Transformations of the transfinite plane

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The results presented here are from a joint work with Jing Zhang.

Conventions

- κ denotes a regular uncountable cardinal;
- θ is some cardinal, $2 \leq \theta \leq \kappa$;
- $E^\kappa_\theta := \{\alpha < \kappa \mid \mathrm{cf}(\alpha) = \theta\}$; $E^\kappa_{\neq\theta}$ and $E^\kappa_{\geq\theta}$ are defined similarly;
- $[\kappa]^2 := \{(\alpha, \beta) \mid \alpha < \beta < \kappa\}$.

For sets of ordinals X, Y, we consider the half-graph

$$X \circledast Y := \{(x, y) \in X \times Y \mid x < y\}.$$

Coloring countable infinite sets

Theorem (Ramsey, 1930)

For every partition of a complete infinite graph into two cells, there exists an infinite set of vertices on which the induced subgraph is either complete or empty.
Equivalently: Any coloring $c : [\omega]^2 \to 2$ *admits an infinite homogeneous.*

X is *homogeneous* for c iff $c \restriction [X]^2$ is constant.

Notation

$\omega \to (\omega)^2_2$.

Alternative notation: infinite $\to$ (infinite)$^2_2$.

Applications of Ramsey's theorem

Corollary (1)

Any infinite partial order $\mathbb{P} = (P, \leq)$ either admits an infinite chain or an infinite (weak) antichain.

Proof

Define a coloring $c : [P]^2 \to 2$ via $c(x, y) := 0$ iff x and y are comparable.

- A 0-homogeneous set is a chain.
- A 1-homogeneous set is a weak antichain. $\square$

Applications of Ramsey's theorem, cont.

Corollary (2)

For partial orders $\mathbb{P}$ and $\mathbb{Q}$ each admitting no infinite antichain, the product poset $\mathbb{P} \times \mathbb{Q}$ admits no infinite antichain.

In the product, (p', q') extends (p, q) iff p' extends p and q' extends q.

Proof

Otherwise, take an infinite antichain in the product $\{(p_n, q_n) \mid n < \omega\}$.
Define $c : [\omega]^2 \to 2$ via $c(n, m) := 0$ iff p_n is incomparable with p_m.

- A 0-homogeneous set gives rise to an infinite antichain in $\mathbb{P}$.
- A 1-homogeneous set gives rise to an infinite antichain in $\mathbb{Q}$. $\square$

Applications of Ramsey's theorem, cont.

Corollary (3)

For every infinite Abelian group $\mathbb{G} = (G, +)$ *and a coloring* $d : G \to 2$*, there exist an infinite* $X \subseteq G$ *and* $i < 2$ *such that, for all* $x \neq y$ *from* X*,* $d(x + y) = i$*.*

Proof

Define a coloring of pairs $c : [G]^2 \to 2$ via $c(x, y) := d(x + y)$. □

Notation

$\mathbb{G} \to (\text{infinite})^{\mathrm{FS}_2}_2$.

I.e., there is an infinite $X \subseteq G$ such that $d \restriction \mathrm{FS}_2(X)$ is constant.
$\mathrm{FS}_n(X) := \{\sum A \mid A \subseteq X, |A| = n\}$, $\mathrm{FS}(X) := \{\sum A \mid A \subseteq X, |A| < \infty\}$.

Coloring countable structures

Theorem (Hindman, 1974)

$\mathbb{N} \to (\mathit{infinite})^{\mathrm{FS}}_2$. *For any coloring* $d : \mathbb{N} \to 2$, *there exists* $\{x_n\}_{n=0}^{\infty} \subseteq \mathbb{N}$ *which is homogeneous with respect to finite sums. That is,*

$$d(x_{i_1} + \cdots + x_{i_n}) = d(x_{j_1} + \cdots + x_{j_m})$$

for all $i_1 < \ldots < i_n$ *and* $j_1 < \ldots < j_m$.

Generalized Hindman theorem

For all Abelian groups $\mathbb{G}$ and positive integers k, $\mathbb{G} \to (\text{infinite})^{\mathrm{FS}}_k$.

May replace Abelian groups by commutative cancellative semigroups.

