A dual of Juhász' question

Oberwolfach conference on Set Theory
10-Jan-2022

Assaf Rinot, *Bar-Ilan University*

Plan for today

I'll be reporting on joint works with my graduate students at BIU:

[RS20] A.R. and Roy Shalev, *A guessing principle from a Souslin tree, with applications to topology*, accepted to Topology Appl.

[GR22] Shira Greenstein and A.R., in preparation.

Motivation

1. A Dowker space is a normal topological space whose product with the unit interval is not normal. Whether such a space exists was asked by C. H. Dowker in a paper from 1951.
2. The first consistent example was soon given by Rudin in 1955, who constructed a Dowker space of size $\aleph_1$, assuming the existence of a Souslin tree.
3. The existence of a Souslin tree was shown to be consistent only at the late 1960's: Tennenbaum using finite forcing, Jech using countable forcing, and Jensen (first, assuming $V = L$, and then) using $\diamondsuit$.

Motivation (cont.)

By now, there are a few constructions of Dowker spaces in ZFC: A space of size $(\aleph_\omega)^{\aleph_0}$ (Rudin, 1972), of size continuum (Balogh, 1996), and of size $\aleph_{\omega+1}$ (Kojman and Shelah, 1998).

The following problem is still standing:

Question

Is there a Dowker space of size $\aleph_1$?

The list of known sufficient conditions include CH (Juhász, Kunen and Rudin, 1976), $\clubsuit$ (de Caux, 1977), a Luzin set (Todorčević, 1989), and a certain tailored instance of a strong club-guessing principle (Hernńdez-Hernńdez and Szeptycki, 2009).

Juhász' question and its dual

Fact

- *Jensen:* $\diamondsuit$ *implies the existence of a Souslin tree;*
- *Devlin:* $\diamondsuit$ *is equivalent to* $\mathsf{CH} + \clubsuit$*;*
- *Jensen:* CH *does not imply the existence of a Souslin tree;*
- *Juhász: Does* $\clubsuit$ *imply the existence of a Souslin tree?*

Juhász' question and its dual

Fact

- *Jensen: $\diamondsuit$ implies the existence of a Souslin tree;*
- *Devlin: $\diamondsuit$ is equivalent to* CH $+ \clubsuit$*;*
- *Jensen:* CH *does not imply the existence of a Souslin tree;*
- *Juhász: Does $\clubsuit$ imply the existence of a Souslin tree?*

Juhász' question remains open for 35 years now.
Here, we propose to look at its dual:

Question

Does the existence of a Souslin tree imply $\clubsuit$?

Juhász' question and its dual

Fact

- *Jensen:* $\diamondsuit$ *implies the existence of a Souslin tree;*
- *Devlin:* $\diamondsuit$ *is equivalent to* CH $+ \clubsuit$*;*
- *Jensen:* CH *does not imply the existence of a Souslin tree;*
- *Juhász: Does* $\clubsuit$ *imply the existence of a Souslin tree?*

Juhász' question remains open for 35 years now.
Here, we propose to look at its dual:

Question

Does the existence of a Souslin tree imply $\clubsuit$?

Answer

No. Just add one Cohen real to any model of CH $+ \neg\diamondsuit$ (such as Jensen's model from above).

Juhász' question and its dual

Fact

- *Jensen:* $\diamondsuit$ *implies the existence of a Souslin tree;*
- *Devlin:* $\diamondsuit$ *is equivalent to* CH $+ \clubsuit$*;*
- *Jensen:* CH *does not imply the existence of a Souslin tree;*
- *Juhász: Does* $\clubsuit$ *imply the existence of a Souslin tree?*

Juhász' question remains open for 35 years now.
Here, we propose to look at its dual:

Question

Does the existence of a Souslin tree imply $\clubsuit$?

Corrected question

Does the existence of a Souslin tree imply a weak form of $\clubsuit$,
strong enough to entail the existence of a Dowker space of size $\aleph_1$?

Combinatorial principles

Let S denote a stationary set of a regular uncountable cardinal κ.

Definition (Jensen, 1972)

$\diamondsuit(S)$ asserts the existence of a sequence $\langle A_\alpha \mid \alpha \in S\rangle$ such that:

1. A_α is a subset of α;
2. For every $B \subseteq \kappa$, there are stationarily many $\alpha \in S$ with $A_\alpha = B \cap \alpha$.

