May the successor of a singular cardinal be Jónsson?

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Perspectives on Set Theory

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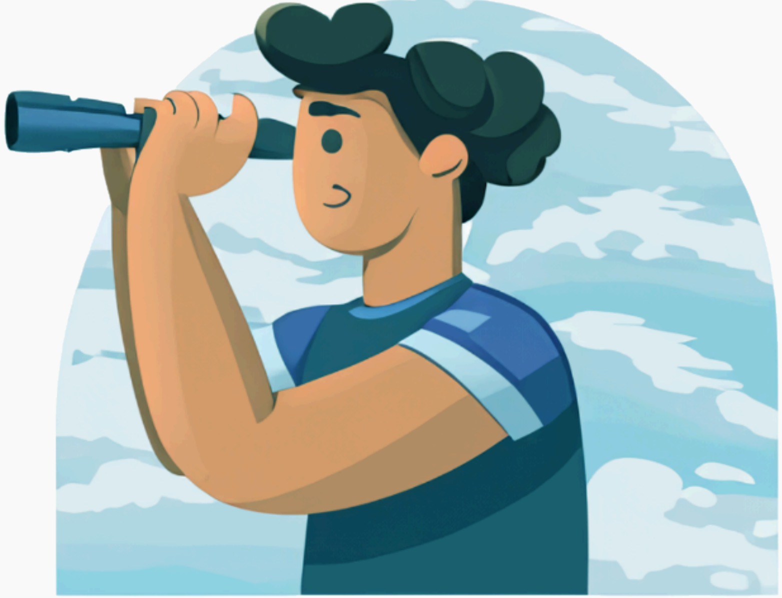
I thank the organizers for the invitation to give a *perspective* talk. Hopefully, the ten questions collected here will be picked up by the community and lead to major advances.



An unusual talk deserves unusual slides. I created this presentation using **typst**.

Introduction

Background

Theorem (Ramsey, 1930). *For every 2-coloring of the unordered pairs of an infinite set* X, $c : [X]^2 \to 2$, *there exists an infinite subset* $Y \subseteq X$ *such that* c *is* constant *over* $[Y]^2$.

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A closely related concept. κ is Jónsson iff $\kappa \to [\kappa]^{<\omega}_\kappa$ holds, i.e., for every coloring $c : [\kappa]^{<\omega} \to \kappa$ there is some $Y \subseteq \kappa$ of full size such that $c \restriction [Y]^{<\omega}$ is not surjective.

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Pump up for successor cardinals. $\lambda^+ \nrightarrow [\lambda^+]^2_\lambda$ *implies* $\lambda^+ \nrightarrow [\lambda^+]^2_{\lambda^+}$.

Strong colorings of successor cardinals

Theorem (Sierpiński, 1932 for $\boldsymbol{\lambda} = \omega$. General case by Erdős-Hajnal-Rado, 1965)
For every infinite cardinal λ such that $2^\lambda = \lambda^+$, $\lambda^+ \nrightarrow [\lambda^+]^2_\lambda$ holds.

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Proof. Assuming $2^\lambda = \lambda^+$, let $\langle a_\alpha \mid \alpha < \lambda^+ \rangle$ enumerate all bounded subsets of λ^+.

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Proof. Assuming $2^\lambda = \lambda^+$, let $\langle a_\alpha \mid \alpha < \lambda^+ \rangle$ enumerate all bounded subsets of λ^+.

For each $\beta < \lambda^+$, let $\mathcal{B}_\beta$ be a disjoint refinement of $\{a_\alpha \mid \alpha < \beta\} \cap [\beta]^\lambda$.

$*$ So $\mathcal{B}_\beta$ is a pairwise disjoint subfamily of $[\beta]^\lambda$ satisfying that for every $\alpha < \beta$ with $a_\alpha \in [\beta]^\lambda$ there is $b \in \mathcal{B}_\beta$ with $b \subseteq a_\alpha$.

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Now, pick a coloring $c : [\lambda^+]^2 \to \lambda$ such that for every $\beta < \lambda^+$, for every $b \in \mathcal{B}_\beta$, $c[b \times \{\beta\}] = \lambda$.