Up to the uncountable

Ramsey statements at uncountable cardinals

Theorem (Sierpiński, 1933)

There is a weakening of the real ordering with no uncountable chains or uncountable (weak) antichains. So, $\mathbb{R} \nrightarrow (\omega_1)^2_2$*. In particular,* $\omega_1 \nrightarrow (\omega_1)^2_2$*.*

Theorem (Kurepa, 1952)

A Souslin tree is a poset $\mathbb{P}$ *with no uncountable antichains, yet,* $\mathbb{P} \times \mathbb{P}$ *does admit an uncountable antichain.*

Theorem (Galvin, 1980. Todorčević, 1986)

Assuming CH (in fact, $\mathfrak{b} = \omega_1$*), for every* $n > 0$*, there is a poset* $\mathbb{P}$ *such that* $\mathbb{P}^n$ *has no uncountable antichains, but* $\mathbb{P}^{n+1}$ *does have.*

Hindman statements for the real line

Quickly after a preprint of Hindman, Leader and Strauss was circulated:

Theorem (Komjáth, 2016. Soukup-Weiss, 2016)

$\mathbb{R} \nrightarrow (\omega_1)^{\mathrm{FS}_2}_2$. *I.e., $\exists$coloring $d : \mathbb{R} \to 2$ such that for every uncountable set of reals X and every $i < 2$, there are $x \neq y$ in X with $d(x+y) = i$.*

Improving upon a result of Galvin and Shelah from 1973:

Theorem (with Fernández-Bretón, 2017)

$\mathbb{R} \nrightarrow [\mathfrak{c}]^{\mathrm{FS}_2}_\omega$. *I.e., $\exists$coloring $d : \mathbb{R} \to \omega$ such that for every $X \subseteq \mathbb{R}$ with $|X| = |\mathbb{R}|$ and every $i < \omega$, there are $x \neq y$ in X with $d(x+y) = i$.*

Improving upon Komjáth-Leader-Russell-Shelah-Soukup-Vidnyánszky:

Theorem (Zhang, 2020)

For every coloring $d : \mathbb{R} \to 2$, there exists an infinite set of reals X such that, for all $x \neq y$ in X, $d(x+y) = d(x+x)$.

Hindman statements for general uncountable groups

Theorem (with Fernández-Bretón, 2017)

For class many regular cardinals κ (including $\aleph_1, \aleph_2, \ldots$), for every Abelian group $\mathbb{G}$ of size κ, $\mathbb{G} \nrightarrow [\kappa]^{\mathsf{FS}_2}_{\kappa}$.

In [Sh:69], Shelah proved that there exists a group of size $\aleph_1$ having no proper uncountable subgroups.

It follows from our theorem that for any Abelian group $\langle G, +\rangle$ of size $\aleph_1$, there is an unary function $d : G \to G$ such that $\langle G, +, d\rangle$ has no proper uncountable substructures.

Ramsey vs. Hindman

Definition

- $\kappa \nrightarrow [\kappa]^2_\theta$ asserts that there exists a coloring $c : [\kappa]^2 \to \theta$ such that for any $X \subseteq \kappa$ of size κ, $c``[X]^2 = \theta$;
- $\mathbb{G} \nrightarrow [\kappa]^{\mathrm{FS}_2}_\theta$ asserts that there exists a coloring $d : G \to \theta$ such that for any $X \subseteq G$ of size κ, $d``\mathrm{FS}_2(X) = \theta$;

By the trick of letting $c(x, y) := d(x + y)$, we infer that $\mathbb{G} \nrightarrow [\kappa]^{\mathrm{FS}_2}_\theta$ for an Abelian group $\mathbb{G}$ of size κ implies $\kappa \nrightarrow [\kappa]^2_\theta$.
It would have been great if we could reverse the arrows and reduce the additive problem into the classic problem.

Question

Does $\kappa \nrightarrow [\kappa]^2_\theta$ imply that for any Abelian group $\mathbb{G}$ of size κ, $\mathbb{G} \nrightarrow [\kappa]^{\mathrm{FS}_2}_\theta$?

Ramsey vs. Hindman, cont.

Theorem (with Fernández-Bretón, 2017, rephrased)

The following are equivalent:

- $\kappa \nrightarrow [\kappa]^2_\theta$;
- $\mathbb{G} \nrightarrow [\kappa]^{\mathsf{FS}_2}_\theta$ *for any commutative cancellative semigroup* $\mathbb{G}$ *of size* κ,

provided that there exists a transformation $\mathbf{t} : [\kappa]^2 \to [\kappa]^2$ *with the property that for every family* $\mathcal{A}$ *consisting of* κ*-many pairwise disjoint finite subsets of* κ*, there is a cofinal* $A \subseteq \kappa$ *such that for all* $\alpha < \beta$ *from* A*, there are* $a < b$ *from* $\mathcal{A}$ *with* $\mathbf{t}[a \times b] = \{(\alpha, \beta)\}$.

