Analytic quasi-orders and two forms of diamond

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50 Years of Set Theory in Toronto

Fields Institute for Research in Mathematical Sciences

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The class of κ-Borel sets

The least collection of sets in κ^κ (resp., 2^κ) containing all open sets that is closed under unions and intersections of length $\leq \kappa$, and complements.

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Borel reductions and continuous reductions

Let E_1 and E_2 be equivalence relations on $X_1, X_2 \in \{2^\kappa, \kappa^\kappa\}$, respectively.

- We say that E_1 is *κ-Borel reducible to E_2* and denote this by $E_1 \hookrightarrow_B E_2$ iff there is a κ-Borel function $f : X_1 \to X_2$ s.t. $\forall \eta, \xi \in X_1$, $(\eta, \xi) \in E_1 \iff (f(\eta), f(\xi)) \in E_2$.

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- If f is moreover continuous, then we say that E_1 *is continuously reducible to E_2* and denote it by $E_1 \hookrightarrow_c E_2$.

Comparing theories

Let T, T' denote complete theories over a countable first-order language.

Counting number of non-isomorphic models of cardinality κ

Say that T' is more complex than T iff $I(T', \kappa)$ is bigger than $I(T, \kappa)$.

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Shelah's Main Gap Theorem implies that this notion is not very informative.

Theorem (Shelah, 1990)

One of the following holds:

1. $I(T,\aleph_\alpha) < \beth_{\omega_1}(|\alpha|)$ *for every nonzero ordinal* α.
2. $I(T,\kappa) = 2^\kappa$ *for every uncountable cardinal* κ.

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One of the following holds:

1. *T is shallow superstable without DOP and without OTOP. In this case, $I(T,\aleph_\alpha) < \beth_{\omega_1}(|\alpha|)$ for every nonzero ordinal α.*
2. *T is not superstable, or superstable and deep or with DOP or with OTOP. In this case, $I(T,\kappa) = 2^\kappa$ for every uncountable cardinal κ.*

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Every element ξ of 2^κ gives rise to a model $\mathfrak{M}_\xi = (\kappa, \ldots)$ over $\mathcal{L}$ via:

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Define a quasi-order $\leq_\kappa$ on complete countable relational theories, letting

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A successful approach

- (Friedman-Hyttinen-Kulikov, 2014)
  If T is unstable and T' is classifiable, then $T \not\leq_\kappa T'$.
- (Hyttinen-Kulikov-Moreno, 2017) Consistently (e.g., under $V = L$), if T is classifiable but T' is not, then $T \leq_\kappa T'$ and $T' \not\leq_\kappa T$.
- (Asperó-Hyttinen-Kulikov-Moreno, 2019) If κ is Π^1_2-indescribable, for every theory T, $T \leq_\kappa$ DLO (dense lin. orders without endpoints).

A natural equivalence relation

Let S denote an arbitrary stationary subset of κ.

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For $\eta, \xi \in \kappa^\kappa$, let $\eta =_S \xi$ iff $\{\alpha \in S \mid \eta(\alpha) \neq \xi(\alpha)\}$ is nonstationary.

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Consistently, if T is a complete non-classifiable theory, then there exists a stationary $S \subseteq \kappa$ for which $=^2_S \hookrightarrow_B \cong_T$.

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So, consistently, for T classifiable and T' non-classifiable, for some $S \subseteq \kappa$:

$$\cong_T \hookrightarrow_c =^2_S \hookrightarrow_B \cong_{T'}$$

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A large antichain (Friedman-Hyttinen-Kulikov, 2014)

Consistently, there exists a collection $\mathcal{S}$ *of* 2^κ *many stationary subsets of* κ *such that, for all* $S \neq S'$ *from* $\mathcal{S}$, $=^2_S \not\hookrightarrow_B =^2_{S'}$.

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We'll say more about this later on.

A simple test case

Let $S^2_0 := \{\alpha < \omega_2 \mid \mathrm{cf}(\alpha) = \omega\}$
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Note that if Clause (1) holds in the F-H-K model, then $=_{S^2_1} \not\hookrightarrow_B =^2_{S^2_1}$.

