In praise of C-sequences

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Bibliography

Much of the results presented here come from joint works with Ari Brodsky and Chris Lambie-Hanson.

I will occasionally provide a reference of the form [n], where n is some positive integer. To obtain the cited paper, simply go to

http://assafrinot.com/paper/n

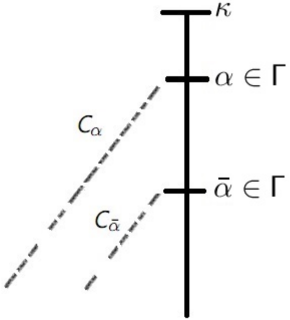
Conventions

- λ denotes an infinite cardinal;
- κ denotes a regular uncountable cardinal (e.g., $\kappa = \lambda^+$);
- $\mathrm{Reg}(\kappa)$ denotes the set of infinite regular cardinals $< \kappa$;
- Γ denotes a stationary subset of κ
  (we often implicitly assume Γ consists only of limit nonzero ordinals);
- $E^\kappa_\chi := \{\alpha < \kappa \mid \mathrm{cf}(\alpha) = \chi\}$;
- $E^\kappa_{>\chi} := \{\alpha < \kappa \mid \mathrm{cf}(\alpha) > \chi\}$;
- etc'...

C-sequences

Definition

A C-sequence over Γ is a sequence $\vec{C} = \langle C_\alpha \mid \alpha \in \Gamma \rangle$ such that, for every $\alpha \in \Gamma$, C_α is a closed subset of α, with $\sup(C_\alpha) = \sup(\alpha)$.



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Thesis

C-sequences successfully capture the combinatorial features of κ.

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Example 1

The type of $\vec{C}$ is the least ordinal ξ satisfying $\mathrm{otp}(C_\alpha) < \xi$ for all $\alpha \in \Gamma$. Clearly, κ is a successor cardinal iff there is a $\vec{C}$ over κ with $\mathrm{type}(\vec{C}) < \kappa$.

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Example 2

The width of $\vec{C}$ is the least cardinal μ satisfying $|\mathcal{G}_\beta(\vec{C})| < \mu$ for all $\beta < \kappa$, where $\mathcal{G}_\beta(\vec{C}) := \{C_\alpha \cap \beta \mid \alpha \in \Gamma, \sup(C_\alpha \cap \beta) = \beta\}$.
Clearly, κ is a strong limit iff for every $\vec{C}$ over κ, $\text{width}(\vec{C}) \leq \kappa$.

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Exercise

Find a C-sequence characterization of the following statements:
κ is Mahlo, κ is weakly compact, κ is ineffable.

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Example 3

$\kappa^{<\kappa} = \kappa$ iff there is a C-sequence $\langle C_\alpha \mid \alpha \in \Gamma \rangle$ satisfying:

- $\{\{\beta < \bar{\alpha} \mid (\beta + 1) \in C_\alpha\} \mid \alpha \in \Gamma, \bar{\alpha} < \alpha\} = [\kappa]^{<\kappa}$.

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Example 4

$\diamondsuit(\Gamma)$ iff there is a C-sequence $\langle C_\alpha \mid \alpha \in \Gamma\rangle$ satisfying the two:

- $\{\{\beta < \bar\alpha \mid (\beta+1) \in C_\alpha\} \mid \alpha \in \Gamma, \bar\alpha < \alpha\} = [\kappa]^{<\kappa}$;
- For every cofinal $A \subseteq \kappa$, there is a nonzero $\alpha \in \Gamma$ with $\{\beta < \alpha \mid (\beta+1) \in C_\alpha\} \subseteq A$.

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C-sequences successfully capture the combinatorial features of κ.

In this series of lectures, we shall present various combinatorial problems, and demonstrate how the C-sequence perspective leads to their solution. We shall also provide a toolbox for manipulating and producing C-sequences.

Graphs

Definition

A graph is a pair $G = (V, E)$, where $E \subseteq [V]^2$.
Elements of V are called the vertices of G;
Elements of E are called the edges of G.

Definition

The chromatic number of $G = (V, E)$, denoted $\mathrm{Chr}(G)$, is the least cardinal θ for which there exists a coloring $f : V \to \theta$ such that:

$$f(x) \neq f(y) \text{ for all } \{x, y\} \in E.$$

Intermediate value theorem?

Suppose that G is a graph of size and chromatic number $\aleph_2$.
Must it contain a subgraph of size and chromatic number $\aleph_1$?

Chain conditions

Definition

Let $\mathbb{P} = (P, \leq)$ denote a partially ordered set (poset).

