpha \in A^0$ implies that $E := \varrho_\alpha[B \setminus \varsigma]$ has size λ. By the choice of d, then, we may pick $\delta \in D \setminus (\chi+1)$ and $\epsilon \in E$ above δ such that $d(\delta,\epsilon) = \tau$. Pick $\gamma \in B \setminus \varsigma$ such that $\epsilon = \varrho(\alpha,\gamma)$. Clearly, $\alpha < \beta_\delta < \gamma$.

Recall that $\Delta(\alpha,\beta_\delta)=\delta>\chi=\Delta(\beta_\delta,\gamma)$. Since $b_\alpha(\delta)=0$, we conclude that $b_{\beta_\delta}(\delta)=1$ and $b_\alpha<_{\mathrm{lex}}b_{\beta_\delta}$. Altogether, $(\alpha,\beta_\delta,\gamma)\in A\circledast A\circledast B$, and

$$c(\alpha,\beta_\delta,\gamma)=d(\Delta(\alpha,\beta_\delta),\varrho(\alpha,\gamma))=d(\delta,\epsilon)=\tau,$$

as sought.

Subcase 2.2: A^1 has size λ. By Condition (iii), find $A'\in[A^1]^\lambda$ and $B'\in[B]^\lambda$ such that $\min(\varrho[A'\circledast B'])>\sigma$. Appeal to Corollary 3.5 with $X:=A'$ and $i:=0$ to pick $\beta\in A'\setminus\upsilon_M$ such that for cofinally many $\delta<\theta$, the two hold:

(1) $b_\beta(\delta)=0$, and
(2) $\{\alpha\in A'\mid\Delta(\alpha,\beta)=\delta\}$ has size λ.

Since $\theta\in E^\lambda_\mu$, $D:=\{\delta<\theta\mid$ Clauses (1) and (2) both hold$\}$ has size μ.

Pick $\gamma\in B'\setminus(\beta+1)$ arbitrarily. As $(\beta,\gamma)\in A'\circledast B'$, the ordinal $\epsilon:=\varrho(\beta,\gamma)$ is bigger than σ. Since e_ϵ is an injection to $\mu=|D|$, we may pick $\delta\in D$ such that $e(\delta,\epsilon)>\max\{e(\chi,\epsilon),e(\sigma,\epsilon)\}$. In addition, since $\mu=\mathrm{cf}(\sigma)$, the following set is bounded below σ:

$$Z:=\{\zeta<\sigma\mid e(\zeta,\epsilon)\le e(\delta,\epsilon)\}.$$

Set $t:=(b_\beta\restriction\delta)^\frown\langle1\rangle$. From $\delta\in D$ we infer that $(A')_t$ has size λ. As $t\in T^{\rightsquigarrow A}\subseteq M$, in particular, $(A^1)_t$ is a set of size λ lying in M. By Condition (iii) and elementarity of M one can find $\alpha_0\neq\alpha_1$ in $(A^1)_t\cap M$ such that $\varrho(\alpha_0,\alpha_1)>\sup(Z)$. As ϱ is subadditive, we may now find $\alpha\in\{\alpha_0,\alpha_1\}$ such that $\varrho(\alpha,\gamma)>\sup(Z)$. Note that, as $\{\alpha,B,\varrho\}\in M$, and as $\alpha\in A^1$, $\varrho_\alpha[B]$ is a set of size no more than μ lying in M, so that $\sup(\varrho_\alpha[B])\in M$. In particular, $\varrho(\alpha,\gamma)<\sigma$.

Claim 5.3.3. *All of the following hold:*

(1) $(\alpha,\beta,\gamma)\in A\circledast A\circledast B$;
(2) $\Delta(\alpha,\beta)>\Delta(\beta,\gamma)$;
(3) $b_\beta<_{\mathrm{lex}}b_\alpha$;
(4) $\min(Z_{(\alpha,\beta,\gamma)})=\sigma$.

Proof. (1) γ was chosen to be in $B\setminus(\beta+1)$. In addition, $\beta\in A'\setminus\upsilon_M$, whereas $\alpha\in M\cap A$.
(2) Since $\Delta(\alpha,\beta)=\Delta(t,b_\beta)=\delta>\chi=\Delta(\beta,\gamma)$.
(3) By the definition of t and since $\delta\in D$.
(4) As $\delta=\Delta(\alpha,\beta)$ and $\varrho(\beta,\gamma)=\epsilon$, we infer that $Z_{(\alpha,\beta,\gamma)}=\{\zeta\in\lambda\setminus\varrho(\alpha,\gamma)\mid e(\delta,\epsilon)\ge e(\zeta,\epsilon)\}$. In particular, $\sigma\in Z_{(\alpha,\beta,\gamma)}$. Now, if $\zeta:=\min(Z_{(\alpha,\beta,\gamma)})$ is below σ, then $\zeta\in Z$, contradicting the fact $\varrho(\alpha,\gamma)>\sup(Z)$. □

By the preceding claim and the definition of c,

$$c(\alpha,\beta,\gamma)=h(\min(Z_{(\alpha,\beta,\gamma)})=h(\sigma)=\tau,$$

as sought. □

6. Countably many colors

The main result of this section asserts that

$$\lambda^+\overset{\sup}{\not\to}[\lambda,\lambda]^3_\omega$$

holds, provided that $\lambda=\mu^+$ for an infinite cardinal $\mu=\mu^{<\mu}$. The idea of the proof is to build on the colorings $c:[\lambda^+]^3\to\lambda$ given by Theorems 5.2 and 5.3 with respect to the subadditive coloring $\rho:[\lambda^+]^2\to\lambda$ given by Fact 2.14. By Clause (iii) of these theorems, we must address the problematic case in which the two sets A,B

of Definition 4.20 do not satisfy that $\rho \restriction [A \cup B]^2$ witnesses $\mathrm{U}(\lambda, \lambda, \lambda, 3)$. Anyone that is familiar with [Tod07, §10] would probably suggest to use the oscillation of [Tod07, §8] in this problematic case, and this indeed works. Unfortunately, to verify that this works in the rectangular context, we had to reopen and tweak the proofs. The experts may want to skip directly to Corollary 6.17. The newcomers may benefit from the modular exposition.

Setup 6.1. For the rest of this section, κ stands for a regular uncountable cardinal, Υ is a large enough regular cardinal (e.g., $(2^\kappa)^+$), and we fix some C-sequence $\vec{C} = \langle C_\beta \mid \beta < \kappa \rangle$. We shall also assume that $\mathrm{otp}(C_\beta) = \mathrm{cf}(\beta)$ for all $\beta < \kappa$, though this will only play a role in the proof of Lemma 6.16 below.

The items of the next definition correspond to Definitions 8.1.4, 6.3.1 and 8.1.1 of [Tod07], where the last item is a non-essential strengthening of the latter.

Definition 6.2. A subset $\Gamma \subseteq \kappa$ with $\mathrm{cf}(\mathrm{otp}(\Gamma)) > \omega$ is said to be:

- *$\vec{C}$-stationary* iff $\bigcup_{\beta \in \Gamma} (\mathrm{acc}(C_\beta) \cup \{\beta\})$ is stationary in $\sup(\Gamma)$;
- *$\vec{C}$-nontrivial* iff for every club $D \subseteq \sup(\Gamma)$, there exists $\alpha \in \Gamma$ such that $D \cap \alpha \nsubseteq C_\beta$ for all $\beta \in \Gamma$;
- *$\vec{C}$-oscillating* iff for every club $D \subseteq \sup(\Gamma)$, there exist $\beta \in \Gamma$ and an increasing sequence $\langle \delta_j \mid j < \omega \rangle$ of ordinals in $D \setminus C_\beta$ such that $(\delta_j, \delta_{j+1}) \cap C_\beta \neq \emptyset$ for all $j < \omega$.

Lemma 6.3. *Suppose that $\Gamma \subseteq \kappa$ is such that $\mathrm{cf}(\mathrm{otp}(\Gamma)) > \omega$.*
If Γ is $\vec{C}$-nontrivial and $\vec{C}$-stationary, then Γ is $\vec{C}$-oscillating.

Proof. Suppose that Γ is $\vec{C}$-nontrivial and $\vec{C}$-stationary. By the latter, $\Delta := \bigcup_{\beta \in \Gamma} (\mathrm{acc}(C_\beta) \cup \{\beta\})$ is a stationary subset of θ. For each $\delta \in \Delta$, pick $\beta_\delta \in \Gamma$ such that $\sup(C_{\beta_\delta} \cap \delta) = \delta$.

Next, to verify that Γ is $\vec{C}$-oscillating, let $D \subseteq \sup(\Gamma)$ be a given club.

