Iterations, Stationary reflection and Prikry-type forcings

האוניברסיטה העברית בירושלים
THE HEBREW UNIVERSITY OF JERUSALEM

Alejandro Poveda

Einstein Institute of Mathematics

Oberwolfach's meeting on Set Theory - January 2022

This is based on a joint work with A. Rinot & D. Sinapova

1. **[PRS1] Sigma-Prikry forcing I: The axioms**, Canadian Journal of Mathematics.
2. **[PRS2] Sigma-Prikry forcing II: Iteration Scheme**, Journal of Mathematical Logic.
3. **[PRS3] Sigma-Prikry forcing III: Down to $\aleph_\omega$**, Submitted on October 2021.

Find the papers here

http://assafrinot.com/t/sigma-prikry

https://homepages.math.uic.edu/~sinapova/

Two applications

The very first application of the Σ-Prikry framework:

Theorem (P., Rinot, Sinapova) ([PRS2])

Assume that $\langle \kappa_n \mid n < \omega \rangle$ is an increasing sequence of supercompact cardinals. Then there is a generic extension where $\kappa = \sup_{n<\omega} \kappa_n$ is a strong limit cardinal, SCH_κ fails and $\mathrm{Refl}(\kappa^+)$ holds (actually, $\mathrm{Refl}({<}\omega, \kappa^+)$ holds).

Initially proved by **Sharon** and later by **Ben-Neria, Hayut and Unger**, and **Gitik**.

Two applications

The very first application of the Σ-Prikry framework:

Theorem (P., Rinot, Sinapova) ([PRS2])

Assume that $\langle \kappa_n \mid n < \omega \rangle$ is an increasing sequence of supercompact cardinals. Then there is a generic extension where $\kappa = \sup_{n<\omega} \kappa_n$ is a strong limit cardinal, SCH_κ fails and $\mathrm{Refl}(\kappa^+)$ holds (actually, $\mathrm{Refl}(<\omega, \kappa^+)$ holds).

Initially proved by **Sharon** and later by **Ben-Neria, Hayut and Unger**, and **Gitik**.

Theorem (P., Rinot, Sinapova) ([PRS3])

Assume that $\langle \kappa_n \mid n < \omega \rangle$ is an increasing sequence of supercompact cardinals. Then there is a generic extension of the set-theoretic universe where the following hold:

1. $2^{\aleph_n} = \aleph_{n+1}$ for all $n < \omega$;
2. $2^{\aleph_\omega} = \aleph_{\omega+2}$;
3. $\mathrm{Refl}(\aleph_{\omega+1})$ holds.

Stationary Reflection

Compactness principles

Compactness Principle

A **Compactness Principle** for a given property φ is a statement of the form:

"If every *small* substructure has property φ then the structure itself has property φ"

Compactness principles

Compactness Principle

A **Compactness Principle** for a given property φ is a statement of the form:

"If every *small* substructure has property φ then the structure itself has property φ"

The dual of **CP** are **Reflection Principles**:

Reflection Principle

A **Reflection Principle** for a given property φ is a statement of the form:

"If a structure has property φ then there is a *small* substructure having property φ"

In practice, ***small*** means "having cardinality $<\kappa$", where κ is some relevant cardinal

Compactness in Set Theory: Stationary reflection

Definition

Let κ be a regular uncountable cardinal.

1. A stationary set $S \subseteq \kappa$ **reflects** if there is $\alpha < \kappa$ with $\mathrm{cf}(\alpha) > \omega$ such that
$$S \cap \alpha \text{ is stationary at } \alpha.$$
2. For a stationary set $S \subseteq \kappa$, $\mathrm{Refl}(S)$ asserts that every stationary set $T \subseteq S$ reflects.

Compactness in Set Theory: Stationary reflection

Definition

Let κ be a regular uncountable cardinal.

1. A stationary set $S \subseteq \kappa$ **reflects** if there is $\alpha < \kappa$ with $\mathrm{cf}(\alpha) > \omega$ such that
$$S \cap \alpha \text{ is stationary at } \alpha.$$
2. For a stationary set $S \subseteq \kappa$, $\mathrm{Refl}(S)$ asserts that every stationary set $T \subseteq S$ reflects.

Question

For which cardinals κ and stationary sets $S \subseteq \kappa$ does $\mathrm{Refl}(S)$ hold?

Compactness in Set Theory: Stationary reflection

Definition

Let κ be a regular uncountable cardinal.

1. A stationary set $S \subseteq \kappa$ **reflects** if there is $\alpha < \kappa$ with $\mathrm{cf}(\alpha) > \omega$ such that
$$S \cap \alpha \text{ is stationary at } \alpha.$$
2. For a stationary set $S \subseteq \kappa$, $\mathrm{Refl}(S)$ asserts that every stationary set $T \subseteq S$ reflects.

Question

For which cardinals κ and stationary sets $S \subseteq \kappa$ does $\mathrm{Refl}(S)$ hold?

We need to separate the discussion into three cases:

1. **Limit cardinals:** Large-cardinal properties use to entail stationary reflection:

Theorem (Tarski (19_), Jensen (1972))

If κ is weakly compact then $\mathrm{Refl}(\kappa)$ holds. Under $V = L$ this is an equivalence.

Theorem (Solovay) (19_)

If κ is supercompact then $\mathrm{Refl}(\mu \cap \mathrm{cf}(<\kappa))$ holds for every regular $\mu \geq \kappa$.

1. **Limit cardinals:** Large-cardinal properties use to entail stationary reflection:

Theorem (Tarski (19_), Jensen (1972))

If κ is weakly compact then $\mathrm{Refl}(\kappa)$ holds. Under $V = L$ this is an equivalence.

Theorem (Solovay) (19_)

If κ is supercompact then $\mathrm{Refl}(\mu \cap \mathrm{cf}(<\kappa))$ holds for every regular $\mu \geq \kappa$.

2. **Successor of a regular:** If $\kappa := \lambda^+$, with $\lambda = \mathrm{cf}(\lambda)$, then $\mathrm{Refl}(\kappa \cap \mathrm{cf}(\lambda))$ fails.

1. **Limit cardinals:** Large-cardinal properties use to entail stationary reflection:

Theorem (Tarski (19_), Jensen (1972))

If κ is weakly compact then $\mathrm{Refl}(\kappa)$ holds. Under $V = L$ this is an equivalence.

Theorem (Solovay) (19_)

If κ is supercompact then $\mathrm{Refl}(\mu \cap \mathrm{cf}(<\kappa))$ holds for every regular $\mu \geq \kappa$.

2. **Successor of a regular:** If $\kappa := \lambda^+$, with $\lambda = \mathrm{cf}(\lambda)$, then $\mathrm{Refl}(\kappa \cap \mathrm{cf}(\lambda))$ fails.

   Despite of this, one can force optimal reflection patterns:

Theorem (Harrington & Shelah) (NDJFL, 1985)

The following are equiconsistent:

- There is a Mahlo cardinal.
- $\mathrm{Refl}(\kappa \cap \mathrm{cf}(<\lambda))$ holds.

3. **Successors of a singular:**

Unlike successors of regulars here one can arrange full stationary reflection:

Theorem (Magidor) (JSL, 1982)

Assume there are ω-many supercompact cardinals and that the GCH holds. Then there is a generic extension where $\mathrm{Refl}(\aleph_{\omega+1})$ holds.

3. **Successors of a singular:**

Unlike successors of regulars here one can arrange full stationary reflection:

Theorem (Magidor) (JSL, 1982)

Assume there are ω-many supercompact cardinals and that the GCH holds. Then there is a generic extension where $\mathrm{Refl}(\aleph_{\omega+1})$ holds.

Theorem (Hayut & Unger) (JSL, 2020)

The principle $\mathrm{Refl}(\aleph_{\omega+1})$ is consistent relative to the existence of a cardinal κ which is κ^+-Π^1_1-subcompact.

The Singular Cardinal Hypothesis

The Singular Cardinal Hypothesis

Definition (simplified version)

The **S**ingular **C**ardinal **H**ypothesis (SCH) asserts that for every singular strong limit cardinal κ, $2^\kappa = \kappa^+$ (i.e., SCH_κ holds).

The Singular Cardinal Hypothesis

Definition (simplified version)

The **S**ingular **C**ardinal **H**ypothesis (SCH) asserts that for every singular strong limit cardinal κ, $2^\kappa = \kappa^+$ (i.e., SCH_κ holds).

Facts

1. The SCH holds above the first strong compact cardinal (Solovay);

The Singular Cardinal Hypothesis

Definition (simplified version)

The **S**ingular **C**ardinal **H**ypothesis (SCH) asserts that for every singular strong limit cardinal κ, $2^\kappa = \kappa^+$ (i.e., SCH_κ holds).

Facts

1. The SCH holds above the first strong compact cardinal (Solovay);
2. The failure of the SCH is consistent modulo large cardinals (Silver & Prikry);

The Singular Cardinal Hypothesis

Definition (simplified version)

The **S**ingular **C**ardinal **H**ypothesis (SCH) asserts that for every singular strong limit cardinal κ, $2^\kappa = \kappa^+$ (i.e., SCH_κ holds).