- $\mathbb{P}$ is said to satisfy the κ-cc iff it has no antichains of size κ, i.e., if for every $A \subseteq P$ of size κ, there exist $a \neq b$ in A that are compatible.
- $\mathbb{P}$ is said to satisfy the κ-Knaster iff for every $A \subseteq P$ of size κ, there is $B \subseteq A$ of size κ such that any two conditions in B are compatible.
- $\mathbb{P}^\theta$ stands for the poset whose elements are functions $f : \theta \to P$, and $f \leq_{\mathbb{P}^\theta} g$ iff $f(\alpha) \leq g(\alpha)$ for all $\alpha < \theta$.

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Fact

Martin's Axiom ($\mathrm{MA}_{\aleph_1}$) *implies that any $\aleph_1$-cc poset $\mathbb{P}$ is $\aleph_1$-Knaster. In particular, $\mathbb{P}^n$ is $\aleph_1$-cc for any positive integer n.*

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Theorem (Shelah, 1997)

There is an $\aleph_2$-cc poset $\mathbb{P}$ such that $\mathbb{P}^2$ does not satisfy $\aleph_2$-cc.

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How about Knaster?

Is there an $\aleph_2$-Knaster poset $\mathbb{P}$ such that $\mathbb{P}^\omega$ does not satisfy $\aleph_2$-cc?

Aronszajn trees

Definition

A (streamlined) κ-Aronszajn tree is a collection $\mathcal{T} \subseteq {}^{<\kappa}2$ satisfying:

- for all $\alpha < \kappa$, the set $\mathcal{T}_\alpha := \{t \in \mathcal{T} \mid \mathrm{dom}(t) = \alpha\}$ has size $< \kappa$;
- for all $\alpha < \kappa$ and $t \in \mathcal{T}$, there is $s \in \mathcal{T}_\alpha$ with $s \cup t \in \mathcal{T}$;
- for all $b : \kappa \to 2$, there is $\alpha < \kappa$ with $b \restriction \alpha \notin \mathcal{T}$.

We think of $\mathcal{T}$ as a set, partially ordered by $\subseteq$.

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A λ^+-Aronszajn tree $\mathcal{T}$ is said to be special iff there exists $f : \mathcal{T} \to \lambda$ such that for every $C \subseteq \mathcal{T}$ linearly ordered by $\subseteq$, $f \restriction C$ is injective.

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Example 5 (Jensen, 1972)

There is a special λ^+-Aronszajn tree iff there is $\vec{C}$ over λ^+ with width$(\vec{C}) \leq \lambda^+$ and type$(\vec{C}) < \lambda^+$.

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Archetypical problem

For an ordinal α, is it consistent with GCH that there is an $\aleph_{\alpha+1}$-Aronszajn tree, but all of them are special?

Fat sets

Definition

A subset $X \subseteq \kappa$ is said to be α-fat iff for every club $D \subseteq \kappa$, there exists a strictly increasing and continuous function $\pi : \alpha \to X \cap D$.
(That is, $X \cap D$ contains a "closed copy" of α.)

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For regular $\theta < \kappa$, if $X \subseteq \kappa$ is $(\theta + 1)$-fat, then $X \cap E^\kappa_\theta$ is stationary.

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Fact

- *(H. Friedman, 1974) A subset of $\aleph_1$ is fat iff it is stationary.*
- *(Ulam, 1930) Every stationary subset of $\aleph_1$ may be split into $\aleph_1$ many stationary sets.*

In particular, any fat subset of $\aleph_1$ may be split into two fat sets.

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How about splitting fat subsets of $\aleph_2$?

Partitioning a fat set

Preliminary: Derived sets

For a cofinal subset $A \subseteq \kappa$, let

- $\mathrm{acc}^+(A) := \{\alpha \in \kappa \mid A \cap \alpha \text{ is unbounded in } \alpha\}$;
- $\mathrm{Tr}(A) := \{\alpha \in E^\kappa_{>\omega} \mid A \cap \alpha \text{ is stationary in } \alpha\}$.

Fact

1. $\mathrm{acc}^+(A)$ *is a club in* κ*;*
2. *If* $A \subseteq E^\kappa_\chi$*, then* $\mathrm{Tr}(A) \subseteq E^\kappa_{>\chi}$*;*
3. *If* D *is a club in* κ*, then* $\mathrm{acc}^+(D) \subseteq D$*;*
4. *If* D *is a club in* κ*, then for every* $\alpha \in \mathrm{acc}^+(D)$*,* $D \cap \alpha$ *is a club in* α*.*

Corollary

$\mathrm{Tr}(A)$ *is stationary in* $\kappa \implies A$ *is stationary in* κ*.*

Proof. Let D be an arbitrary club in κ. We shall prove that $D \cap A \neq \emptyset$.
As $\mathrm{Tr}(A)$ is stationary in κ, let us pick $\alpha \in \mathrm{Tr}(A) \cap \mathrm{acc}^+(D)$.
As $\alpha \in \mathrm{Tr}(A)$, $A \cap \alpha$ is stat. in α; As $\alpha \in \mathrm{acc}^+(D)$, $D \cap \alpha$ is a club in α.
Altogether, $A \cap D \cap \alpha \neq \emptyset$. □

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4. *If* D *is a club in* κ*, then for every* $\alpha \in \mathrm{acc}^+(D)$*,* $D \cap \alpha$ *is a club in* α*.*

Exercise

For every cofinal $A \subseteq \kappa$ and every club $B \subseteq \kappa$, there exists a cofinal $A' \subseteq A$ such that $\mathrm{acc}^+(A') \subseteq B$.