Stationary reflection principles and 2^κ

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For every stationary $S \subseteq \kappa \cap \mathrm{cof}(\omega)$, there exists a strictly increasing and continuous map $\varphi : \omega_1 \to S$.

Observation

Friedman's problem implies $=^2_{S^2_0} \hookrightarrow_c =^2_{S^2_1}$.

Proof. Define a reduction $f : 2^{\omega_2} \to 2^{\omega_2}$, as follows.
Given $\xi : \omega_2 \to 2$ and $\delta < \omega_2$, we let $f(\xi)(\delta) = 1$ iff

$$\delta \in S^2_1 \text{ and } \{\alpha < \delta \mid \xi(\alpha) = 0\} \text{ is nonstationary in } \delta.$$

Note that $f(\xi)(\delta)$ depends only on $\xi \restriction \delta$, so that f is continuous.

- Suppose $\xi =^2_{S^2_0} \eta$. Pick a club $C \subseteq \omega_2$ s.t. $\forall \alpha \in C \cap S^2_0 [\xi(\alpha) = \eta(\alpha)]$.

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Consider the club D of accumulation points of C. For any $\delta \in D \cap S^2_1$:

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Consequently, $f(\xi) =^2_{S^2_1} f(\eta)$.

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Stationary reflection principles and κ^κ

Friedman's problem yields a reduction on the κ-antor space, 2^κ.
Moreno proved that, for κ accessible, $=^2_S \hookrightarrow_c =^2_{S'}$ entails $=_S \hookrightarrow_c =_{S'}$.
The latter (i.e., reduction in κ^κ) already follows from vanilla reflection:

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More generally:

Theorem (Asperó-Hyttinen-Kulikov-Moreno, 2019)

Suppose X, S are stationary subsets of κ, with $S \subseteq \mathrm{cof}(>\omega)$.
If every stationary subset of X reflects in S, then $=_X \hookrightarrow_c =_S$.

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Bear with me, as I overcomplicate their proof...

Reflection principles and reductions on κ^κ

Suppose X, S are stationary subsets of κ, with $S \subseteq \mathrm{cof}({>}\omega)$.
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Suppose X, S are stationary subsets of κ, and
$\vec{\mathcal{F}} = \langle \mathcal{F}_\delta \mid \delta \in S \rangle$ is a sequence such that each $\mathcal{F}_\delta$ is a filter on δ.

Filter reflection (aka, Fake reflection)

We say that X *$\vec{\mathcal{F}}$-reflects to S* iff the two hold:

1. For every stat. $Y \subseteq X$, there are stat. many $\delta \in S$ s.t. $Y \cap \delta \in \mathcal{F}_\delta^+$;
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- If $\xi =_X \eta$, then, by Clause (2), $f(\xi) =_S f(\eta)$.

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- If $\xi \neq_X \eta$, then, by Clause (1), $f(\xi) \neq_S f(\eta)$. $\square$

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Consistency strength

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Just force it

Unlike stationary reflection, fake reflection at the levels of successor cardinals $\kappa = \lambda^+$ may be forced without appealing to large cardinals! Furthermore, forcing over models of GCH preserves the cardinals structure.

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Let us take a closer look at this principle...

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About Clause (1):

- Sequences $\vec{\mathcal{F}}$ satisfying just Clause (1) exist in ZFC, where each $\mathcal{F}_\delta$ is principal.

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- A consistent non-principal example is given by Moore's work on *trace reflection* in models of MRP.

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1. For every stat. $Y \subseteq X$, there are stat. many $\delta \in S$ s.t. $Y \cap \delta \in \mathcal{F}_\delta^+$;
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About Clause (2):

- Sequences $\vec{\mathcal{F}}$ satisfying just Clause (2) exist in ZFC, letting $\mathcal{F}_\delta := \mathrm{CUB}(\delta)$.
- If S is ineffable and Clause (2) holds, then $\mathcal{F}_\delta \supseteq \mathrm{CUB}(\delta)$.
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- If each $\mathcal{F}_\delta$ has a small base, then Clause (2) is a strong club guessing principle. Such a sequence is easily derivable from $\diamondsuit^*_S$.