Claim 6.3.1. *There exists $\delta \in \Delta$ such that $\sup(D \cap \delta \setminus C_{\beta_\delta}) = \delta$.*

Proof. Suppose not. Then, for every $\delta \in \Delta$, $\epsilon_\delta := \sup(D \cap \delta \setminus C_{\beta_\delta})$ is smaller than δ. Fix $\epsilon < \sup(\Gamma)$ for which $S := \{\delta \in \Delta \mid \epsilon_\delta < \epsilon < \delta\}$ is stationary. Now, consider the club $D' := D \setminus \epsilon$. Then, for every $\alpha \in \Gamma$, letting $\delta := \min(S \setminus \alpha)$, it is the case that $D' \cap \alpha \subseteq D \cap [\epsilon, \delta) \subseteq C_{\beta_\delta}$. This contradicts the fact that Γ is $\vec{C}$-nontrivial. $\square$

Let δ be given by the claim. As $\sup(D \cap \delta \setminus C_{\beta_\delta}) = \delta = \sup(C_{\beta_\delta} \cap \delta)$, it is easy to recursively construct an increasing sequence $\langle \delta_j \mid j < \omega \rangle$ of ordinals in $D \cap \delta \setminus C_{\beta_\delta}$ such that $(\delta_j, \delta_{j+1}) \cap C_{\beta_\delta} \neq \emptyset$ for all $j < \omega$. $\square$

Definition 6.4. For two disjoint sets of ordinals y and z, we say that P is a *y-convex subset of z* iff one of the following occurs:

- $P = \{\zeta \in z \mid \zeta < \alpha\}$ and $\alpha = \min(y)$;
- $P = \{\zeta \in z \mid \beta < \zeta\}$ and $\beta = \max(y)$;
- $P = \{\zeta \in z \mid \alpha < \zeta < \beta\}$ and $\alpha < \beta$ are two consecutive elements of y.

Note that if P and Q are nonempty y-convex subsets of z, then either $P < Q$ or $Q < P$.

Definition 6.5 (Todorčević, [Tod07, §8])**.** For an ordinal $\varepsilon < \kappa$, define a function $\mathrm{Osc}_\varepsilon : [\mathcal{P}(\kappa)]^2 \to \mathcal{P}(\mathcal{P}(\kappa))$ via

$$\mathrm{Osc}_\varepsilon(x, y) := \begin{cases} \{P \mid P \text{ is a nonempty } y\text{-convex subset of } x \setminus \varepsilon\}, & \text{if } y \cap x \subseteq \varepsilon; \\ \emptyset, & \text{otherwise.} \end{cases}$$

Then the *oscillation mapping* $\mathrm{osc}_\varepsilon : [\mathcal{P}(\kappa)]^2 \to \mathrm{CARD}(\kappa + 1)$ is defined via $\mathrm{osc}_\varepsilon(x, y) := |\mathrm{Osc}_\varepsilon(x, y)|$.

Remark 6.6.
1. If we omit the subscript ε, then $\mathrm{Osc}(x, y)$ and $\mathrm{osc}(x, y)$ are understood to be $\mathrm{Osc}_\varepsilon(x, y)$ and $\mathrm{osc}_\varepsilon(x, y)$ for $\varepsilon := \mathrm{ssup}(x \cap y)$.
2. For all $\varepsilon < \alpha < \beta < \kappa$ such that $C_\alpha \cap C_\beta \subseteq \varepsilon$, $C_\alpha \setminus \varepsilon$ and C_β have no common accumulation points, and hence $\mathrm{Osc}_\varepsilon(C_\alpha, C_\beta)$ is finite. In this case, we shall identify $\mathrm{Osc}_\varepsilon(C_\alpha, C_\beta)$ with its increasing enumeration $\langle P_0, \ldots, P_n\rangle$.

The next lemma makes explicit some of the features that are present in the proof of [Tod07, Lemma 8.1.2].

Lemma 6.7 (Todorčević)**.** *Suppose that Γ is a cofinal subset of some $\theta \leq \kappa$ of uncountable cofinality, and that Γ is $\vec{C}$-oscillating. For every cofinal $E \subseteq \theta$, there exists $\beta \in \Gamma$ such that for every positive integer n, there are $\alpha \in \Gamma \cap \beta$ and $\varepsilon \in E \cap \alpha$ such that all of the following hold:*

- $\mathrm{osc}_\varepsilon(C_\alpha, C_\beta) = n$;
- *for every $j < n$, there is a pair $\epsilon < \epsilon'$ of ordinals in $E \setminus C_\alpha$ for which*
$$\mathrm{Osc}_\varepsilon(C_\alpha, C_\beta)(j) = C_\alpha \cap (\epsilon, \epsilon').$$

Proof. Set $\mu := \mathrm{cf}(\theta)$, and fix a map $\psi : \mu \to \theta$ whose image is cofinal in θ. Let $\mathcal{M}$ be a continuous $\in$-chain of length μ consisting of elementary submodels $M \prec H_\Upsilon$ with $M \cap \mu \in \mu$ and $\{\psi, \vec{C}, \Gamma, E\} \in M$. It follows that $D := \{\sup(M \cap \theta) \mid M \in \mathcal{M}\}$ constitutes a club in θ. Recalling that Γ is $\vec{C}$-oscillating, pick $\beta \in \Gamma$ and an increasing sequence $\langle \delta_j \mid j < \omega\rangle$ of ordinals in $D \setminus C_\beta$ such that $(\delta_j, \delta_{j+1}) \cap C_\beta \neq \emptyset$ for all $j < \omega$. By possibly replacing δ_j by δ_{j+1}, we may assume that $C_\beta \cap \delta_0$ is nonempty. For every $j < \omega$, since $\delta_j \in D \setminus C_\beta$, pick $M_j \in \mathcal{M}$ such that $\sup(M_j \cap \theta) = \delta_j$, and note that $\gamma_j := \sup(C_\beta \cap \delta_j)$ and $\Omega_j := \min(M_j \cap \theta \setminus \gamma_j)$ are both smaller than δ_j. So, for every $j < \omega$:

$$0 < \sup(C_\beta \cap \delta_j) = \gamma_j \leq \Omega_j < \delta_j < \gamma_{j+1} < \beta.$$

For each $k < \omega$, let $\mathcal{I}_k$ denote the collection of all increasing sequences $\vec{I} = \langle I_j \mid j \leq k\rangle$ of closed intervals in θ. Now, let n be a positive integer and we shall find $\alpha \in \Gamma \cap \beta$ and $\varepsilon \in E \cap \alpha$ as in the conclusion of the lemma.

Define a sequence of collections $\langle \mathcal{F}_{n-i} \mid i \leq n\rangle$ by recursion on $i \leq n$, as follows:

▶ For $i = 0$, let $\mathcal{F}_n$ be the set of all $\langle I_j \mid j \leq n\rangle \in \mathcal{I}_n$ such that the following two hold:

1. $I_0 = [0, \Omega_0]$;
2. $\alpha := \max(I_n)$ belongs to Γ, $C_\alpha \subseteq I_0 \cup \cdots \cup I_n$, and $C_\alpha \cap I_j \neq \emptyset$ for every $j \leq n$.

▶ For every $i < n$ such that $\mathcal{F}_{n-i}$ has already been defined, let $\mathcal{F}_{n-i-1}$ be the collection of all $\vec{I} \in \mathcal{I}_{n-i-1}$ with the property that for every $\epsilon < \theta$ there exists a closed interval $I \subseteq (\epsilon, \theta)$ such that $\vec{I}^\frown\langle I\rangle \in \mathcal{F}_{n-i}$.

Claim 6.7.1. $\langle [0, \Omega_0]\rangle \in \mathcal{F}_0$.

Proof. For every $j \leq n$, define:

$$I_j := \begin{cases} [0, \Omega_0], & \text{if } j = 0; \\ [\delta_{j-1}, \Omega_j], & \text{if } 0 < j < n; \\ [\delta_{n-1}, \beta], & \text{otherwise.} \end{cases}$$

We shall prove by induction on $i \leq n$ that $\langle I_j \mid j \leq n-i\rangle \in \mathcal{F}_{n-i}$. The base case is immediate, since $\langle I_j \mid j \leq n\rangle$ satisfies requirements (1) and (2), with β playing the role of α.

Next, suppose that we are given $i < n$ for which $\langle I_j \mid j \leq n-i\rangle \in \mathcal{F}_{n-i}$ has been established. Note:

- $\langle \mathcal{F}_k \mid k \leq n\rangle \in M_0 \subseteq M_{n-i-1}$;
- $\langle I_j \mid j \leq n-i-1\rangle \in M_{n-i-1} \cap \mathcal{F}_{n-i}$;
- $I_{n-i} \in M_{n-i} \setminus M_{n-i-1}$.

So, by elementarity of M_{n-i-1}, $\langle I_j \mid j \leq n-i-1\rangle \in \mathcal{F}_{n-i-1}$. □

It follows that we may recursively construct a sequence $\langle I_j \mid j \leq n\rangle$ such that:

(3) $I_0 = [0, \Omega_0]$, so that $\langle I_0\rangle \in \mathcal{F}_0 \cap M_0$;
(4) $\langle I_j \mid j \leq k+1\rangle \in \mathcal{F}_{k+1} \cap M_k$ for every $k < n$;
(5) $I_{j+1} \subseteq (\min(E \setminus \Omega_j + 1), \theta)$ for every $j < n$.

