Splitting a stationary set: Is there another way?

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This talk is based on a joint work with Maxwell Levine.

Conventions

- κ denotes a regular uncountable cardinal;
- λ denotes an infinite cardinal;
- $\mathrm{Reg}(\kappa) := \{\lambda < \kappa \mid \aleph_0 \leq \mathrm{cf}(\lambda) = \lambda\}$;
- $E^\kappa_\lambda := \{\alpha < \kappa \mid \mathrm{cf}(\alpha) = \lambda\}$;
- $E^\kappa_{\neq\lambda}$, $E^\kappa_{\geq\lambda}$ and $E^\kappa_{>\lambda}$ are defined analogously;
- $\mathrm{acc}^+(A) := \{\alpha < \sup(A) \mid \sup(A \cap \alpha) = \alpha > 0\}$.

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Solovay's theorem has countless applications in Set Theory.
For instance, it plays a role in the proof of strong negative partition relations of the form $\kappa \nrightarrow [\kappa]^2_\kappa$, and variations of it are missing for the sought proof that successors of a singular cardinals cannot be Jónsson.

Variations of Solovay's theorem

Variation I (Brodsky-Rinot, 2019)

For every $\theta \leq \kappa$ and a sequence $\langle S_i \mid i < \theta \rangle$ of stationary subsets of κ, there exists a cofinal $I \subseteq \theta$ and pairwise disjoint stationary sets $\langle T_i \mid i \in I \rangle$ such that $T_i \subseteq S_i$ for all $i \in I$.

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Variation II (Magidor?, 1970's)

If $\square_\lambda$ holds, then for every stationary $S \subseteq \lambda^+$, there is a partition $\langle S_i \mid i < \lambda^+ \rangle$ of S into stationary sets such that, for all $i < \lambda^+$, S_i does not reflect.

Variations of Solovay's theorem

Definition

For $S \subseteq \kappa$, let $\mathrm{Tr}(S) := \{\beta \in E^{\kappa}_{>\omega} \mid S \cap \beta \text{ is stationary in } \beta\}$.

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Theorem (Shelah, 1991)

If $\kappa > \aleph_2$, and $E^\kappa_{\geq\aleph_2}$ admits a nonreflecting stationary set, then there exists a κ-cc poset whose square is not κ-cc.

Variations of Solovay's theorem

Variation III (Brodsky-Rinot, 2019)

If $\square(\kappa)$ holds, then for every fat $F \subseteq \kappa$, there is a partition $\langle F_i \mid i < \kappa\rangle$ of F into fat sets such that, for all $i < j < \kappa$, $\mathrm{Tr}(F_i) \cap \mathrm{Tr}(F_j) = \emptyset$.

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Definition

$\Pi(S, \theta)$ asserts the existence of a partition $\langle S_i \mid i < \theta \rangle$ of S such that $\bigcap_{i<\theta} \mathrm{Tr}(S_i)$ is stationary.

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Definition

$\Pi(S, \theta, T)$ asserts the existence of a partition $\langle S_i \mid i < \theta \rangle$ of S such that $\bigcap_{i<\theta} \mathrm{Tr}(S_i) \cap T$ is stationary.

Singular cardinals combinatorics

Scales

Definition

Suppose that λ is a singular cardinal, and $\vec{\lambda} = \langle \lambda_i \mid i < \mathrm{cf}(\lambda) \rangle$ is a strictly increasing sequence of regular cardinals, converging to λ. For any two functions $f, g \in \prod \vec{\lambda}$ and $i < \mathrm{cf}(\lambda)$, we write $f <^i g$ to express that $f(j) < g(j)$ whenever $i \leq j < \mathrm{cf}(\lambda)$.

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Suppose that λ is a singular cardinal; $\vec{f} = \langle f_\beta \mid \beta < \lambda^+ \rangle$ is said to be a <u>scale for λ</u> iff there exists a sequence $\vec{\lambda}$ as above, such that:

- for every $\beta < \lambda^+$, $f_\beta \in \prod \vec{\lambda}$;
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An ordinal $\alpha \in E^{\lambda^+}_{>\mathrm{cf}(\lambda)}$ is said to be good if there exist $i < \mathrm{cf}(\lambda)$ and a cofinal $A \subseteq \alpha$ such that, for all $\delta < \gamma$ from A, $f_\delta <^i f_\gamma$.

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For every regular θ with $\mathrm{cf}(\lambda) < \theta < \lambda$, $G(\vec{f}) \cap E^{\lambda^+}_{\theta}$ is stationary.