Strong colorings of successor cardinals

Theorem (Sierpiński, 1932 for $\lambda = \omega$. General case by Erdős-Hajnal-Rado, 1965)
For every infinite cardinal λ such that $2^\lambda = \lambda^+$, $\lambda^+ \not\to [\lambda^+]^2_\lambda$ holds.

Proof. Assuming $2^\lambda = \lambda^+$, let $\langle a_\alpha \mid \alpha < \lambda^+ \rangle$ enumerate all bounded subsets of λ^+.

For each $\beta < \lambda^+$, let $\mathcal{B}_\beta$ be a disjoint refinement of $\{a_\alpha \mid \alpha < \beta\} \cap [\beta]^\lambda$.

Now, pick a coloring $c : [\lambda^+]^2 \to \lambda$ such that for every $\beta < \lambda^+$, for every $b \in \mathcal{B}_\beta$, $c[b \times \{\beta\}] = \lambda$.

This works because given $Y \subseteq \lambda^+$ of full size, we may find an $\alpha < \lambda^+$ with $a_\alpha \in [Y]^\lambda$, and then find a large enough $\beta \in Y$ to satisfy $(a_\alpha \cup \alpha) \subseteq \beta$.
Pick $b \in \mathcal{B}_\beta$ with $b \subseteq a_\alpha$. Then $\lambda = c[b \times \{\beta\}] \subseteq c\left[[Y]^2\right]$.

qed

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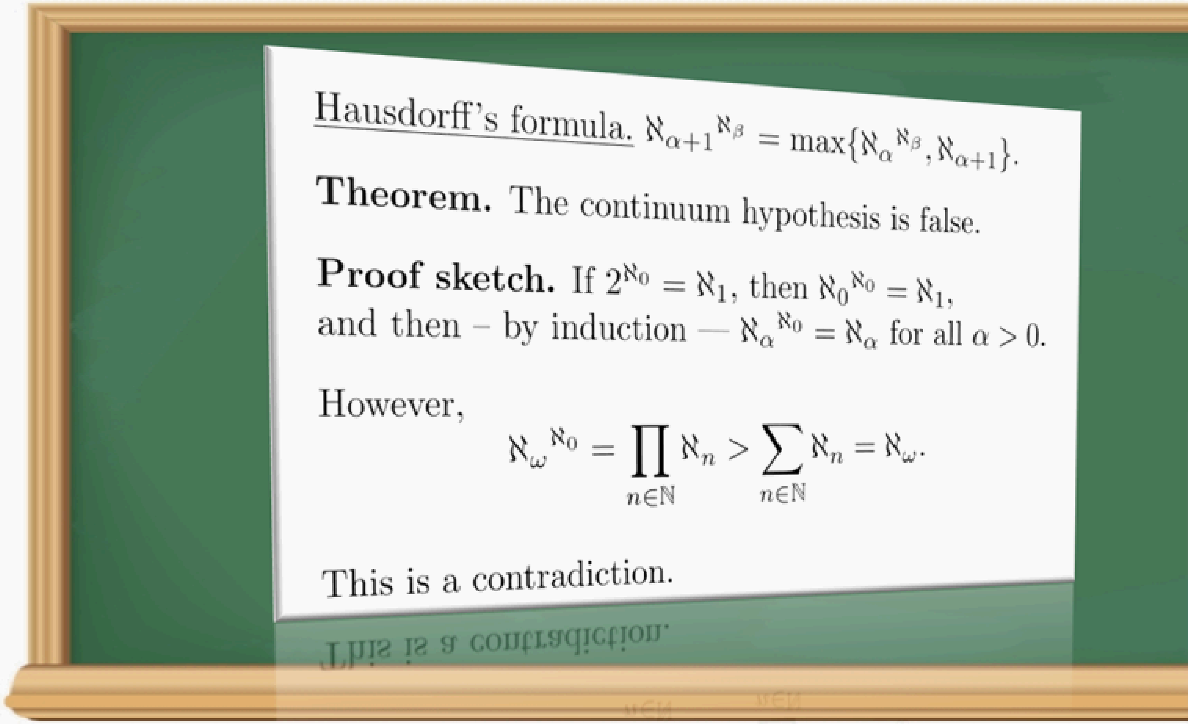
What about successors of singulars?