Facts

1. The SCH holds above the first strong compact cardinal (Solovay);
2. The failure of the SCH is consistent modulo large cardinals (Silver & Prikry);
3. The SCH can fail at the first singular cardinal.

Theorem (Magidor) (Ann. Math, 1977)

Assume there is a supercompact cardinal along with a huge cardinal on top. Then there is a generic extension where $\mathrm{GCH}_{<\aleph_\omega}$ holds but $\mathrm{SCH}_{\aleph_\omega}$ fails.

Tension between $\neg\mathsf{SCH}_\kappa$ and $\mathrm{Refl}(\kappa^+)$

General fact

Getting $\neg$SCH usually involves singularizing cardinals, which entails $\square$-sequences (**Džamonja-Shelah** and **Gitik**). These latter are at odds with stationary reflection.

Tension between $\neg\mathsf{SCH}_\kappa$ and $\mathrm{Refl}(\kappa^+)$

General fact

Getting $\neg$SCH usually involves singularizing cardinals, which entails $\square$-sequences (**Džamonja-Shelah** and **Gitik**). These latter are at odds with stationary reflection.

Let us illustrate this tension by analysing the most basic example: The Prikry model.

Tension between $\neg\mathrm{SCH}_\kappa$ and $\mathrm{Refl}(\kappa^+)$

General fact

Getting $\neg$SCH usually involves singularizing cardinals, which entails $\square$-sequences (**Džamonja-Shelah** and **Gitik**). These latter are at odds with stationary reflection.

Let us illustrate this tension by analysing the most basic example: The Prikry model.

Theorem (Cummings, Foreman and Magidor) (JML, 2001)

Assume that κ is a κ^+-supercompact cardinal. Then $\mathrm{Refl}(\kappa^+ \cap \mathrm{cf}^V(<\kappa))$ holds in any Prikry generic extension derived from a normal measure on κ.

Tension between $\neg\mathsf{SCH}_\kappa$ and $\mathrm{Refl}(\kappa^+)$

Theorem (Cummings, Foreman and Magidor) (JML, 2001)

Assume that κ is a κ^+-supercompact cardinal. Then $\mathrm{Refl}(\kappa^+ \cap \mathrm{cf}^V(<\kappa))$ holds in any Prikry extension derived from a mormal measure on κ.

Proof sketch:

Tension between $\neg\mathsf{SCH}_\kappa$ and $\mathrm{Refl}(\kappa^+)$

Theorem (Cummings, Foreman and Magidor) (JML, 2001)

Assume that κ is a κ^+-supercompact cardinal. Then $\mathrm{Refl}(\kappa^+ \cap \mathrm{cf}^V(<\kappa))$ holds in any Prikry extension derived from a mormal measure on κ.

Proof sketch:

1. Assume $\dot{T}$ is a $\mathbb{P}$-name for a non-reflecting stationary subset of $S := \kappa^+ \cap \mathrm{cf}^V(<\kappa)$.

Tension between $\neg\mathsf{SCH}_\kappa$ and $\mathrm{Refl}(\kappa^+)$

Theorem (Cummings, Foreman and Magidor) (JML, 2001)

Assume that κ is a κ^+-supercompact cardinal. Then $\mathrm{Refl}(\kappa^+ \cap \mathrm{cf}^V(<\kappa))$ holds in any Prikry extension derived from a mormal measure on κ.

Proof sketch:

1. Assume $\dot{T}$ is a $\mathbb{P}$-name for a non-reflecting stationary subset of $S := \kappa^+ \cap \mathrm{cf}^V(<\kappa)$.
2. Let $G \subseteq \mathbb{P}$ and $\vec{\kappa}$ be the induced Prikry sequence. For each $s \sqsubseteq \vec{\kappa}$, put

$$T_s := \{\alpha \in S \mid \exists A \in U\ (s, A) \Vdash_{\mathbb{P}} \check{\alpha} \in \dot{T}\}.$$

Tension between $\neg\mathsf{SCH}_\kappa$ and $\mathrm{Refl}(\kappa^+)$

Theorem (Cummings, Foreman and Magidor) (JML, 2001)

Assume that κ is a κ^+-supercompact cardinal. Then $\mathrm{Refl}(\kappa^+ \cap \mathrm{cf}^V(<\kappa))$ holds in any Prikry extension derived from a mormal measure on κ.

Proof sketch:

1. Assume $\dot{T}$ is a $\mathbb{P}$-name for a non-reflecting stationary subset of $S := \kappa^+ \cap \mathrm{cf}^V(<\kappa)$.
2. Let $G \subseteq \mathbb{P}$ and $\vec{\kappa}$ be the induced Prikry sequence. For each $s \sqsubseteq \vec{\kappa}$, put

$$T_s := \{\alpha \in S \mid \exists A \in U\ (s, A) \Vdash_{\mathbb{P}} \check{\alpha} \in \dot{T}\}.$$

3. Note that $T_s \in V$ and that for some (many) s, T_s is stationary.

Tension between $\neg\mathrm{SCH}_\kappa$ and $\mathrm{Refl}(\kappa^+)$

Theorem (Cummings, Foreman and Magidor) (JML, 2001)

Assume that κ is a κ^+-supercompact cardinal. Then $\mathrm{Refl}(\kappa^+ \cap \mathrm{cf}^V(<\kappa))$ holds in any Prikry extension derived from a mormal measure on κ.

Proof sketch:

1. Assume $\dot{T}$ is a $\mathbb{P}$-name for a non-reflecting stationary subset of $S := \kappa^+ \cap \mathrm{cf}^V(<\kappa)$.
2. Let $G \subseteq \mathbb{P}$ and $\vec{\kappa}$ be the induced Prikry sequence. For each $s \sqsubseteq \vec{\kappa}$, put

$$T_s := \{\alpha \in S \mid \exists A \in U\ (s, A) \Vdash_{\mathbb{P}} \check{\alpha} \in \dot{T}\}.$$

3. Note that $T_s \in V$ and that for some (many) s, T_s is stationary.
4. Since κ is κ^+-supercompact, T_s reflects (in V) at an ordinal α with $\mathrm{cf}^V(\alpha) \in (\omega, \kappa)$.

Tension between $\neg\mathsf{SCH}_\kappa$ and $\mathrm{Refl}(\kappa^+)$

Theorem (Cummings, Foreman and Magidor) (JML, 2001)

Assume that κ is a κ^+-supercompact cardinal. Then $\mathrm{Refl}(\kappa^+ \cap \mathrm{cf}^V(<\kappa))$ holds in any Prikry extension derived from a mormal measure on κ.

Proof sketch:

1. Assume $\dot{T}$ is a $\mathbb{P}$-name for a non-reflecting stationary subset of $S := \kappa^+ \cap \mathrm{cf}^V(<\kappa)$.
2. Let $G \subseteq \mathbb{P}$ and $\vec{\kappa}$ be the induced Prikry sequence. For each $s \sqsubseteq \vec{\kappa}$, put

$$T_s := \{\alpha \in S \mid \exists A \in U\ (s, A) \Vdash_{\mathbb{P}} \check{\alpha} \in \dot{T}\}.$$

3. Note that $T_s \in V$ and that for some (many) s, T_s is stationary.
4. Since κ is κ^+-supercompact, T_s reflects (in V) at an ordinal α with $\mathrm{cf}^V(\alpha) \in (\omega, \kappa)$.
5. Let $C \subseteq \alpha$ with $\mathrm{otp}(C) = \mathrm{cf}^V(\alpha)$. Note that $|T_s \cap C| < \kappa$.

Tension between $\neg\mathsf{SCH}_\kappa$ and $\mathrm{Refl}(\kappa^+)$

Theorem (Cummings, Foreman and Magidor) (JML, 2001)

Assume that κ is a κ^+-supercompact cardinal. Then $\mathrm{Refl}(\kappa^+ \cap \mathrm{cf}^V(<\kappa))$ holds in any Prikry extension derived from a mormal measure on κ.

Proof sketch:

1. Assume $\dot{T}$ is a $\mathbb{P}$-name for a non-reflecting stationary subset of $S := \kappa^+ \cap \mathrm{cf}^V(<\kappa)$.
2. Let $G \subseteq \mathbb{P}$ and $\vec{\kappa}$ be the induced Prikry sequence. For each $s \sqsubseteq \vec{\kappa}$, put
$$T_s := \{\alpha \in S \mid \exists A \in U\ (s, A) \Vdash_{\mathbb{P}} \check{\alpha} \in \dot{T}\}.$$
3. Note that $T_s \in V$ and that for some (many) s, T_s is stationary.
4. Since κ is κ^+-supercompact, T_s reflects (in V) at an ordinal α with $\mathrm{cf}^V(\alpha) \in (\omega, \kappa)$.
5. Let $C \subseteq \alpha$ with $\mathrm{otp}(C) = \mathrm{cf}^V(\alpha)$. Note that $|T_s \cap C| < \kappa$.
6. Use the κ-closure of $\langle \mathbb{P}, \leq^* \rangle$ to obtain some $A^* \in U$ such that
$$(s, A^*) \Vdash_{\mathbb{P}} \text{“}T_s \cap C \subseteq \dot{T} \cap \alpha\text{”}.$$

Tension between $\neg\mathsf{SCH}_\kappa$ and $\mathrm{Refl}(\kappa^+)$

Still, the tension between $\neg\mathsf{SCH}_\kappa$ and $\mathrm{Refl}(\kappa^+)$ prevails:

Theorem (Cummings, Foreman and Magidor) (JML, 2001)

Assume that κ is a κ^+-supercompact cardinal. Then $\mathrm{Refl}(\kappa^+ \cap \mathrm{cf}^V(<\kappa))$ holds in any Prikry generic extension derived from a normal measure on κ. But, $\mathrm{Refl}(\kappa^+ \cap \mathrm{cf}^V(\kappa))$ fails.