In [Rinot, 2012], by extending works of Eisworth, such a transformation was shown to exist for any κ which is the successor of a singular cardinal.

Transformations of $[\kappa]^2$

Definition (with Zhang, 2020)

$\mathrm{P}\ell_1(\kappa)$ asserts the existence of a transformation $\mathbf{t} : [\kappa]^2 \rightarrow [\kappa]^2$ satisfying:

1. for every $(\alpha, \beta) \in [\kappa]^2$, if $\mathbf{t}(\alpha, \beta) = (\alpha^*, \beta^*)$, then $\alpha^* \leq \alpha < \beta^* \leq \beta$;
2. for every family $\mathcal{A}$ consisting of κ-many pairwise disjoint finite subsets of κ, there is a stationary $S \subseteq \kappa$ such that, for every $\alpha^* < \beta^*$ from S, there are $a < b$ from $\mathcal{A}$ with $\mathbf{t}[a \times b] = \{(\alpha^*, \beta^*)\}$.

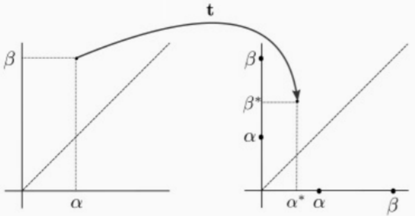
The yellow requirements are not needed for the problem stated before, but are useful in studying the problem of productivity of the chain condition:

Theorem (with Zhang, 2020)

If $\mathrm{P}\ell_1(\kappa)$ holds, then, for every $n > 0$, there is a poset $\mathbb{P}$ such that $\mathbb{P}^n$ has no antichains of size κ, but $\mathbb{P}^{n+1}$ does.

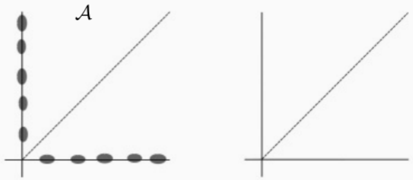
P$\ell_1(\kappa)$

(i)



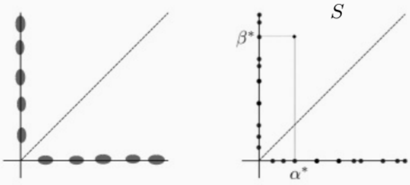
for every $(\alpha, \beta) \in [\kappa]^2$, if $\mathbf{t}(\alpha, \beta) = (\alpha^*, \beta^*)$, then $\alpha^* \leq \alpha < \beta^* \leq \beta$;

(ii)



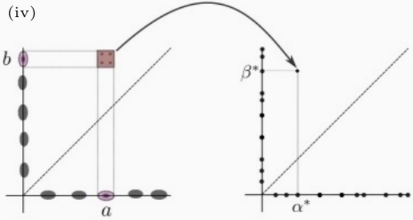
for every family $\mathcal{A}$ consisting of κ many pairwise disjoint finite subsets of κ,

(iii)



there exists a stationary $S \subseteq \kappa$ such that, for every pair $\alpha^* < \beta^*$ of elements of S,

(iv)



there exists a pair $a < b$ of elements of $\mathcal{A}$ with $\mathbf{t}[a \times b] = \{(\alpha^*, \beta^*)\}$.

Baby case: Squares vs. Rectangles

Definition

- $\kappa \nrightarrow [\mathrm{Stat}(\kappa)]^2_\theta$ asserts that there is a coloring $c : [\kappa]^2 \rightarrow \theta$ such that for any stationary $S \subseteq \kappa$, $c``[S]^2 = \theta$;
- $\kappa \nrightarrow [\kappa;\kappa]^2_\theta$ asserts that there is a coloring $c : [\kappa]^2 \rightarrow \theta$ such that for any cofinal $X, Y \subseteq \kappa$, $c``X \circledast Y = \theta$.