For each $j < n$, denote $\epsilon_j := \min(E \setminus \Omega_j + 1)$, and note that since I_{j+1} and E are in M_j, $\epsilon'_j := \min(E \setminus \max(I_{j+1}) + 1)$ is $< \delta_j$. Denote $\gamma'_j := \min(C_\beta \setminus \gamma_j + 1)$ so that $\gamma_j < \gamma'_j$ are two consecutive elements of C_β. Since $\sup(C_\beta \cap \delta_j) = \gamma_j$, altogether,

$$I_{j+1} \subseteq (\epsilon_j, \epsilon'_j) \subseteq (\Omega_j, \delta_j) \subseteq (\gamma_j, \gamma'_j) \subseteq (\gamma_j, \gamma_{j+1}).$$

Now, put $\alpha := \max(I_n)$. Then $\alpha \in \Gamma \cap \delta_{n-1} \subseteq \Gamma \cap \beta$, $C_\alpha \subseteq I_0 \cup \cdots \cup I_n$ and $C_\alpha \cap I_j \neq \emptyset$ for every $j \leq n$. So $(C_\alpha \setminus I_0) \subseteq \bigcup_{j<n}(\Omega_j, \delta_j)$. On the other hand, $(C_\beta \setminus I_0) \cap (\bigcup_{j<n}(\Omega_j, \delta_j)) = \emptyset$. Therefore, for $\varepsilon := \epsilon_0$, we get that

$$C_\alpha \cap C_\beta \subseteq (\Omega_0 + 1) \subseteq \varepsilon.$$

Claim 6.7.2. $\{\epsilon_j, \epsilon'_j \mid j < n\} \cap C_\alpha = \emptyset$.

Proof. Suppose not, and fix $j < n$ such that $\{\epsilon_j, \epsilon'_j\} \cap C_\alpha \neq \emptyset$. As $\max(I_0) = \Omega_0 \leq \Omega_j < \epsilon_j < \epsilon'_j$, we may fix some $i < n$ such that $\{\epsilon_j, \epsilon'_j\} \cap I_{i+1} \neq \emptyset$. Recalling that $I_{j+1} \subseteq (\epsilon_j, \epsilon'_j)$, it must be the case that $i \neq j$. Note:

► If $i < j$, then $\gamma_{i+1} \leq \Omega_{i+1} \leq \Omega_j < \epsilon_j < \epsilon'_j$.
► If $i > j$, then $\epsilon_j < \epsilon'_j < \delta_j < \gamma_{j+1} \leq \gamma_i$.

So, both options contradict the fact that $I_{i+1} \subseteq (\gamma_i, \gamma_{i+1})$. □

By Clause (2), for every $j \leq n$, $C_\alpha \cap I_{j+1} \neq \emptyset$. Altogether, for every $\varsigma \in (\Omega_0, \varepsilon]$:

$$\begin{aligned} \mathrm{Osc}_\varsigma(C_\alpha, C_\beta) &= \langle C_\alpha \cap I_{j+1} \mid j < n\rangle \\ &= \langle C_\alpha \cap (\gamma_j, \gamma'_j) \mid j < n\rangle \\ &= \langle C_\alpha \cap (\epsilon_j, \epsilon'_j) \mid j < n\rangle. \end{aligned}$$

In particular, $\mathrm{osc}_\varepsilon(C_\alpha, C_\beta) = n$. □

Remark 6.8. In the preceding proof, in the special case that $\kappa = \theta$ or $\kappa = (\mathrm{cf}(\theta))^+$, one can secure that Ω_0 be equal to γ_0. So, in this case, we would get that $\max(C_\alpha \cap C_\beta) = \Omega_0$, meaning that the conclusion of the lemma remains valid also after omitting the subscript ε.

Definition 6.9. Define $\chi : [\kappa]^3 \to \omega$ by letting for all $\alpha < \beta < \kappa$:

$$\chi(\alpha, \beta, \gamma) := \max\{k < \omega \mid \mathrm{Tr}(\alpha, \gamma)(k) = \mathrm{Tr}(\beta, \gamma)(k)\}.$$

Definition 6.10 ([Tod07, Definition 10.3.1])**.** A subset $A \subseteq \kappa$ is said to be *stable* if $\chi``[A]^3$ is finite. Otherwise, we say that A is *unstable*.

Similar to [Tod07, Definition 10.3.3], we use χ to derive the following stepping-up of the two-dimensional oscillation.

Definition 6.11. The *three-dimensional oscillation mapping*, $\overline{\mathrm{osc}} : [\kappa]^3 \to \omega$ is defined on the basis of the two-dimensional oscillation defined in Definition 6.5 via:

$$\overline{\mathrm{osc}}(\alpha, \beta, \gamma) := \mathrm{osc}_\alpha(C_{\mathrm{Tr}(\alpha,\beta)(\chi(\alpha,\beta,\gamma))}, C_{\mathrm{Tr}(\alpha,\gamma)(\chi(\alpha,\beta,\gamma))}).$$

We now verify a rectangular version of [Tod07, Lemma 10.3.4]:

Lemma 6.12 (Todorčević)**.** *Suppose that B is a cofinal subset of some $\theta \leq \kappa$ of uncountable cofinality, and that every cofinal subset of B is unstable.*

Then, for every cofinal $A \subseteq \theta$ and every positive integer n, there exists $(\alpha, \beta, \gamma) \in A \circledast B \circledast B$ such that $\overline{\mathrm{osc}}(\alpha, \beta, \gamma) = n$.

Proof. For each $\delta < \theta$, let $\beta_\delta := \min(B \setminus (\delta + 1))$ and $\Lambda_\delta := \lambda_2(\delta, \beta_\delta)$. By Fodor's lemma, fix $\Lambda < \theta$, $k < \omega$ and a stationary $S \subseteq \mathrm{acc}(\theta)$ such that, for all $\delta \in S$:

(1) $\Lambda_\delta \leq \Lambda$;
(2) $\rho_2(\eth_{\delta,\beta_\delta}, \beta_\delta) = k$;
(3) for every $\bar{\delta} < \delta$, $\beta_{\bar{\delta}} < \delta$.

Claim 6.12.1. *For every $\delta \in S$ and every ordinal α with $\Lambda < \alpha < \eth_{\delta,\beta_\delta}$:*

- $\mathrm{Tr}(\alpha, \beta_\delta) \restriction (k + 1) = \mathrm{Tr}(\delta, \beta_\delta) \restriction (k + 1)$*, and*
- $\mathrm{Tr}(\alpha, \beta_\delta)(k) = \eth_{\delta,\beta_\delta}$.

Proof. By Remark 2.13. □

Let $\Gamma := \{\eth_{\delta,\beta_\delta} \mid \delta \in S\}$. For each $\xi \in \Gamma$, pick $\delta(\xi) \in S$ such that $\xi = \eth_{\delta(\xi),\beta_{\delta(\xi)}}$. Note that $\delta(\xi) \leq \xi \leq \beta_{\delta(\xi)}$.

Claim 6.12.2. Γ *is $\vec{C}$-oscillating.*

Proof. As $\delta \in \mathrm{acc}(\eth_{\delta,\beta_\delta}) \cup \{\eth_{\delta,\beta_\delta}\}$ for every $\delta \in S$, we have $S \subseteq \bigcup_{\xi \in \Gamma} \mathrm{acc}(C_\xi) \cup \{\xi\}$. So Γ is $\vec{C}$-stationary. By Lemma 6.3, it thus suffices to prove that Γ is $\vec{C}$-nontrivial. Towards a contradiction, suppose this is not so, and fix a club $D \subseteq \theta$ such that, for every $\alpha \in \Gamma$ there exists $\beta \in \Gamma$ with $D \cap \alpha \subseteq C_\beta$. As $\sup(\Gamma) = \theta$, we may then recursively construct a sparse enough cofinal subset $X \subseteq \Gamma$ with the property that for every pair $\xi < \xi'$ of ordinals from X, all of the following hold:

- $\Lambda < \delta(\xi)$;
- $D \cap (\beta_{\delta(\xi)}, \delta(\xi')) \neq \emptyset$;
- $D \cap \beta_{\delta(\xi)} \subseteq C_{\xi'}$.

As $B' := \{\beta_{\delta(\xi)} \mid \xi \in X\}$ is a cofinal subset of B, it must be unstable. We shall reach a contradiction by showing that $\chi``[B']^3 = \{k\}$. To this end, let $\alpha < \beta < \gamma$ be a triple of ordinals from B'. Fix a triple $\xi < \xi' < \xi''$ of ordinals from X such that $\alpha = \beta_{\delta(\xi)}$, $\beta = \beta_{\delta(\xi')}$, and $\gamma = \beta_{\delta(\xi'')}$. Then:

- $\Lambda < \delta(\xi) < \alpha < \delta(\xi') < \beta < \delta(\xi'') \leq \xi'' \leq \gamma$;
- $D \cap (\alpha, \delta(\xi')) \neq \emptyset$;

- $D \cap \beta \subseteq C_{\xi''}$.

