Chain conditions, unbounded colorings and the C-sequence spectrum

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Bibliography

Most results are taken from the following joint papers with Chris Lambie-Hanson:

1. **Knaster and friends I: Closed colorings and precalibers**, *Algebra Universalis*, 79(4), Art. 90, 39 pp., 2018.
2. **Knaster and friends II: The C-sequence number**, *to be submitted*.
3. **Knaster and friends III: Subadditive colorings and stationarily layered posets**, *in preparation*.

Conventions

- κ and λ denote infinite cardinals;
- $\mathrm{Reg}(\kappa) := \{\theta < \kappa \mid \mathrm{cf}(\theta) = \theta \geq \aleph_0\}$;
- $E^{\kappa}_{\geq\chi} := \{\alpha < \kappa \mid \mathrm{cf}(\alpha) \geq \chi\}$ and $E^{\kappa}_{>\chi} := \{\alpha < \kappa \mid \mathrm{cf}(\alpha) > \chi\}$;
- $[A]^{\chi} := \{a \subseteq A \mid |a| = \chi\}$ and $[A]^{<\chi} := \{a \subseteq A \mid |a| < \chi\}$;
- For a, b, nonempty sets of ordinals, $a < b$ means that $\sup(a) < \min(b)$.

Chain conditions

Let $\mathbb{P} := \langle P, \leq \rangle$ denote a poset.

Definition

For a subset $X \subseteq P$, we write $\bigwedge X := \{z \in P \mid \forall x \in X(z \leq x)\}$.
We say that $x, y \in P$ are compatible iff $\bigwedge\{x, y\} \neq \emptyset$.

Definition

- $\mathbb{P}$ satisfies the κ-cc iff $\forall A \in [P]^\kappa\ \exists X \in [A]^2\ \bigwedge X \neq \emptyset$;
- $\mathbb{P}$ is κ-Knaster iff $\forall A \in [P]^\kappa\ \exists B \in [A]^\kappa\ \forall X \in [B]^2\ \bigwedge X \neq \emptyset$;
- $\mathbb{P}$ has precaliber κ iff $\forall A \in [P]^\kappa\ \exists B \in [A]^\kappa\ \forall X \in [B]^{<\omega}\ \bigwedge X \neq \emptyset$.
- $\mathbb{P}$ is κ-stationarily layered iff $\{Q \in [P]^{<\kappa} \mid \langle Q, \leq \rangle$ is a regular suborder of $\mathbb{P}\}$ is stationary in $[P]^{<\kappa}$.

The product order (aka, coordinatewise order)

Given posets $\langle P_1, \leq_1\rangle, \langle P_2, \leq_2\rangle$, consider their product $\langle P_1 \times P_2, \trianglelefteq\rangle$, where $(x, y) \trianglelefteq (x', y')$ iff $x \leq_1 x'$ and $y \leq_2 y'$. (Longer products are defined analogously.)

Question

Suppose that $\langle P_1, \leq_1\rangle, \langle P_2, \leq_2\rangle$ satisfy the κ-cc. Must their product satisfy the κ-cc?

Sufficient condition

If one of the posets is moreover κ-Knaster, then "yes".

Definition

Let $\mathcal{C}_\kappa$ denote the assertion that the product of any two κ-cc posets is again κ-cc.

Note: It suffices to consider squares

$\mathcal{C}_\kappa$ iff $\mathbb{P}^2$ is κ-cc for every κ-cc poset $\mathbb{P}$.

Basic facts

Fact 1. $\mathcal{C}_\kappa$ holds for $\kappa = \aleph_0$.

We moreover show that every κ-cc poset $\langle P, \leq \rangle$ is κ-Knaster.
Given $A \in [P]^\kappa$, define a coloring $c : [A]^2 \to 2$ via $c(x, y) = 1$ iff $\bigwedge\{x, y\} \neq \emptyset$.
By Ramsey's theorem, there exists $B \in [A]^\kappa$ which is c-homogeneous.
As $|B| = \kappa$ and $\langle P, \leq \rangle$ satisfies the κ-cc, there exists $X \in [B]^2$ with $\bigwedge X \neq \emptyset$. But B is c-homogeneous, and hence, for every $X \in [B]^2$, $\bigwedge X \neq \emptyset$, so that B is as sought. $\square$

Fact 2. $\mathcal{C}_\kappa$ holds for κ weakly compact.

κ is weakly compact iff $\kappa > \aleph_0$ and for every $c : [\kappa]^2 \to 2$, there exists $B \in [\kappa]^\kappa$ which is homogeneous for c. $\square$

Fact 3. $\mathcal{C}_\kappa$ holds for κ singular strong limit.

