A tutorial on Club-Guessing
Part I

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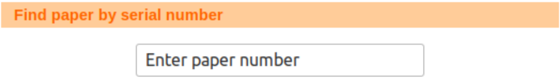
Bibliography

Club guessing played an important role in our joint works with Ari Brodsky, Chris Lambie-Hanson, Tanmay Inamdar, Jing Zhang and others.

Occasionally, I'll be providing a numeral reference to one of these works. To fetch the cited paper, enter the said numeral to the following textbox of my webpage (scroll down to find the box):



Conventions

Throughout this series of talks:

- $\theta < \kappa$ is a pair of infinite regular cardinals;
- E^κ_θ stands for $\{\alpha < \kappa \mid \mathrm{cf}(\alpha) = \theta\}$;
- S denotes a stationary subset of κ.
  Typically, S consists of limits ordinals;
- For $D \subseteq \kappa$, $\mathrm{acc}(D) := \{\delta \in D \mid \sup(D \cap \delta) = \delta > 0\}$,
  and $\mathrm{nacc}(D) := D \setminus \mathrm{acc}(D)$.

Some variations: $E^\kappa_{<\theta}, E^\kappa_{\leq\theta}, E^\kappa_{\neq\theta}, E^\kappa_{>\theta}, E^\kappa_{\geq\theta}$ and
$\mathrm{acc}^+(X) := \{\delta < \sup(X) \mid \sup(X \cap \delta) = \delta > 0\}$.

Motivation

Our starting point is Jensen's diamond principle:

Definition (Jensen, 1972)

$\diamondsuit(S)$ asserts the existence of a sequence $\vec{A} = \langle A_\delta \mid \delta \in S \rangle$ s.t.:

1. for every $\delta \in S$, A_δ is a subset of δ;
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Note that $\diamondsuit(S) \implies \diamondsuit(S')$ whenever $S \subseteq S' \subseteq \kappa$.

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$\diamondsuit$ has numerous applications, e.g., $\diamondsuit(\omega_1) \implies \exists$ a Souslin tree. However, $\diamondsuit(S) \implies \kappa^{<\kappa} = \kappa$ (in fact, $[\kappa]^{<\kappa} \subseteq \{A_\delta \mid \delta \in S\}$) and hence $\diamondsuit(\omega_1)$ fails in Cohen's model.

A weakening of diamond

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Definition (Shelah, 1990's)

CG(S) asserts the existence of a sequence $\vec{C} = \langle C_\delta \mid \delta \in S \rangle$ s.t.:

1. for every $\delta \in S$, C_δ is a club in δ;
2. for every club $D \subseteq \kappa$, the set $\{\delta \in S \mid C_\delta \subseteq D \cap \delta\}$ is stationary in κ.

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As before, $\mathrm{CG}(S) \implies \mathrm{CG}(S')$ whenever $S \subseteq S' \subseteq \kappa$.
Note it is harmless to replace C_δ by some cofinal (not necessarily closed) subset of it.

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1. for every $\delta \in S$, C_δ is a club in δ;
2. for every club $D \subseteq \kappa$, the set $\{\delta \in S \mid C_\delta \subseteq D \cap \delta\}$ is stationary in κ.

Unlike $\diamondsuit$, instances of CG are provable in ZFC. It is also harder to destroy them, as witnesses to CG(S) are preserved by κ-cc forcing.

Why should you care about club-guessing

At the 3rd installment of the Young Set Theory Workshop (Raach, February 2010), I met a fellow brilliant student from Budapest, Dani Soukup. He told me about Gruenhage's problem related to van Douwen's *D-spaces*, asking whether scattered aD-spaces are D. Dani outlined a consistent construction of a counterexample, using MAD families and diamond. I told him of two club guessing theorems of Shelah, suggesting to try to replace $\diamondsuit$ by CG.