Combinatorial principles

Let S denote a stationary set of a regular uncountable cardinal κ.

Definition (Jensen, 1972)

$\diamondsuit(S)$ asserts the existence of a sequence $\langle A_\alpha \mid \alpha \in S\rangle$ such that:

1. A_α is a subset of α;
2. For every $B \subseteq \kappa$, there are stationarily many $\alpha \in S$ with $A_\alpha = B \cap \alpha$.

Definition (Ostaszewski, 1976)

$\clubsuit(S)$ asserts the existence of a sequence $\langle A_\alpha \mid \alpha \in S\rangle$ such that:

1. A_α is a cofinal subset of α;
2. For every $B \in [\kappa]^\kappa$, there are stationarily many $\alpha \in S$ with $A_\alpha \subseteq B \cap \alpha$.

Combinatorial principles

Let S denote a stationary set of a regular uncountable cardinal κ.

A trivial weakening

$\clubsuit_{-}(S)$ asserts the existence of a sequence $\langle A_\alpha \mid \alpha \in S\rangle$ such that:

1. A_α is a cofinal subset of α;
2. For every $B \in [\kappa]^\kappa$, there are stationarily many $\alpha \in S$ with $\sup(A_\alpha \cap B) = \alpha$.

Definition (Ostaszewski, 1976)

$\clubsuit(S)$ asserts the existence of a sequence $\langle A_\alpha \mid \alpha \in S\rangle$ such that:

1. A_α is a cofinal subset of α;
2. For every $B \in [\kappa]^\kappa$, there are stationarily many $\alpha \in S$ with $A_\alpha \subseteq B \cap \alpha$.

Combinatorial principles

Let S denote a stationary set of a regular uncountable cardinal κ.

A trivial weakening

$\clubsuit_-(S)$ asserts the existence of a sequence $\langle A_\alpha \mid \alpha \in S\rangle$ such that:

1. A_α is a cofinal subset of α;
2. For every $B \in [\kappa]^\kappa$, there are stationarily many $\alpha \in S$ with $\sup(A_\alpha \cap B) = \alpha$.

Definition ([RS20])

$\clubsuit_{\mathrm{AD}}(S)$ asserts the existence of a sequence $\langle A_\alpha \mid \alpha \in S\rangle$ s.t.:

1. A_α is a cofinal subset of α;
2. For every $B \in [\kappa]^\kappa$, there are stationarily many $\alpha \in S$ with $\sup(A_\alpha \cap B) = \alpha$;
3. For all $\alpha \neq \alpha'$ from S, $\sup(A_\alpha \cap A_{\alpha'}) < \alpha$.

Combinatorial principles

Let S denote a stationary set of a regular uncountable cardinal κ.

Theorem ([RS20])

- PFA *refutes* $\clubsuit_{\mathrm{AD}}(S)$ *for any stationary* $S \subseteq \omega_1$*;*
- *If* κ *is weakly compact, then* $\clubsuit_{\mathrm{AD}}(S)$ *fails for every* S *such that* $\mathrm{Reg}(\kappa) \subseteq S \subseteq \kappa$*. So* $\diamondsuit(\mathrm{Reg}(\kappa)) \not\Longrightarrow \clubsuit_{\mathrm{AD}}(\mathrm{Reg}(\kappa))$.

Definition ([RS20])

$\clubsuit_{\mathrm{AD}}(S)$ asserts the existence of a sequence $\langle A_\alpha \mid \alpha \in S\rangle$ s.t.:

1. A_α is a cofinal subset of α;
2. For every $B \in [\kappa]^\kappa$, there are stationarily many $\alpha \in S$ with $\sup(A_\alpha \cap B) = \alpha$;
3. For all $\alpha \neq \alpha'$ from S, $\sup(A_\alpha \cap A_{\alpha'}) < \alpha$.

Combinatorial principles

Let S denote a stationary set of a regular uncountable cardinal κ.