Exercise

For κ weakly compact, A is stationary $\implies$ $\mathrm{Tr}(A)$ is stationary.

Magidor's 1982 model

Proposition

Assuming the consistency of a weakly compact cardinal, it is consistent that $\aleph_2$ cannot be split into two fat sets.

Proof. Starting from a weakly compact cardinal, Magidor constructed a model in which for every stationary $S \subseteq E^{\omega_2}_{\omega}$, there exists a club $D \subseteq \omega_2$ such that $\mathrm{Tr}(S) = \{\delta \in E^{\omega_2}_{\omega_1} \mid S \cap \delta \text{ is stationary in } \delta\}$ covers $D \cap E^{\omega_2}_{\omega_1}$.
Work in this model, and let F_0, F_1 be arbitrary fat subsets of ω_2.
As F_0 is $(\omega + 1)$-fat, $S_0 := F_0 \cap E^{\omega_2}_{\omega}$ is a stationary subset of $E^{\omega_2}_{\omega}$.
So, let D be a club subset of ω_2 such that $D \cap E^{\omega_2}_{\omega_1} \subseteq \mathrm{Tr}(S_0)$.
As F_1 is $(\omega_1 + 1)$-fat, let $\pi : \omega_1 + 1 \to F_1 \cap D$ be a strictly increasing and continuous function.
Put $\delta := \pi(\omega_1)$, and $C := \pi[\omega_1]$, so that $\delta \in D \cap E^{\omega_2}_{\omega_1}$ and C is a club in δ.
As $D \cap E^{\omega_2}_{\omega_1} \subseteq \mathrm{Tr}(S_0)$, we have $\delta \in \mathrm{Tr}(S_0)$. That is, $S_0 \cap \delta$ is stationary.
Consequently, $S_0 \cap C \neq \emptyset$. In particular, $F_0 \cap F_1 \neq \emptyset$. $\square$

Amenable C-sequences

Definition [29]

A C-sequence $\vec{C} = \langle C_\alpha \mid \alpha \in \Gamma \rangle$ is said to be amenable iff for every club $D \subseteq \kappa$, the set $\{\alpha \in \Gamma \mid D \cap \alpha \subseteq C_\alpha\}$ is nonstationary.

Example

Any $\vec{C}$ over a subset of κ with $\mathrm{type}(\vec{C}) < \kappa$, is amenable.
In particular, any successor cardinal carries an amenable C-sequence.

(for every $\xi < \kappa$ and club $D \subseteq \kappa$, $\{\alpha < \kappa \mid \mathrm{otp}(D \cap \alpha) > \xi\}$ is a club in κ)

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Note: If $\vec{C}$ is amenable, then $\vec{C} \restriction \Gamma'$ is amenable for every stationary $\Gamma' \subseteq \Gamma$.

Exercise

If $V = L$, then the following are equivalent for all regular uncountable κ:

- κ carries an amenable C-sequence;
- There is a κ-Kurepa tree.

Amenable C-sequences (cont.)

Recall: An amenable C-sequence over Γ is a seq. $\vec{C} = \langle C_\alpha \mid \alpha \in \Gamma \rangle$ s.t.:

1. for every limit ordinal $\alpha \in \Gamma$, C_α is a club subset of α;
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Proposition

Every stationary $\Gamma \subseteq \kappa$ admits a stationary subset $\Gamma' \subseteq \Gamma$ that carries an amenable C-sequence.

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Claim 1. $\Gamma' := \Gamma \setminus \mathrm{Tr}(\Gamma)$ is stationary.

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Proof. Fix an arbitrary club $D \subseteq \kappa$. We shall show that $\Gamma' \cap D \neq \emptyset$.

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Claim 1. $\Gamma' := \Gamma \setminus \mathrm{Tr}(\Gamma)$ is stationary.
Proof. Fix an arbitrary club $D \subseteq \kappa$. We shall show that $\Gamma' \cap D \neq \emptyset$.
As Γ is stationary, we may let $\alpha := \min(\mathrm{acc}^+(D) \cap \Gamma)$.