Filter reflection

Filter reflection with diamond

We say that X $\vec{\mathcal{F}}$*-reflects to* S *with* $\diamondsuit$ iff there is $\langle Y_\delta \mid \delta \in S\rangle$ such that:

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If X $\vec{\mathcal{F}}$-reflects to S with $\diamondsuit$, then $=_X \hookrightarrow_c =^2_S$.

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Fake it till you make it

Fake reflection makes sense even for S that concentrates on points of countable cofinality, so we consistently get a complete cycle of reductions:

$$=_{S^2_1} \hookrightarrow_c =^2_{S^2_0} \hookrightarrow_c =_{S^2_0} \hookrightarrow_c =^2_{S^2_1} \hookrightarrow_c =_{S^2_1} .$$

Authentic reflection with diamond

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Theorem (Sun, 1993)

If κ is ineffable, then $\diamondsuit^1_\kappa$ holds. I.e., there is a sequence $\langle Y_\delta \mid \delta < \kappa\rangle$ s.t., for every $Y \subseteq \kappa$, $\{\delta < \kappa \mid Y_\delta = Y \cap \delta\}$ is a Π^1_1-indescribable set.

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Theorem (Sun, 1993)

If S is ineffable, then $\diamondsuit^1_S$ holds. In particular, κ $\diamondsuit$-reflects to S.

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The special case X $\mathrm{C\vec{U}B}$-reflects to S with $\diamondsuit$ appears in the work of [Friedman-Hyttinen-Kulikov, 2014] under the name X $\diamondsuit$-reflects to S. They proved it implies $=^2_X \hookrightarrow_c =^2_S$, but we now know that it furthermore implies $=_X \hookrightarrow_c =^2_S$.

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If S is ineffable, then $\diamondsuit^1_S$ holds. In particular, κ $\diamondsuit$-reflects to S.

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1. *If S is weakly compact and $\diamondsuit^*_S$ holds, then so does $\diamondsuit^1_S$.*

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Theorem

1. *If S is weakly compact and $\diamondsuit^*_S$ holds, then so does $\diamondsuit^1_S$.*
2. *If κ is weakly compact, $\lambda \in \mathrm{Reg}(\kappa)$ and GCH holds, then after forcing with $\mathrm{Col}(\lambda, {<}\kappa)$, $\lambda^+ \cap \mathrm{cof}({<}\lambda)$ $\diamondsuit$-reflects to $\lambda^+ \cap \mathrm{cof}(\lambda)$.*

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3. *Assuming MM (Martin's Maximum), if $\diamondsuit_{\kappa\cap\mathrm{cof}(\omega)}$ holds, then $\kappa \cap \mathrm{cof}(\omega)$ $\diamondsuit$-reflects to $\kappa \cap \mathrm{cof}(\omega_1)$.*

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4. *Whenever $\mathrm{Add}(\kappa, 1)$ forces that every stationary subset of X reflects in S, it moreover forces that X $\diamondsuit$-reflects to S.*

The non-ineffable case

Definition (Devlin, 1982)

$\diamondsuit^{\sharp}_{S}$ asserts the existence of a sequence $\langle N_\delta \mid \delta \in S\rangle$ satisfying (1)–(3):

1. each N_δ is a p.r.-closed transitive set of size $|\delta| + \aleph_0$ & $\delta + 1 \subseteq N_\delta$;

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Note: Clauses (1) and (2) amount to $\diamondsuit^+_S$. Unlike $\diamondsuit^+_S$, for every $\vec{N}$ witnessing $\diamondsuit^\sharp_S$, there is a stationary $T \subseteq S$ such that $\vec{N} \restriction T$ fails to witness $\diamondsuit^\sharp_T$.