Pick $\iota \in D \cap (\alpha, \delta(\xi')) \neq \emptyset$, so that $\iota \in D \cap (\alpha, \beta) \subseteq C_{\xi''}$. Appealing to Claim 6.12.1 with $\delta'' := \delta(\xi'')$, we infer that:

- $\mathrm{Tr}(\alpha, \gamma) \restriction (k+1) = \mathrm{Tr}(\delta'', \gamma) \restriction (k+1) = \mathrm{Tr}(\beta, \gamma) \restriction (k+1)$, and
- $\mathrm{Tr}(\alpha, \gamma)(k) = \xi'' = \mathrm{Tr}(\beta, \gamma)(k)$.

Therefore

$$\mathrm{Tr}(\alpha, \gamma)(k+1) = \min(C_{\xi''} \setminus \alpha) \leq \iota < \beta \leq \min(C_{\xi''} \setminus \beta) = \mathrm{Tr}(\beta, \xi)(k+1).$$

Recalling Definition 6.9, this indeed means that $\chi(\alpha, \beta, \gamma) = k$. □

Now, given a cofinal $A \subseteq \theta$ and a positive integer n, appeal to Lemma 6.7 with $E := \mathrm{acc}^+(A \setminus \Lambda)$ to find a pair $(\xi, \zeta) \in \Gamma \circledast \Gamma$ and an ordinal $\varepsilon \in E \cap \xi$ such that:

- $\mathrm{osc}_\varepsilon(C_\xi, C_\zeta) = n + 1$, and
- for every $j < n+1$, there is a pair $\epsilon < \epsilon'$ of ordinals in $E \setminus C_\xi$ for which
$$\mathrm{Osc}_\varepsilon(C_\xi, C_\zeta)(j) = C_\xi \cap (\epsilon, \epsilon').$$

Let $\epsilon < \epsilon'$ be a pair of ordinals witnessing the case $j = 0$ of the preceding. Clearly,

$$\mathrm{Osc}_\varepsilon(C_\xi, C_\zeta)(0) = C_\xi \cap [\epsilon, \epsilon'].$$

Since $\mathrm{osc}(C_\xi, C_\zeta) > 1$ and $\xi < \zeta$, we may fix two consecutive elements $\bar{\alpha} < \bar{\beta}$ of C_ζ such that

$$\mathrm{Osc}_\varepsilon(C_\xi, C_\zeta)(1) = C_\xi \cap (\bar{\alpha}, \bar{\beta}).$$

So, $\epsilon < \epsilon' \leq \bar{\alpha} < \xi$.

Since C_ξ is a closed subset of ξ, and $\epsilon' \in \mathrm{acc}^+(A \setminus \Lambda) \cap (\xi \setminus C_\xi)$, we may pick a large enough $\alpha \in A \cap (\Lambda, \epsilon')$ such that

$$\mathrm{Osc}_\varepsilon(C_\xi, C_\zeta)(0) \subseteq (\epsilon, \alpha).$$

In particular, $C_\xi \cap C_\zeta \subseteq \varepsilon \subseteq \alpha$, and

$$\mathrm{osc}_\alpha(C_\xi, C_\zeta) = \mathrm{osc}_\varepsilon(C_\xi, C_\zeta) - 1 = n.$$

Denote $\bar{\delta} := \delta(\xi)$ and $\delta := \delta(\zeta)$. Then $(\bar{\delta}, \delta) \in [S]^2$, $\xi = \eth_{\bar{\delta}, \beta_{\bar{\delta}}}$ and $\zeta = \eth_{\delta, \beta_\delta}$. Set $\beta := \beta_{\bar{\delta}}$ and $\gamma := \beta_\delta$, so that $(\beta, \gamma) \in [B]^2$. Note that

$$\Lambda < \alpha < \epsilon' \leq \bar{\alpha} < \xi \leq \beta < \delta \leq \zeta \leq \gamma.$$

By Claim 6.12.1, then:

- $\mathrm{Tr}(\alpha, \gamma) \restriction (k+1) = \mathrm{Tr}(\delta, \gamma) \restriction (k+1) = \mathrm{Tr}(\beta, \gamma) \restriction (k+1)$;
- $\mathrm{Tr}(\alpha, \gamma)(k) = \zeta = \mathrm{Tr}(\beta, \gamma)(k)$;
- $\mathrm{Tr}(\alpha, \beta)(k) = \xi$.

Therefore,

$$\mathrm{Tr}(\alpha, \gamma)(k+1) = \min(C_\zeta \setminus \alpha) \leq \bar{\alpha} < \beta \leq \min(C_\zeta \setminus \beta) = \mathrm{Tr}(\beta, \gamma)(k+1),$$

and $\chi(\alpha, \beta, \gamma) = k$.

Summing all up, $(\alpha, \beta, \gamma) \in A \circledast B \circledast B$, and

$$\begin{aligned}
\overline{\mathrm{osc}}(\alpha, \beta, \gamma) &= \mathrm{osc}_\alpha(C_{\mathrm{Tr}(\alpha,\beta)(\chi(\alpha,\beta,\gamma))}, C_{\mathrm{Tr}(\alpha,\gamma)(\chi(\alpha,\beta,\gamma))}) \\
&= \mathrm{osc}_\alpha(C_{\mathrm{Tr}(\alpha,\beta)(k)}, C_{\mathrm{Tr}(\alpha,\gamma)(k)}) \\
&= \mathrm{osc}_\alpha(C_\xi, C_\zeta) = n,
\end{aligned}$$

as sought. □

The ending of the proof of Claim 6.12.2 makes it clear that the following hold.

Observation 6.13. *Suppose:*

- $\lambda_2(\delta,\gamma) < \alpha < \beta < \delta < \gamma < \kappa$;
- $C_{\eth_{\delta,\gamma}} \cap [\alpha,\beta)$ *is nonempty.*

Then $\chi(\alpha,\beta,\gamma) = \rho_2(\eth_{\delta,\gamma},\gamma)$. □

The next lemma extracts features present in the proof of [Tod07, Lemma 10.3.2].

Lemma 6.14. *Suppose that X is a stable cofinal subset of some $\theta \leq \kappa$ of uncountable cofinality. Then there exist a cofinal $X' \subseteq X$, a club $D \subseteq \theta$, and a positive integer k satisfying all of the following:*

1. *for every $(\delta,\gamma) \in D \circledast X'$, $D \cap \delta \subseteq C_{\eth_{\delta,\gamma}}$;*
2. *for every $(\delta,\alpha,\delta',\beta,\delta'',\gamma) \in D \circledast \theta \circledast D \circledast \theta \circledast D \circledast X'$:*
   - *$\chi(\alpha,\beta,\gamma) = k$, and*
   - *$D \cap \delta'' \subseteq C_{\mathrm{Tr}(\alpha,\gamma)(k)}$.*

Proof. Set $\mu := \mathrm{cf}(\theta)$, and fix a map $\psi : \mu \to \theta$ whose image is cofinal in θ. Let $\mathcal{M}$ be a continuous $\in$-chain of length μ consisting of elementary submodels $M \prec H_\Upsilon$ with $M \cap \mu \in \mu$ and $\{\psi, \vec{C}, X\} \in M$. For each $M \in \mathcal{M}$, denote $\theta_M := \sup(M \cap \theta)$, so that $E := \{\theta_M \mid M \in \mathcal{M}\}$ is a club in θ.

Claim 6.14.1. *Let $N \prec \mathcal{H}_\Upsilon$ be such that $\{\psi, \vec{C}, X, \mathcal{M}\} \in N$ and $N \cap \mu \in \mu$. Denote $\theta_N := \sup(N \cap \theta)$ and let $\gamma \in X \setminus (\theta_N + 1)$.*[7]

If $\sup(E \cap \theta_N \setminus C_{\eth_{\theta_N,\gamma}}) = \theta_N$, then there exists $\gamma' \in X \setminus (\theta_N + 1)$ with $\rho_2(\eth_{\theta_N,\gamma'},\gamma') > \rho_2(\eth_{\theta_N,\gamma},\gamma)$.