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Claim 1. $\Gamma' := \Gamma \setminus \mathrm{Tr}(\Gamma)$ is stationary.
Proof. Fix an arbitrary club $D \subseteq \kappa$. We shall show that $\Gamma' \cap D \neq \emptyset$.
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Amenable C-sequences (cont.)

Recall: An amenable C-sequence over Γ is a seq. $\vec{C} = \langle C_\alpha \mid \alpha \in \Gamma \rangle$ s.t.:

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Suppose $\mathrm{cf}(\alpha) > \omega$. Then also $\mathrm{acc}^+(D \cap \alpha)$ is closed and unbounded in α.
By minimality of α, the club $\mathrm{acc}^+(D \cap \alpha) = \mathrm{acc}^+(D) \cap \alpha$ is disjoint from Γ. In particular, $\alpha \notin \mathrm{Tr}(\Gamma)$, and we are done. $\boxtimes$

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Let D be an arbitrary club in κ. As Γ is stationary, let $\beta := \min(D \cap \Gamma)$.

As $\mathrm{otp}(D) = \kappa$, we may fix $\gamma \in D$ such that $\mathrm{otp}(D \cap \gamma) = \beta + \omega + 1$.

Then $A := \{\alpha \in \Gamma' \mid D \cap \alpha \subseteq C_\alpha\}$ is bounded below γ; For $\alpha \in A \setminus \gamma$:

- If $\mathrm{cf}(\alpha) = \omega$, then $\mathrm{otp}(D \cap \alpha) > \omega = \mathrm{otp}(C_\alpha)$.
- If $\mathrm{cf}(\alpha) > \omega$, then $\beta \in D \cap \alpha \setminus C_\alpha$. ⊠□

Utility of amenability

Lemma

Suppose that $\vec{C} = \langle C_\alpha \mid \alpha \in \Gamma \rangle$ is an amenable C-sequence.
For every stationary $\Omega \subseteq \Gamma$, there exists $i < \kappa$ such that, for all $\tau < \kappa$, $\Omega_{i,\tau} := \{\alpha \in \Omega \mid \min(C_\alpha \setminus i) \geq \tau\}$ is stationary.

Proof. Suppose not. Fix a stationary $\Omega \subseteq \Gamma$ and a function $f : \kappa \to \kappa$ such that, for each $i < \kappa$, $\Omega_{i,f(i)}$ is disjoint from some club, say, D_i.
Consider the club $D := \{\alpha \in \triangle_{i<\kappa} D_i \mid f[\alpha] \subseteq \alpha\}$.
As $\vec{C} \restriction (\Omega \cap D)$ is amenable, we may fix $\alpha \in \Omega \cap D$ with $D \cap \alpha \nsubseteq C_\alpha$.
Pick $\beta \in D \cap \alpha \setminus C_\alpha$. Evidently, $\beta < \alpha$ and $f[\beta] \subseteq \beta$.
For all $i < \beta$, as $\alpha \in D$, we have $\alpha \in D_i$ and hence $\min(C_\alpha \setminus i) < f(i) < \beta$.
So $\{\min(C_\alpha \setminus i) \mid i < \beta\}$ is unbounded in β, while $\beta \notin C_\alpha$.
This is a contradiction. $\square$

Splitting a stationary set

Corollary

Every stationary $\Omega \subseteq \kappa$ may be split into κ many stationary sets.

Proof. By shrinking, we may assume that Ω carries an amenable C-sequence $\langle C_\alpha \mid \alpha \in \Omega \rangle$. Let $i < \kappa$ be given by the preceding Lemma. Then, by Fodor's lemma, for every $\tau < \kappa$, there exists $\tau' \in [\tau, \kappa)$ such that $\{\alpha \in \Omega \mid \min(C_\alpha \setminus i) = \tau'\}$ is stationary. Thus, there is an increasing $h : \kappa \to \kappa$ with $\{\alpha \in \Omega \mid \min(C_\alpha \setminus i) = h(j)\}$ stationary for all $j < \kappa$. $\square$

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Define $\Phi : \mathcal{P}(\kappa) \to \mathcal{P}(\kappa)$ by stipulating:

$$\Phi(x) := \begin{cases} (x \setminus i), & \text{if } \sup(x) > i; \\ x, & \text{otherwise.} \end{cases}$$

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Postprocessing functions

$K(\kappa)$ denotes the set of all $x \in \mathcal{P}(\kappa)$ s.t. x is a club subset of $\sup(x)$.