Theorem

- *(Sierpiński, 1933)* $\omega_1 \nrightarrow [\omega_1]^2_2$
- *(Erdős-Hajnal-Rado, 1965)* $\omega_1 \nrightarrow [\omega_1;\omega_1]^2_{\omega_1}$, *assuming CH*
- *(Galvin-Shelah, 1973)* $\omega_1 \nrightarrow [\omega_1]^2_4$
- *(Todorčević, 1981)* $\omega_1 \nrightarrow [\mathrm{Stat}(\omega_1);\mathrm{Stat}(\omega_1)]^2_{\omega_1}$
- *(Todorčević, 1987)* $\omega_1 \nrightarrow [\omega_1]^2_{\omega_1}$
- *(Moore, 2006)* $\omega_1 \nrightarrow [\omega_1;\omega_1]^2_{\omega_1}$

Transformations to the rescue

Proposition

Assuming $\mathrm{P}\ell_1(\kappa)$, the following are equivalent:

1. $\kappa \nrightarrow [\mathrm{Stat}(\kappa)]^2_\theta$;
2. $\kappa \nrightarrow [\kappa; \kappa]^2_\theta$.

Proof

Take $\mathbf{t}$ witnessing $\mathrm{P}\ell_1(\kappa)$ and c witnessing $\kappa \nrightarrow [\mathrm{Stat}(\kappa)]^2_\theta$.
Define $d : [\kappa]^2 \rightarrow \theta$ via $d := c \circ \mathbf{t}$.
Given cofinal $X, Y \subseteq \kappa$, find $\{x_i \mid i < \kappa\} \subseteq X$ and $\{y_i \mid i < \kappa\} \subseteq Y$ such that $x_i < y_i < x_j$ for all $i < j < \kappa$.
Then $\mathcal{A} = \{\{x_i, y_i\} \mid i < \kappa\}$ consists of κ-many pwd finite subsets of κ.
By the choice of $\mathbf{t}$, find a stationary $S \subseteq \kappa$ such that for all $\alpha < \beta$ from S, there are $a < b$ from $\mathcal{A}$ with $\mathbf{t}[a \times b] = \{(\alpha, \beta)\}$.
Given a prescribed color $\tau < \theta$, find $\alpha < \beta$ in S such that $c(\alpha, \beta) = \tau$.
Find $a < b$ from $\mathcal{A}$ with $\mathbf{t}[a \times b] = \{(\alpha, \beta)\}$, so that $a = \{x_i, y_i\}$, $b = \{x_j, y_j\}$ with $i < j$. Then $(x_i, y_j) \in X \circledast Y$ and $d(x_i, y_j) = \tau$. $\square$

Main results (joint with Zhang)

Definition

$\mathrm{P}\ell_1(\kappa,\chi)$ asserts the existence of a function $\mathbf{t} : [\kappa]^2 \rightarrow [\kappa]^2$ satisfying:

- for all $(\alpha,\beta) \in [\kappa]^2$, if $\mathbf{t}(\alpha,\beta) = (\alpha^*,\beta^*)$, then $\alpha^* \leq \alpha < \beta^* \leq \beta$;
- for all $\sigma < \chi$ and a family $\mathcal{A} \subseteq [\kappa]^\sigma$ consisting of κ many pairwise disjoint sets, there exists a stationary $S \subseteq \kappa$ such that, for every $\alpha^* < \beta^*$ from S, there are $a < b$ from $\mathcal{A}$ with $\mathbf{t}[a \times b] = \{(\alpha^*,\beta^*)\}$.

Theorem

For a regular cardinal $\chi \leq \kappa$, $\mathrm{P}\ell_1(\kappa,\chi)$ holds in any of the following cases:

1. $\chi^+ < \kappa$ *and* $\square(\kappa)$ *holds;*
2. $\chi^+ = \kappa$ *and* $\square(\kappa)$ *and GCH both hold;*
3. $\chi = \omega$, $\kappa = \omega_1$ *and there is a free Souslin tree;*
4. $\chi^+ < \kappa$ *and* $E^\kappa_{\geq\chi}$ *admits a stationary set that does not reflect;*
5. κ *is inaccessible, and* $E^\kappa_{\geq\chi}$ *admits a stationary set that does not reflect at inaccessibles;*
6. $\chi = \kappa$ *and* $\diamondsuit$ *holds over a nonreflecting stationary subset of* $\mathrm{Reg}(\kappa)$.

Walks on ordinals

Walk along a C-sequence

Fix a sequence $\vec{C} = \langle C_\alpha \mid \alpha < \kappa \rangle$ such that each C_α is a closed subset of α with $\sup(C_\alpha) = \sup(\alpha)$. In the next definitions, we assume $\alpha < \beta < \kappa$.