Tension between $\neg\mathsf{SCH}_\kappa$ and $\mathrm{Refl}(\kappa^+)$

Still, the tension between $\neg\mathsf{SCH}_\kappa$ and $\mathrm{Refl}(\kappa^+)$ prevails:

Theorem (Cummings, Foreman and Magidor) (JML, 2001)

Assume that κ is a κ^+-supercompact cardinal. Then $\mathrm{Refl}(\kappa^+ \cap \mathrm{cf}^V(<\kappa))$ holds in any Prikry generic extension derived from a normal measure on κ. But, $\mathrm{Refl}(\kappa^+ \cap \mathrm{cf}^V(\kappa))$ fails.

- Assume $S := \kappa^+ \cap \mathrm{cf}^V(\kappa)$ reflects in some Prikry-extension $V[G]$.
- Let $\alpha < \kappa^+$ with $\omega < \mathrm{cf}^{V[G]}(\alpha)$ witnessing this, and note that $\mathrm{cf}^V(\alpha) < \kappa$.
- There is a V-club $C \subseteq \alpha$ (hence, a $V[G]$-club) such that $C \cap S = \emptyset$.

Tension between $\neg$SCH$_\kappa$ and $\mathrm{Refl}(\kappa^+)$

Still, the tension between $\neg$SCH$_\kappa$ and $\mathrm{Refl}(\kappa^+)$ prevails:

Theorem (Cummings, Foreman and Magidor) (JML, 2001)

Assume that κ is a κ^+-supercompact cardinal. Then $\mathrm{Refl}(\kappa^+ \cap \mathrm{cf}^V(<\kappa))$ holds in any Prikry generic extension derived from a normal measure on κ. But, $\mathrm{Refl}(\kappa^+ \cap \mathrm{cf}^V(\kappa))$ fails.

- Assume $S := \kappa^+ \cap \mathrm{cf}^V(\kappa)$ reflects in some Prikry-extension $V[G]$.
- Let $\alpha < \kappa^+$ with $\omega < \mathrm{cf}^{V[G]}(\alpha)$ witnessing this, and note that $\mathrm{cf}^V(\alpha) < \kappa$.
- There is a V-club $C \subseteq \alpha$ (hence, a $V[G]$-club) such that $C \cap S = \emptyset$.

The above argument applies to fairly any Prikry-type forcing singularizing κ. Even more, it applies to any such forcing singularizing an interval of cardinals $[\kappa, \lambda)$.

Tension between $\neg\mathsf{SCH}_\kappa$ and $\mathrm{Refl}(\kappa^+)$

Still, the tension between $\neg\mathsf{SCH}_\kappa$ and $\mathrm{Refl}(\kappa^+)$ prevails:

Theorem (Cummings, Foreman and Magidor) (JML, 2001)

Assume that κ is a κ^+-supercompact cardinal. Then $\mathrm{Refl}(\kappa^+ \cap \mathrm{cf}^V(<\kappa))$ holds in any Prikry generic extension derived from a normal measure on κ. But, $\mathrm{Refl}(\kappa^+ \cap \mathrm{cf}^V(\kappa))$ fails.

- Assume $S := \kappa^+ \cap \mathrm{cf}^V(\kappa)$ reflects in some Prikry-extension $V[G]$.
- Let $\alpha < \kappa^+$ with $\omega < \mathrm{cf}^{V[G]}(\alpha)$ witnessing this, and note that $\mathrm{cf}^V(\alpha) < \kappa$.
- There is a V-club $C \subseteq \alpha$ (hence, a $V[G]$-club) such that $C \cap S = \emptyset$.

The above argument applies to fairly any Prikry-type forcing singularizing κ. Even more, it applies to any such forcing singularizing an interval of cardinals $[\kappa, \lambda)$.

In particular, $\mathrm{Refl}(\aleph_{\omega+1})$ fails in Magidor's model for $\neg\mathsf{SCH}_{\aleph_\omega}$.

Tension between $\neg\mathsf{SCH}_\kappa$ and $\mathrm{Refl}(\kappa^+)$

Suggestion to get $\neg\mathsf{SCH}_\kappa + \mathrm{Refl}(\kappa^+)$

Start with a cardinal κ for which $\mathrm{Refl}(\kappa^+)$ holds in the ground model: e.g., a cardinal κ that is the limit of an increasing sequence $\langle \kappa_n \mid n < \omega \rangle$ of supercompact cardinals.

Tension between $\neg$SCH$_\kappa$ and $\mathrm{Refl}(\kappa^+)$

Suggestion to get $\neg$SCH$_\kappa + \mathrm{Refl}(\kappa^+)$

Start with a cardinal κ for which $\mathrm{Refl}(\kappa^+)$ holds in the ground model: e.g., a cardinal κ that is the limit of an increasing sequence $\langle \kappa_n \mid n < \omega \rangle$ of supercompact cardinals.

- There are Prirkry-type forcings that allow to violate SCH$_\kappa$ - even when κ is singular (e.g., the **E**xtender **B**ased **P**rikry **F**orcing (Gitik-Magidor))

Tension between $\neg\mathsf{SCH}_\kappa$ and $\mathrm{Refl}(\kappa^+)$

Suggestion to get $\neg\mathsf{SCH}_\kappa + \mathrm{Refl}(\kappa^+)$

Start with a cardinal κ for which $\mathrm{Refl}(\kappa^+)$ holds in the ground model: e.g., a cardinal κ that is the limit of an increasing sequence $\langle \kappa_n \mid n < \omega \rangle$ of supercompact cardinals.

- There are Prirkry-type forcings that allow to violate SCH_κ - even when κ is singular (e.g., the **E**xtender **B**ased **P**rikry **F**orcing (Gitik-Magidor))
- What is the reflection pattern in the EBPF-extensions?

Tension between $\neg\mathrm{SCH}_\kappa$ and $\mathrm{Refl}(\kappa^+)$

Suggestion to get $\neg\mathrm{SCH}_\kappa + \mathrm{Refl}(\kappa^+)$

Start with a cardinal κ for which $\mathrm{Refl}(\kappa^+)$ holds in the ground model: e.g., a cardinal κ that is the limit of an increasing sequence $\langle \kappa_n \mid n < \omega \rangle$ of supercompact cardinals.

- There are Prirkry-type forcings that allow to violate SCH_κ - even when κ is singular

  (e.g., the **E**xtender **B**ased **P**rikry **F**orcing (Gitik-Magidor))
- What is the reflection pattern in the EBPF-extensions?

Lemma (Sharon) (2005)

Assume $\langle \kappa_n \mid n < \omega \rangle$ is an increasing sequence of Laver-indestructible supercompact cardinals, and put $\kappa := \sup_{n<\omega} \kappa_n$. Then, in any generic extension by the EBPF

$$\mathrm{Refl}(\kappa^+ \cap \mathrm{cf}^V(>\omega)) \text{ holds.}$$

Tension between $\neg\mathsf{SCH}_\kappa$ and $\mathrm{Refl}(\kappa^+)$

Lemma (Sharon) (2005)

Assume $\langle \kappa_n \mid n < \omega \rangle$ is an increasing sequence of Laver-indestructible supercompact cardinals, and put $\kappa := \sup_{n<\omega} \kappa_n$. Then, in any generic extension by the EBPF

$$\mathrm{Refl}(\kappa^+ \cap \mathrm{cf}^V(>\omega)) \text{ holds.}$$

Proof sketch: Mimic Cummings, Foreman and Magidor:

Tension between $\neg\mathsf{SCH}_\kappa$ and $\mathrm{Refl}(\kappa^+)$

Lemma (Sharon) (2005)

Assume $\langle \kappa_n \mid n < \omega \rangle$ is an increasing sequence of Laver-indestructible supercompact cardinals, and put $\kappa := \sup_{n<\omega} \kappa_n$. Then, in any generic extension by the EBPF

$$\mathrm{Refl}(\kappa^+ \cap \mathrm{cf}^V(>\omega)) \text{ holds.}$$

Proof sketch: Mimic Cummings, Foreman and Magidor:

- Assume $\dot{T}$ is a $\mathbb{P}$-name for a stationary subset of $S := \kappa^+ \cap \mathrm{cf}^V(> \omega)$.