Proof. Denote $\bar\gamma := \eth_{\theta_N,\gamma}$, $n := \rho_2(\bar\gamma,\gamma)$, and

$$\mathcal{M}^\gamma := \{M \in \mathcal{M} \cap N \mid \theta_M \notin C_{\bar\gamma}\}.$$

Now, assuming that $\sup(E \cap \theta_N \setminus C_{\bar\gamma}) = \theta_N$, we infer that

$$\sup\{\theta_M \mid M \in \mathcal{M}^\gamma\} = \theta_N.$$

Pick $M \in \mathcal{M}^\gamma$ with $\theta_M > \lambda_2(\theta_N,\gamma)$. Since $\theta_M \notin C_{\bar\gamma}$, it is the case that $\rho_2(\theta_M,\bar\gamma) > 1$. So, by Remark 2.13,

$$\mathrm{tr}(\theta_M,\gamma) = \mathrm{tr}(\bar\gamma,\gamma)^\frown \mathrm{tr}(\theta_M,\bar\gamma),$$

and

$$\rho_2(\theta_M,\gamma) > \rho_2(\bar\gamma,\gamma) + 1.$$

Thus, $\mathrm{Im}(\mathrm{tr}(\theta_M,\gamma))$ contains not only $\mathrm{Im}(\mathrm{tr}(\bar\gamma,\gamma))$ but also $\min(C_{\bar\gamma} \setminus \theta_M)$ which is strictly above θ_M, therefore, $\rho_2(\eth_{\theta_M,\gamma},\gamma) > n$. Pick $\Lambda \in M \cap \theta$ above $\lambda_2(\eth_{\theta_M,\gamma},\gamma)$. Then, the following set belongs to M and γ witnesses that it has θ_M as an element:

$$S := \{\delta < \theta \mid \exists \gamma' \in X \setminus (\theta + 1)\, [\rho_2(\eth_{\delta,\gamma'},\gamma') > n \ \&\ \lambda_2(\delta,\gamma') \leq \Lambda]\},$$

so that S is stationary in θ. Pick $\delta \in S$ above θ_N, along with a witnessing γ'. As $\lambda_2(\delta,\gamma') \leq \Lambda < \theta_M < \theta_N < \delta$, we get from Remark 2.13 that

$$\mathrm{tr}(\theta_N,\gamma') = \mathrm{tr}(\eth_{\delta,\gamma'},\gamma')^\frown \mathrm{tr}(\theta_N,\eth_{\delta,\gamma'}),$$

and hence $\rho_2(\theta_N,\gamma') \geq \rho_2(\eth_{\delta,\gamma'},\gamma') + 1 > n + 1$. Therefore $\rho_2(\eth_{\theta_N,\gamma'},\gamma') > n$, as sought. □

[7] As $\mathrm{cf}(\theta) = \mu > |N|$, $X \setminus \theta_N$ is co-bounded in X.

Claim 6.14.2. $\Delta := \{\delta < \theta \mid \exists\gamma \in X \setminus (\delta+1)\, [E \cap \delta \subseteq^* C_{\eth_{\delta,\gamma}}]\}$ *covers a club in* θ.

Proof. Let S be a stationary subset of θ, and we shall prove that $S \cap \Delta \neq \emptyset$. Let $N \prec \mathcal{H}_\Upsilon$ be such that $\{\psi, \vec{C}, X, \mathcal{M}\} \in N$, $N \cap \mu \in \mu$, with $\theta_N := \sup(N \cap \theta)$ in S. Using the fact that X is stable, fix $m < \omega$ such that $\chi``[X]^3 \subseteq m$. Now, if $\theta_N \notin \Delta$, then by iterating Claim 6.14.1 finitely many times, we may find a $\gamma \in X \setminus (\theta_N + 1)$ such that $\rho_2(\eth_{\theta_N,\gamma}, \gamma) > m$. Pick $\alpha, \beta \in X$ with $\lambda_2(\theta_N, \gamma) < \alpha < \beta < \theta_N$ such that $C_{\eth_{\theta_N,\gamma}} \cap (\alpha, \beta) \neq \emptyset$. By Observation 6.13, $\chi(\alpha, \beta, \gamma) > m$. This is a contradiction. □

For each $\delta \in \Delta$, fix $\gamma_\delta \in X \setminus (\delta + 1)$, $\epsilon_\delta < \delta$ and $k_\delta < \omega$ such that:

- $E \cap \delta \setminus \epsilon_\delta \subseteq C_{\eth_{\delta,\gamma_\delta},\gamma_\delta}$;
- $\lambda_2(\delta, \gamma_\delta) \leq \epsilon_\delta$;
- $\rho_2(\eth_{\delta,\gamma_\delta}, \gamma_\delta) = k_\delta$.

Find $\epsilon < \theta$ and $k < \omega$ for which the following set is stationary:

$$S := \{\delta \in \Delta \mid \epsilon_\delta = \epsilon \ \&\ k_\delta = k\}.$$

Consider the club $D := \{\delta \in \operatorname{acc}(E \setminus \epsilon) \mid \forall\bar{\delta} \in \Delta \cap \delta\, (\gamma_{\bar{\delta}} < \delta)\}$, and the set $X' := \{\gamma_\delta \mid \delta \in S \cap D\}$. We shall verify that X', D and k satisfy the requirements of the two clauses.

(1) Let $\delta \in D$ and $\gamma \in X' \setminus (\delta + 1)$. Pick $\delta^* \in D$ such that $\gamma = \gamma_{\delta^*}$. If $\delta^* = \delta$, then $D \cap \delta^* \subseteq E \cap \delta \setminus \epsilon \subseteq C_{\eth_{\delta^*,\gamma}}$. Otherwise, $\lambda_2(\delta^*, \gamma) \leq \epsilon < \delta < \delta^* < \gamma = \gamma_{\delta^*}$. So, by Remark 2.13,

$$\operatorname{tr}(\delta, \gamma) = \operatorname{tr}(\eth_{\delta^*,\gamma}, \gamma)^\frown \operatorname{tr}(\delta, \eth_{\delta^*,\gamma}),$$

with $E \cap \delta^* \setminus \epsilon \subseteq C_{\eth_{\delta^*,\gamma}}$. Since $\delta \in \operatorname{acc}(E) \cap (\epsilon, \delta^*)$, it is the case that $\delta \in \operatorname{acc}(C_{\eth_{\delta^*,\gamma}})$. Altogether,

$$\operatorname{tr}(\eth_{\delta,\gamma}, \gamma) = \operatorname{tr}(\eth_{\delta^*,\gamma}, \gamma).$$

In particular, $D \cap \delta \subseteq E \cap (\epsilon, \delta^*) \subseteq C_{\eth_{\delta^*,\gamma}}$.

(2) Let $(\delta, \alpha, \delta', \beta, \delta'', \gamma) \in D \circledast \theta \circledast D \circledast \theta \circledast D \circledast X'$. Fix $\delta^* \in S \cap D$ such that $\gamma = \gamma_{\delta^*}$. As $\delta^* \in \Delta$ and $\gamma_{\bar{\delta}} < \delta'' < \gamma$ for every $\bar{\delta} \in \Delta \cap \delta''$, it must be the case that $\delta^* \geq \delta''$. So $\lambda_2(\delta^*, \gamma) \leq \epsilon < \alpha < \beta < \delta^* < \gamma$ and $\delta' \in D \cap (\alpha, \beta) \subseteq C_{\eth_{\delta^*,\gamma}} \cap (\alpha, \beta)$. Then, Observation 6.13 implies that $\chi(\alpha, \beta, \gamma) = \rho_2(\eth_{\delta^*,\gamma}, \gamma) = k$. □

Remark 6.15. While we will not be needing this fact, we point out that the proofs of Lemmas 6.12 and 6.14 together show that for every $\theta \leq \kappa$ of uncountable cofinality, and every cofinal subset $X \subseteq \theta$, the following are equivalent:

- Every cofinal subset of X is unstable;
- For every stationary $S \subseteq \theta$ and every $\langle \beta_\delta \mid \delta \in S\rangle$ in $\prod_{\delta \in S}(X \setminus (\delta + 1))$, the set $\Gamma := \{\eth_{\delta,\beta_\delta} \mid \delta \in S\}$ is $\vec{C}$-nontrivial.

The following is an easy strengthening of [Tod07, Lemma 10.3.2]:

Lemma 6.16 (Todorčević)**.** *Suppose that* $\kappa = \lambda^+$ *for some regular uncountable cardinal* λ*, and let* $\rho : [\kappa]^2 \to \lambda$ *be the corresponding map given by Fact 2.14.*

Suppose also that X *is a stable subset of* κ *of order-type* λ*. Then there exists a cofinal subset* $X' \subseteq X$ *such that* $\rho \restriction [X']^2$ *witnesses* $\mathrm{U}(\lambda, \lambda, \lambda, \omega)$.

Proof. Denote $\theta := \sup(X)$. Let $X' \subseteq X$ and $D \subseteq \theta$ be given by Lemma 6.14. To see that $\rho \restriction [X']^2$ witnesses $\mathrm{U}(\lambda, \lambda, \lambda, \omega)$, let $\mathcal{A} \in [X']^\sigma$ be some λ-sized pairwise disjoint, with $\sigma < \omega$, and let $\tau < \lambda$.