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If $V = L$, then for every regular uncountable cardinal κ, $\diamondsuit^\sharp_\kappa$ holds iff κ is not ineffable.

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Theorem (Devlin, 1982)

If $V = L$, then for every regular uncountable cardinal κ, $\diamondsuit^\sharp_\kappa$ holds iff κ is not ineffable. In fact, for every stationary subset S of a regular uncountable cardinal, $\diamondsuit^\sharp_S$ holds iff S is not ineffable.

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Definition (Devlin, 1982)

$\diamondsuit_S^\sharp$ asserts the existence of a sequence $\langle N_\delta \mid \delta \in S\rangle$ satisfying (1)–(3):

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Proposition

If $\diamondsuit_S^\sharp$ holds, then, κ $\vec{\mathcal{F}}$-reflects to S with $\diamondsuit$ (for some choice of $\vec{\mathcal{F}}$).

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Recalling Sun's theorem about ineffable sets, we infer:

In L, for every regular uncountable cardinal κ, and every stationary $S \subseteq \kappa$, there is a sequence of filters $\vec{\mathcal{F}}_S$ for which κ $\vec{\mathcal{F}}_S$-reflects to S with $\diamondsuit$.

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Corollary

In L, for all two stationary $S, S' \subseteq \kappa$, $=_S \hookrightarrow_c =^2_{S'}$.

Recalling Sun's theorem about ineffable sets, we infer:

In L, for every regular uncountable cardinal κ, and every stationary $S \subseteq \kappa$, there is a sequence of filters $\vec{\mathcal{F}}_S$ for which κ $\vec{\mathcal{F}}_S$-reflects to S with $\diamondsuit$.

$\diamondsuit^\sharp$ is a GDST's best friend

It takes more work, yet, all of the above generalize to arbitrary Σ^1_1-equivalence relations (projections of closed sets in $(\kappa^\kappa)^3$ or $(2^\kappa)^3$), and to Σ^1_1-quasi-orders (reflexive+transitive).

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Natural quasi-orders on κ^κ and 2^κ

- For $\eta, \xi \in \kappa^\kappa$, let $\eta \leq_S \xi$ iff $\{\alpha \in S \mid \eta(\alpha) > \xi(\alpha)\}$ is nonstationary.

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Theorem

$\diamondsuit^\sharp_S$ implies that $Q \hookrightarrow_c \subseteq_S$ for every Σ^1_1-quasi-order Q on κ^κ.
In particular, $\diamondsuit^\sharp_S$ implies that $=^2_S$ is a Σ^1_1-complete equivalence relation.

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By [Friedman-Wu-Zdomskyy, 2015], it is consistent with GCH that for a successor cardinal κ, for all stationary $S \subseteq \kappa$, $=^2_S$ is Δ^1_1.

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Theorem

$\diamondsuit^\sharp_S$ implies that $Q \hookrightarrow_c \subseteq_S$ for every Σ^1_1-quasi-order Q on κ^κ. In particular, $\diamondsuit^\sharp_S$ implies that $=^2_S$ is a Σ^1_1-complete equivalence relation, while $\diamondsuit^+_S$ does not.

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Comparing theories, revisited

Devlin's $\diamondsuit^\sharp_S$ may become an essential tool for the model theorist...

Theorem

Let T be a complete first-order countable relational theory. In any of the following cases, $\cong_T$ is Σ^1_1-complete:

- $\kappa=\lambda^+$, $\lambda^{<\lambda}=\lambda$, $\diamondsuit^\sharp_{\kappa\cap\mathrm{cof}(\lambda)}$ *holds and T is unstable;*
- κ *is inaccessible,* $\diamondsuit^\sharp_{\kappa\cap\mathrm{cof}((2^{\aleph_0})^+)}$ *and T is superstable with S-DOP;*
- κ *is $\aleph_0$-inaccessible,* $\diamondsuit^\sharp_{\kappa\cap\mathrm{cof}(\aleph_0)}$ *holds, and T is stable unsuperstable.*