Definition [29]

$\Phi : \mathcal{K}(\kappa) \to \mathcal{K}(\kappa)$ is a postprocessing function iff for every $x \in \mathcal{K}(\kappa)$:

- $\Phi(x)$ is a club in $\sup(x)$;
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A monoid acting on the class of C-sequences

- The identity map $\mathrm{Id} : \mathcal{K}(\kappa) \to \mathcal{K}(\kappa)$ is a postprocessing function;
- The composition of postprocessing function is a postprocessing func.
- If $\vec{C} = \langle C_\alpha \mid \alpha \in \Gamma \rangle$ is a C-sequence, so is $\vec{C}^\Phi := \langle \Phi(C_\alpha) \mid \alpha \in \Gamma \rangle$. Furthermore, $\mathrm{type}(\vec{C}^\Phi) \leq \mathrm{type}(\vec{C})$ and $\mathrm{width}(\vec{C}^\Phi) \leq \mathrm{width}(\vec{C})$;
- $\vec{C}$ is amenable iff $\vec{C}^\Phi$ is amenable.

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The monoid of postprocessings is closed under various mixing operations.

Example

If $\vec{\Phi} = \langle \Phi_\tau \mid \tau \in T \rangle$ is a sequence of postprocessing functions, then $\mathrm{mix}(\vec{\Phi})$, defined by

$$\mathrm{mix}(\vec{\Phi})(x) = \begin{cases} x, & \text{if } \min(x) \notin T; \\ \Phi_\tau(x), & \text{if } \min(x) = \tau, \end{cases}$$

is a postprocessing function.

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Recall that we have shown

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Suppose that $\Omega \subseteq \Gamma$ is stationary.
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Postprocessing functions

A theorem on disjoint refinements [29]

Suppose that $\vec{C} = \langle C_\alpha \mid \alpha \in \Gamma\rangle$ is an amenable C-sequence.
Suppose that $\langle \Omega_\tau \mid \tau < \lambda\rangle$ is a sequence of stationary subsets of Γ, $\lambda \leq \kappa$.
Then there exists a postprocessing function $\Phi : \mathcal{K}(\kappa) \to \mathcal{K}(\kappa)$ such that, for cofinally many $\tau < \lambda$, $\{\alpha \in \Omega_\tau \mid \min(\Phi(C_\alpha)) = \tau\}$ is stationary.

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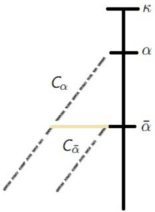
The latter follows from the former by invoking it with a constant κ-sequence.

Square sequences

Definition

$\square(\kappa)$ asserts the existence of a C-sequence $\langle C_\alpha \mid \alpha \in \mathrm{acc}^+(\kappa)\rangle$ such that:

1. for every $\alpha \in \mathrm{acc}^+(\kappa)$ and every $\bar\alpha \in \mathrm{acc}^+(C_\alpha)$, $C_{\bar\alpha} = C_\alpha \cap \bar\alpha$;



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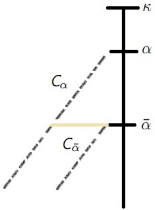

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Fact (Todorcevic, 1987)

If $V \models \neg\square(\kappa)$, then $L \models \kappa$ is weakly compact.

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2. for every club $D \subseteq \kappa$, there exists $\bar\alpha \in \mathrm{acc}^+(D)$ with $C_{\bar\alpha} \neq D \cap \bar\alpha$.

Note: If $\vec{C}$ is $\square(\kappa)$-sequence, then so is $\vec{C}^\Phi$. $\mathrm{width}(\vec{C}^\Phi) \leq \mathrm{width}(\vec{C}) = 2$.
If $D \subseteq \kappa$ is a club satisfying $\Phi(C_{\bar\alpha}) = D \cap \bar\alpha$ for all $\bar\alpha \in \mathrm{acc}^+(D)$, then for all $\bar\alpha < \alpha$ from $\mathrm{acc}^+(D)$, $\bar\alpha \in \mathrm{acc}^+(D \cap \alpha) = \mathrm{acc}^+(\Phi(C_\alpha)) \subseteq \mathrm{acc}^+(C_\alpha)$, and hence $C_{\bar\alpha} = C_\alpha \cap \bar\alpha$.

Square sequences

Recall: $\Phi : \mathcal{K}(\kappa) \to \mathcal{K}(\kappa)$ is a postprocessing function iff for every x:

- $\Phi(x)$ is a club in $\sup(x)$;
- $\mathrm{acc}^+(\Phi(x)) \subseteq \mathrm{acc}^+(x)$;
- $\Phi(x) \cap \bar\alpha = \Phi(x \cap \bar\alpha)$ for every $\bar\alpha \in \mathrm{acc}^+(\Phi(x))$.

Definition

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So $\langle C_{\bar\alpha} \mid \bar\alpha \in \mathrm{acc}^+(D)\rangle$ converges to a club contradicting Clause (2). $\Box$

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Exercise

Any $\square(\kappa)$-sequence is amenable.