- Erdős and Tarski (1943): If κ is a singular cardinal and a poset $\mathbb{P}$ satisfies the κ-cc, then $\mathbb{P}$ satisfies the λ-cc for some $\lambda < \kappa$.
- Kurepa (1963): If $\mathbb{P}$ satisfies the λ^+-cc, then $\mathbb{P}^2$ satisfies the $(2^\lambda)^+$-cc. $\square$

The case $\kappa = \aleph_1$.

Question (Marczewski, 1947)

Is $\mathcal{C}_{\aleph_1}$ (aka, "productivity of the *ccc*") true?

Answers

- (Kurepa, 1952): $\mathcal{C}_{\aleph_1}$ entails Souslin's hypothesis.
- (Kunen;Rowbottom;Solovay;Hajnal-Juhász;Juhász, 1970's) $\mathrm{MA}_{\aleph_1}$ entails $\mathcal{C}_{\aleph_1}$;
- (Todorcevic-Velickovic, 1987) $\mathrm{MA}_{\aleph_1}$ iff every ccc poset has precaliber $\aleph_1$;
- (Roitman, 1979): After adding random/Cohen real, $\mathcal{C}_{\aleph_1}$ fails;
- (Fleissner, 1978): After adding κ many Cohen reals, there exists a ccc poset $\mathbb{P}$, such that $\mathbb{P}^2$ has antichain of size κ;
- (Galvin, 1980) after (Laver, unpublished): $\mathfrak{c} = \aleph_1$ refutes $\mathcal{C}_{\aleph_1}$.
- (Todorcevic, 1988): $\mathfrak{b} = \aleph_1$ refutes $\mathcal{C}_{\aleph_1}$.

Open problem

Is $\mathrm{MA}_{\aleph_1}$ equivalent to $\mathcal{C}_{\aleph_1}$?

The case $\kappa > \aleph_1$. Counterexamples in ZFC

Theorem (Todorcevic, 1985)

$\mathcal{C}_{\mathrm{cf}(\beth_{\alpha+1})}$ fails for every limit ordinal α.
Moreover, if λ is a cardinal for which there exists a linear order of size 2^λ with a dense subset of size λ, then $\mathcal{C}_\kappa$ fails, for $\kappa = \mathrm{cf}(2^\lambda)$.

Theorem (Todorcevic, 1986)

$\mathcal{C}_{\lambda^+}$ fails whenever λ singular, and $\theta^{\mathrm{cf}(\lambda)} < \lambda$ for all $\theta < \lambda$.

Theorem (Todorcevic, 1989)

$\mathcal{C}_{\lambda^+}$ fails whenever λ singular, and $2^{\mathrm{cf}(\lambda)} < \lambda$.

Theorem (Shelah, 1994)

$\mathcal{C}_{\lambda^+}$ fails whenever λ singular.

More counterexamples in ZFC

Theorem (Shelah, 1990–1997)

$\mathcal{C}_{\lambda^+}$ fails whenever λ is a regular cardinal $\geq \aleph_1$. Specifically:

- [Sh:280]: $\lambda > 2^{\aleph_0}$;
- [Sh:327]: $\lambda > \aleph_1$;
- [Sh:572]: $\lambda = \aleph_1$.

Corollary

$\mathcal{C}_\kappa$ fails for every successor cardinal $\kappa > \aleph_1$.

Conjecture (Todorcevic, 1980's)

For every regular cardinal $\kappa > \aleph_1$, $\mathcal{C}_\kappa$ iff κ is weakly compact.

Theorem (2014)

For every regular cardinal $\kappa > \aleph_1$, $\mathcal{C}_\kappa$ entails $(\kappa \text{ is weakly compact})^L$.
In fact, $\mathcal{C}_\kappa$ entails $\neg\Box(\kappa)$ and that every stationary subset of κ reflects.

Longer products and stronger chain conditions

Shortly after our work on Todorcevic's conjecture, Lücke and his colleagues addressed analogous questions involving stronger variations of the κ-cc. We mention three results:

Characterization theorem (Cox and Lücke, 2016)

For every regular uncountable cardinal κ:
κ is weakly compact iff every κ-cc poset is moreover κ-stationarily layered.

Non-characterization theorem (Cox and Lücke, 2016)

Suppose κ is weakly compact. In some cofinality-preserving forcing extension:
For every $\theta < \kappa$, the class of κ-Knaster posets is closed under θ-support products, yet, κ is not weakly compact.