Definition (Todorčević, 1987)

- $\mathrm{Tr}(\alpha, \beta) \in {}^\omega\kappa$ is defined by recursion on $n < \omega$:

$$\mathrm{Tr}(\alpha,\beta)(n) := \begin{cases} \beta & n = 0 \\ \min(C_{\mathrm{Tr}(\alpha,\beta)(n-1)} \setminus \alpha) & n > 0 \ \&\ \mathrm{Tr}(\alpha,\beta)(n-1) > \alpha \\ \alpha & \text{otherwise} \end{cases}$$

- $\rho_2(\alpha, \beta) := \min\{n < \omega \mid \mathrm{Tr}(\alpha,\beta)(n) = \alpha\}$;
- $\lambda(\alpha, \beta) := \sup\{\sup(C_{\mathrm{Tr}(\alpha,\beta)(i)} \cap \alpha) \mid i < \rho_2(\alpha, \beta)\}$.

Definition

For $\eta < \kappa$, let $\eta_{\alpha,\beta} := \min\{m < \omega \mid \eta \in C_{\mathrm{Tr}(\alpha,\beta)(m)} \text{ or } m = \rho_2(\alpha,\beta)\} + 1$.

Defining the transformations

To define a transformation $\mathbf{t} : [\kappa]^2 \rightarrow [\kappa]^2$, first:

a. Make an educated choice of the sequence $\vec{C} = \langle C_\alpha \mid \alpha < \kappa \rangle$.
   club-guessing/swallowing club-guessing/coherent/avoiding a stationary set/witnessing a Brodsky-Rinot proxy principle.
b. Fix a wild oscillation map $o : [\kappa]^2 \rightarrow \omega$.

Then, given $(\alpha, \beta) \in [\kappa]^2$:

1. Let $n := o(\alpha, \beta)$;
2. Walk from β down to α, and stop at $\beta^* := \mathrm{Tr}(\alpha, \beta)(n)$;
3. Compute the lower trace $\eta := \lambda(\beta^*, \beta)$ and let $\varepsilon := \eta + 1$;
4. Let $m := \eta_{\varepsilon,\alpha}$;
   $\eta_{\varepsilon,\alpha} := \min\{m < \omega \mid \eta \in C_{\mathrm{Tr}(\varepsilon,\alpha)(m)} \text{ or } m = \rho_2(\varepsilon, \alpha)\} + 1$.
5. Walk from α down to ε, and stop at $\alpha^* := \mathrm{Tr}(\varepsilon, \alpha)(m)$.

If nothing broke down, let $\mathbf{t}(\alpha, \beta) := (\alpha^*, \beta^*)$; o.w., $\mathbf{t}(\alpha, \beta) := (\alpha, \beta)$.

Example: Inferring $\mathrm{P}\ell_1(\kappa,\chi)$ from square

Suppose $\chi < \kappa$ regular with $\chi^+ < \kappa$, and $\square(\kappa)$ holds.

Lemma

There is a sequence $\vec{C} = \langle C_\alpha \mid \alpha < \kappa \rangle$ satisfying the following:

1. $C_{\alpha+1} = \{0, \alpha\}$ *for every* $\alpha < \kappa$;
2. *for every* $\alpha \in \mathrm{acc}(\kappa)$ *and* $\bar{\alpha} \in \mathrm{acc}(C_\alpha)$, $C_{\bar{\alpha}} = C_\alpha \cap \bar{\alpha}$;
3. *for every* $\gamma \geq \chi^+$, $\{\delta \in E^\kappa_\chi \mid \min(C_\delta) = \gamma\}$ *is empty;*
4. *for every* $\gamma < \chi^+$, $\{\delta \in E^\kappa_\chi \mid \min(C_\delta) = \gamma\}$ *is stationary;*
5. *for every club* $D \subseteq \kappa$, *there exists* $\delta \in E^\kappa_\chi$ *with* $\sup(\mathrm{nacc}(C_\delta) \cap D) = \delta$.

$\mathrm{acc}(C) := \{\alpha \in C \mid \sup(C \cap \alpha) = \alpha > 0\}$. $\mathrm{nacc}(C) := C \setminus \mathrm{acc}(C)$.

To simplify

We shall hereafter assume that $\chi = \omega$, so that $\chi^+ = \omega_1$.