Tension between $\neg\mathsf{SCH}_\kappa$ and $\mathrm{Refl}(\kappa^+)$

Lemma (Sharon) (2005)

Assume $\langle \kappa_n \mid n < \omega\rangle$ is an increasing sequence of Laver-indestructible supercompact cardinals, and put $\kappa := \sup_{n<\omega} \kappa_n$. Then, in any generic extension by the EBPF

$$\mathrm{Refl}(\kappa^+ \cap \mathrm{cf}^V(>\omega)) \text{ holds.}$$

Proof sketch: Mimic Cummings, Foreman and Magidor:

- Assume $\dot{T}$ is a $\mathbb{P}$-name for a stationary subset of $S := \kappa^+ \cap \mathrm{cf}^V(>\omega)$.
- For each $n < \omega$, define $\mathbb{P}_n := \{p \in \mathbb{P} \mid \ell(p) = n\}$ and put

$$\dot{T}_n := \{(\check{\alpha}, p) \mid (\alpha, p) \in S \times \mathbb{P}_n \wedge (p \Vdash_{\mathbb{P}} \check{\alpha} \in \dot{T})\}.$$

Tension between $\neg\mathsf{SCH}_\kappa$ and $\mathrm{Refl}(\kappa^+)$

Lemma (Sharon) (2005)

Assume $\langle \kappa_n \mid n < \omega \rangle$ is an increasing sequence of Laver-indestructible supercompact cardinals, and put $\kappa := \sup_{n<\omega} \kappa_n$. Then, in any generic extension by the EBPF

$$\mathrm{Refl}(\kappa^+ \cap \mathrm{cf}^V(>\omega)) \text{ holds.}$$

<u>Proof sketch:</u> Mimic Cummings, Foreman and Magidor:

- Assume $\dot{T}$ is a $\mathbb{P}$-name for a stationary subset of $S := \kappa^+ \cap \mathrm{cf}^V(> \omega)$.
- For each $n < \omega$, define $\mathbb{P}_n := \{p \in \mathbb{P} \mid \ell(p) = n\}$ and put

$$\dot{T}_n := \{(\check{\alpha}, p) \mid (\alpha, p) \in S \times \mathbb{P}_n \wedge (p \Vdash_{\mathbb{P}} \check{\alpha} \in \dot{T})\}.$$

- Show that $p \Vdash_{\mathbb{P}_{\ell(p)}}$ "$\dot{T}_{\ell(p)}$ is stationary" for densely many $p \in \mathbb{P}$.

Tension between $\neg\mathrm{SCH}_\kappa$ and $\mathrm{Refl}(\kappa^+)$

Lemma (Sharon) (2005)

Assume $\langle \kappa_n \mid n < \omega\rangle$ is an increasing sequence of Laver-indestructible supercompact cardinals, and put $\kappa := \sup_{n<\omega} \kappa_n$. Then, in any generic extension by the EBPF

$$\mathrm{Refl}(\kappa^+ \cap \mathrm{cf}^V(>\omega)) \text{ holds.}$$

Proof sketch: Mimic Cummings, Foreman and Magidor:

- Assume $\dot{T}$ is a $\mathbb{P}$-name for a stationary subset of $S := \kappa^+ \cap \mathrm{cf}^V(>\omega)$.
- For each $n < \omega$, define $\mathbb{P}_n := \{p \in \mathbb{P} \mid \ell(p) = n\}$ and put

$$\dot{T}_n := \{(\check{\alpha}, p) \mid (\alpha, p) \in S \times \mathbb{P}_n \wedge (p \Vdash_{\mathbb{P}} \check{\alpha} \in \dot{T})\}.$$

- Show that $p \Vdash_{\mathbb{P}_{\ell(p)}}$ "$\dot{T}_{\ell(p)}$ is stationary" for densely many $p \in \mathbb{P}$.
- If κ_n is Laver-indestructible, $V^{\mathbb{P}_n} \models \mathrm{Refl}(\kappa^+ \cap \mathrm{cf}(<\kappa_n))$, hence $\dot{T}_n$ reflects in $V^{\mathbb{P}_n}$.

Tension between $\neg\mathrm{SCH}_\kappa$ and $\mathrm{Refl}(\kappa^+)$

Lemma (Sharon) (2005)

Assume $\langle \kappa_n \mid n < \omega\rangle$ is an increasing sequence of Laver-indestructible supercompact cardinals, and put $\kappa := \sup_{n<\omega} \kappa_n$. Then, in any generic extension by the EBPF

$$\mathrm{Refl}(\kappa^+ \cap \mathrm{cf}^V(>\omega)) \text{ holds.}$$

<u>Proof sketch:</u> Mimic Cummings, Foreman and Magidor:

- Assume $\dot{T}$ is a $\mathbb{P}$-name for a stationary subset of $S := \kappa^+ \cap \mathrm{cf}^V(> \omega)$.
- For each $n < \omega$, define $\mathbb{P}_n := \{p \in \mathbb{P} \mid \ell(p) = n\}$ and put

$$\dot{T}_n := \{(\check{\alpha}, p) \mid (\alpha, p) \in S \times \mathbb{P}_n \wedge (p \Vdash_{\mathbb{P}} \check{\alpha} \in \dot{T})\}.$$

- Show that $p \Vdash_{\mathbb{P}_{\ell(p)}}$ "$\dot{T}_{\ell(p)}$ is stationary" for densely many $p \in \mathbb{P}$.
- If κ_n is Laver-indestructible, $V^{\mathbb{P}_n} \models \mathrm{Refl}(\kappa^+ \cap \mathrm{cf}(<\kappa_n))$, hence $\dot{T}_n$ reflects in $V^{\mathbb{P}_n}$.
- As before, argue that some condition q forces "$\dot{T}$ reflects".

Tension between $\neg\mathsf{SCH}_\kappa$ and $\mathrm{Refl}(\kappa^+)$

Yet again, the tension prevails:

Theorem (Sharon) (2005)

In the EBPF-models there is a stationary set $S \subseteq \kappa^+ \cap \mathrm{cf}^V(\omega)$ that fails to reflect.

Tension between $\neg\mathrm{SCH}_\kappa$ and $\mathrm{Refl}(\kappa^+)$

Yet again, the tension prevails:

Theorem (Sharon) (2005)

In the EBPF-models there is a stationary set $S \subseteq \kappa^+ \cap \mathrm{cf}^V(\omega)$ that fails to reflect.

- Getting $\mathrm{Refl}(\kappa^+)$ will require to (iterative) kill all the non-reflecting stationary sets.

Tension between $\neg\mathsf{SCH}_\kappa$ and $\mathrm{Refl}(\kappa^+)$

Yet again, the tension prevails:

Theorem (Sharon) (2005)

In the EBPF-models there is a stationary set $S \subseteq \kappa^+ \cap \mathrm{cf}^V(\omega)$ that fails to reflect.

- Getting $\mathrm{Refl}(\kappa^+)$ will require to (iterative) kill all the non-reflecting stationary sets.
- Note that such iteration may introduce new counterexamples for $\mathrm{Refl}(\kappa^+ \cap \mathrm{cf}^V(>\omega))$.

Tension between $\neg\mathrm{SCH}_\kappa$ and $\mathrm{Refl}(\kappa^+)$

Yet again, the tension prevails:

Theorem (Sharon) (2005)

In the EBPF-models there is a stationary set $S \subseteq \kappa^+ \cap \mathrm{cf}^V(\omega)$ that fails to reflect.

- Getting $\mathrm{Refl}(\kappa^+)$ will require to (iterative) kill all the non-reflecting stationary sets.
- Note that such iteration may introduce new counterexamples for $\mathrm{Refl}(\kappa^+ \cap \mathrm{cf}^V(>\omega))$.

Our strategy

1. Isolate a general class of Prikry-type forcings (Σ-Prikry) that encompasses both the EBPF and all the stages of the eventual iteration;

Tension between $\neg\mathsf{SCH}_\kappa$ and $\mathrm{Refl}(\kappa^+)$

Yet again, the tension prevails:

Theorem (Sharon) (2005)

In the EBPF-models there is a stationary set $S \subseteq \kappa^+ \cap \mathrm{cf}^V(\omega)$ that fails to reflect.

- Getting $\mathrm{Refl}(\kappa^+)$ will require to (iterative) kill all the non-reflecting stationary sets.
- Note that such iteration may introduce new counterexamples for $\mathrm{Refl}(\kappa^+ \cap \mathrm{cf}^V(>\omega))$.

Our strategy

1. Isolate a general class of Prikry-type forcings (Σ-Prikry) that encompasses both the EBPF and all the stages of the eventual iteration;
2. Prove that any such forcing yields the very same reflection pattern as the EBPF did;

Tension between $\neg\mathrm{SCH}_\kappa$ and $\mathrm{Refl}(\kappa^+)$

Yet again, the tension prevails:

Theorem (Sharon) (2005)

In the EBPF-models there is a stationary set $S \subseteq \kappa^+ \cap \mathrm{cf}^V(\omega)$ that fails to reflect.