Interlude: the birth of singular cardinals

At the 3rd ICM meeting in Heidelberg, 1904
all other parallel sessions were canceled
to allow everyone including Cantor and Hilbert
to attend Julius König's sensational lecture.



<u>Hausdorff's formula.</u> ${\aleph_{\alpha+1}}^{\aleph_\beta} = \max\{{\aleph_\alpha}^{\aleph_\beta}, \aleph_{\alpha+1}\}$.

Theorem. The continuum hypothesis is false.

Proof sketch. If $2^{\aleph_0} = \aleph_1$, then ${\aleph_0}^{\aleph_0} = \aleph_1$, and then – by induction — ${\aleph_\alpha}^{\aleph_0} = \aleph_\alpha$ for all $\alpha > 0$.

However,

$$ {\aleph_\omega}^{\aleph_0} = \prod_{n \in \mathbb{N}} \aleph_n > \sum_{n \in \mathbb{N}} \aleph_n = \aleph_\omega. $$

This is a contradiction.

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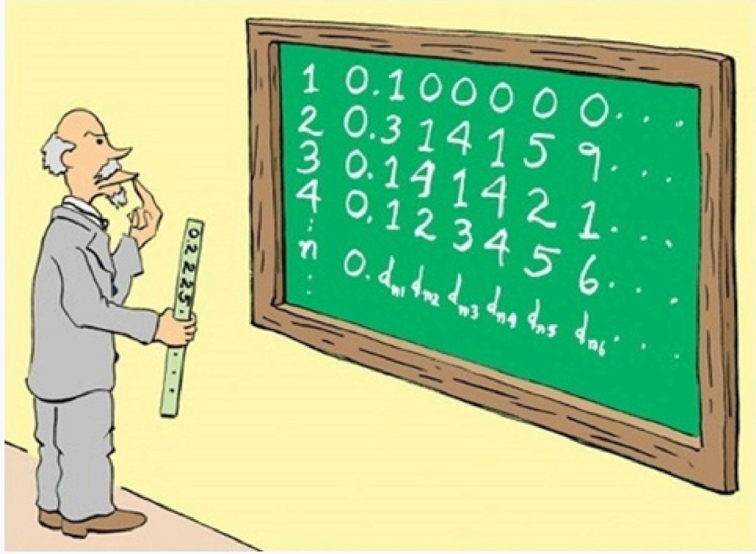
At the end of the lecture, Cantor said how grateful he was to have lived to see his conjecture answered,



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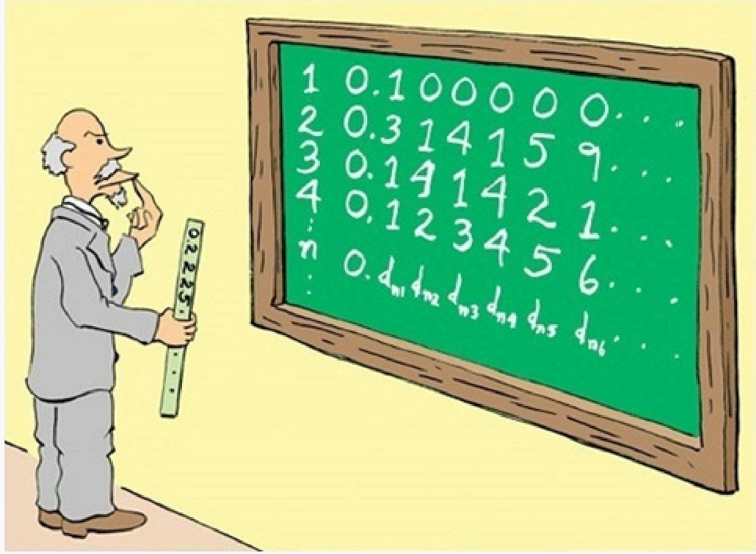

even if the answer was negative.

But König's proof had a flaw (that goes back to Bernstein) overlooking ***singular*** cardinals.