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The set of good points is robust

If $\vec{f}, \vec{g}$ are scales in $\prod \vec{\lambda}$, then $G(\vec{f}) \triangle G(\vec{g})$ is nonstationary.

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An ordinal $\alpha \in E^{\lambda^+}_{>\mathrm{cf}(\lambda)}$ is said to be <u>very good</u> if there exist $i < \mathrm{cf}(\lambda)$ and a ~~cofinal~~ club $A \subseteq \alpha$ such that, for all $\delta < \gamma$ from A, $f_\delta <^i f_\gamma$.

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Recall

If $\vec{f}, \vec{g}$ are scales in $\prod \vec{\lambda}$, then $G(\vec{f}) \triangle G(\vec{g})$ is nonstationary.

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If $\vec{f}, \vec{g}$ are scales in $\prod \vec{\lambda}$, then $G(\vec{f}) \triangle G(\vec{g})$ is nonstationary.

Theorem (Cummings-Foreman, 2010)

If $V = L$, then there are scales $\vec{f}, \vec{g}$ in $\prod_{n<\omega} \aleph_n$ for which $V(\vec{f}) = E^{\aleph_{\omega+1}}_{>\omega}$ and $V(\vec{g}) = \emptyset$.

Very good points are not robust

The following is implicit in the proof of the above-mentioned theorem of Cummings-Foreman concerning $V = L$:

Proposition

Suppose λ is singular, $T \subseteq \lambda^+$ is stationary and $\Pi(\lambda^+, \mathrm{cf}(\lambda), T)$. Suppose $\vec{f}$ is a scale for λ, living in some product $\prod_{i<\mathrm{cf}(\lambda)} \lambda_i$. Then $T \setminus V(\vec{g})$ is stationary for some scale $\vec{g}$ in $\prod_{i<\mathrm{cf}(\lambda)} \lambda_i$.

Proof.

Fix a partition $\langle S_i \mid i < \mathrm{cf}(\lambda)\rangle$ of λ^+, with $T' := T \cap \bigcap_{i<\mathrm{cf}(\lambda)} \mathrm{Tr}(S_i)$ stationary. Define $\langle g_\beta \mid \beta < \lambda^+\rangle$ by letting $g_\beta(i) := 0$ for $\beta \in S_i$, and $g_\beta(i) := f_\beta(i)$, otherwise.
Let $\alpha \in T'$ be arbitrary. To see that $\alpha \notin V(\vec{g})$, fix an arbitrary club $C \subseteq \alpha$ and an index $i < \mathrm{cf}(\lambda)$.
Let $\delta := \min(C \cap S_i)$ and $\gamma := \min(C \cap S_i \setminus (\delta + 1))$.
Then $\delta < \gamma$ is a pair of elements of C, while $g_\delta(i) = 0 = g_\gamma(i)$. $\square$

Very good scales

Definition

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Conclusion

Suppose λ is a singular cardinal and $\Pi(\lambda^+, \mathrm{cf}(\lambda), E^{\lambda^+}_{>\mathrm{cf}(\lambda)})$ holds. Then any product $\prod_{i<\mathrm{cf}(\lambda)} \lambda_i$ admitting a scale for λ, admits yet another scale which is not very good.

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Note
There are numerous ways to consistently get instances of $\Pi(S, \theta, T)$. For instance, in a model of Magidor (1982), $\Pi(S, \aleph_1, T)$ holds for all stationary $S \subseteq E^{\aleph_2}_{\aleph_0}$ and $T \subseteq E^{\aleph_2}_{\aleph_1}$. The main point here is to prove instances of $\Pi(S, \theta, T)$ in ZFC.

ZFC results

Main result

Theorem

Suppose that $\mu < \theta$ are infinite regular cardinals $< \lambda$.

1. *If λ is inaccessible, then $\Pi(\lambda, \theta, \lambda)$ and $\Pi(\lambda^+, \lambda, \lambda^+)$ hold;*

This is trivial

Simply take $\langle E^{\lambda}_{\mu} \mid \mu \in \mathrm{Reg}(\aleph_{\theta+1})\rangle$ and $\langle E^{\lambda^+}_{\mu} \mid \mu \in \mathrm{Reg}(\lambda)\rangle$.

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1. *If λ is inaccessible, then $\Pi(\lambda, \theta, \lambda)$ and $\Pi(\lambda^+, \lambda, \lambda^+)$ hold;*
2. *If λ is regular, then $\Pi(E^{\lambda^+}_{\mu}, \theta, E^{\lambda^+}_{\theta})$ holds;*

This is optimal

If $\Pi(S, \theta, T)$ holds, then $\{\alpha \in T \mid \mathrm{cf}(\alpha) \geq \theta\}$ must be stationary.