Theorem ([RS20])

- PFA *refutes* $\clubsuit_{\mathrm{AD}}(S)$ *for any stationary* $S \subseteq \omega_1$;
- *If* κ *is weakly compact, then* $\clubsuit_{\mathrm{AD}}(S)$ *fails for every* S *such that* $\mathrm{Reg}(\kappa) \subseteq S \subseteq \kappa$. *So* $\diamondsuit(\mathrm{Reg}(\kappa)) \not\Longrightarrow \clubsuit_{\mathrm{AD}}(\mathrm{Reg}(\kappa))$.

Compare with the fact that $\diamondsuit(E^\kappa_\lambda) \implies \clubsuit(E^\kappa_\lambda) \implies \clubsuit_{\mathrm{AD}}(E^\kappa_\lambda)$.

Definition ([RS20])

$\clubsuit_{\mathrm{AD}}(S)$ asserts the existence of a sequence $\langle A_\alpha \mid \alpha \in S\rangle$ s.t.:

1. A_α is a cofinal subset of α;
2. For every $B \in [\kappa]^\kappa$, there are stationarily many $\alpha \in S$ with $\sup(A_\alpha \cap B) = \alpha$;
3. For all $\alpha \neq \alpha'$ from S, $\sup(A_\alpha \cap A_{\alpha'}) < \alpha$.

Combinatorial principles

Let S denote a stationary set of a regular uncountable cardinal κ.

Strengthening in a different axis

$\clubsuit_{\mathrm{AD}}(S,\mu)$ asserts the existence of a sequence $\langle \mathcal{A}_\alpha \mid \alpha \in S\rangle$ s.t.:

1. $\mathcal{A}_\alpha$ is a pairwise disjoint family of μ many cofinal subsets of α;
2. For every $B \in [\kappa]^\kappa$, there are stationarily many $\alpha \in S$ with $\sup(A \cap B) = \alpha$ for every $A \in \mathcal{A}_\alpha$;
3. For all $A \neq A'$ from $\bigcup_{\alpha\in S} \mathcal{A}_\alpha$, $\sup(A \cap A') < \sup(A)$.

Definition ([RS20])

$\clubsuit_{\mathrm{AD}}(S)$ asserts the existence of a sequence $\langle A_\alpha \mid \alpha \in S\rangle$ s.t.:

1. A_α is a cofinal subset of α;
2. For every $B \in [\kappa]^\kappa$, there are stationarily many $\alpha \in S$ with $\sup(A_\alpha \cap B) = \alpha$;
3. For all $\alpha \neq \alpha'$ from S, $\sup(A_\alpha \cap A_{\alpha'}) < \alpha$.

Combinatorial principles

Let S denote a stationary set of a regular uncountable cardinal κ.

Strengthening in a different axis

$\clubsuit_{\mathrm{AD}}(S, \mu, \theta)$ asserts the existence of a sequence $\langle \mathcal{A}_\alpha \mid \alpha \in S \rangle$ s.t.:

1. $\mathcal{A}_\alpha$ is a pairwise disjoint family of μ many cofinal subsets of α;
2. For every $\mathcal{B} \subseteq [\kappa]^\kappa$ of size θ, there are stat. many $\alpha \in S$ with $\sup(A \cap B) = \alpha$ for all $A \in \mathcal{A}_\alpha$ and $B \in \mathcal{B}$;
3. For all $A \neq A'$ from $\bigcup_{\alpha \in S} \mathcal{A}_\alpha$, $\sup(A \cap A') < \sup(A)$.

Definition ([RS20])

$\clubsuit_{\mathrm{AD}}(S)$ asserts the existence of a sequence $\langle A_\alpha \mid \alpha \in S \rangle$ s.t.:

1. A_α is a cofinal subset of α;
2. For every $B \in [\kappa]^\kappa$, there are stationarily many $\alpha \in S$ with $\sup(A_\alpha \cap B) = \alpha$;
3. For all $\alpha \neq \alpha'$ from S, $\sup(A_\alpha \cap A_{\alpha'}) < \alpha$.

In full generality

Let $\mathcal{S}$ denote a nonempty collection of stationary subsets of a regular uncountable cardinal κ.