The core of König's proof is correct

Suppose that σ is a limit ordinal > 0. Given a strictly increasing sequence $\vec{\lambda} = \langle \lambda_i \mid i < \sigma \rangle$ of regular uncountable cardinals, define a quasi-ordering $<^*$ of the product $\prod \vec{\lambda}$ by letting

$f <^* g$ iff $\{i < \sigma \mid f(i) \geq g(i)\}$ is bounded in σ.

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Definition.

- $\mathfrak{b}(\vec{\lambda})$ denotes the least size of an unbounded family in $(\prod \vec{\lambda}, <^*)$.
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Lemma (König)

$\mathfrak{b}(\vec{\lambda})$ *is a regular cardinal greater than* $\sup(\vec{\lambda})$.

PCF theory

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Theorem (Shelah, 1990's)
For every singular cardinal λ, *there is an increasing sequence* $\vec{\lambda} = \langle \lambda_i \mid i < \mathrm{cf}(\lambda) \rangle$ *of regular uncountable cardinals converging to* λ *such that* $\mathfrak{d}(\vec{\lambda}) = \lambda^+$. So, by König, also $\mathfrak{b}(\vec{\lambda}) = \lambda^+$.

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*If $\vec{\lambda}$ is as in Shelah's theorem **and** $\lambda_i \nrightarrow [\lambda_i]^2_{\lambda_i}$ for every $i < \mathrm{cf}(\lambda)$, then $\lambda^+ \nrightarrow [\lambda^+]^2_{\lambda}$.*

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Question 6.7 of Gilton's 2022 preprint PCF theory and the Tukey spectrum is equivalent to asking whether Todorčević's hypothesis of $\mathfrak{d}(\vec{\lambda}) = \lambda^+$ may be reduced to $\mathfrak{b}(\vec{\lambda}) = \lambda^+$.

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Define $c : [\lambda^+]^2 \to \lambda$ by letting $c(\alpha, \beta) := c_i\left(f_{\gamma(i)}, f_{\delta(i)}\right)$ whenever $d(\alpha, \beta) = (\gamma, \delta, i)$. qed

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Definition. *Weakly compact failure of* SCH_λ asserts that for every $\vec{\lambda} = \langle \lambda_i \mid i < \mathrm{cf}(\lambda) \rangle$ converging to λ with $\mathfrak{b}(\vec{\lambda}) = \lambda^+$, it is the case that almost all λ_i's are weakly compact.

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Ben-Neria: An affirmative answer seems to emerge from Merimovich's work on supercompact extender based Prikry forcing.

Compactness and incompactness

Stationary reflection (compactness)

Theorem (Todorčević, 1987)
For a regular uncountable κ, if $\kappa \nrightarrow [\kappa]^2_\kappa$ fails, then every stationary subset of κ reflects.

Theorem (Eisworth, 2012)
If λ is a singular cardinal for which $\lambda^+ \nrightarrow [\lambda^+]^2_\lambda$ fails, then every family of less than $\mathrm{cf}(\lambda)$ many stationary subsets of λ^+ reflect simultaneously.

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A few years ago, a model satisfying the above form of simultaneous reflection together with $\neg\,\mathrm{SCH}_\lambda$ was obtained by Poveda, Rinot and Sinapova using iterated Prikry-type forcing, and by Ben-Neria, Hayut and Unger using iterated ultrapowers and then simplified by Gitik.

Aronszajn trees (incompactness)

Theorem (Jensen, 1972. Shore 1974)
If there is a κ-Souslin tree, then $\kappa \nrightarrow [\kappa]^2_\kappa$ holds.

Theorem (R., 2014)
If $\square(\kappa)$ holds, then so does $\kappa \nrightarrow [\kappa]^2_\kappa$.
Remains true assuming weak variants of square.

Note that both a κ-Souslin tree and $\square(\kappa)$ are particular sorts of κ-Aronszajn trees.

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Question 2. Suppose that λ is a singular cardinal and there exists a λ^+-Aronszajn tree. Does $\lambda^+ \nrightarrow [\lambda^+]^2_\lambda$ hold?