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2. *If λ is regular, then $\Pi(E^{\lambda^+}_{\mu}, \theta, E^{\lambda^+}_{\theta})$ holds;*
3. *If $2^\theta \leq \lambda$ and $\theta \neq \mathrm{cf}(\lambda)$, then $\Pi(E^{\lambda^+}_{\mu}, \theta, E^{\lambda^+}_{\theta})$ holds;*

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3. *If $2^\theta \leq \lambda$ and $\theta \neq \mathrm{cf}(\lambda)$, then $\Pi(E^{\lambda^+}_\mu, \theta, E^{\lambda^+}_\theta)$ holds;*
4. *If λ is singular and $\theta^{++} \neq \mathrm{cf}(\lambda)$, then $\Pi(E^{\lambda^+}_\mu, \theta, E^{\lambda^+}_{\theta^{++}})$ holds;*

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5. *If λ is singular and $\theta^{++} = \mathrm{cf}(\lambda)$, then $\Pi(E^{\lambda^+}_\mu, \theta, E^{\lambda^+}_{\theta^{+3}})$ holds.*

Remark

This follows from Clause (4).

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Remark

Our proof at the level of successors of singulars is indeed different from the standard proofs for partitioning a stationary set. We build on the fact that any singular cardinal admits a scale and that the set of good points of a scale is stationary relative to any cofinality; we also use a combination of Ulam matrices with club-guessing to avoid any cardinal arithmetic hypotheses (Clauses (4) and (5)).

A special case with a simplified proof

Theorem

Let λ be a singular cardinal. Let $\mu < \theta$ be regular cardinals with $\mathrm{cf}(\lambda) < \mu < \theta < \lambda$. Then $\Pi(E^{\lambda^+}_{\mu}, \theta, E^{\lambda^+}_{\theta^{++}})$ holds.

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Proof. Fix a scale $\vec{f}$ for λ in some product $\prod_{i<\mathrm{cf}(\lambda)} \lambda_i$.
By Shelah's theorem, $T_0 := E^{\lambda^+}_{\theta^{++}} \cap G(\vec{f})$ is stationary.

Claim 1

There exist $i < \mathrm{cf}(\lambda)$, $\zeta \in E^{\lambda}_{\theta^{++}}$, a stationary $T_1 \subseteq T_0$, and a sequence $\langle S^1_\alpha \mid \alpha \in T_1 \rangle$ such that, for all $\alpha \in T_1$:

- *S^1_α is a stationary subset of E^{α}_{μ};*
- *$\langle f_\beta(i) \mid \beta \in S^1_\alpha \rangle$ is strictly increasing and converging to ζ.*

Proof. By Fodor's lemma, it suffices to prove that for each $\alpha \in T_0$, there is $i < \mathrm{cf}(\lambda)$ and a stationary $S \subseteq E^{\alpha}_{\mu}$ on which $\beta \mapsto f_\beta(i)$ is strictly increasing.

Proof of Claim 1

Let $\alpha \in T_0$ be arbitrary. We shall find $i < \mathrm{cf}(\lambda)$ and a stationary $S \subseteq E^{\alpha}_{\mu}$ on which $\beta \mapsto f_\beta(i)$ is strictly increasing.
For each $\gamma < \beta < \alpha$, pick $i_{\gamma,\beta} < \mathrm{cf}(\lambda)$ such that $f_\gamma <^{i_{\gamma,\beta}} f_\beta$.
As $\alpha \in T_0$ is a good point, let us also fix $i' < \mathrm{cf}(\lambda)$ and a cofinal $A \subseteq \alpha$ such that, for all $\delta < \gamma$ from A, $f_\delta <^{i'} f_\gamma$.
Consider $S' := \mathrm{acc}^+(A) \cap E^{\alpha}_{\mu}$, which is a stationary subset of E^{α}_{μ}.
As $\mu > \mathrm{cf}(\lambda)$, for each $\beta \in S'$, we may pick a cofinal $a_\beta \subseteq A \cap \beta$ and $i_\beta < \mathrm{cf}(\lambda)$ such that, for all $\gamma \in a_\beta$, $i_{\gamma,\beta} = i_\beta$.
As $\theta^{++} > \mathrm{cf}(\lambda)$, we may pick a stationary $S \subseteq S'$ and $i < \mathrm{cf}(\lambda)$ such that, for all $\beta \in S$, $\max\{i_\beta, i', i_{\beta,\min(A\setminus\beta+1)}\} = i$.
To see that i and S are as sought, let $\epsilon < \beta$ be arbitrary elements of S. Consider $\delta := \min(A \setminus \epsilon + 1)$ and $\gamma := \min(a_\beta \setminus \delta + 1)$.
Clearly, $\epsilon < \delta < \gamma < \beta$ and $f_\epsilon <^{i_{\epsilon,\min(A\setminus\epsilon+1)}} f_\delta <^{i'} f_\gamma <^{i_\beta} f_\beta$.
In particular, $f_\epsilon <^{i} f_\beta$, so that $f_\epsilon(i) < f_\beta(i)$, as sought. $\square$

Fix i, ζ, and $\langle S^1_\alpha \mid \alpha \in T_1 \rangle$ as in Claim 1.