As $\mathcal{A}$ consists of λ-many pairwise disjoint subsets of X' and $\mathrm{otp}(X') = \lambda$, for every $\delta \in D$, we may pick $b_\delta \in \mathcal{A}$ with $\min(b_\delta) > \delta$. Then put $\Lambda_\delta := \max\{\lambda_2(\delta, \beta) \mid \beta \in b_\delta\}$. Recalling that X' and D were given by Lemma 6.14, for all $\delta \in D$ and $\beta \in b_\delta$,

$$D \cap \delta \subseteq C_{\eth_{\delta,\beta}}.$$

Next, fix some $\Lambda < \theta$ and a stationary set $S \subseteq D$ such that for every $\delta \in S$:

(1) $\Lambda_\delta \leq \Lambda < \delta$,
(2) $\mathrm{otp}(D \cap \delta) > \tau$, and
(3) For every $\gamma \in D \cap \delta$, $\sup(b_\gamma) < \delta$.

Evidently, $\mathcal{B} := \{b_\delta \mid \delta \in S\}$ is a λ-sized subset of $\mathcal{A}$. Now, given $a \neq b$ in $\mathcal{B}$, we may find a pair $\gamma \neq \delta$ of ordinals from S such that $a = b_\gamma$ and $b = b_\delta$. Without loss of generality, $\gamma < \delta$. Let $(\alpha, \beta) \in a \times b$. Then

$$\Lambda_\delta < \gamma < \alpha < \delta < \beta,$$

so by Remark 2.13, $\eth_{\delta,\beta} \in \mathrm{Im}(\mathrm{tr}(\alpha, \beta))$. As the map ρ was given by Fact 2.14,

$$\rho(\alpha, \beta) \geq \mathrm{otp}(C_{\eth_{\delta,\beta}} \cap \alpha) \geq \mathrm{otp}(C_{\eth_{\delta,\beta}} \cap \gamma) \geq \mathrm{otp}(D \cap \gamma) > \tau,$$

as sought. $\square$

It is clear that every subset of a stable set is stable. The next corollary addresses the question of closure under unions.

Corollary 6.17. *Suppose that A, B are cofinal stable subsets of some $\theta \leq \kappa$ of uncountable cofinality. Then, there exist cofinal subsets $A' \subseteq A$ and $B' \subseteq B$ such that:*

- *$A' \cup B'$ is stable, and*
- *for every $(\alpha, \beta, \gamma) \in [A' \cup B']^3$, if $(\beta, \gamma) \in [A']^2 \cup [B']^2$, then $\overline{\mathrm{osc}}(\alpha, \beta, \gamma) = 0$.*

Proof. Appeal to Lemma 6.14 with A to get a cofinal $A' \subseteq A$, a club $D_1 \subseteq \theta$ and an integer k_1. Likewise, appeal to Lemma 6.14 with B to get a cofinal $B' \subseteq B$, a club $D_2 \subseteq \theta$ and an integer k_2. Consider the club $D := D_1 \cap D_2$. Pick sparse enough cofinal subsets $A'' \subseteq A'$ and $B'' \subseteq B'$ such that, letting $X := A'' \cup B''$, for every $(\alpha, \beta) \in [X]^2$, there are $\iota < \theta$ and $\delta, \delta', \delta'' \in D$ with $\delta < \alpha < \delta' < \iota < \delta'' < \beta$.

Claim 6.17.1. *X is stable. Furthermore,* $\max(\chi``[X]^3) = k_2$.

Proof. Let $(\alpha, \beta, \gamma) \in [X]^3$. Pick $\delta, \delta', \delta''$ such that

$$(\delta, \alpha, \delta', \beta, \delta'', \gamma) \in D \circledast X \circledast D \circledast X \circledast D \circledast X.$$

Then $\chi(\alpha, \beta, \gamma) = k_1$ if $\gamma \in A'$, and $\chi(\alpha, \beta, \gamma) = k_2$ otherwise. $\square$

Claim 6.17.2. *Let $Y \in \{A'', B''\}$ and $(\alpha, \beta, \gamma) \in X \circledast Y \circledast Y$. Then $\overline{\mathrm{osc}}(\alpha, \beta, \gamma) = 0$.*

Proof. If $Y = A''$, then denote $k := k_1$. Otherwise, denote $k := k_2$. Now, pick $\iota, \delta, \delta', \delta'', \delta'''$ such that

$$(\delta, \alpha, \delta', \iota, \delta'', \beta, \delta''', \gamma) \in D \circledast X \circledast D \circledast \theta \circledast D \circledast Y \circledast D \circledast Y.$$

In particular,

$$\{(\delta, \alpha, \delta', \iota, \delta'', \beta), (\delta, \alpha, \delta'', \beta, \delta''', \gamma)\} \subseteq D \circledast \theta \circledast D \circledast \theta \circledast D \circledast X.$$

Then $D \cap \delta'' \subseteq C_{\mathrm{Tr}(\alpha,\beta)(k)}$ and $D \cap \delta''' \subseteq C_{\mathrm{Tr}(\alpha,\gamma)(k)}$. In addition, $\chi(\alpha, \beta, \gamma) = k$, so that

$$\delta' \in D \cap (\alpha, \delta'') \subseteq C_{\mathrm{Tr}(\alpha,\beta)(\chi(\alpha,\beta,\gamma))} \cap C_{\mathrm{Tr}(\alpha,\gamma)(\chi(\alpha,\beta,\gamma))}.$$

Recalling Definition 6.5, this means that

$$\mathrm{Osc}_\alpha(C_{\mathrm{Tr}(\alpha,\beta)(\chi(\alpha,\beta,\gamma))}, C_{\mathrm{Tr}(\alpha,\gamma)(\chi(\alpha,\beta,\gamma))}) = \emptyset,$$

and hence $\overline{\mathrm{osc}}(\alpha,\beta,\gamma) = 0$. □

So A'' and B'' are as sought. □

Corollary 6.18. *Suppose that:*

(1) $\mu < \lambda < \lambda^+ = \kappa$ are infinite regular cardinals;
(2) E^λ_μ admits a nonreflecting stationary set;
(3) there exists a weak μ-Kurepa tree with at least κ-many branches.

Then $\kappa \overset{\sup}{\not\to} [\lambda,\lambda]^3_\omega$ holds.

Proof. Let $c : [\kappa]^3 \to \lambda$ be the map given by Theorem 5.2 with respect to the subadditive coloring $\rho : [\kappa]^2 \to \lambda$ of Fact 2.14. Define $c_\omega : [\kappa]^3 \to \omega$ via

$$c_\omega(\alpha,\beta,\gamma) := \begin{cases} c(\alpha,\beta,\gamma), & \text{if } c(\alpha,\beta,\gamma) < \omega; \\ 0, & \text{otherwise.} \end{cases}$$

Let $T \subseteq {}^{<\mu}2$ be a weak μ-Kurepa tree with at least κ-many branches, and let $\langle b_\xi \mid \xi < \kappa\rangle$ be an injective sequence consisting of elements of $\mathcal{B}(T)$. For notational simplicity, we shall write $\Delta(\alpha,\beta)$ for $\Delta(b_\alpha, b_\beta)$. Define a coloring $d : [\kappa]^3 \to \omega$ by letting for all $\alpha < \beta < \gamma < \kappa$:

$$d(\alpha,\beta,\gamma) := \begin{cases} \overline{\mathrm{osc}}(\alpha,\beta,\gamma) - 1, & \text{if } \overline{\mathrm{osc}}(\alpha,\beta,\gamma) > 0 \text{ and } \Delta(\alpha,\beta) < \Delta(\beta,\gamma); \\ c_\omega(\alpha,\beta,\gamma), & \text{otherwise.} \end{cases}$$

To see that d witnesses $\kappa \overset{\sup}{\not\to} [\lambda,\lambda]^3_\omega$, let A, B be disjoint subsets of κ of order-type λ with $\sup(A) = \sup(B)$, and let $n < \omega$. Using Lemma 3.4 and by possibly passing to cofinal subsets, we may assume the existence of $s \in T$ and $i \neq i'$ such that $s^\frown\langle i\rangle \sqsubseteq b_\alpha$ for all $\alpha \in A$, and $s^\frown\langle i'\rangle \sqsubseteq b_\beta$ for all $\beta \in B$.

Claim 6.18.1. *Let $(\alpha,\beta,\gamma) \in [A \cup B]^3 \setminus ([A]^3 \cup [B]^3)$.*
Then $(\beta,\gamma) \in ([A]^2 \cup [B]^2)$ iff $\Delta(\alpha,\beta) < \Delta(\beta,\gamma)$.

Proof. For every $(\epsilon,\delta) \in (A \circledast B) \cup (B \circledast A)$, $\Delta(\epsilon,\delta) = \Delta(s^\frown\langle i\rangle, s^\frown\langle i'\rangle) = \mathrm{dom}(s)$. For every $(\epsilon,\delta) \in (A \circledast A) \cup (B \circledast B)$, $\Delta(\epsilon,\delta) \geq \mathrm{dom}(s+1) > \mathrm{dom}(s)$.