By July, he had the ZFC proof. . . :

Constructing *aD*, non-*D*-spaces

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ABSTRACT

We introduce a general method to construct 0-dimensional, scattered T_2 spaces which are not linearly D. The construction is used to show that there are *aD*, non-*D*-spaces, answering a question of Arhangel'skii. The latter example is achieved using Shelah's club guessing principles.

The first theorem

Theorem (Shelah)

If $\aleph_1 < \theta^+ < \kappa$, then $\mathsf{CG}(E^\kappa_\theta)$ holds.

Proof. Let $\vec{C}$ be any *θ-bounded C-sequence over E^κ_θ*, i.e., $\vec{C} = \langle C_\delta \mid \delta \in E^\kappa_\theta \rangle$ with each C_δ a club in δ of order-type θ.

If we are lucky and $\vec{C}$ witnesses $\mathsf{CG}(E^\kappa_\theta)$, then our work is done 😄.

Otherwise, pick a club $D \subseteq \kappa$ such that $\{\delta \in E^\kappa_\theta \mid C_\delta \subseteq D\}$ is nonstationary in κ. For every $\delta \in E^\kappa_\theta \cap \mathrm{acc}(D)$, both C_δ and $D \cap \delta$ are clubs in δ, so it is easy to come up with a corrected sequence $\vec{C}' := \langle C_\delta \cap D \mid \delta \in E^\kappa_\theta \cap \mathrm{acc}(D) \rangle$.

If we are lucky and $\vec{C}'$ witnesses $\mathsf{CG}(E^\kappa_\theta \cap \mathrm{acc}(D))$, then... 😄😄.

Otherwise, pick a club $D' \subseteq \kappa$ such that $\{\delta \in E^\kappa_\theta \cap \mathrm{acc}(D) \mid C_\delta \cap D \subseteq D'\}$ is nonstationary in κ, and let $\vec{C}'' := \langle C_\delta \cap D \cap D' \mid \delta \in E^\kappa_\theta \cap \mathrm{acc}(D \cap D') \rangle$.

If we are lucky and $\vec{C}''$ witnesses $\mathsf{CG}(E^\kappa_\theta \cap \ldots)$, then... 😄😄😄.

Otherwise 😡😡😡😡😡

The first theorem (cont.)

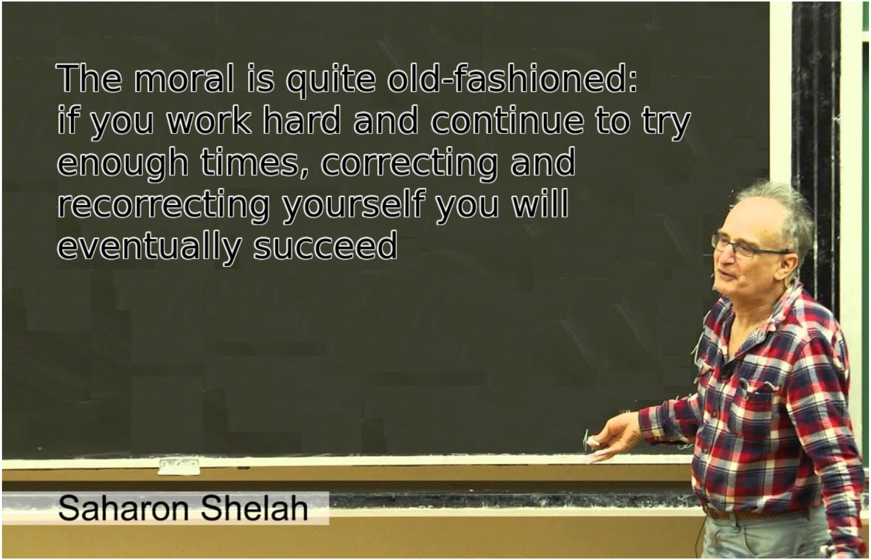
Claim

There exists a club $D \subseteq \kappa$ such that $\langle C_\delta \cap D \mid \delta \in E^\kappa_\theta \cap \mathrm{acc}(D)\rangle$ witnesses $\mathrm{CG}(E^\kappa_\theta \cap \mathrm{acc}(D))$.