- Getting $\mathrm{Refl}(\kappa^+)$ will require to (iterative) kill all the non-reflecting stationary sets.
- Note that such iteration may introduce new counterexamples for $\mathrm{Refl}(\kappa^+ \cap \mathrm{cf}^V(>\omega))$.

Our strategy

1. Isolate a general class of Prikry-type forcings (Σ-Prikry) that encompasses both the EBPF and all the stages of the eventual iteration;
2. Prove that any such forcing yields the very same reflection pattern as the EBPF did;
3. Prove an iteration theorem for the class of Σ-Prikry forcings and devise an iteration aimed to kill all non-reflecting stationary sets $S \subseteq \kappa^+ \cap \mathrm{cf}^V(\omega)$.

Σ-Prikry forcings

Σ-Prikry forcings in a nutshell

$\Sigma = \langle \kappa_n \mid n < \omega \rangle$ is non-decreasing seq. of regular uncountable cardinals. Set $\kappa := \sup(\Sigma)$.

A Σ-Prikry poset is a triple $(\mathbb{P}, \ell, c)$ such that:

Σ-Prikry forcings in a nutshell

$\Sigma = \langle \kappa_n \mid n < \omega \rangle$ is non-decreasing seq. of regular uncountable cardinals. Set $\kappa := \sup(\Sigma)$.

A Σ-Prikry poset is a triple $(\mathbb{P}, \ell, c)$ such that:

1. $\ell \colon \mathbb{P} \to \omega$ is a "canonical notion of length";

Σ-Prikry forcings in a nutshell

$\Sigma = \langle \kappa_n \mid n < \omega \rangle$ is non-decreasing seq. of regular uncountable cardinals. Set $\kappa := \sup(\Sigma)$.

A Σ-Prikry poset is a triple $(\mathbb{P}, \ell, c)$ such that:

1. $\ell \colon \mathbb{P} \to \omega$ is a "canonical notion of length";
2. $c \colon \mathbb{P} \to \mu$ witnesses a **strong form of μ^+-Linkedness**, where $\mathbb{1} \Vdash_{\mathbb{P}} \check{\mu} = \check{\kappa}^+$.

$$c(p) = c(q) \Longrightarrow \exists r \in \mathbb{P}\, (r \leq^0 p, q).$$

Σ-Prikry forcings in a nutshell

$\Sigma = \langle \kappa_n \mid n < \omega \rangle$ is non-decreasing seq. of regular uncountable cardinals. Set $\kappa := \sup(\Sigma)$.

A Σ-Prikry poset is a triple $(\mathbb{P}, \ell, c)$ such that:

1. $\ell \colon \mathbb{P} \to \omega$ is a "canonical notion of length";
2. $c \colon \mathbb{P} \to \mu$ witnesses a **strong form of μ^+-Linkedness**, where $\mathbb{1} \Vdash_{\mathbb{P}} \check{\mu} = \check{\kappa}^+$.
$$c(p) = c(q) \Longrightarrow \exists r \in \mathbb{P}\,(r \leq^0 p, q).$$
3. $\mathbb{P}$ is a forcing poset such that:
   - For each $n < \omega$, $\mathbb{P}_n$ (contains a dense subforcing that) is κ_n-directed-closed;
   - $\mathbb{P}$ has the **Complete Prikry Property**.

Complete Prikry property

For each $p \in \mathbb{P}$, $n < \omega$ and a 0-open set $D \subseteq \mathbb{P}$ there is a condition $q \leq^0 p$ such that either $P^q_n \subseteq D$ or $P^q_n \cap D = \emptyset$ (Here, $P^q_n := \{r \leq q \mid \ell(r) = \ell(q) + n\}$).

An example: Prikry forcing

Prikry forcing $\mathbb{P}(\mathcal{U})$ can be regarded as a Σ-Prikry triple $(\mathbb{P}(\mathcal{U}), \ell, c)$:

An example: Prikry forcing

Prikry forcing $\mathbb{P}(\mathcal{U})$ can be regarded as a Σ-Prikry triple $(\mathbb{P}(\mathcal{U}), \ell, c)$:

1. There is a natural notion of length associated to each condition $(s, A) \in \mathbb{P}(\mathcal{U})$:

$$\ell(s, A) := |s|.$$

An example: Prikry forcing

Prikry forcing $\mathbb{P}(\mathcal{U})$ can be regarded as a Σ-Prikry triple $(\mathbb{P}(\mathcal{U}), \ell, c)$:

1. There is a natural notion of length associated to each condition $(s, A) \in \mathbb{P}(\mathcal{U})$:

$$\ell(s, A) := |s|.$$

2. Let $c : \mathbb{P}(\mathcal{U}) \to \kappa^+$ be the map defined by $c(s, A) := s$.

$$c(s, A) = c(t, B) \implies (s, A \cap B) \leq^0 (s, A), (t, B).$$

An example: Prikry forcing

Prikry forcing $\mathbb{P}(\mathcal{U})$ can be regarded as a Σ-Prikry triple $(\mathbb{P}(\mathcal{U}), \ell, c)$:

1. There is a natural notion of length associated to each condition $(s, A) \in \mathbb{P}(\mathcal{U})$:

$$\ell(s, A) := |s|.$$

2. Let $c : \mathbb{P}(\mathcal{U}) \to \kappa^+$ be the map defined by $c(s, A) := s$.

$$c(s, A) = c(t, B) \implies (s, A \cap B) \leq^0 (s, A), (t, B).$$

3. Set $\Sigma := \langle \kappa \mid n < \omega \rangle$. Then, $\mathbb{P}_n = \{(s, A) \in \mathbb{P}(\mathcal{U}) \mid |s| = n\}$ is κ-directed-closed.

An example: Prikry forcing

Prikry forcing $\mathbb{P}(\mathcal{U})$ can be regarded as a Σ-Prikry triple $(\mathbb{P}(\mathcal{U}), \ell, c)$:

1. There is a natural notion of length associated to each condition $(s, A) \in \mathbb{P}(\mathcal{U})$:
$$\ell(s, A) := |s|.$$
2. Let $c : \mathbb{P}(\mathcal{U}) \to \kappa^+$ be the map defined by $c(s, A) := s$.
$$c(s, A) = c(t, B) \implies (s, A \cap B) \leq^0 (s, A), (t, B).$$
3. Set $\Sigma := \langle \kappa \mid n < \omega \rangle$. Then, $\mathbb{P}_n = \{(s, A) \in \mathbb{P}(\mathcal{U}) \mid |s| = n\}$ is κ-directed-closed.
4. $\mathbb{P}(\mathcal{U})$ has the CPP.

Some other (more sophisticated) examples

1. Diagonal Prikry forcing;
2. Supercompact Prikry forcing;
3. Gitik-Sharon forcing;
4. AIM forcing;
5. **Extender Based Prikry forcing**;
6. Extender Based Prikry forcing with a single extender;
7. Supercompact Extender Based Prikry forcing;

Iterations of Σ-Prikry forcings

Iterating Σ-Prikry forcings

The philosophy of the Σ-Prikry iterations

Let $\mathbb{Q}$ be a Σ-Prikry forcing along with a *problem* $\sigma \in V^{\mathbb{Q}}$. Want to find a Σ-Prikry forcing $\mathbb{A}$ that projects onto $\mathbb{Q}$ and settles the problem raised by σ. Instead of $\mathbb{A} := \mathbb{Q} * \dot{\mathbb{P}}(\sigma)$ for a notion of forcing $\dot{\mathbb{P}}(\sigma)$ that solves σ, we will be doing something different.

Iterating Σ-Prikry forcings

The philosophy of the Σ-Prikry iterations

Let $\mathbb{Q}$ be a Σ-Prikry forcing along with a *problem* $\sigma \in V^{\mathbb{Q}}$. Want to find a Σ-Prikry forcing $\mathbb{A}$ that projects onto $\mathbb{Q}$ and settles the problem raised by σ. Instead of $\mathbb{A} := \mathbb{Q} * \dot{\mathbb{P}}(\sigma)$ for a notion of forcing $\dot{\mathbb{P}}(\sigma)$ that solves σ, we will be doing something different.

The above is achieved by invoking a solving-problem functor $\mathbb{A}(\cdot, \cdot)$ such that

0. $\mathbb{A} := \mathbb{A}(\mathbb{Q}, \sigma)$, "solves the problem raised by σ"

Iterating Σ-Prikry forcings

The philosophy of the Σ-Prikry iterations

Let $\mathbb{Q}$ be a Σ-Prikry forcing along with a *problem* $\sigma \in V^{\mathbb{Q}}$. Want to find a Σ-Prikry forcing $\mathbb{A}$ that projects onto $\mathbb{Q}$ and settles the problem raised by σ. Instead of $\mathbb{A} := \mathbb{Q} * \dot{\mathbb{P}}(\sigma)$ for a notion of forcing $\dot{\mathbb{P}}(\sigma)$ that solves σ, we will be doing something different.