Step 2: Find a function g

Claim 2

There are $g : E^{\lambda^+}_{\mu} \to \theta^{++}$ and a sequence $\langle S^2_\alpha \mid \alpha \in T_1\rangle$ such that, for all $\alpha \in T_1$:

- *S^2_α is a stationary subset of S^1_α (hence, of E^α_μ);*
- *$\langle g(\beta) \mid \beta \in S^2_\alpha\rangle$ is strictly increasing (hence, cofinal in θ^{++}).*

Proof. Fix a club z in ζ with $\mathrm{otp}(z) = \theta^{++}$. Define $g : E^{\lambda^+}_{\mu} \to \theta^{++}$ by letting $g(\beta) := \mathrm{otp}(f_\beta(i) \cap z)$ if $f_\beta(i) < \zeta$ and $g(\beta) := 0$, o.w.; To see that g is as sought, let $\alpha \in T_1$ be arbitrary. Let $\pi : \theta^{++} \to \alpha$ be the inverse collapse of some club in α. Clearly, $\bar{S} := \{\bar\beta < \theta^{++} \mid \pi(\bar\beta) \in S^1_\alpha \ \&\ (g \circ \pi)``\bar\beta \subseteq \bar\beta\}$ is stationary. Let $\bar{B} := \{\bar\beta \in \bar{S} \mid (g \circ \pi)(\bar\beta) < \bar\beta\}$. For all $\bar\epsilon < \bar\beta$ from $\bar{S} \setminus \bar{B}$, we have $g(\pi(\bar\epsilon)) < \bar\beta \leq g(\pi(\bar\beta))$. Thus, it suffices to show that $S^2_\alpha := \pi[\bar{S} \setminus \bar{B}]$ (which is a subset of S^1_α) is stationary. Suppose not. In particular, $\bar{B}$ is stationary. But then, Fodor's lemma entails a stationary $\hat{B} \subseteq \bar{B}$ on which $g \circ \pi$ is constant, contradicting the fact that $\langle f_{\pi(\bar\beta)}(i) \mid \bar\beta \in \hat{B}\rangle$ converges to ζ. $\square$

Step 3: An Ulam Matrix

Let $g : E_\mu^{\lambda^+} \to \theta^{++}$ and $\langle S_\alpha^2 \mid \alpha \in T_1 \rangle$ be given by Claim 2.
Now, fix an Ulam matrix $\langle A_{\xi,\eta} \mid \xi < \theta^{++}, \eta < \theta^+ \rangle$ over θ^{++}, i.e.,

- for all $\xi < \theta^{++}$, $|\theta^{++} \setminus \bigcup_{\eta<\theta^+} A_{\xi,\eta}| \leq \theta^+$;
- for all $\eta < \theta^+$ and $\xi < \xi' < \theta^{++}$, $A_{\xi,\eta} \cap A_{\xi',\eta} = \emptyset$.

Claim 3

For every $\alpha \in T_1$, there are $\eta < \theta^+$ and $x \in [\theta^{++}]^{\theta^{++}}$ such that, for all $\xi \in x$, $g^{-1}[A_{\xi,\eta}] \cap \alpha$ is stationary in α.

Proof. Suppose not. Then, for all $\eta < \theta^+$, the set
$x_\eta := \{\xi < \theta^{++} \mid g^{-1}[A_{\xi,\eta}] \cap \alpha \text{ is stationary in } \alpha\}$ has size $\leq \theta^+$.
So $X := \bigcup_{\eta<\theta^+} x_\eta$ has size $\leq \theta^+$, and we may fix $\xi \in \theta^{++} \setminus X$.
It follows that for all $\eta < \theta^+$, $g^{-1}[A_{\xi,\eta}] \cap \alpha$ is nonstationary in α.
Consequently, $g^{-1}[\bigcup_{\eta<\theta^+} A_{\xi,\eta}] \cap \alpha$ is nonstationary in α.
However, $\bigcup_{\eta<\theta^+} A_{\xi,\eta}$ contains a tail of θ^{++}, contradicting the fact that $\langle g(\beta) \mid \beta \in S_\alpha^2 \rangle$ is strictly increasing and cofinal in θ^{++}. $\square$

Step 4: Club-guessing

By Shelah's club-guessing theorem, we now fix a sequence $\langle C_\iota \mid \iota \in E^{\theta^{++}}_\theta \rangle$ such that, for every club $C \subseteq \theta^{++}$, there exists $\iota \in E^{\theta^{++}}_\theta$ such that $C_\iota \subseteq C \cap \iota$ and $\mathrm{otp}(C_\iota) = \theta$.