Theorem (Lambie-Hanson and Lücke, 2018)

Suppose $\theta < \kappa$ are infinite and regular. If the class of κ-Knaster posets is closed under θ-support products, then $\neg\square(\kappa)$, so that (κ is weakly comapct)^L.

How to cook up a counterexample

Hereafter, κ denotes a regular uncountable cardinal.

Galvin (1980) gave a consistent construction of an anti-Ramsey coloring $c : [\kappa]^2 \to 2$ from which he derived a κ-cc poset whose square is not κ-cc.
In 1997, Shelah constructed a ZFC example of such a coloring for $\kappa = \aleph_2$.

Lambie-Hanson and Lücke (2018) gave a consistent construction of non-special κ-tree from which they derived a κ-Knaster poset whose infinite power is not κ-cc.
They proved that such a tree exists, assuming $\square(\kappa)$.

We would like to obtain the conclusions of Lambie-Hanson and Lücke from ZFC, e.g., getting a ZFC example of an $\aleph_2$-Knaster poset whose ω^{th}-power is not $\aleph_2$-cc.

For this, let us revisit Galvin's approach.

Colorings

From a coloring $c : [\kappa]^2 \to \theta$ with $\theta \in \mathrm{Reg}(\kappa)$, we derive posets:

- $\mathbb{P} := \{(x, i) \mid x \in [\kappa]^{<\omega}, c``[x]^2 \subseteq \{i\}\}$;

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Key feature

- $\mathbb{P}^2$ fails to have the κ-cc;
- $\mathbb{Q}^\theta$ fails to have the κ-cc.

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Key feature

- $\mathbb{P}^2$ fails to have the κ-cc, e.g., $\{\langle(\{\alpha\}, 0), (\{\alpha\}, 1)\rangle \mid \alpha < \kappa\}$.
- $\mathbb{Q}^\theta$ fails to have the κ-cc.

About $\mathbb{P}^2$.

For $\alpha < \beta < \kappa$ and $i := c(\alpha, \beta)$, $(\{\alpha\}, 1 - i)$ and $(\{\beta\}, 1 - i)$ are $\mathbb{P}$-incompatible. $\square$

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Key feature

- $\mathbb{P}^2$ fails to have the κ-cc, e.g., $\{\langle(\{\alpha\}, i) \mid i < 2\rangle \mid \alpha < \kappa\}$.
- $\mathbb{Q}^\theta$ fails to have the κ-cc, e.g., $\{\langle(\{\alpha\}, i) \mid i < \theta\rangle \mid \alpha < \kappa\}$.

About $\mathbb{P}^2$.

For $\alpha < \beta < \kappa$ and $i := c(\alpha, \beta)$, $(\{\alpha\}, 1 - i)$ and $(\{\beta\}, 1 - i)$ are $\mathbb{P}$-incompatible. $\square$

About $\mathbb{Q}^\theta$.

For $\alpha < \beta < \kappa$ and $i := c(\alpha, \beta)$, $(\{\alpha\}, i + 1)$ and $(\{\beta\}, i + 1)$ are $\mathbb{Q}$-incompatible. $\square$

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Key feature

- $\mathbb{P}^2$ fails to have the κ-cc;
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The heart of the matter is to construct c for which the corresponding $\mathbb{P}$ be κ-cc, or $\mathbb{Q}^\tau$ be κ-Knaster for all $\tau < \theta$.

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The heart of the matter is to construct c for which the corresponding $\mathbb{P}$ be κ-cc, or $\mathbb{Q}^\tau$ be κ-Knaster for all $\tau < \theta$.
By a simple reverse-engineering process, one arrives at a reformulation of these features in the language of the coloring c.
The poset $\mathbb{P}$ was analyzed by Galvin. Today, we shall focus on the poset $\mathbb{Q}$.

Unbounded functions

Suppose $\mathbb{Q} := \{(x,i) \mid x \in [\kappa]^{<\omega}, c``[x]^2 \cap i = \emptyset\}$ is derived from $c : [\kappa]^2 \to \theta$. Assuming $\theta \in \mathrm{Reg}(\kappa)$, $\mathbb{Q}$ is κ-Knaster iff it has precaliber κ iff c witnesses $\mathrm{U}(\kappa,\theta)$:

Definition

$\mathrm{U}(\kappa,\theta)$ asserts that there exists a coloring $c : [\kappa]^2 \to \theta$ such that for every family $\mathcal{A} \subseteq [\kappa]^{<\omega}$ consisting of κ-many pairwise disjoint sets, and every $i < \theta$, there is $\mathcal{B} \in [\mathcal{A}]^\kappa$ such that $\min(c[a \times b]) \geq i$ for every pair $a < b$ from $\mathcal{B}$.