The above is achieved by invoking a solving-problem functor $\mathbb{A}(\cdot,\cdot)$ such that

0. $\mathbb{A} := \mathbb{A}(\mathbb{Q}, \sigma)$, "solves the problem raised by σ"

and for which there is a pair of maps $(\pitchfork, \pi)$ such that:

1. π **defines a projection** from $\mathbb{A}$ to $\mathbb{Q}$
2. $\pitchfork$ **is a canonical operation** to move from $\mathbb{Q}$ back to $\mathbb{A}$, which coheres with π

Iterating Σ-Prikry forcings

The philosophy of the Σ-Prikry iterations

Let $\mathbb{Q}$ be a Σ-Prikry forcing along with a *problem* $\sigma \in V^{\mathbb{Q}}$. Want to find a Σ-Prikry forcing $\mathbb{A}$ that projects onto $\mathbb{Q}$ and settles the problem raised by σ. Instead of $\mathbb{A} := \mathbb{Q} * \dot{\mathbb{P}}(\sigma)$ for a notion of forcing $\dot{\mathbb{P}}(\sigma)$ that solves σ, we will be doing something different.

The above is achieved by invoking a solving-problem functor $\mathbb{A}(\cdot, \cdot)$ such that

0. $\mathbb{A} := \mathbb{A}(\mathbb{Q}, \sigma)$, "solves the problem raised by σ"

and for which there is a pair of maps $(\pitchfork, \pi)$ such that:

1. π **defines a projection** from $\mathbb{A}$ to $\mathbb{Q}$
2. $\pitchfork$ **is a canonical operation** to move from $\mathbb{Q}$ back to $\mathbb{A}$, which coheres with π

Upshot

Provided (1) & (2) of the above hold then $\mathbb{A}$ **is not so far from being Σ-Prikry.**

Why $\mathbb{A}$ is Σ-Prikry?

Upshot

Provided (1) & (2) of the above hold then $\mathbb{A}$ **is not so far from being Σ-Prikry.**

The two main concerns are the **Chain Condition** and the **Complete Prikry property**:

Why $\mathbb{A}$ is Σ-Prikry?

Upshot

Provided (1) & (2) of the above hold then $\mathbb{A}$ **is not so far from being Σ-Prikry.**

The two main concerns are the **Chain Condition** and the **Complete Prikry property**:

- $\mathbb{A}$ has a good chain condition: This is a consequence of the following:

$$c_{\mathbb{A}}(a) = c_{\mathbb{A}}(a') \implies \exists p \leq \pi(a), \pi(a') \ (\pitchfork(a)(p) = \pitchfork(a')(p)).$$

Why $\mathbb{A}$ is Σ-Prikry?

Upshot

Provided (1) & (2) of the above hold then $\mathbb{A}$ **is not so far from being Σ-Prikry.**

The two main concerns are the **Chain Condition** and the **Complete Prikry property**:

- $\mathbb{A}$ has a good chain condition: This is a consequence of the following:

$$c_{\mathbb{A}}(a) = c_{\mathbb{A}}(a') \implies \exists p \leq \pi(a), \pi(a') \ (\pitchfork(a)(p) = \pitchfork(a')(p)).$$

- $\mathbb{A}$ has the CPP: The crux of the matter is **Diagonalizability**:

For all $a \in \mathbb{A}$ and $D \subseteq \mathbb{A}$ dense open there is $a^* \trianglelefteq^0 a$ that "diagonalizes" D:

$(\mathcal{D})$ If some $b \trianglelefteq a^*$ is in D then $w(a^*, b) \in D$, as well

Why $\mathbb{A}$ is Σ-Prikry?

Upshot

Provided (1) & (2) of the above hold then $\mathbb{A}$ **is not so far from being Σ-Prikry.**

The two main concerns are the **Chain Condition** and the **Complete Prikry property**:

- $\mathbb{A}$ has a good chain condition: This is a consequence of the following:

$$c_{\mathbb{A}}(a) = c_{\mathbb{A}}(a') \implies \exists p \leq \pi(a), \pi(a') \; (\pitchfork(a)(p) = \pitchfork(a')(p)).$$

- $\mathbb{A}$ has the CPP: The crux of the matter is **Diagonalizability**:

  For all $a \in \mathbb{A}$ and $D \subseteq \mathbb{A}$ dense open there is $a^* \trianglelefteq^0 a$ that "diagonalizes" D:

  $(\mathcal{D})$ If some $b \trianglelefteq a^*$ is in D then $w(a^*, b) \in D$, as well

This is achieved by:

1. Assuming that $\mathbb{P}$ enjoys Diagonalizability;
2. Showing that the **forking projection** $(\pitchfork, \pi)$ has the Mixing Property;

   (Allows to lift diagonalizability witnesses from $\mathbb{P}$ to $\mathbb{A}$)

A model for $\neg\mathsf{SCH}_\kappa + \mathrm{Refl}(\kappa^+)$

Towards a model of $\neg\mathsf{SCH}_\kappa + \mathrm{Refl}(<\omega, \kappa^+)$

Set up

1. Let $\Sigma := \langle \kappa_n \mid n < \omega \rangle$ be a strictly increasing sequence of Laver-indestructible supercompact cardinals κ_n. Put $\kappa := \sup(\Sigma)$;
2. Let $\mathbb{P}$ be the Extender-Based Prikry forcing associated to Σ;
3. Assuming $2^{2^\kappa} = \kappa^{++}$, we fix a bookkeeping function $\psi : \kappa^{++} \to H_{\kappa^{++}}$.

The first step: What stationary sets do reflect?

Proposition (P., Rinot & Sinapova) ([PRS1])

Let $\mathbb{Q}$ be a Σ-Prikry forcing not collapsing κ^+. Then $V^{\mathbb{Q}} \models \mathrm{Refl}(<\omega, \kappa^+ \cap \mathrm{cf}^V(> \omega))$.

The first step: What stationary sets do reflect?

Proposition (P., Rinot & Sinapova) ([PRS1])

Let $\mathbb{Q}$ be a Σ-Prikry forcing not collapsing κ^+. Then $V^{\mathbb{Q}} \models \mathrm{Refl}(<\omega, \kappa^+ \cap \mathrm{cf}^V(> \omega))$.

Strategy

Define a Σ-Prikry forcing iteration $\mathbb{P}_{\kappa^{++}}$ such that

1. $\mathbb{P}_{\kappa^{++}}$ is Σ-Prikry and does not collapse κ^+,
2. $V^{\mathbb{P}_{\kappa^{++}}} \models \mathrm{Refl}(\kappa^+ \cap \mathrm{cf}^V(\omega))$,
3. $\mathbb{P}_{\kappa^{++}}$ projects to $\mathbb{P}$.

If $\mathbb{P}_{\kappa^{++}}$ fulfills the above conditions it will yield the desired generic extension.

An iteration to get $\mathrm{Refl}(\kappa^+) + \neg\mathsf{SCH}_\kappa$

1. Set $\mathbb{P}_0 := (\{\emptyset\}, \leq)$ and $\mathbb{P}_1 := {}^1\mathbb{P}$;

An iteration to get $\mathrm{Refl}(\kappa^+) + \neg\mathsf{SCH}_\kappa$

1. Set $\mathbb{P}_0 := (\{\emptyset\}, \leq)$ and $\mathbb{P}_1 := {}^1\mathbb{P}$;
2. $\mathbb{P}_{\alpha+1}$: If $\psi(\alpha) = (\beta, r, \sigma)$ with $\beta < \alpha$, $r \in P_\beta$, $\sigma \in V^{\mathbb{P}_\beta}$ and

$$r \Vdash_{\mathbb{P}_\beta} \sigma \text{ is a non-reflecting stationary set of } \kappa^+ \cap \mathrm{cf}^V(\omega),$$

then $\mathbb{P}_{\alpha+1} := \mathbb{A}(\mathbb{P}_\alpha, \sigma)$, where $\mathbb{A}(\cdot,\cdot)$ is a functor that *destroys the stationarity of* σ.

(Sharon's functor)

An iteration to get $\mathrm{Refl}(\kappa^+) + \neg\mathsf{SCH}_\kappa$

1. Set $\mathbb{P}_0 := (\{\emptyset\}, \leq)$ and $\mathbb{P}_1 := {}^1\mathbb{P}$;
2. $\mathbb{P}_{\alpha+1}$: If $\psi(\alpha) = (\beta, r, \sigma)$ with $\beta < \alpha$, $r \in P_\beta$, $\sigma \in V^{\mathbb{P}_\beta}$ and

$$r \Vdash_{\mathbb{P}_\beta} \sigma \text{ is a non-reflecting stationary set of } \kappa^+ \cap \mathrm{cf}^V(\omega),$$

   then $\mathbb{P}_{\alpha+1} := \mathbb{A}(\mathbb{P}_\alpha, \sigma)$, where $\mathbb{A}(\cdot, \cdot)$ is a functor that *destroys the stationarity of* σ.

   (Sharon's functor)
3. $\mathbb{P}_\alpha$ is the $\leq \kappa$-supported inverse limit of $\langle \mathbb{P}_\beta \mid \beta < \alpha \rangle$.