By the hypothesis on (α,β,γ), there are three cases to consider:

► If $(\alpha,\beta,\gamma) \in A \circledast B \circledast B$, then $\Delta(\alpha,\beta) = \mathrm{dom}(s) < \Delta(\beta,\gamma)$.
► If $(\alpha,\beta,\gamma) \in B \circledast A \circledast A$, then $\Delta(\alpha,\beta) = \mathrm{dom}(s) < \Delta(\beta,\gamma)$.
► If $(\beta,\gamma) \in (A \circledast B) \cup (B \circledast A)$, then $\Delta(\beta,\gamma) = \mathrm{dom}(s) \leq \Delta(\alpha,\beta)$. □

There are three cases to consider:

► Suppose that there exist $A' \in [A]^\lambda$ and $B' \in [B]^\lambda$ such that A' and B' are stable. Then, by Corollary 6.17, we may moreover assume that $A' \cup B'$ is stable, and that for every $(\alpha,\beta,\gamma) \in [A' \cup B']^3$ with $(\beta,\gamma) \in [A']^2 \cup [B']^2$, $\overline{\mathrm{osc}}(\alpha,\beta,\gamma) = 0$. By Lemma 6.16, then, $\rho \restriction [A' \cup B']^2$ witnesses $\mathrm{U}(\lambda,\lambda,\lambda,3)$. As $c : [\kappa]^3 \to \lambda$ was given by Theorem 5.2, we may find $(\alpha,\beta,\gamma) \in [A' \cup B']^3 \setminus ([A']^3 \cup [B']^3)$ such that $c(\alpha,\beta,\gamma) = n$. Now, if $(\beta,\gamma) \in [A']^2 \cup [B']^2$, then $\overline{\mathrm{osc}}(\alpha,\beta,\gamma) = 0$, and if $(\beta,\gamma) \notin [A']^2 \cup [B']^2$, then $\Delta(\alpha,\beta) \geq \Delta(\beta,\gamma)$. It thus follows that $d(\alpha,\beta,\gamma) = c_\omega(\alpha,\beta,\gamma) = c(\alpha,\beta,\gamma) = n$, as sought.

- ▶ Suppose that every cofinal subset of B is unstable. By appealing to Lemma 6.12 with A and B, we may find $(\alpha,\beta,\gamma) \in A \circledast B \circledast B$ such that $\overline{\mathrm{osc}}(\alpha,\beta,\gamma) = n+1$. As $(\beta,\gamma) \in [B]^2$, it is the case that $\Delta(\alpha,\beta) < \Delta(\beta,\gamma)$. So $d(\alpha,\beta,\gamma) = \overline{\mathrm{osc}}(\alpha,\beta,\gamma) - 1 = n$.
- ▶ Otherwise. So, every cofinal subset of A is unstable, and then the argument is similar to that of the previous case. □

We are finally in conditions to prove the rectangular extension of [Tod07, Theorem 10.3.6].

Corollary 6.19. *Suppose that $\lambda = \mu^+$ for an infinite cardinal $\mu = \mu^{<\mu}$.*
Then $\lambda^+ \overset{\sup}{\nrightarrow} [\lambda,\lambda]^3_\omega$ holds.

Proof. Denote $\kappa := \lambda^+$. If $2^\mu > \mu^+$, then since $\mu^{<\mu} = \mu$, there is a weak μ-Kurepa tree with κ-many branches, and E^λ_μ constitutes a nonreflecting stationary set. So, by Corollary 6.18, we may assume here that $2^\mu = \mu^+$. In particular, $\lambda \nrightarrow [\mu;\lambda]^2_\lambda$ holds by a theorem of Sierpiński. Let $c : [\kappa]^3 \to \lambda$ be the map given by Theorem 5.3 with respect to the subadditive coloring $\rho : [\kappa]^2 \to \lambda$ of Fact 2.14. Derive c_ω from c as in the proof of Theorem 6.18. Also, denote $T := {}^{<\lambda}2$, and let $\langle b_\xi \mid \xi < \kappa\rangle$ be the injective sequence of elements of $\mathcal{B}(T)$ used in the proof of Theorem 5.3 to define the coloring c. For notational simplicity, we shall write $\Delta(\alpha,\beta)$ for $\Delta(b_\alpha, b_\beta)$. Likewise, for $B \subseteq \kappa$, we write $T^{\rightsquigarrow B}$ for $T^{\rightsquigarrow\{b_\beta \mid \beta \in B\}}$.

Finally, define a coloring $d : [\kappa]^3 \to \omega$ by letting for all $\alpha < \beta < \gamma < \kappa$:[8]

$$d(\alpha,\beta,\gamma) := \begin{cases} \overline{\mathrm{osc}}(\alpha,\beta,\gamma) - 1, & \text{if } \overline{\mathrm{osc}}(\alpha,\beta,\gamma) > 0 \text{ and } \Delta(\alpha,\beta) < \Delta(\beta,\gamma); \\ c_\omega(\alpha,\beta,\gamma), & \text{otherwise.} \end{cases}$$

To see that d witnesses $\kappa \overset{\sup}{\nrightarrow} [\lambda,\lambda]^3_\omega$, let A, B be disjoint subsets of κ of order-type λ with $\sup(A) = \sup(B)$, and let $n < \omega$. Using Corollary 6.17 and by possibly passing to cofinal subsets, we may assume that one of the following holds:

- (I) $A \cup B$ is stable, and for every $(\alpha,\beta,\gamma) \in [A\cup B]^3$ with $(\beta,\gamma) \in [A]^2 \cup [B]^2$, $\overline{\mathrm{osc}}(\alpha,\beta,\gamma) = 0$;
- (II) every cofinal subset of B is unstable;
- (III) every cofinal subset of A is unstable.

Let us dispose of Case (I) right away. In this case, by Lemma 6.16, $\rho \restriction [A\cup B]^2$ witnesses $\mathrm{U}(\lambda,\lambda,\lambda,3)$. So by the choice of c, we may find $(\alpha,\beta,\gamma) \in [A\cup B]^3 \setminus ([A]^3 \cup [B]^3)$ such that $c(\alpha,\beta,\gamma) = n$. Going over the division into cases in the proof of Theorem 5.3, we see that $\Delta(\alpha,\beta) \geq \Delta(\beta,\gamma)$ in all subcases but to Subcase 1.1. So, in all of these cases, $d(\alpha,\beta,\gamma) = c_\omega(\alpha,\beta,\gamma) = c(\alpha,\beta,\gamma) = n$. Finally, looking at Subcase 1.1, we see that the provided triple (α,β,γ) is an element of $A \circledast B \circledast B$ (or an element of $B \circledast A \circledast A$, once lifting the initial "without loss of generality" assumption). So $(\beta,\gamma) \in [A]^2 \cup [B]^2$, and hence $\overline{\mathrm{osc}}(\alpha,\beta,\gamma) = 0$. Therefore, again $d(\alpha,\beta,\gamma) = c_\omega(\alpha,\beta,\gamma) = n$.

Moving on to handling Cases (II) and (III), we shall need the following claim.

Claim 6.19.1. *There are cofinal subsets $A' \subseteq A$ and $B' \subseteq B$ such that, for every $(\alpha,\beta,\gamma) \in (A \circledast B \circledast B) \cup (B \circledast A \circledast A)$, $\Delta(\alpha,\beta) < \Delta(\beta,\gamma)$.*

[8]It may appear that this is the same map from the proof of Corollary 6.18. Note, however, that there Δ was a map from $[\kappa]^2$ to μ, whereas here Δ is a map from $[\kappa]^2$ to λ.

Proof. By possibly passing to a cofinal subset of B, we may assume that $B = B'$ in the sense of Lemma 3.7. So let $\theta_B \leq \lambda$ be such that $T^{\rightsquigarrow B'}$ is a normal tree in $\mathcal{T}(\lambda, \theta_B)$ for every $B' \in [B]^\lambda$. Likewise, we may assume that $A = A'$ in the sense of Lemma 3.7, and let $\theta_A \leq \lambda$ be such that $T^{\rightsquigarrow A'}$ is a normal tree in $\mathcal{T}(\lambda, \theta_A)$ for every $A' \in [A]^\lambda$.