There is also a χ-closed variation: $\{(x,i) \mid x \in [\kappa]^{<\chi}, c``[x]^2 \cap i = \emptyset\}$. For this, we need:

Definition

$\mathrm{U}(\kappa,\theta,\chi)$ asserts there is a coloring $c : [\kappa]^2 \to \theta$ such that for every $\chi' < \chi$, every family $\mathcal{A} \subseteq [\kappa]^{\chi'}$ consisting of κ-many pairwise disjoint sets, and every $i < \theta$, there is $\mathcal{B} \in [\mathcal{A}]^\kappa$ such that $\min(c[a \times b]) \geq i$ for every pair $a < b$ from $\mathcal{B}$.

The coloring axiom

Definition

$\mathsf{U}(\kappa, \mu, \theta, \chi)$ asserts there is a coloring $c : [\kappa]^2 \rightarrow \theta$ such that for every $\chi' < \chi$, every family $\mathcal{A} \subseteq [\kappa]^{\chi'}$ consisting of κ-many pairwise disjoint sets, and every $i < \theta$, there is $\mathcal{B} \in [\mathcal{A}]^{\mu}$ such that $\min(c[a \times b]) \geq i$ for every pair $a < b$ from $\mathcal{B}$.

Note that $\mathrm{Pr}_1(\kappa, \kappa, \theta, \chi)$ entails $\mathsf{U}(\kappa, 2, \theta, \chi)$.

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Proposition

Suppose $\chi, \theta \in \mathrm{Reg}(\kappa)$ *and that* κ *is* $({<}\chi)$*-inaccessible. For every coloring* $c : [\kappa]^2 \to \theta$ *witnessing* $\mathsf{U}(\kappa, \mu, \theta, \chi)$*, the corresponding poset* $\mathbb{Q}$ *satisfies the following:*

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- $\mathbb{Q}^{\theta}$ *is not* κ*-cc;*

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Suppose $\chi, \theta \in \mathrm{Reg}(\kappa)$ and that κ is $({<}\chi)$-inaccessible. For every coloring $c : [\kappa]^2 \to \theta$ witnessing $\mathsf{U}(\kappa,\mu,\theta,\chi)$, the corresponding poset $\mathbb{Q}$ satisfies the following:

- *$\mathbb{Q}^{\theta}$ is not κ-cc;*
- *if $\mu = 2$, then $\mathbb{Q}^{\tau}$ is κ-cc for all $\tau < \min\{\chi, \theta\}$;*

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- *$\mathbb{Q}^{\theta}$ is not κ-cc;*
- *if $\mu = 2$, then $\mathbb{Q}^{\tau}$ is κ-cc for all $\tau < \min\{\chi, \theta\}$;*
- *if $\mu = \kappa$, then $\mathbb{Q}^{\tau}$ has precaliber κ for all $\tau < \min\{\chi, \theta\}$;*

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- $\mathbb{Q}^{\theta}$ *is not* κ*-cc;*
- *if* $\mu = 2$*, then* $\mathbb{Q}^{\tau}$ *is* κ*-cc for all* $\tau < \min\{\chi, \theta\}$*;*
- *if* $\mu = \kappa$*, then* $\mathbb{Q}^{\tau}$ *has precaliber* κ *for all* $\tau < \min\{\chi, \theta\}$*;*
- $\mathbb{Q}$ *is well-met and* χ*-directed-closed with greatest lower bounds.*

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Conjecture

For κ regular uncountable, κ is weakly compact iff $\mathrm{U}(\kappa, 2, \omega, 2)$ fails.

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For κ regular uncountable, κ is weakly compact iff $\mathrm{U}(\kappa, 2, \omega, 2)$ fails.
In other words, we ask whether the existence of a κ-Aronszajn tree gives rise to a coloring $c : [\kappa]^2 \to \omega$ with the property that $\sup(c\text{“}[A]^2) = \omega$ for every $A \in [\kappa]^{\kappa}$.

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Partial answer 1

The existence of a κ-Aronszajn tree with an ω-ascent path entails $\mathrm{U}(\kappa,2,\omega,\omega)$.

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Partial answer 1

The existence of a κ-Aronszajn tree with an ω-ascent path entails U$(\kappa, 2, \omega, \omega)$.