By Claim 3, for every $\alpha \in T_1$, let us fix $\eta_\alpha < \theta^+$ and $x_\alpha \in [\theta^{++}]^{\theta^{++}}$ such that, for all $\xi \in x_\alpha$, $g^{-1}[A_{\xi,\eta_\alpha}] \cap \alpha$ is stationary in α.
Then, fix $\iota_\alpha \in E^{\theta^{++}}_\theta$ such that $C_{\iota_\alpha} \subseteq \mathrm{acc}^+(x_\alpha) \cap \iota_\alpha$ and $\mathrm{otp}(C_{\iota_\alpha}) = \theta$.
By Fodor's lemma, fix a stationary $T_2 \subseteq T_1$, $\eta < \theta^+$ and $\iota \in E^{\theta^{++}}_\theta$ such that, for all $\alpha \in T_2$, $\eta_\alpha = \eta$ and $\iota_\alpha = \iota$.
As the elements of $\langle A_{\xi,\eta} \mid \xi < \theta^{++} \rangle$ are pairwise disjoint, we may fix a function $h : E^{\lambda^+}_\mu \to \theta$ such that, for all $\beta < \lambda^+$:

$$(g(\beta) \in A_{\xi,\eta} \;\&\; \xi < \iota) \implies h(\delta) = \sup(\mathrm{otp}(C_\iota \cap \xi)).$$

Step 5: Verification

For each $i < \theta$, let $S_i := h^{-1}\{i\}$.
We claim that $\langle S_i \mid i < \theta \rangle$ witnesses $\Pi(E^{\lambda^+}_{\mu}, \theta, E^{\lambda^+}_{\theta^{++}})$. Furthermore:

Claim 4

$\bigcap_{i<\theta} \mathrm{Tr}(S_i) \cap E^{\lambda^+}_{\theta^{++}}$ *covers the stationary set* T_2.

Proof. Fix arbitrary $\alpha \in T_2$ and $i < \theta$. We shall find a stationary subset $S' \subseteq E^{\alpha}_{\mu}$ such that $h[S'] = \{i\}$.
As $i < \theta = \mathrm{otp}(C_\iota)$, let ξ' denote the unique element of C_ι such that $\mathrm{otp}(C_\iota \cap \xi') = i$. Then, put $\xi := \min(x_\alpha \setminus (\xi' + 1))$.
As $C_\iota \subseteq \mathrm{acc}^+(x_\alpha)$, we have that $[\xi', \xi) \cap C_\iota = \{\xi'\}$.
Consequently, $\mathrm{otp}(C_\iota \cap \xi) = \mathrm{otp}(C_\iota \cap (\xi' + 1)) = i + 1$.
As $\eta = \eta_\alpha$ and $\xi \in x_\alpha$, the set $S' := g^{-1}[A_{\xi,\eta}] \cap \alpha$ is a stationary subset of E^{α}_{μ}. Finally, for each $\beta \in S'$, we have $g(\beta) \in A_{\xi,\eta}$, meaning that $h(\beta) = \sup(\mathrm{otp}(C_\iota \cap \xi)) = \sup(i + 1) = i$, as sought.

qed

A finer result

We also have a finer result that apply for arbitrary stationary $S \subseteq \lambda^+$ (rather than $S = E^{\lambda^+}_{\mu}$).

Theorem

Suppose $\theta < \lambda$ are infinite cardinals with $\theta \neq \mathrm{cf}(\lambda)$ and $2^\theta \leq \lambda$. For all subsets S, T of λ^+ with a stationary $\mathrm{Tr}(S) \cap T \cap E^{\lambda^+}_{\theta}$, any of the following implies that $\Pi(S, \theta, T)$ holds:

1. *λ is regular;*
2. *λ is a singular cardinal admitting a good scale.*

Good scale

A scale $\vec{f}$ for λ such that club many $\alpha \in E^{\lambda^+}_{>\mathrm{cf}(\lambda)}$ are good for $\vec{f}$.