The above iteration scheme is successful

Fact

1. $\mathbb{P}_{\kappa^{++}}$ is Σ-Prikry and does not collapse κ^+.

Proof

1. Corollary of our iteration theorem.

The above iteration scheme is successful

Fact

1. $\mathbb{P}_{\kappa^{++}}$ is Σ-Prikry and does not collapse κ^+.
2. $V^{\mathbb{P}_{\kappa^{++}}} \models \mathrm{Refl}(\kappa^+ \cap \mathrm{cf}^V(\omega))$.

Proof

1. Corollary of our iteration theorem.
2. By the κ^{++}-cc of $\mathbb{P}_{\kappa^{++}}$ and the usual "catch our tail" argument.

The above iteration scheme is successful

Fact

1. $\mathbb{P}_{\kappa^{++}}$ is Σ-Prikry and does not collapse κ^+.
2. $V^{\mathbb{P}_{\kappa^{++}}} \models \mathrm{Refl}(\kappa^+ \cap \mathrm{cf}^V(\omega))$.
3. $\mathbb{P}_{\kappa^{++}}$ projects to $\mathbb{P}$.

Proof

1. Corollary of our iteration theorem.
2. By the κ^{++}-cc of $\mathbb{P}_{\kappa^{++}}$ and the usual "catch our tail" argument.
3. Essentially, by our assumption over the functors.

A model for $\neg \mathsf{SCH}_{\aleph_\omega} + \mathrm{Refl}(\aleph_{\omega+1})$

Failure of the SCH and reflection at the first singular cardinal

Two classical results of Magidor:

Theorem (Magidor) (Ann. Math, 1977) ($\aleph_\omega$ may be non-compact)

Assuming strong enough large cardinals there is a generic extension of the set-theoretic universe where $\mathrm{GCH}_{<\aleph_\omega}$ holds but $\mathrm{SCH}_{\aleph_\omega}$ fails.

Theorem (Magidor) (JSL, 1982) ($\aleph_\omega$ may be compact)

Assuming strong enough large cardinals there is a generic extension of the set-theoretic universe where $\mathrm{Refl}(\aleph_{\omega+1})$ holds.

Failure of the SCH and reflection at the first singular cardinal

Two classical results of Magidor:

Theorem (Magidor) (Ann. Math, 1977) ($\aleph_\omega$ may be non-compact)

Assuming strong enough large cardinals there is a generic extension of the set-theoretic universe where $\mathsf{GCH}_{<\aleph_\omega}$ holds but $\mathsf{SCH}_{\aleph_\omega}$ fails.

Theorem (Magidor) (JSL, 1982) ($\aleph_\omega$ may be compact)

Assuming strong enough large cardinals there is a generic extension of the set-theoretic universe where $\mathrm{Refl}(\aleph_{\omega+1})$ holds.

Question

Is it possible to combine these two rather contradictory features of $\aleph_\omega$?

Failure of the SCH and reflection at the first singular cardinal

Two classical results of Magidor:

Theorem (Magidor) (Ann. Math, 1977) ($\aleph_\omega$ may be non-compact)

Assuming strong enough large cardinals there is a generic extension of the set-theoretic universe where $\mathrm{GCH}_{<\aleph_\omega}$ holds but $\mathrm{SCH}_{\aleph_\omega}$ fails.

Theorem (Magidor) (JSL, 1982) ($\aleph_\omega$ may be compact)

Assuming strong enough large cardinals there is a generic extension of the set-theoretic universe where $\mathrm{Refl}(\aleph_{\omega+1})$ holds.

Question

Is it possible to combine these two rather contradictory features of $\aleph_\omega$? (Yes)

The theorem

Theorem (P., Rinot & Sinapova) ([PRS3])

Assuming the consistency of infinitely many supercompact cardinals, there is a generic extension where $\mathsf{GCH}_{<\aleph_\omega}$ holds, $\mathsf{SCH}_{\aleph_\omega}$ fails and $\mathrm{Refl}(\aleph_{\omega+1})$ holds.

The theorem

Theorem (P., Rinot & Sinapova) ([PRS3])

Assuming the consistency of infinitely many supercompact cardinals, there is a generic extension where $\mathsf{GCH}_{<\aleph_\omega}$ holds, $\mathsf{SCH}_{\aleph_\omega}$ fails and $\mathrm{Refl}(\aleph_{\omega+1})$ holds.

New components of the proof:

1. Need to generalize the Σ-Prikry class to the broader one of $(\Sigma, \vec{\mathbb{S}})$-Prikry.

   (Necessary to enable interleaved collapses)

The theorem

Theorem (P., Rinot & Sinapova) ([PRS3])

Assuming the consistency of infinitely many supercompact cardinals, there is a generic extension where $\mathsf{GCH}_{<\aleph_\omega}$ holds, $\mathsf{SCH}_{\aleph_\omega}$ fails and $\mathrm{Refl}(\aleph_{\omega+1})$ holds.

New components of the proof:

1. Need to generalize the Σ-Prikry class to the broader one of $(\Sigma, \vec{\mathbb{S}})$-Prikry.

   (Necessary to enable interleaved collapses)

2. Verify that Gitik's EBPFC falls into the $(\Sigma, \vec{\mathbb{S}})$-Prikry framework;

The theorem

Theorem (P., Rinot & Sinapova) ([PRS3])

Assuming the consistency of infinitely many supercompact cardinals, there is a generic extension where $\mathsf{GCH}_{<\aleph_\omega}$ holds, $\mathsf{SCH}_{\aleph_\omega}$ fails and $\mathrm{Refl}(\aleph_{\omega+1})$ holds.

New components of the proof:

1. Need to generalize the Σ-Prikry class to the broader one of $(\Sigma, \vec{\mathbb{S}})$-Prikry.

   (Necessary to enable interleaved collapses)

2. Verify that Gitik's EBPFC falls into the $(\Sigma, \vec{\mathbb{S}})$-Prikry framework;
3. Generalize the iteration scheme to the more general setting of $(\Sigma, \vec{\mathbb{S}})$-Prikry forcings.

The theorem

Theorem (P., Rinot & Sinapova) ([PRS3])

Assuming the consistency of infinitely many supercompact cardinals, there is a generic extension where $\mathsf{GCH}_{<\aleph_\omega}$ holds, $\mathsf{SCH}_{\aleph_\omega}$ fails and $\mathrm{Refl}(\aleph_{\omega+1})$ holds.

New components of the proof:

1. Need to generalize the Σ-Prikry class to the broader one of $(\Sigma, \vec{\mathbb{S}})$-Prikry.

   (Necessary to enable interleaved collapses)
2. Verify that Gitik's EBPFC falls into the $(\Sigma, \vec{\mathbb{S}})$-Prikry framework;
3. Generalize the iteration scheme to the more general setting of $(\Sigma, \vec{\mathbb{S}})$-Prikry forcings.
4. Generalize Sharon's functor $\mathbb{A}(\cdot, \cdot)$ into a $(\Sigma, \vec{\mathbb{S}})$-Prikry functor for killing "fragile" stationary sets;

The theorem

Theorem (P., Rinot & Sinapova) ([PRS3])

Assuming the consistency of infinitely many supercompact cardinals, there is a generic extension where $\mathsf{GCH}_{<\aleph_\omega}$ holds, $\mathsf{SCH}_{\aleph_\omega}$ fails and $\mathrm{Refl}(\aleph_{\omega+1})$ holds.

New components of the proof:

1. Need to generalize the Σ-Prikry class to the broader one of $(\Sigma, \vec{\mathbb{S}})$-Prikry.

   (Necessary to enable interleaved collapses)
2. Verify that Gitik's EBPFC falls into the $(\Sigma, \vec{\mathbb{S}})$-Prikry framework;
3. Generalize the iteration scheme to the more general setting of $(\Sigma, \vec{\mathbb{S}})$-Prikry forcings.
4. Generalize Sharon's functor $\mathbb{A}(\cdot, \cdot)$ into a $(\Sigma, \vec{\mathbb{S}})$-Prikry functor for killing "fragile" stationary sets;
5. Find a scenario in which the iteration to kill all fragile stationary sets moreover produces a model of $\mathrm{Refl}(\aleph_{\omega+1})$.

The theorem

Theorem (P., Rinot & Sinapova) ([PRS3])

Assuming the consistency of infinitely many supercompact cardinals, there is a generic extension where $\mathsf{GCH}_{<\aleph_\omega}$ holds, $\mathsf{SCH}_{\aleph_\omega}$ fails and $\mathrm{Refl}(\aleph_{\omega+1})$ holds.

Proof sketch:

1. Start with GCH and ω-many supercompact cardinals $\langle\kappa_n \mid n<\omega\rangle$.

The theorem

Theorem (P., Rinot & Sinapova) ([PRS3])

Assuming the consistency of infinitely many supercompact cardinals, there is a generic extension where $\mathsf{GCH}_{<\aleph_\omega}$ holds, $\mathsf{SCH}_{\aleph_\omega}$ fails and $\mathrm{Refl}(\aleph_{\omega+1})$ holds.