Partial answer 2 (with Todorcevic)

The existence of a coherent κ-Aronszajn tree entails U$(\kappa, 2, \omega, \omega)$ but not U$(\kappa, \kappa, \omega, \omega)$.

Inspecting the parameters

Definition

$\mathrm{U}(\kappa, \mu, \theta, \chi)$ asserts there is a coloring $c : [\kappa]^2 \to \theta$ such that for every $\chi' < \chi$, every family $\mathcal{A} \subseteq [\kappa]^{\chi'}$ consisting of κ-many pairwise disjoint sets, and every $i < \theta$, there is $\mathcal{B} \in [\mathcal{A}]^{\mu}$ such that $\min(c[a \times b]) \geq i$ for every pair $a < b$ from $\mathcal{B}$.

About the second parameter

- $\mathrm{U}(\kappa, 2, \theta, \chi)$ iff $\mathrm{U}(\kappa, \omega, \theta, \chi)$;
- Suppose $c \models \mathrm{U}(\kappa, 2, \theta, \chi)$. If c is closed, then $c \models \mathrm{U}(\kappa, \kappa, \theta, \chi)$.

Definition

$c : [\kappa]^2 \to \theta$ is closed iff $\{\alpha < \beta \mid c(\alpha, \beta) \leq i\}$ is closed below β for all $\beta < \kappa$, $i < \theta$.

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$\mathrm{U}(\kappa,\mu,\theta,\chi)$ asserts there is a coloring $c:[\kappa]^2\rightarrow\theta$ such that for every $\chi'<\chi$, every family $\mathcal{A}\subseteq[\kappa]^{\chi'}$ consisting of κ-many pairwise disjoint sets, and every $i<\theta$, there is $\mathcal{B}\in[\mathcal{A}]^{\mu}$ such that $\min(c[a\times b])\geq i$ for every pair $a<b$ from $\mathcal{B}$.

About the third parameter

- $\mathrm{U}(\kappa,\kappa,\kappa,\kappa)$ holds;
- $\mathrm{U}(\kappa,\mu,\theta,\chi)$ iff $\mathrm{U}(\kappa,\mu,\mathrm{cf}(\theta),\chi)$;

Therefore, hereafter, we shall focus on $\theta\in\mathrm{Reg}(\kappa)$.

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Definition

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About the third parameter

- $\mathrm{U}(\kappa,\kappa,\kappa,\kappa)$ holds;
- $\mathrm{U}(\kappa,\mu,\theta,\chi)$ iff $\mathrm{U}(\kappa,\mu,\mathrm{cf}(\theta),\chi)$;
- Lack of monotonicity: If λ is the singular limit of strongly compact cardinals, then, for every $\theta\leq\lambda$, $\mathrm{U}(\lambda^+,\lambda^+,\theta,\lambda)$ iff $\mathrm{cf}(\theta)=\mathrm{cf}(\lambda)$.

Inspecting the parameters

Definition

$\mathrm{U}(\kappa, \mu, \theta, \chi)$ asserts there is a coloring $c : [\kappa]^2 \to \theta$ such that for every $\chi' < \chi$, every family $\mathcal{A} \subseteq [\kappa]^{\chi'}$ consisting of κ-many pairwise disjoint sets, and every $i < \theta$, there is $\mathcal{B} \in [\mathcal{A}]^{\mu}$ such that $\min(c[a \times b]) \geq i$ for every pair $a < b$ from $\mathcal{B}$.

About the fourth parameter

- $\mathrm{U}(\kappa, \kappa, \theta, 3)$ iff $\mathrm{U}(\kappa, \kappa, \theta, \omega)$;
- $\mathrm{U}(\lambda^+, 2, \theta, 2)$ iff $\mathrm{U}(\lambda^+, 2, \theta, \mathrm{cf}(\lambda))$;

The above is optimal: If λ is the limit of strongly compact cardinals, $\theta \in \mathrm{Reg}(\lambda)$ with $\theta \neq \mathrm{cf}(\lambda)$, then $\mathrm{U}(\lambda^+, 2, \theta, \chi)$ holds for $\chi := \mathrm{cf}(\lambda)$, but fails for $\chi := \mathrm{cf}(\lambda)^+$.

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- $\mathrm{U}(\lambda^+, 2, \theta, 2)$ iff $\mathrm{U}(\lambda^+, 2, \theta, \mathrm{cf}(\lambda))$;
- There are κ, θ and colorings c, $c \models \mathrm{U}(\kappa, \kappa, \theta, 2)$, but $c \not\models \mathrm{U}(\kappa, 2, \theta, 3)$;
- If there is a closed witness to $\mathrm{U}(\lambda^+, \lambda^+, \theta, 2)$, then there is for $\mathrm{U}(\lambda^+, \lambda^+, \theta, \mathrm{cf}(\lambda))$.