Partitioning a fat set

Recall: $F \subseteq \kappa$ is fat iff it is $(\theta+1)$-fat for every $\theta \in \mathrm{Reg}(\kappa)$.

Theorem [29]

Suppose that $\square(\kappa)$ holds, $\kappa \geq \aleph_2$, and that $F \subseteq \kappa$ is fat.
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For each $\tau < \lambda$, let $\Omega_\tau := \{\alpha \in F \cap E^\kappa_{\theta_\tau} \mid F \cap \alpha \text{ contains a club in } \alpha\}$.
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$$\Phi(x) = \begin{cases} x, & \text{if } \min(x) \notin \Theta; \\ \Phi_\theta(x), & \text{if } \min(x) = \theta. \end{cases}$$

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As $\mathrm{cf}(\alpha) = \theta \geq \aleph_1$, e is a club in α.

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As $\mathrm{cf}(\alpha) = \theta \geq \aleph_1$, e is a club in α. Altogether, $e \cup \{\alpha\} \subseteq F_\tau \cap D$. $\square$

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Corollary

The following are equiconsistent:

- *There exists a weakly compact cardinal;*
- *$\aleph_2$ cannot be partitioned into two fat sets.*

Homework: Shelah's club-guessing

Suppose $\vec{C} = \langle C_\alpha \mid \alpha \in \Gamma \rangle$ is a C-sequence with $\Gamma \subseteq \kappa$ stationary. Show:

1. If $|\,\mathrm{type}(\vec{C})|^+ < \kappa$, then there is a postprocessing $\Phi : \mathcal{K}(\kappa) \to \mathcal{K}(\kappa)$ such that for every club $D \subseteq \kappa$, for some $\gamma \in \Gamma$, $\Phi(C_\gamma) \subseteq D$.
2. If $\vec{C}$ is amenable, then there is a postprocessing $\Phi : \mathcal{K}(\kappa) \to \mathcal{K}(\kappa)$ s.t. for every club $D \subseteq \kappa$, for some $\gamma \in \Gamma$, $\sup(\mathrm{nacc}(\Phi(C_\gamma)) \cap D) = \gamma$.

Here, $\mathrm{nacc}(x) := x \setminus \mathrm{acc}^+(x)$.

Productivity of chain conditions

Deriving posets from colorings

From a coloring $d : [\kappa]^2 \to \theta$ with $\theta \in \mathrm{Reg}(\kappa)$, we derive two posets:

- $\mathbb{P} := \{(x, i) \mid x \in [\kappa]^{<\omega}, d``[x]^2 \subseteq \{i\}\}$;
- $\mathbb{Q} := \{(x, i) \mid x \in [\kappa]^{<\omega}, d``[x]^2 \cap i = \emptyset\}$.

(y, j) extends (x, i) iff $y \supseteq x$ and $j = i$.

Key feature

- $\mathbb{P}^2$ fails to have the κ-cc;
- $\mathbb{Q}^\theta$ fails to have the κ-cc.

Deriving posets from colorings

From a coloring $d : [\kappa]^2 \to \theta$ with $\theta \in \mathrm{Reg}(\kappa)$, we derive two posets:

- $\mathbb{P} := \{(x, i) \mid x \in [\kappa]^{<\omega}, d``[x]^2 \subseteq \{i\}\}$;
- $\mathbb{Q} := \{(x, i) \mid x \in [\kappa]^{<\omega}, d``[x]^2 \cap i = \emptyset\}$.

(y, j) extends (x, i) iff $y \supseteq x$ and $j = i$.

Key feature

- $\mathbb{P}^2$ fails to have the κ-cc, e.g., $\{\langle(\{\alpha\}, 0), (\{\alpha\}, 1)\rangle \mid \alpha < \kappa\}$.
- $\mathbb{Q}^\theta$ fails to have the κ-cc.

▶ for $\alpha < \beta < \kappa$, if $(\{\alpha\}, 0)$ and $(\{\beta\}, 0)$ are compatible in $\mathbb{P}$, then $d(\alpha, \beta) = 0$, so that $(\{\alpha\}, 1)$ and $(\{\beta\}, 1)$ are incompatible.

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(y,j) extends (x,i) iff $y \supseteq x$ and $j = i$.

Key feature

- $\mathbb{P}^2$ fails to have the κ-cc, e.g., $\{\langle(\{\alpha\},i) \mid i < 2\rangle \mid \alpha < \kappa\}$.
- $\mathbb{Q}^\theta$ fails to have the κ-cc, e.g., $\{\langle(\{\alpha\},i) \mid i < \theta\rangle \mid \alpha < \kappa\}$.

► for $\alpha < \beta < \kappa$ if $d(\alpha,\beta) = i$, then $(\{\alpha\}, i+1)$ and $(\{\beta\}, i+1)$ are incompatible in $\mathbb{Q}$.