Definition ([RS20])

$\clubsuit_{\mathsf{AD}}(\mathcal{S}, \mu, \theta)$ asserts there is a sequence $\langle \mathcal{A}_\alpha \mid \alpha \in \bigcup \mathcal{S} \rangle$ s.t.:

1. $\mathcal{A}_\alpha$ is a pairwise disjoint family of μ many cofinal subsets of α;
2. For every $\mathcal{B} \subseteq [\kappa]^\kappa$ of size θ and every $S \in \mathcal{S}$, there are stat. many $\alpha \in S$ with $\sup(A \cap B) = \alpha$ for all $A \in \mathcal{A}_\alpha$, $B \in \mathcal{B}$;
3. For all $A \neq A'$ from $\bigcup_{S \in \mathcal{S}} \bigcup_{\alpha \in S} \mathcal{A}_\alpha$, $\sup(A \cap A') < \sup(A)$.

Back to the Dowker space problem

Theorem ([RS20])

For a regular uncountable κ, assume any of the following:

1. $\clubsuit_{\mathsf{AD}}(\mathcal{S},1,2)$ *holds for some infinite partition $\mathcal{S}$ of a nonreflecting stationary subset of κ;*
2. $\clubsuit_{\mathsf{AD}}(\{E^{\kappa}_{\lambda}\},\lambda,1)$ *holds, where $\kappa=\lambda^+$, λ is regular.*

Then there exists a Dowker space of size κ.

A new invariant of trees

Recall that a κ-Souslin tree is a poset $\mathcal{T} = (T, <)$ such that:

- $|T| = \kappa$;
- $\mathcal{T}$ has no chains or antichains of size κ;
- For every $x \in T$, $x_\downarrow := \{y \in T \mid y < x\}$ is well-ordered by $<$, and we write $\mathrm{ht}(x) := \mathrm{otp}(x_\downarrow, <)$.

A new invariant of trees

Recall that a κ-Souslin tree is a poset $\mathcal{T} = (T, <)$ such that:

- $|T| = \kappa$;
- $\mathcal{T}$ has no chains or antichains of size κ;
- For every $x \in T$, $x_\downarrow := \{y \in T \mid y < x\}$ is well-ordered by $<$, and we write $\mathrm{ht}(x) := \mathrm{otp}(x_\downarrow, <)$.

Definition

- $B \subseteq T$ is an α-branch iff it is a chain and $\{\mathrm{ht}(x) \mid x \in B\} = \alpha$.
- An α-branch is vanishing iff it has no upper bound in $\mathcal{T}$.

A new invariant of trees

Definition

- $B \subseteq T$ is an α-branch iff it is a chain and $\{\text{ht}(x) \mid x \in B\} = \alpha$.
- An α-branch is vanishing iff it has no upper bound in $\mathcal{T}$.

Definition (The vanishing levels of the tree)

$V(\mathcal{T})$ stands for the set of $\alpha \in \text{acc}(\kappa)$ such that, for every $x \in T$ with $\text{ht}(x) < \alpha$, there exists a vanishing α-branch containing x.

A new invariant of trees

The vanishing levels is an invariant of κ-Souslin trees:
$\mathcal{T} \cong \mathcal{T}' \implies V(\mathcal{T}) = V(\mathcal{T}')$. $\quad \mathcal{T} \cong_{cub} \mathcal{T}' \implies V(\mathcal{T}) \equiv_{cub} V(\mathcal{T}')$.

Definition

- $B \subseteq T$ is an α-branch iff it is a chain and $\{\mathrm{ht}(x) \mid x \in B\} = \alpha$.
- An α-branch is vanishing iff it has no upper bound in $\mathcal{T}$.

Definition (The vanishing levels of the tree)

$V(\mathcal{T})$ stands for the set of $\alpha \in \mathrm{acc}(\kappa)$ such that, for every $x \in T$ with $\mathrm{ht}(x) < \alpha$, there exists a vanishing α-branch containing x.

A new invariant of trees

The vanishing levels is an invariant of κ-Souslin trees:
$\mathcal{T} \cong \mathcal{T}' \implies V(\mathcal{T}) = V(\mathcal{T}')$. $\mathcal{T} \cong_{cub} \mathcal{T}' \implies V(\mathcal{T}) \equiv_{cub} V(\mathcal{T}')$.

- If $\mathcal{T}$ is a λ-complete λ^+-Souslin tree, then $V(\mathcal{T}) = E^{\lambda^+}_{\lambda}$;
- If $\mathcal{T}$ is a uniformly coherent κ-Souslin tree, $V(\mathcal{T}) \equiv_{cub} E^{\kappa}_{\omega}$.