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Question 3. Suppose that λ is a singular cardinal and there exists a λ^+-Aronszajn tree. Does there exist a λ^+-Souslin tree? Here, I don't mind assuming the GCH.

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If there is a κ-Souslin tree, then $\kappa \nrightarrow [\kappa]^2_\kappa$ holds.

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If $\square(\kappa)$ holds, then so does $\kappa \nrightarrow [\kappa]^2_\kappa$.
Remains true assuming weak variants of square.

Question 3. Suppose that λ is a singular cardinal and there exists a λ^+-Aronszajn tree. Does there exist a λ^+-Souslin tree? Here, I don't mind assuming the GCH.

► Results from More notions of forcing add a Souslin tree (with Brodsky, 2019) show that — in the context of GCH — singularizations of a regular λ tend to introduce λ^+-Souslin trees.

Putting it all together

The tree property: $\mathrm{TP}(\kappa)$ asserts that there are no κ-Aronszajn trees.

Theorem (Neeman, 2009)
Starting with infinitely many supercompact cardinals, it is consistent that for some singular strong limit cardinal λ of countable cofinality, SCH_λ fails and $\mathrm{TP}(\lambda^+)$ holds.

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Question 4. Is the conjunction of the following consistent for some singular cardinal λ?

i) Weakly compact failure of SCH_λ;

ii) $\mathrm{TP}(\lambda^+)$;

iii) every finite family of stationary subsets of λ^+ reflect simultaneously.

Reductions and approximations

Reduction 1

Let λ denote a singular cardinal.

Theorem (Eisworth, 2013)
If $\lambda^+ \nrightarrow [\lambda^+]^2_\theta$ *holds for arbitrarily large* $\theta < \lambda$, *then* $\lambda^+ \nrightarrow [\lambda^+]^2_\lambda$ *holds.*

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$\lambda^+ \nrightarrow [\lambda^+]^2_\theta$ *holds for* $\theta = \mathrm{cf}(\lambda)$.

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$\lambda^+ \nrightarrow [\lambda^+]^2_\theta$ *holds for* $\theta = \mathrm{cf}(\lambda)$.

Question 5. Does $\lambda^+ \nrightarrow [\lambda^+]^2_\theta$ hold for $\theta = \mathrm{cf}(\lambda)^+$?

Reduction 2

Let λ denote a singular cardinal.

Theorem (R., 2012)
If there are a cardinal $\mu < \lambda$ and a coloring $c : [\lambda^+]^2 \to \theta$ such that $c\left[[S]^2\right] = \theta$ for every stationary $S \subseteq \lambda^+ \cap \mathrm{cof}(> \mu)$, then $\lambda^+ \nrightarrow [\lambda^+]^2_\theta$ holds.

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Question 6. Identify interesting ideals J over λ^+ for which **ZFC** proves the existence of a coloring $c : [\lambda^+]^2 \rightarrow \lambda$ satisfying $c\left[[B]^2\right] = \lambda$ for every $B \in J^+$.

Reduction 3

Given a coloring $c : [\lambda^+]^2 \to \theta$, let $\mathbb{P}_{c,\mu} := \left(\left\{x \in [\lambda^+]^{<\mu} \mid c \restriction [x]^2 \text{ is constant}\right\}, \supseteq\right)$.
This poset adds a large homogeneous set, thus ensuring c ceases to witness $\lambda^+ \not\to [\lambda^+]^2_\theta$.

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The good (R., 2012)
Suppose that λ is a singular cardinal. If $\lambda^+ \nrightarrow [\lambda^+]^2_\theta$ holds, then it may be witnessed by a coloring $c : [\lambda^+]^2 \to \theta$ for which $\mathbb{P}_{c,\omega}$ has the λ^+-cc.

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The bad (R.-Zhang, 2024)

Suppose that λ is a singular cardinal. Let $c : [\lambda^+]^2 \to 2$ be any coloring.

- $\mathbb{P}_{c,\lambda}$ has an antichain of size λ^+ consisting of pairwise disjoint sets;
- If λ is the limit of strongly compacts, then this is true already for $\mathbb{P}_{c,\,\mathrm{cf}(\lambda)^+}$.