Example: Inferring $\mathrm{P}\ell_1(\kappa, \chi)$ from square, cont.

We have already made our choice of the sequence $\vec{C} = \langle C_\alpha \mid \alpha < \kappa \rangle$, so we now need to find an oscillation map $o : [\kappa]^2 \to \omega$.

Projected walk

For $(\alpha, \beta) \in [\kappa]^2$, define $\tau(\alpha, \beta) \in {}^{<\omega}\omega_1$ via

$$\tau(\alpha, \beta) := \langle \min(C_{\mathrm{Tr}(\alpha,\beta)(n)}) \mid n < \rho_2(\alpha, \beta) \rangle.$$

We shall use a wild $d : {}^{<\omega}\omega_1 \to \omega$, and let $o(\alpha, \beta) := d(\tau(\alpha, \beta))$.

Lemma (2014)

There is a map $d : {}^{<\omega}\omega_1 \to \omega$, such that, for every $\langle (u_i, v_i, \sigma_i) \mid i \in I \rangle$:

1. *I is a cofinal subset of ω_1,*
2. *u_i and v_i are finite subsets of ${}^{<\omega}\omega_1$,*
3. *$i \in \mathrm{Im}(\varrho)$ for all $\varrho \in u_i$, and $\sigma_i {}^\frown \langle i \rangle \sqsubseteq \sigma$ for all $\sigma \in v_i$,*

there are $i < j$ in A such that, for all $\varrho \in u_i$ and $\sigma \in v_j$, $d(\varrho {}^\frown \sigma) = \ell(\varrho)$.

Verifying this works

Given a family $\mathcal{A}$ consisting of κ many pairwise disjoint finite subsets of κ, fix $\{x_\beta \mid \beta < \kappa\} \subseteq \mathcal{A}$ with $\min(x_\beta) > \beta$.

Lemma

There are a stationary $S \subseteq \kappa$ and some $\eta < \kappa$ such that, for every $\epsilon \in S$ and every $\varsigma < \kappa$, there is a cofinal $I \subseteq \omega_1$ and a sequence $\langle \beta_i \mid i \in I\rangle \in \prod_{i\in I} \kappa \setminus \varsigma$, such that, for all $i \in I$ and $\beta \in x_{\beta_i}$:

(i) $i \in \mathrm{Im}(\tau(\epsilon, \beta))$;

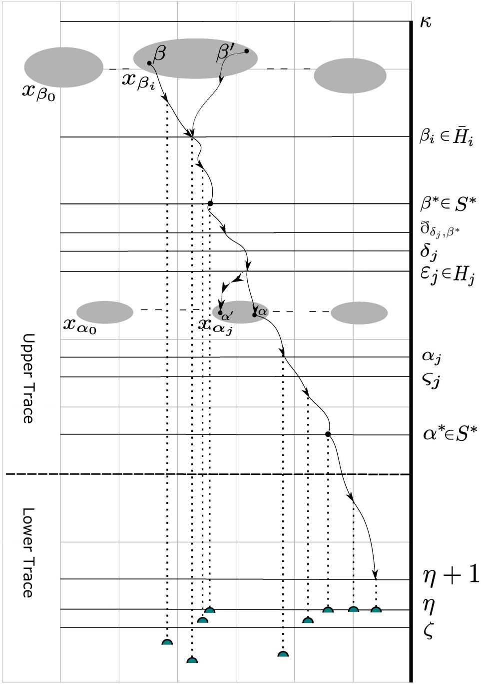
(ii) $\lambda(\epsilon, \beta) = \eta$;

(iii) $\rho_2(\epsilon, \beta) = \eta_{\epsilon,\beta}$.

Fix such η and S. Let $S^* := S \cap E$ for some sparse enough club E.

Lemma

For every $\alpha^* < \beta^*$ in S^*, there are $a < b$ in $\mathcal{A}$ with $\mathbf{t}[a \times b] = \{(\alpha^*, \beta^*)\}$.


κ
β
β'
x_{β_0}
x_{β_i}
$\beta_i \in \bar{H}_i$
$\beta^* \in S^*$
$\eth_{\delta_j,\beta^*}$
δ_j
$\varepsilon_j \in H_j$
α'
α
x_{α_0}
x_{α_j}
α_j
ς_j
$\alpha^* \in S^*$
Upper Trace
Lower Trace
$\eta+1$
η
ζ