Further findings

Theorem

For every regular λ *and* $\theta \in \mathrm{Reg}(\lambda^+)$, *there is* $c : [\lambda^+]^2 \to \theta$ *witnessing* $\mathsf{U}(\lambda^+, \lambda^+, \theta, \lambda)$ *which is moreover closed.*

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In case you wondered

The corresponding tree $\mathcal{T}(c) := \{c(\cdot, \gamma) \restriction \beta \mid \beta \leq \gamma < \lambda^+\}$ may consistently be special λ^+-Aronszajn tree / almost Souslin λ^+-Aronszajn tree.

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Corollary

There exists an $\aleph_2$-Knaster poset whose ω^{th}-power is not $\aleph_2$-cc.

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More generally

Suppose that $\theta \le \chi \le \lambda$ are regular, with $\lambda^{<\chi} = \lambda$. Then $\exists \chi$-directed-closed poset $\mathbb{Q}$:

- *$\mathbb{Q}^\tau$ has precaliber λ^+ for all $\tau < \theta$;*
- *$\mathbb{Q}^\theta$ is not λ^+-cc.*

Further findings

Theorem

For every regular λ *and* $\theta \in \mathrm{Reg}(\lambda^+)$*, there is* $c : [\lambda^+]^2 \to \theta$ *witnessing* $\mathsf{U}(\lambda^+, \lambda^+, \theta, \lambda)$ *which is moreover closed.*

Corollary

There exists an $\aleph_2$*-Knaster poset whose* ω^{th}*-power is not* $\aleph_2$*-cc.*
CH entails a σ*-directed-closed* $\aleph_2$*-Knaster poset whose* ω^{th}*-power is not* $\aleph_2$*-cc.*

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Open problem

Does CH entail a σ-closed $\aleph_2$-cc poset whose square is not $\aleph_2$-cc?

Further findings (cont.)

Theorem

For every singular λ and $\theta \in \mathrm{Reg}(\lambda)$, any of the following entail the existence of a closed witness to $\mathrm{U}(\lambda^+, \lambda^+, \theta, \mathrm{cf}(\lambda))$:

- $2^\lambda = \lambda^+$;
- $\mathrm{Refl}({<}\,\mathrm{cf}(\lambda), \lambda^+)$ *fails;*
- $\theta = \omega$ *or* $\theta = \mathrm{cf}(\lambda)$;
- $\theta < \nu < \nu^+ = \mathrm{cf}(\lambda)$;
- $\theta < \mathrm{cf}(\lambda)$ *and* $\mathrm{cf}(\mathrm{NS}_{\mathrm{cf}(\lambda)}, \subseteq) < \lambda$.

Further findings (cont.)

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Corollary

If the class of κ*-Knaster posets is closed under* ω *powers, then* κ *is inaccessible.*

Further findings (cont.)

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Theorem

For every $\theta, \chi \in \mathrm{Reg}(\kappa)$, any of the following entails a closed witness to $\mathrm{U}(\kappa, \kappa, \theta, \chi)$:

- $\square(\kappa, {<}\omega)$ *or* $\square^{\mathrm{ind}}(\kappa, \theta)$;
- $\exists$*stationary* $S \subseteq E^\kappa_{\geq\chi}$ *with* $S \cap \alpha$ *nonstationary for all* $\alpha \in E^\kappa_{>\omega}$;
- $\exists$*stationary* $S \subseteq E^\kappa_{\geq\chi}$ *with* $S \cap \alpha$ *nonstationary for all* $\alpha \in \mathrm{Reg}(\kappa)$, *and* κ *is inacc.*

A new cardinal invariant

Theorem (Todorcevic, 1987)

For every strongly inaccessible cardinal κ, the following are equivalent:

1. κ is weakly compact;
2. For every C-sequence $\langle C_\beta \mid \beta < \kappa \rangle$, there exist $\Delta \in [\kappa]^\kappa$ and $b : \kappa \to \kappa$ such that $\Delta \cap \alpha = C_{b(\alpha)} \cap \alpha$ for every $\alpha < \kappa$.

Recall

$\langle C_\beta \mid \beta < \kappa \rangle$ is a C-sequence iff each C_β is closed subset of β with $\sup(C_\beta) = \sup(\beta)$.