Proof. Suppose not. Construct by recursion a sequence of clubs $\langle D_i \mid i < \theta^+\rangle$ such that:

1. $D_0 := \kappa$.
2. For each $i < \kappa$, $D_{i+1} \subseteq D_i$ and $\{\delta \in E^\kappa_\theta \cap \mathrm{acc}(D_i) \mid C_\delta \cap D_i \subseteq D_{i+1}\}$ is disjoint from D_{i+1};
3. For each $i \in \mathrm{acc}(\theta^+)$, $D_i = \bigcap_{j<i} D_j$.

As $\theta^+ < \kappa$, $D^* := \bigcap_{i<\theta^+} D_i$ is a club in κ. Pick $\delta \in E^\kappa_\theta \cap \mathrm{acc}(D^*)$.
Note that $\langle C_\delta \cap D_i \mid i < \theta^+\rangle$ is a descending chain of subsets of C_δ.
As $\mathrm{otp}(C_\delta) = \theta$, for a large enough $i < \theta^+$, $C_\delta \cap D_i = C_\delta \cap D_{i+1}$.
However, $\delta \in E^\kappa_\theta \cap \mathrm{acc}(D^*) \subseteq E^\kappa_\theta \cap \mathrm{acc}(D_i) \cap D_{i+1}$.
This is a contradiction to Clause (2). □


The moral is quite old-fashioned:
if you work hard and continue to try
enough times, correcting and
recorrecting yourself you will
eventually succeed
Saharon Shelah

A closer look

Let $\mathcal{K}(\kappa)$ denote the collection of all subsets $x \subseteq \kappa$ such that $\mathrm{otp}(x) \in \mathrm{acc}(\kappa)$ and such that x is closed below its sup. Note: every member of a C-sequence $\langle C_\delta \mid \delta \in S\rangle$ belongs to $\mathcal{K}(\kappa)$.

Given a club $D \subseteq \kappa$, define a map $\Phi_D : \mathcal{K}(\kappa) \to \mathcal{K}(\kappa)$ via:

$$\Phi_D(x) := \begin{cases} x \cap D, & \text{if } \sup(x \cap D) = \sup(x); \\ x \setminus \sup(x \cap D), & \text{otherwise.} \end{cases}$$

The above argument shows

If S is a stationary subset of $E^\kappa_{>\omega}$, and $\vec{C} = \langle C_\delta \mid \delta \in S\rangle$ is a C-sequence such that $\sup_{\delta \in S} |C_\delta|^+ < \kappa$, then there exists a club $D \subseteq \kappa$ such that $\langle \Phi_D(C_\delta) \mid \delta \in S\rangle$ witnesses $\mathrm{CG}(S)$.

Postprocessing functions

$$\Phi_D(x) := \begin{cases} x \cap D, & \text{if } \sup(x \cap D) = \sup(x); \\ x \setminus \sup(x \cap D), & \text{otherwise.} \end{cases}$$

Definition ([29])

$\Phi : \mathcal{K}(\kappa) \to \mathcal{K}(\kappa)$ is a postprocessing function if for all $x \in \mathcal{K}(\kappa)$:

1. $\Phi(x)$ is a club in $\sup(x)$;
2. $\mathrm{acc}(\Phi(x)) \subseteq \mathrm{acc}(x)$;
3. $\Phi(x) \cap \bar{\alpha} = \Phi(x \cap \bar{\alpha})$ for every $\bar{\alpha} \in \mathrm{acc}(\Phi(x))$.

It is conservative if $\Phi(x) \subseteq x$ for all x.

Note: The second requirement implies that $\mathrm{otp}(\Phi(x)) \leq \mathrm{otp}(x)$.

Postprocessing functions homework

Suppose that $\langle C_\delta \mid \delta \in S\rangle$ is a C-sequence such that, for every club $D \subseteq \kappa$, the set $\{\delta \in S \mid C_\delta \subseteq^* D \cap \delta\}$ is stationary.
Must $\mathsf{CG}(S)$ hold?