Deriving posets from colorings

From a coloring $d : [\kappa]^2 \rightarrow \theta$ with $\theta \in \mathrm{Reg}(\kappa)$, we derive two posets:

- $\mathbb{P} := \{(x, i) \mid x \in [\kappa]^{<\chi}, d\text{``}[x]^2 \subseteq \{i\}\}$;
- $\mathbb{Q} := \{(x, i) \mid x \in [\kappa]^{<\chi}, d\text{``}[x]^2 \cap i = \emptyset\}$.

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Key feature

- $\mathbb{P}^2$ fails to have the κ-cc, and is χ-closed;
- $\mathbb{Q}^\theta$ fails to have the κ-cc, and is χ-closed.

Deriving posets from colorings

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(y, j) extends (x, i) iff $y \supseteq x$ and $j = i$.

Key feature

- $\mathbb{P}^2$ fails to have the κ-cc, and is χ-closed;
- $\mathbb{Q}^\theta$ fails to have the κ-cc, and is χ-closed.

The heart of the matter is to construct d for which the corresponding $\mathbb{P}$ be κ-cc, or $\mathbb{Q}^\tau$ be κ-Knaster for all $\tau < \theta$.
By a simple reverse-engineering process, one arrives at a reformulation of these features in the language of the coloring d.
For the poset $\mathbb{P}$, see [18]. Today, we shall focus on the poset $\mathbb{Q}$.

Unbounded functions

Suppose $\mathbb{Q} := \{(x, i) \mid x \in [\kappa]^{<\omega}, d``[x]^2 \cap i = \emptyset\}$ derived from $d : [\kappa]^2 \to \theta$. Assuming $\theta \in \mathrm{Reg}(\kappa)$, $\mathbb{Q}$ is κ-Knaster iff d witnesses $\mathsf{U}(\kappa, \theta)$:

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Definition

$\mathrm{U}(\kappa, \theta)$ asserts the existence of a coloring $d : [\kappa]^2 \to \theta$ such that for every family $\mathcal{A} \subseteq [\kappa]^{<\omega}$ consisting of κ-many pairwise disjoint sets, and every $i < \theta$, there is $\mathcal{B} \in [\mathcal{A}]^\kappa$ such that $\min(d[a \times b]) > i$ for all $a < b$ from $\mathcal{B}$.

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Sometimes, one would prefer the χ-closed variation:
$\mathbb{Q} := \{(x, i) \mid x \in [\kappa]^{<\chi}, d\text{``}[x]^2 \cap i = \emptyset\}$.

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$\mathbb{Q} := \{(x, i) \mid x \in [\kappa]^{<\chi}, d``[x]^2 \cap i = \emptyset\}$.

Definition [34]

$\mathrm{U}(\kappa, \theta, \chi)$ asserts there is a coloring $d : [\kappa]^2 \to \theta$ s.t. for every $\chi' < \chi$, every family $\mathcal{A} \subseteq [\kappa]^{\chi'}$ consisting of κ-many pairwise disjoint sets, and every $i < \theta$, there is $\mathcal{B} \in [\mathcal{A}]^\kappa$ s.t. $\min(d[a \times b]) > i$ for all $a < b$ from $\mathcal{B}$.

Unbounded functions

The higher the χ is, the harder it gets.
$\mathrm{U}(\kappa,\theta,2)$ simply asserts there is a coloring $d:[\kappa]^2\rightarrow\theta$ such that for every $A\in[\kappa]^\kappa$ and $i<\theta$, there is $B\in[A]^\kappa$ with $d(\alpha,\beta)>i$ for all $(\alpha,\beta)\in[B]^2$.

Definition [34]

$\mathrm{U}(\kappa,\theta,\chi)$ asserts there is a coloring $d:[\kappa]^2\rightarrow\theta$ s.t. for every $\chi'<\chi$, every family $\mathcal{A}\subseteq[\kappa]^{\chi'}$ consisting of κ-many pairwise disjoint sets, and every $i<\theta$, there is $\mathcal{B}\in[\mathcal{A}]^\kappa$ s.t. $\min(d[a\times b])>i$ for all $a<b$ from $\mathcal{B}$.

Unbounded functions

Exercise

Suppose that $\theta \leq \chi$ are regular cardinals, and $\mu^{<\chi} < \kappa$ for all $\mu < \kappa$.
If $\mathrm{U}(\kappa, \theta, \chi)$ holds, then there exists a χ-closed poset $\mathbb{Q}$ such that:

1. $\mathbb{Q}^\tau$ is κ-Knaster for all $\tau < \theta$.
2. $\mathbb{Q}^\theta$ fails to have the κ-cc.