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The cardinal invariant that we introduce suggests a way to measure how far an inaccessible cardinal κ is from being weakly compact, though, as we will see, it is of interest for successor cardinals as well.

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If κ is weakly compact, then let $\chi(\kappa) := 0$.

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Definition (The C-sequence number of κ)

If κ is weakly compact, then let $\chi(\kappa) := 0$.
Otherwise, let $\chi(\kappa)$ denote the least $\chi \leq \kappa$ s.t., for every C-sequence $\langle C_\beta \mid \beta < \kappa \rangle$, there exist $\Delta \in [\kappa]^\kappa$ and $b : \kappa \to [\kappa]^\chi$ with $\Delta \cap \alpha \subseteq \bigcup_{\beta \in b(\alpha)} C_\beta$ for all $\alpha < \kappa$.

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Note that $\chi(\kappa)$ is well-defined. In fact, $\chi(\kappa) \leq \sup(\mathrm{Reg}(\kappa))$.

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A new cardinal invariant

Todorcevic's analysis of *the number of steps* function readily establishes the following.

The C-sequence number and yoU

$\mathrm{U}(\kappa,\kappa,\omega,\chi(\kappa))$ *holds, as witnessed by the closed function* ρ_2.

However, it is consistent that $\mathrm{U}(\kappa,\kappa,\omega,\chi)$ holds with $\chi \gg \chi(\kappa)$.

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However, it is consistent that U$(\kappa, \kappa, \omega, \chi)$ holds with $\chi \gg \chi(\kappa)$.

Corollary

If the class of κ-Knaster posets is closed under taking ω powers, then $\chi(\kappa) < \omega$.

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A new cardinal invariant

Questions

- Is "$\chi(\kappa) < \omega$" a large cardinal property?
- How about "$\chi(\kappa) < \sup(\mathrm{Reg}(\kappa))$"?
- Could $\chi(\kappa)$ be singular?

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Increasing the C-sequence number

Kunen (1978) showed that by forcing over a model with a weakly compact cardinal κ, one obtains a model V having a κ-Souslin tree $\mathbb{S}$ such that $V^{\mathbb{S}} \models \kappa$ is weakly compact.

Proposition

In Kunen's model, $\chi(\kappa) = 1$.

Proof. The κ-Souslin tree witnesses that κ is not weakly compact, so $\chi(\kappa) \neq 0$.
Now, let $\vec{C} = \langle C_\beta \mid \beta < \kappa \rangle$ be an arbitrary C-sequence.
In $V^{\mathbb{S}}$, $\vec{C}$ is a C-sequence over a weakly compact cardinal κ, and hence there is $\Delta \in [\kappa]^\kappa$ and $b : \kappa \to \kappa$ such that $\Delta \cap \alpha = C_{b(\alpha)} \cap \alpha$ for each $\alpha < \kappa$.
Clearly, Δ is a club. As $\mathbb{S}$ is κ-cc, there is a club $D \subseteq \kappa$ in V, with $D \subseteq \Delta$.
Then $D \cap \alpha \subseteq C_{b(\alpha)} \cap \alpha$ for each $\alpha < \kappa$. $\square$

Theorem

Suppose $\chi(\kappa) = 0$. For every $\theta \in \mathrm{Reg}(\kappa^+)$, there is a cofinality-preserving forcing extension in which κ remains strongly inaccessible, and $\chi(\kappa) = \theta$.

Increasing the C-sequence number (cont.)

Observation

$\mathrm{cf}(\lambda) \leq \chi(\lambda^+) \leq \lambda$.

Theorem

If λ is a singular limit of supercompact cardinals, then $\chi(\lambda^+) = \mathrm{cf}(\lambda)$.

Theorem

If λ is a singular limit of supercompact cardinals, and $\theta \in \mathrm{Reg}(\lambda)$ with $\theta \geq \mathrm{cf}(\lambda)$, then, in some cofinality-preserving forcing extension, $\chi(\lambda^+) = \theta$.

Theorem

$\chi(\aleph_{\omega+1}) = \aleph_\omega$ is consistent, and so is $\chi(\aleph_{\omega+1}) = \omega$.[1]

[1] The latter assumes the consistency of a supercompact.