Given $\varepsilon < \kappa$, define a postprocessing $\Phi^\varepsilon : \mathcal{K}(\kappa) \to \mathcal{K}(\kappa)$ via:

$$\Phi^\varepsilon(x) := \begin{cases} x \setminus \varepsilon, & \text{if } \sup(x) > \varepsilon; \\ x, & \text{otherwise.} \end{cases}$$

Exercise: Prove that there exists some $\varepsilon < \kappa$ such that $\langle \Phi^\varepsilon(C_\delta) \mid \delta \in S\rangle$ witnesses $\mathsf{CG}(S)$.

Back to where we were

If S is a stationary subset of $E^{\kappa}_{>\omega}$, and $\vec{C} = \langle C_\delta \mid \delta \in S\rangle$ is a C-sequence such that $\sup_{\delta\in S}|C_\delta|^+ < \kappa$, then there exists a club $D \subseteq \kappa$ such that $\langle \Phi_D(C_\delta) \mid \delta \in S\rangle$ witnesses $\mathrm{CG}(S)$.

Thus, $\mathrm{CG}(S)$ holds provided that $S \subseteq E^{\kappa}_{>\omega} \cap E^{\kappa}_{\leq\theta}$ and $\theta^+ < \kappa$. In particular, $\mathrm{ZFC} \vdash \bigwedge_{\kappa\geq\aleph_3} \mathrm{CG}(\kappa)$.

Baumgartner devised a proper forcing for shooting a club through $\aleph_1$, and this shows that $\mathrm{PFA} \implies \neg\,\mathrm{CG}(\aleph_1)$.

- What about $\mathrm{CG}(E^{\aleph_2}_{\aleph_0})$?
- What about $\mathrm{CG}(E^{\aleph_2}_{\aleph_1})$?
- What about $\mathrm{CG}(\mathrm{Reg}(\kappa))$ for some Mahlo cardinal κ?

Taking care of $E^{\aleph_2}_{\aleph_0}$

For every $\delta \in E^{\aleph_2}_{\aleph_0}$, fix a surjection $\psi_\delta : \omega_1 \to \delta$.

Claim

There exists $i < \omega_1$ such that, for every club $D \subseteq \aleph_2$, $\{\delta \in E^{\aleph_2}_{\aleph_0} \mid \sup(\psi_\delta[i] \cap D) = \delta\}$ is stationary in $\aleph_2$.

Proof. Suppose not. It follows that for every $i < \omega_1$, we may fix a sparse enough club $D_i \subseteq \aleph_2$ such that

$$\{\delta \in E^{\aleph_2}_{\aleph_0} \mid \sup(\psi_\delta[i] \cap D_i) = \delta\} \cap D_i = \emptyset.$$

Consider the club $D^* := \bigcap_{i<\omega_1} D_i$ and pick $\delta \in E^{\aleph_2}_{\aleph_0} \cap \mathrm{acc}(D^*)$.
Let d be some cofinal subset of $D^* \cap \delta$ of order-type ω.
Then there exists a large enough $i < \omega_1$ such that $\psi_\delta[i] \supseteq d$.
As $D_i \supseteq D^* \supseteq d$, we infer that $\sup(\psi_\delta[i] \cap D_i) = \sup(d) = \delta$, contradicting the fact that $\delta \in D^* \subseteq D_i$. □

Taking care of $E^{\aleph_2}_{\aleph_0}$ (cont.)

Reformulating the result of the previous slide:
There is a sequence of countable sets $\langle a_\delta \mid \delta \in E^{\aleph_2}_{\aleph_0} \rangle$ such that for every club $D \subseteq \aleph_2$, $\{\delta \in E^{\aleph_2}_{\aleph_0} \mid \sup(a_\delta \cap D) = \delta\}$ is stationary in $\aleph_2$.