Reduction 3

Question 7. Given a coloring $c : [\lambda^+]^2 \to \theta$ witnessing $\lambda^+ \nrightarrow [\lambda^+]^2_\theta$, is there a cofinality-preserving notion of forcing for killing c? Identify features of c that enable a YES answer.

Reduction 4 and two ZFC approximations

Theorem (Inamdar-R., 2023)

Suppose a singular cardinal λ *is a strong limit or satisfies* $\aleph_\lambda > \lambda$.
If there exists a coloring $c : \lambda \times \lambda^+ \to \lambda$ *such that for every* $Y \subseteq \lambda^+$ *of full size, there is* $i < \lambda$ *with* $c[\{i\} \times Y] = \lambda$, *then* $\lambda^+ \nrightarrow [\lambda^+]^2_\lambda$ *holds.*

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Theorem (Inamdar-R., 202⌛)
For every singular cardinal λ, for every $\theta < \lambda$, there is a coloring $c : \lambda \times \lambda^+ \to \theta$
such that for every $Y \subseteq \lambda^+$ of full size, there is $i < \lambda$ with $c[\{i\} \times Y] = \theta$.

Reduction 4 and two ZFC approximations

Theorem (Inamdar-R., 2023)
Suppose a singular cardinal λ is a strong limit or satisfies $\aleph_\lambda > \lambda$.
If there exists a coloring $c : \lambda \times \lambda^+ \to \lambda$ such that for every $Y \subseteq \lambda^+$ of full size, there is $i < \lambda$ with $c[\{i\} \times Y] = \lambda$, then $\lambda^+ \not\to [\lambda^+]^2_\lambda$ holds.

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For every singular cardinal λ, for every $\theta < \lambda$, there is a coloring $c : \lambda \times \lambda^+ \to \theta$ such that for every $Y \subseteq \lambda^+$ of full size, there is $i < \lambda$ with $c[\{i\} \times Y] = \theta$.

One cannot get $\theta = \lambda$ in **ZFC**, as we proved it fails in a model of [GaSh:949].

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For every singular cardinal λ, *there is a coloring* $c : \lambda \times \lambda^+ \to \lambda$ *such that for every* $Y \subseteq \lambda^+$ *of full size, there is* $i < \lambda$ *with* $\mathrm{otp}(c[\{i\} \times Y]) = \lambda$.

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Suppose a singular cardinal λ is a strong limit or satisfies $\aleph_\lambda > \lambda$.
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Curiously, the analogous assertion for λ regular is equivalent to $\mathfrak{b}_\lambda = \lambda^+$.

Club guessing

Club guessing

Consider $S := \{\delta < \lambda^+ \mid \mathrm{cf}(\delta) = \mathrm{cf}(\lambda)\}$ for a given singular cardinal λ.

Suppose that $\vec{C} = \langle C_\delta \mid \delta \in S\rangle$ is a sequence such that each C_δ is a club in δ.

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- $\vec{C}$ is guessing clubs iff for every club $D \subseteq \lambda^+$, there is some $\delta \in S$ with $C_\delta \subseteq D$;
-

Theorem (Shelah, 1990's)
There is a $\vec{C} = \langle C_\delta \mid \delta \in S\rangle$ that guesses clubs with $\mathrm{otp}(C_\delta) = \mathrm{cf}(\lambda)$ for all $\delta \in S$.

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- $\vec{C}$ is guessing clubs iff for every club $D \subseteq \lambda^+$, there is some $\delta \in S$ with $C_\delta \subseteq D$;
- $\vec{C}$ is uninhibited iff for club many $\delta \in S$, for every $\mu < \lambda$, $\sup(\mathrm{nacc}(C_\delta) \cap \mathrm{cof}(> \mu)) = \delta$.

Remark. $\mathrm{nacc}(C_\delta)$ stands for the non-accumulation points of C_δ.

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Theorem (Eisworth-Shelah, 2009)
If λ has uncountable cofinality, then it admits an uninhibited club guessing sequence.

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Theorem (Eisworth-Shelah, 2009)
If λ has uncountable cofinality, then it admits an uninhibited club guessing sequence.