Proof sketch:

1. Start with GCH and ω-many supercompact cardinals $\langle \kappa_n \mid n < \omega \rangle$.
2. Make each κ_n indestructible under κ_n-directed-closed forcing that preserve the GCH.

The theorem

Theorem (P., Rinot & Sinapova) ([PRS3])

Assuming the consistency of infinitely many supercompact cardinals, there is a generic extension where $\mathsf{GCH}_{<\aleph_\omega}$ holds, $\mathsf{SCH}_{\aleph_\omega}$ fails and $\mathrm{Refl}(\aleph_{\omega+1})$ holds.

Proof sketch:

1. Start with GCH and ω-many supercompact cardinals $\langle \kappa_n \mid n < \omega \rangle$.
2. Make each κ_n indestructible under κ_n-directed-closed forcing that preserve the GCH.
3. Define a $(\Sigma, \vec{\mathbb{S}})$-Prikry-style iteration $\langle \mathbb{P}_\alpha \mid \alpha \leq \kappa^{++} \rangle$ with $\leq \kappa$ support such that:
   - $\mathbb{P}_1$ is Gitik's EBPFC, which makes $\kappa = \aleph_\omega$ and $2^{\aleph_\omega} = \aleph_{\omega+2}$, and preserves $\mathsf{GCH}_{<\aleph_\omega}$.
   - $\mathbb{P}_{\alpha+1} := \mathbb{A}(\mathbb{P}_\alpha, \dot{T})$ is the generalization of Sharon's functor killing the stationarity of $\dot{T}$.

The theorem

Theorem (P., Rinot & Sinapova) ([PRS3])

Assuming the consistency of infinitely many supercompact cardinals, there is a generic extension where $\mathsf{GCH}_{<\aleph_\omega}$ holds, $\mathsf{SCH}_{\aleph_\omega}$ fails and $\mathrm{Refl}(\aleph_{\omega+1})$ holds.

Proof sketch:

1. Start with GCH and ω-many supercompact cardinals $\langle \kappa_n \mid n < \omega\rangle$.
2. Make each κ_n indestructible under κ_n-directed-closed forcing that preserve the GCH.
3. Define a $(\Sigma, \vec{\mathbb{S}})$-Prikry-style iteration $\langle \mathbb{P}_\alpha \mid \alpha \leq \kappa^{++}\rangle$ with $\leq \kappa$ support such that:
   - $\mathbb{P}_1$ is Gitik's EBPFC, which makes $\kappa = \aleph_\omega$ and $2^{\aleph_\omega} = \aleph_{\omega+2}$, and preserves $\mathsf{GCH}_{<\aleph_\omega}$.
   - $\mathbb{P}_{\alpha+1} := \mathbb{A}(\mathbb{P}_\alpha, \dot{T})$ is the generalization of Sharon's functor killing the stationarity of $\dot{T}$.
4. Show that this is **successful**: to wit, $\mathbb{P}_{\kappa^{++}}$ preserves all cardinals, is κ^{++}-cc and kills all non-reflecting stationary subsets of κ^+.

A glimpse to the main issues

Disclaimer

Now our posets $\mathbb{P}_\alpha$ fail to satisfy that its *layers* $(\mathbb{P}_\alpha)_n$ contain a sufficiently closed dense subforcing (because of the interleaved collapses).

A glimpse to the main issues

Disclaimer

Now our posets $\mathbb{P}_\alpha$ fail to satisfy that its *layers* $(\mathbb{P}_\alpha)_n$ contain a sufficiently closed dense subforcing (because of the interleaved collapses).

- It turns out that the former Σ-Prikry machinery can be naturally adapted to handle the preservation of cardinals and the chain condition calculations;

A glimpse to the main issues

Disclaimer

Now our posets $\mathbb{P}_\alpha$ fail to satisfy that its *layers* $(\mathbb{P}_\alpha)_n$ contain a sufficiently closed dense subforcing (because of the interleaved collapses).

- It turns out that the former Σ-Prikry machinery can be naturally adapted to handle the preservation of cardinals and the chain condition calculations;
- The main issue is to secure that all non-reflecting stationary sets are killed:

A glimpse to the main issues

Disclaimer

Now our posets $\mathbb{P}_\alpha$ fail to satisfy that its *layers* $(\mathbb{P}_\alpha)_n$ contain a sufficiently closed dense subforcing (because of the interleaved collapses).

- It turns out that the former Σ-Prikry machinery can be naturally adapted to handle the preservation of cardinals and the chain condition calculations;
- The main issue is to secure that all non-reflecting stationary sets are killed:

Key lemma in the non-collapsing scenario

Assume $\Sigma := \langle \kappa_n \mid n < \omega \rangle$ is a sequence of Laver-indestructible supercompact cardinals. If $\mathbb{P}$ is a Σ-Prikry forcing and $\dot{T}$ is a $\mathbb{P}$-name for a non-reflecting stationary set then $\dot{T}$ is **fragile** (hence, it can be killed in a Σ-Prikry fashion).

A glimpse to the main issues

Disclaimer

Now our posets $\mathbb{P}_\alpha$ fail to satisfy that its *layers* $(\mathbb{P}_\alpha)_n$ contain a sufficiently closed dense subforcing (because of the interleaved collapses).

- It turns out that the former Σ-Prikry machinery can be naturally adapted to handle the preservation of cardinals and the chain condition calculations;
- The main issue is to secure that all non-reflecting stationary sets are killed:

Key lemma in the non-collapsing scenario

Assume $\Sigma := \langle \kappa_n \mid n < \omega \rangle$ is a sequence of Laver-indestructible supercompact cardinals. If $\mathbb{P}$ is a Σ-Prikry forcing and $\dot{T}$ is a $\mathbb{P}$-name for a non-reflecting stationary set then $\dot{T}$ is **fragile** (hence, it can be killed in a Σ-Prikry fashion).

A key ingredient: For each $n < \omega$, $V^{\mathbb{P}_n} \models \mathrm{Refl}(\kappa^+ \cap \mathrm{cf}(<\kappa_n))$.

A glimpse to the main issues

Disclaimer

Now our posets $\mathbb{P}_\alpha$ fail to satisfy that its *layers* $(\mathbb{P}_\alpha)_n$ contain a sufficiently closed dense subforcing (because of the interleaved collapses).

- It turns out that the former Σ-Prikry machinery can be naturally adapted to handle the preservation of cardinals and the chain condition calculations;
- The main issue is to secure that all non-reflecting stationary sets are killed:

Key lemma in the non-collapsing scenario

Assume $\Sigma := \langle \kappa_n \mid n < \omega \rangle$ is a sequence of Laver-indestructible supercompact cardinals. If $\mathbb{P}$ is a Σ-Prikry forcing and $\dot{T}$ is a $\mathbb{P}$-name for a non-reflecting stationary set then $\dot{T}$ is **fragile** (hence, it can be killed in a Σ-Prikry fashion).

A key ingredient: For each $n < \omega$, $V^{\mathbb{P}_n} \models \mathrm{Refl}(\kappa^+ \cap \mathrm{cf}(<\kappa_n))$.

(But now $V^{\mathbb{P}_n}$ does not contain large cardinals!)

How to get reflection in $V^{\mathbb{P}_n}$:

Let $\mathbb{P}$ be some stage of the iteration and $\dot{T}$ be a $\mathbb{P}_n$-name for a stationary:

How to get reflection in $V^{\mathbb{P}_n}$:

Let $\mathbb{P}$ be some stage of the iteration and $\dot{T}$ be a $\mathbb{P}_n$-name for a stationary:

1. $\mathbb{Q}_n := \mathbb{P}'_n \times \mathbb{S}_n$ projects to $\mathbb{P}_n$, where
   - $\mathbb{P}'_n$ is κ^+_{n-1}-directed closed and GCH-preserving (hence preserve supercomp. of κ_{n-1});
   - $\mathbb{S}_n$ is the product of the first n-many Lévy collapses in Gitik's EBPFC;

How to get reflection in $V^{\mathbb{P}_n}$:

Let $\mathbb{P}$ be some stage of the iteration and $\dot{T}$ be a $\mathbb{P}_n$-name for a stationary:

1. $\mathbb{Q}_n := \mathbb{P}'_n \times \mathbb{S}_n$ projects to $\mathbb{P}_n$, where
   - $\mathbb{P}'_n$ is κ^+_{n-1}-directed closed and GCH-preserving (hence preserve supercomp. of κ_{n-1});
   - $\mathbb{S}_n$ is the product of the first n-many Lévy collapses in Gitik's EBPFC;
2. To prove reflection in $V^{\mathbb{P}_n}$ we need:
   - An outer elementary embedding $j \colon V^{\mathbb{Q}_n} \to N \subseteq V^{\mathbb{Q}_n * \dot{\mathbb{R}}_n}$;
   - A stationary-preservation lemma between $V^{\mathbb{Q}_n}$ and $V^{\mathbb{Q}_n * \dot{\mathbb{R}}_n}$;
   - A further stationary-preservation lemma between $V^{\mathbb{P}_n}$ and $V^{\mathbb{Q}_n}$.

Thank you very much for your attention!