So, now we are in conditions to run the familiar argument, i.e., find a club $D \subseteq \aleph_2$ such that $\langle \Phi_D(a_\delta) \mid \delta \in S \rangle$ witnesses $\mathrm{CG}(S)$.

Corollary (Shelah)

$\mathrm{CG}(\kappa)$ *holds for every regular* $\kappa \geq \aleph_2$.

Let us record a feature that the above proof approach secures and is not present in the other proof approach.

Relative club guessing

Theorem (folklore)

For every uncountable cardinal λ, for every stationary $S \subseteq E^{\lambda^+}_{<\lambda}$, for every stationary $T \subseteq \lambda^+$, there exists a C-sequence $\langle C_\delta \mid \delta \in S\rangle$ s.t. for every club $D \subseteq \lambda^+$, the following set is stationary in λ^+:

$$\{\delta \in S \mid C_\delta \subseteq D \ \& \ \mathrm{nacc}(C_\delta) \subseteq T\}.$$

Application: Ulam-type matrices

Ulam vs. Solovay

By a celebrated 1971 theorem of Solovay, every stationary subset S of every regular uncountable cardinal κ may be decomposed into κ many stationary sets, $\langle S_i \mid i < \kappa \rangle$.

We would like to point out that for a successor cardinal $\kappa = \lambda^+$, Solovay's theorem follows from the existence of an Ulam matrix.

Theorem (Ulam, 1930)

There exists a matrix $\langle U_{\eta,\tau} \mid \eta < \lambda, \tau < \lambda^+ \rangle$ such that:

1. *For every $\eta < \lambda$, $\langle U_{\eta,\tau} \mid \tau < \lambda^+ \rangle$ consists of pairwise disjoint subsets of λ^+;*
2. *For every $\tau < \lambda^+$, $\bigcup_{\eta<\lambda} U_{\eta,\tau}$ is co-bounded in λ^+.*

The proof of Ulam's theorem is very simple. Fix a coloring $c : \lambda \times \lambda^+ \to \lambda^+$ such that, for every $\beta \in [\lambda, \lambda^+)$, the fiber map $c(\cdot, \beta) : \lambda \to \beta$ is a bijection. Then let:

$$U_{\eta,\tau} := \{\beta < \lambda^+ \mid c(\eta, \beta) = \tau\}.$$

Ulam vs. Solovay (cont.)

Lemma

Suppose J is a κ-complete ideal over $\kappa = \lambda^+$, extending $[\kappa]^{<\kappa}$. Then, for every $B \in J^+$, there exists some $\eta < \lambda$ such that the following set has size κ:

$$T_\eta(B) := \{\tau < \kappa \mid B \cap U_{\eta,\tau} \in J^+\}.$$

Proof. Suppose not. Then, for every $\eta < \lambda$, $|T_\eta(B)| \leq \lambda$, and hence $|\bigcup_{\eta<\lambda} T_\eta(B)| \leq \lambda$. Pick $\tau \in \lambda^+ \setminus \bigcup_{\eta<\lambda} T_\eta(B)$.
As $\bigcup_{\eta<\lambda} U_{\eta,\tau}$ is co-bounded, it is in the dual of J, so that $B \cap \bigcup_{\eta<\lambda} U_{\eta,\tau}$ is in J^+. Since J is κ-complete, there must exist some $\eta < \lambda$ such that $B \cap U_{\eta,\tau} \in J^+$. So, $\tau \in \bigcup_{\eta<\lambda} T_\eta(B)$, contradicting the choice of τ. □

Is it possible to do better than that? I.e., getting $T_\eta(B) = \kappa$?