Definition

• $B \subseteq T$ is an α-branch iff it is a chain and $\{\mathrm{ht}(x) \mid x \in B\} = \alpha$.
• An α-branch is vanishing iff it has no upper bound in $\mathcal{T}$.

Definition (The vanishing levels of the tree)

$V(\mathcal{T})$ stands for the set of $\alpha \in \mathrm{acc}(\kappa)$ such that, for every $x \in T$ with $\mathrm{ht}(x) < \alpha$, there exists a vanishing α-branch containing x.

Deriving $\clubsuit_{\mathrm{AD}}$ from a Souslin tree

Main Theorem ([RS20])

Suppose that $\mathcal{T}$ is a κ-Souslin tree for a regular uncountable κ. Then $\clubsuit_{\mathrm{AD}}(S)$ holds for some $S \equiv_{cub} V(\mathcal{T})$.

Deriving $\clubsuit_{\mathsf{AD}}$ from a Souslin tree

Main Theorem ([RS20])

Suppose that $\mathcal{T}$ is a κ-Souslin tree for a regular uncountable κ. Then $\clubsuit_{\mathsf{AD}}(\mathcal{S}, \mu, {<}\theta)$ holds, provided that $\mu < \kappa = \kappa^{<\theta}$, and $\mathcal{S}$ is <u>any</u> partition of $V(\mathcal{T}) \cap E^{\kappa}_{\geq\theta}$ (modulo a club) into stationary sets.

Recall

$\clubsuit_{\mathsf{AD}}(\mathcal{S}, \mu, {<}\theta)$ asserts there is a sequence $\langle \mathcal{A}_\alpha \mid \alpha \in \bigcup \mathcal{S} \rangle$:

1. $\mathcal{A}_\alpha$ is a pairwise disjoint family of μ many cofinal subsets of α;
2. For every $\mathcal{B} \subseteq [\kappa]^\kappa$ of size $< \theta$ and every $S \in \mathcal{S}$, there are stat. many $\alpha \in S$ with $\sup(A \cap B) = \alpha$ for all $A \in \mathcal{A}_\alpha$, $B \in \mathcal{B}$;
3. For all $A \neq A'$ from $\bigcup_{S \in \mathcal{S}} \bigcup_{\alpha \in S} \mathcal{A}_\alpha$, $\sup(A \cap A') < \sup(A)$.

Deriving $\clubsuit_{\mathsf{AD}}$ from a Souslin tree

Main Theorem ([RS20])

Suppose that $\mathcal{T}$ is a κ-Souslin tree for a regular uncountable κ. Then $\clubsuit_{\mathsf{AD}}(\mathcal{S},\mu,{<}\theta)$ holds, provided that $\mu<\kappa=\kappa^{<\theta}$, and $\mathcal{S}$ is any partition of $V(\mathcal{T})\cap E^{\kappa}_{\geq\theta}$ (modulo a club) into stationary sets.

Whenever $V(\mathcal{T})$ has uniform cofinality, we can waive the club, e.g.:

Corollary ([RS20])

If there exists a Souslin tree, then for every partition $\mathcal{S}$ of $\mathrm{acc}(\omega_1)$ into stationary sets, $\clubsuit_{\mathsf{AD}}(\mathcal{S},\omega,{<}\omega)$ holds.

The Dowker space we construct from $\clubsuit_{\mathsf{AD}}(\{\mathrm{acc}(\omega_1)\},\omega,1)$ is an hereditary separable refinement of the order topology on ω_1, hence, we altogether obtained a new proof of Rudin's 1974 result that a Souslin tree yields an S-space which is Dowker.

Deriving $\clubsuit_{\mathrm{AD}}$ from a Souslin tree

A problem arises

In 1999, answering a question of Kunen, Shelah gave a forcing construction over a strongly inaccessible Mahlo cardinal κ of a full κ-Souslin tree $\mathcal{T}$. 'Full' means that for every $\alpha \in \mathrm{acc}(\kappa)$, the set of vanishing α-branches through $\mathcal{T}$ is no more than a singleton!
In particular, $V(\mathcal{T}) \equiv_{cub} \emptyset$.