Question 8. What about singular cardinals of countable cofinality?

Club guessing ideals

Given a sequence of local clubs $\vec{C} = \langle C_\delta \mid \delta \in S \rangle$, consider the following ideal:

$$J := \{A \subseteq \lambda^+ \mid \exists \text{ club } D \subseteq \lambda^+ \forall \delta \in S \exists \mu < \lambda [\sup(\mathrm{nacc}(C_\delta) \cap \mathrm{cof}(> \mu) \cap D \cap A) < \delta]\}.$$

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Shelah [Sh:365] proved that if there is $B \in J^+$ with $B \subseteq \{\beta < \lambda^+ \mid \mathrm{cf}(\beta) \text{ is not Jónsson}\}$, then λ^+ is not Jónsson.

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The club guessing ideal J is σ-indecomposable for every regular cardinal $\sigma \in \lambda \setminus \{\mathrm{cf}(\lambda)\}$.
$*$ An ideal is *σ-indecomposable* iff it is closed under increasing unions of length σ.

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The club guessing ideal J is σ-indecomposable for every regular cardinal $\sigma \in \lambda \setminus \{\mathrm{cf}(\lambda)\}$. The extent of the failure of weak saturation of indecomposable ideals is studied in Part III of our series Was Ulam Right? (joint work with Inamdar).

Guessing with large order-type

Proposition

Suppose λ *is a singular cardinal, and* $\vec{C} = \langle C_\delta \mid \delta \in S \rangle$ *is a club guessing sequence such that* $\mathrm{otp}(C_\delta) = \lambda$ *for all* $\delta \in S$. *Then* $\lambda^+ \nrightarrow [\lambda^+]^2_\lambda$ *holds.*

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Proof. Fix a partition $S = \biguplus_{\tau<\lambda} S_\tau$ such that $\vec{C} \restriction S_\tau$ guesses clubs for each $\tau < \lambda$.

$*$ This follows from a general partition theorem, see A club guessing toolbox I (w/ Inamdar).

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Recall we may assume λ is the limit of inaccessibles, so $\lambda = \aleph_\lambda$ and we may find a pairwise disjoint sequence $\langle K_\tau \mid \tau < \lambda\rangle$ of cofinal subsets of $\{\mu < \lambda \mid \mathrm{cf}(\mu) = \mu\}$ of order-type $\mathrm{cf}(\lambda)$.

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For all $\tau < \lambda$ and $\delta \in S_\tau$, let $D_\delta := \{C_\delta(i) \mid i \in \mathrm{cl}(K_\tau)\}$. Consider the corresponding ideal:

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For every $\tau < \lambda$, $B_\tau := \{\beta < \lambda^+ \mid \mathrm{cf}(\beta) \in K_\tau\}$ is in J^+. If $\tau \neq \tau'$, then $B_\tau \cap B_{\tau'} = \emptyset$.

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So J admits λ many pairwise disjoint positive sets, and hence $\lambda^+ \nrightarrow [\lambda^+]^2_\lambda$ holds. qed

Guessing with large order-type (cont.)

Consider $S := \{\delta < \lambda^+ \mid \mathrm{cf}(\delta) = \mathrm{cf}(\lambda)\}$ for a given singular cardinal λ.

Question 10. Is there a club guessing sequence $\vec{C} = \langle C_\delta \mid \delta \in S \rangle$ such that $\mathrm{otp}(C_\delta) = \lambda$ for all $\delta \in S$?

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► By Abraham and Shelah [AbSh:182] it is consistent for a regular λ to have λ^{++} many clubs in λ^+ such that the intersection of any λ^+ many of them has size $< \lambda$.

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- By Abraham and Shelah [AbSh:182] it is consistent for a regular λ to have λ^{++} many clubs in λ^+ such that the intersection of any λ^+ many of them has size $< \lambda$.
- See [Sh:186] , [Sh:667] and Gitik-R. (2012) for related work (consistency results on the failure of diamond at successors of singulars).
- An affirmative answer to the 2nd part was shown to follow from $2^\lambda = \lambda^+$ in R. (2015).

Thank you for
Your attention!