Sierpiński vs. Ulam

Theorem (Sierpiński, 1934)

If $2^\lambda = \lambda^+ = \kappa$, then there is a coloring $c : \lambda \times \lambda^+ \to \lambda^+$ such that the induced matrix $\langle U_{\eta,\tau} \mid \eta < \lambda, \tau < \lambda^+ \rangle$ satisfies that for every κ-complete ideal J over κ, extending $[\kappa]^{<\kappa}$, for every $B \in J^+$, there exists some $\eta < \lambda$ such that the following set is equal to κ:

$$T_\eta(B) := \{\tau < \kappa \mid B \cap U_{\eta,\tau} \in J^+\}.$$

A proof of Sierpiński's theorem

Proof. Fix an enumeration $\langle f_\alpha \mid \alpha < \lambda^+\rangle$ of all functions from λ to λ^+. For every $\beta < \lambda^+$, let $c(\cdot, \beta) : \lambda \to \lambda^+$ be such that, for every $\alpha < \beta$, for some $\eta = \eta_{\alpha,\beta} < \lambda$, $c(\eta, \beta) = f_\alpha(\eta)$.
Towards a contradiction, suppose J is a κ-complete ideal extending $[\kappa]^{<\kappa}$ and $B \in J^+$ is such that, for every $\eta < \lambda$, $T_\eta(B) \neq \kappa$.
So, we may define a function $f : \lambda \to \lambda^+$ via:

$$f(\eta) := \min(\kappa \setminus T_\eta(\beta)).$$

Find $\alpha < \lambda^+$ such that $f = f_\alpha$. For every $\beta \in (\alpha, \kappa)$, there exists $\eta_{\alpha,\beta} < \lambda$ such that $c(\eta_{\alpha,\beta}, \beta) = f_\alpha(\eta_{\alpha,\beta})$. As J is κ-complete, there is $\eta < \lambda$ such that $B' := \{\beta \in B \setminus (\alpha + 1) \mid \eta_{\alpha,\beta} = \eta\}$ in J^+.
Denote $\tau := f(\eta)$. Then, for every $\beta \in B'$,

$$c(\eta, \beta) = f_\alpha(\eta) = f(\eta) = \tau,$$

meaning that $B' \subseteq B \cap U_{\eta,\tau}$. As $B' \in J^+$, it follows that $\tau \in T_\eta(\beta)$, contradicting the fact that $f(\eta) = \tau$. □

Recap

Both Ulam's and Sierpiński's matrices provide us with λ many λ^+-partitions of λ^+, but with Sierpiński we moreover know how the final decomposition of a positive set would look like. In contrast, after using Ulam's, we still have to do some further thinning out. This makes a difference, for instance, if we want the partition to lie in some inner model.

The Sierpiński-type improvement of the Solovay-Ulam theorems boils down to the following problem:

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The Sierpiński-type improvement of the Solovay-Ulam theorems boils down to the following problem:
Find a coloring $c : \lambda \times \lambda^+ \to \lambda^+$ such that, for every $B \in [\lambda^+]^{\lambda^+}$, for some $\eta < \lambda$, $c[\{\eta\} \times B] = \lambda^+$.

Recap

Both Ulam's and Sierpiński's matrices provide us with λ many λ^+-partitions of λ^+, but with Sierpiński we moreover know how the final decomposition of a positive set would look like. In contrast, after using Ulam's, we still have to do some further thinning out. This makes a difference, for instance, if we want the partition to lie in some inner model.

The Sierpiński-type improvement of the Solovay-Ulam theorems boils down to the following problem:
Find the θ's for which there is a coloring $c : \lambda \times \lambda^+ \to \theta$ such that, for every $B \in [\lambda^+]^{\lambda^+}$, for some $\eta < \lambda$, $c[\{\eta\} \times B] = \theta$.

A Sierpiński theorem in ZFC

Theorem ([53])

Suppose that λ is a singular cardinal. For every cardinal $\theta < \lambda$, there is a coloring $c : \lambda \times \lambda^+ \to \theta$ such that, for every $B \in [\lambda^+]^{\lambda^+}$, for some $\eta < \lambda$, $c[\{\eta\} \times B] = \theta$.

We shall present the proof in the next talk.

In [53], it is also proved that the case $\theta = \lambda$ may consistently fail.