Deriving $\clubsuit_{\mathsf{AD}}$ from a Souslin tree

Theorem (unpublished)

Suppose there is a κ-Souslin tree for a regular uncountable κ. Then $\clubsuit_{\mathsf{AD}}(\mathcal{S}, 1, 1)$ holds for some κ-sized collection $\mathcal{S}$ of pairwise disjoint stationary subsets of κ.

A problem arises

In 1999, answering a question of Kunen, Shelah gave a forcing construction over a strongly inaccessible Mahlo cardinal κ of a full κ-Souslin tree $\mathcal{T}$. 'Full' means that for every $\alpha \in \mathrm{acc}(\kappa)$, the set of vanishing α-branches through $\mathcal{T}$ is no more than a singleton!
In particular, $V(\mathcal{T}) \equiv_{cub} \emptyset$.

The spectrum of $V(\mathcal{T})$

A full κ-Souslin tree is a rather bizarre object. Could there be more than one (up to, say, club-isomorphism)?

The spectrum of $V(\mathcal{T})$

A full κ-Souslin tree is a rather bizarre object. Could there be more than one (up to, say, club-isomorphism)?
In a joint work with Greenstein, we found a proof of Shelah's result that is based on the Brodsky-Rinot *microscopic approach to Souslin-tree constructions*.

A construction á la microscopic approach

#include <NormalTree.h>
#include <SealAntichain.h>
#include <SealAutomorphism.h>
//#include <Specialize.h>
//#include <SealProductTree.h>

The spectrum of $V(\mathcal{T})$

A full κ-Souslin tree is a rather bizarre object. Could there be more than one (up to, say, club-isomorphism)?
In a joint work with Greenstein, we found a proof of Shelah's result that is based on the Brodsky-Rinot *microscopic approach to Souslin-tree constructions*. By the modular nature of this approach and by plugging in one of the already existing modules, we get:

Theorem ([GR22])

Work in L, and suppose that κ is a Mahlo cardinal that is not weakly compact.
Then there exists a family of 2^κ many full κ-Souslin trees such that the product of any finitely many of them is again κ-Souslin.
In particular, the trees are pairwise not club-isomorphic.

The spectrum of $V(\mathcal{T})$

Denote $\mathrm{Vspec}(\kappa) = \{V(\mathcal{T}) \mid \mathcal{T} \text{ is a } \kappa\text{-Souslin tree}\}$.

Theorem ([GR22])

Work in L, and suppose that κ is an inaccessible cardinal that is not weakly compact.
Then $\mathrm{Vspec}(\kappa)$ is dense in $(\mathrm{NS}_\kappa)^+$, and every stationary $S \subseteq \kappa$ of uniform cofinality is $\equiv_{cub}$ to some set in $\mathrm{Vspec}(\kappa)$.

Theorem ([GR22])

Work in L, and suppose that κ is a Mahlo cardinal that is not weakly compact.
Then there exists a family of 2^κ many full κ-Souslin trees such that the product of any finitely many of them is again κ-Souslin.
In particular, the trees are pairwise not club-isomorphic.

Open problems

1. We have seen that PFA kills $\clubsuit_{\mathrm{AD}}(\omega_1)$. What about $\mathrm{MA}_{\aleph_1}$?
   ** It is open whether* $\mathrm{MA} + \neg\,\mathrm{CH}$ *is compatible with existence of a Dowker space of size* $\aleph_1$.
2. Are there any (e.g., topological) applications of $\clubsuit_{\mathrm{AD}}(\mathcal{S}, 1, 1)$?
3. Is $\clubsuit_{\mathrm{AD}}(\mathrm{Reg}(\kappa))$ refuted by κ being a generic large cardinal?
4. In 1977, Shelah proved the consistency of $\diamondsuit(S) + \neg\diamondsuit(\omega_1 \setminus S)$ for some stationary co-stationary $S \subseteq \omega_1$.
   Is the analogous statement for $\clubsuit_{\mathrm{AD}}(\omega_1)$ consistent?
   ** It follows from our main result that the consistency of* $\clubsuit(S) + \neg\clubsuit_{\mathrm{AD}}(\omega_1 \setminus S)$ *for some stationary co-stationary* $S \subseteq \omega_1$ *would provide a negative answer to Juhász' question.*