If $\min\{\theta_A, \theta_B\} < \lambda$, then the proof of Claim 5.3.2 provides $\chi < \min\{\theta_A, \theta_B\}$ and a pair $(t, t') \in (T^{\rightsquigarrow A})_{\chi+1} \times (T^{\rightsquigarrow B})_{\chi+1}$ such that $\Delta(t, t') = \chi$. Thus letting $A' := A_t$ and $B' := B_{t'}$, we see that $\Delta(\alpha, \beta) = \chi$ whenever $(\alpha, \beta) \in (A' \circledast B') \cup (B' \circledast A')$, and $\Delta(\alpha, \beta) > \chi$ whenever $(\alpha, \beta) \in (A' \circledast A') \cup (B' \circledast B')$. Thus, we may assume that $\theta_A = \theta_B = \lambda$, so that $T^{\rightsquigarrow A}, T^{\rightsquigarrow B} \in \mathcal{T}(\lambda, \lambda)$. Now, if $(T^{\rightsquigarrow A}) \nsubseteq (T^{\rightsquigarrow B})$ and $(T^{\rightsquigarrow A}) \nsubseteq (T^{\rightsquigarrow B})$, then by normality of the two trees there must exist $\chi < \lambda$, $t \in (T^{\rightsquigarrow A})_\chi \setminus (T^{\rightsquigarrow B})_\chi$ and $t' \in (T^{\rightsquigarrow B})_\chi \setminus (T^{\rightsquigarrow A})_\chi$. Clearly, $A' := A_t$ and $B' := B_{t'}$ are as sought.

Thus, the only nontrivial case is in which $\theta_A = \theta_B = \lambda$ and $T^{\rightsquigarrow A} \subseteq T^{\rightsquigarrow B}$ or $T^{\rightsquigarrow A} \subseteq T^{\rightsquigarrow B}$. Without loss of generality, assume that $T^{\rightsquigarrow A} \subseteq T^{\rightsquigarrow B}$. Now, there are two options:

▶ If $\mathcal{B}(T^{\rightsquigarrow A})$ is nonempty, then there is $b : \lambda \to 2$ that constitutes a branch through both $T^{\rightsquigarrow A}$ and $T^{\rightsquigarrow B}$. In this case, it is easy to recursively simultaneously construct cofinal subsets $A' \subseteq A$ and $B' \subseteq B$ such that for all triple $\alpha < \beta < \gamma$ of ordinals from $A' \cup B'$, it is the case that $\Delta(\alpha, \beta) < \Delta(\beta, \gamma)$.

▶ Otherwise. So $T^{\rightsquigarrow A}$ is a λ-Aronszajn tree. In particular, we may find $\chi < \lambda$ and $t \neq t'$ in $(T^{\rightsquigarrow A})_\chi$. Altogether, $t \in (T^{\rightsquigarrow A})_\chi$, $t' \in (T^{\rightsquigarrow B})_\chi$ and $\Delta(t, t') < \chi$. Then $A' := A_t$ and $B' := B_{t'}$ are as sought. □

At this point, the proof is similar to that of Theorem 6.18. Succinctly, in Case (II), we appeal to Lemma 6.12 with A and B, to find $(\alpha, \beta, \gamma) \in A \circledast B \circledast B$ such that $\overline{\mathrm{osc}}(\alpha, \beta, \gamma) = n+1$. By the preceding claim, it is the case that $\Delta(\alpha, \beta) < \Delta(\beta, \gamma)$. So $d(\alpha, \beta, \gamma) = \overline{\mathrm{osc}}(\alpha, \beta, \gamma) - 1 = n$. The handling of Case (III) is similar. □

7. Connecting the dots

The next result implies Theorem A′.

Theorem 7.1. *For every infinite cardinal μ satisfying $\mu^{<\mu} < \mu^+ < 2^\mu$:*

(1) $S_3(\mu^{++}, \mu^+, \omega)$ holds;
(2) $G \nrightarrow [\mu^+]^{\mathrm{FS}_3}_\omega$ holds for every Abelian group $(G, +)$ of size μ^{++}.

Proof. Suppose that μ is an infinite cardinal satisfying $\mu^{<\mu} < \mu^+ < 2^\mu$. Denote $\lambda := \mu^+$ and $\kappa := \lambda^+$. Then, $T := {}^{<\mu}2$ is a weak μ-Kurepa tree with at least κ-many branches. So, by Corollary 6.18, $\kappa \overset{\sup}{\nrightarrow} [\lambda, \lambda]^3_\omega$ holds. As T witnesses that $\kappa \in T(\lambda, \mu)$, by Lemma 4.23, $\mathrm{Extract}_3(\kappa, \lambda, \mu, \omega)$ holds, as well. Together with Lemma 4.22, this yields Clause (1). Then, Clause (2) follows from Proposition 4.19 and the fact (see [FBR17, Lemma 2.2]) that every Abelian group is a well-behaved magma. □

By [FBR17, Theorem 3.8], for every infinite cardinal $\mu = 2^{<\mu}$, $S_2(2^\mu, 2^\mu, \omega)$ holds. By the upcoming corollary, for every infinite cardinal $\mu = 2^{<\mu}$, $S_2(2^\mu, \mu^+, 2)$ holds. While it is easy to get $S_2(2^\mu, \mu^+, \mu)$ from $2^\mu = \mu^+$, Remark 4.4 shows that $S_2(2^\mu, \mu^+, \mu)$ is also compatible with $2^\mu > \mu^+$. Note, however, that by a theorem of Shelah [She88, Theorem 2.1], one cannot prove $S_2(2^\mu,\mu^+,3)$ in ZFC. Thus, in view of the number of colors, the following corollary is optimal.

Corollary 7.2. *For every infinite cardinal μ, $S_2(\mu^\theta,\mu^+,2)$ holds, for $\theta := \log_\mu(\mu^+)$.*
In particular, $S_2(2^\mu,\mu^+,2)$ holds for every strong limit cardinal μ.

Proof. Appeal to the upcoming theorem with $\lambda := \mu^+$ and $\kappa := \mu^\theta$. □

Lemma 7.3. *Suppose that $\theta < \lambda \leq \kappa$ are infinite cardinals, with λ being regular. If $\kappa \in T(\lambda,\theta)$, then:*

(1) $\kappa \overset{\sup}{\not\to} [\lambda,\lambda]^2_2$ holds;
(2) $S_2(\kappa,\lambda,2)$ holds.

Proof. Suppose that $\kappa \in T(\lambda,\theta)$, and fix $T \in \mathcal{T}(\lambda,\theta)$ admitting an injective sequence $\langle b_\xi \mid \xi < \kappa\rangle$ consisting of elements of $\mathcal{B}(T)$.

(1) Consider the Sierpiński map $c : [\kappa]^2 \to 2$ defined by letting, for all $\alpha < \beta < \kappa$:

$$c(\alpha,\beta) := 1 \text{ iff } b_\alpha <_{\text{lex}} b_\beta.$$

Claim 7.3.1. *c witnesses $\kappa \overset{\sup}{\not\to} [\lambda,\lambda]^2_2$.*

Proof. Suppose that we are given two disjoint subsets A, B of κ with $\text{otp}(A) = \text{otp}(B) = \lambda$ and $\sup(A) = \sup(B)$. By Lemma 3.4(2) (using $\mu := \lambda$), we may find $s \in T$ and $i \neq i'$ such that $A' := \{\alpha \in A \mid s^\frown\langle i\rangle \sqsubseteq b_\alpha\}$ and $B' := \{\beta \in B \mid s^\frown\langle i'\rangle \sqsubseteq b_\beta\}$ are both of size λ. As $\sup(A') = \sup(B')$, we may now fix $(\alpha,\beta,\gamma) \in A' \circledast B' \circledast A'$. To see that $\{c(\alpha,\beta), c(\beta,\gamma)\} = 2$, consider the following cases:

- ▶ If $i < i'$, then $b_\alpha, b_\gamma <_{\text{lex}} b_\beta$ and hence $c(\alpha,\beta) = 1 > 0 = c(\beta,\gamma)$;
- ▶ If $i' < i$, then $b_\beta < b_\alpha, b_\gamma$ and hence $c(\alpha,\beta) = 0 < 1 = c(\beta,\gamma)$. □

(2) By Lemma 4.9 (again, using $\mu := \lambda$), in particular, $\text{Extract}_2(\kappa,\lambda,\omega,\omega)$ holds. So, by Lemma 4.22, Clause (1) implies Clause (2). □

Corollary 7.4. *If there exists a weak μ-Kurepa tree with κ-many branches, then $S_2(\kappa,\mu^+,2)$ holds.* □

Theorem B now follows (using $\mu := 2$):

Corollary 7.5. *For every infinite set G, for every map $\varphi : G \to [G]^{<\omega}$, and for every pair of cardinals μ,θ such that $\mu^{<\theta} < |G| \leq \mu^\theta$, there exists a corresponding coloring $c : G \to 2$ satisfying the following.*

For every binary operation $*$ on G, if φ witnesses that $(G,*)$ is well-behaved, then for every $X \subseteq G$ of size $(\mu^{<\theta})^+$ and every $i \in \{0,1\}$, there are $x \neq y$ in X such that $c(x*y) = i$.

Proof. Given G, φ, μ and θ as above, denote $\kappa := |G|$ and $\lambda := (\mu^{<\theta})^+$, so that $\lambda \leq \kappa \leq \mu^\theta$. Evidently, $T := {}^{<\theta}\mu$ witnesses that $\kappa \in T(\lambda,\theta)$, so $S_2(\kappa,\lambda,2)$ holds by Lemma 7.3. Now, appeal to Proposition 4.19. □

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