How large

Theorem

1. $\mathrm{Refl}({<}\omega, E^{\kappa}_{>\chi(\kappa)})$;
2. *If* $\chi(\kappa) < \omega$, *then* $\chi(\kappa) \in \{0, 1\}$;
3. *If* κ *is inaccessible and* $\chi(\kappa) < \kappa$, *then* κ *is* ω*-Mahlo;*
4. *If* $\chi(\kappa) = 1$, *then* $\square(\kappa, {<}\mu)$ *fails for all* $\mu < \kappa$;
5. *If* $\chi(\kappa) = 1$, *then, for every sequence* $\langle S_i \mid i < \kappa \rangle$ *of stationary subsets of* κ, *there exists an inaccessible* $\beta < \kappa$ *such that* $S_i \cap \beta$ *is stationary in* β *for all* $i < \beta$.

Corollary

- *In* L, *either* $\chi(\kappa) = 0$ *or* $\chi(\kappa) = \sup(\mathrm{Reg}(\kappa))$;
- $\square(\kappa, {<}\omega)$ *entails* $\chi(\kappa) = \sup(\mathrm{Reg}(\kappa))$;
- *If* $\chi(\kappa) = 1$, *then* κ *is greatly Mahlo.*
- *If the class of* κ*-Knaster posets is closed under* ω *powers, then* κ *is greatly Mahlo.*

The C-sequence spectrum

Definition

For a C-sequence $\vec{C} = \langle C_\beta \mid \beta < \kappa \rangle$, let $\chi(\vec{C})$ denote the least cardinal $\chi \leq \kappa$ such that there exist $\Delta \in [\kappa]^\kappa$ and $b : \kappa \to [\kappa]^\chi$ with $\Delta \cap \alpha \subseteq \bigcup_{\beta \in b(\alpha)} C_\beta$ for every $\alpha < \kappa$.

Definition

$\mathrm{Cspec}(\kappa) := \{\chi(\vec{C}) \mid \vec{C}$ is a C-sequence over $\kappa\} \setminus \omega$.

Theorem

1. *If* $\mathrm{Cspec}(\kappa) \neq \emptyset$, *then* $\min(\mathrm{Cspec}(\kappa)) = \omega$ *and* $\chi(\kappa) = \max(\mathrm{Cspec}(\kappa))$*;*
2. $\chi \in \mathrm{Cspec}(\kappa) \implies \mathrm{cf}(\chi) \in \mathrm{Cspec}(\kappa)$, *but not* $\Longleftarrow$ *.*

Open problem

Is $\mathrm{Cspec}(\kappa)$ an interval? Is it a closed set?
Is every limit uncountable cardinal in $\mathrm{Cspec}(\kappa)$ an accumulation point of $\mathrm{Cspec}(\kappa)$?

Unexpected equivalency

Theorem

For every $\theta \in \mathrm{Reg}(\kappa)$, the following are equivalent:

- $\theta \in \mathrm{Cspec}(\kappa)$;
- *There exists a closed witness to* $\mathrm{U}(\kappa, \kappa, \theta, \theta)$.

The forward implication also works for θ singular; the backward does not.

Corollary

- *If κ is a successor of a regular cardinal, then* $\mathrm{Reg}(\kappa) \subseteq \mathrm{Cspec}(\kappa)$;
- *If κ is a non-Mahlo inaccessible, then* $\mathrm{Reg}(\kappa) \subseteq \mathrm{Cspec}(\kappa)$;
- *If $\square(\kappa, {<}\omega)$ holds, then* $\mathrm{Reg}(\kappa) \subseteq \mathrm{Cspec}(\kappa)$;
- *If $E^{\kappa}_{\geq\chi}$ admits a non-reflecting stationary subset, then* $\mathrm{Reg}(\chi^{+}) \subseteq \mathrm{Cspec}(\kappa)$.

Conjectures

1. If κ is inaccessible and $1 < \chi(\kappa) < \kappa$, $\exists\kappa$-Aronszajn tree with a $\chi(\kappa)$-ascent path.
2. Any instance $\mathsf{U}(\kappa, \kappa, \ldots)$ may be witnessed by a closed coloring.
3. If $\chi(\kappa) = 1$, then, there exists a coherent κ-Aronszajn tree.
4. If $\chi(\kappa) = 1$, then, in some set-forcing extension, $\chi(\kappa) = 0$.
5. If $\chi(\kappa)$ is singular, then $\mathrm{cf}(\chi(\kappa)) = \mathrm{cf}(\sup(\mathrm{Reg}(\kappa)))$.
6. $\mathrm{Reg}(\mathrm{cf}(\lambda)^+) \subseteq \mathrm{Cspec}(\lambda^+)$ for every singular λ.
7. For all $\theta, \chi \in \mathrm{Cspec}(\kappa)$, $\mathsf{U}(\kappa, \kappa, \theta, \chi)$ holds.