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$\mathrm{U}(\kappa, \theta, \chi)$ asserts there is a coloring $d : [\kappa]^2 \rightarrow \theta$ s.t. for every $\chi' < \chi$, every family $\mathcal{A} \subseteq [\kappa]^{\chi'}$ consisting of κ-many pairwise disjoint sets, and every $i < \theta$, there is $\mathcal{B} \in [\mathcal{A}]^\kappa$ s.t. $\min(d[a \times b]) > i$ for all $a < b$ from $\mathcal{B}$.

Unbounded functions

Exercise

Suppose that $\theta \leq \chi$ are regular cardinals, and $\mu^{<\chi} < \kappa$ for all $\mu < \kappa$.
If $\mathsf{U}(\kappa, \theta, \chi)$ holds, then there exists a χ-closed poset $\mathbb{Q}$ such that:

1. $\mathbb{Q}^\tau$ is κ-Knaster for all $\tau < \theta$.
2. $\mathbb{Q}^\theta$ fails to have the κ-cc.

The higher the χ is, the harder it gets.
If κ is weakly compact, then $\mathsf{U}(\kappa, \theta, \chi)$ fails already for $\chi = 2$.

Definition [34]

$\mathsf{U}(\kappa, \theta, \chi)$ asserts there is a coloring $d : [\kappa]^2 \rightarrow \theta$ s.t. for every $\chi' < \chi$, every family $\mathcal{A} \subseteq [\kappa]^{\chi'}$ consisting of κ-many pairwise disjoint sets, and every $i < \theta$, there is $\mathcal{B} \in [\mathcal{A}]^\kappa$ s.t. $\min(d[a \times b]) > i$ for all $a < b$ from $\mathcal{B}$.

The C-sequence number

Theorem (Todorcevic, 1987)

For every strongly inaccessible cardinal κ, the following are equivalent:

1. κ is weakly compact;
2. For every C-sequence $\langle C_\beta \mid \beta < \kappa\rangle$, there exist $\Delta \in [\kappa]^\kappa$ and $b : \kappa \to \kappa$ such that $\Delta \cap \alpha = C_{b(\alpha)} \cap \alpha$ for every $\alpha < \kappa$.

The cardinal invariant that we introduce suggests a way to measure how far an inaccessible cardinal κ is from being weakly compact, though, as we will see, it is of interest for successor cardinals as well.

The C-sequence number of κ [35]

If κ is weakly compact, then let $\chi(\kappa) := 0$. Otherwise, let $\chi(\kappa)$ denote the least cardinal $\chi \leq \kappa$ such that, for every C-sequence $\langle C_\beta \mid \beta < \kappa\rangle$, there exist $\Delta \in [\kappa]^\kappa$ and $b : \kappa \to [\kappa]^\chi$ with $\Delta \cap \alpha \subseteq \bigcup_{\beta \in b(\alpha)} C_\beta$ for every $\alpha < \kappa$.

Increasing the C-sequence number

Kunen showed that by forcing over a model with a weakly compact cardinal κ, one obtains a model V having a κ-Souslin tree $\mathbb{S}$ such that $V^{\mathbb{S}} \models \kappa$ is weakly compact.

Proposition

In Kunen's model, $\chi(\kappa) = 1$.

Proof. The κ-Souslin tree witnesses that κ is not weakly compact, so $\chi(\kappa) \neq 0$. Now, let $\vec{C} = \langle C_\beta \mid \beta < \kappa \rangle$ be an arbitrary C-sequence. In $V^{\mathbb{S}}$, $\vec{C}$ is a C-sequence over a weakly compact cardinal κ, and hence there is $\Delta \in [\kappa]^\kappa$ and $b : \kappa \to \kappa$ s.t. $\Delta \cap \alpha = C_{b(\alpha)} \cap \alpha$ for each $\alpha < \kappa$. Clearly, Δ is a club. As $\mathbb{S}$ is κ-cc, there is a club $D \subseteq \kappa$ in V, with $D \subseteq \Delta$. Then $D \cap \alpha \subseteq C_{b(\alpha)} \cap \alpha$ for each $\alpha < \kappa$. □

Theorem [35]

Suppose κ is weakly compact. For every regular cardinal $\theta \leq \kappa$, there is a forcing extension in which κ remains strongly inaccessible, and $\chi(\kappa) = \theta$.

The C-sequence number and yoU

Recall: If κ is not weakly compact, then $\chi(\kappa)$ denotes the least cardinal $\chi \leq \kappa$ such that, for every C-sequence $\langle C_\beta \mid \beta < \kappa \rangle$, there exist $\Delta \in [\kappa]^\kappa$ and $b : \kappa \rightarrow [\kappa]^\chi$ with $\Delta \cap \alpha \subseteq \bigcup_{\beta \in b(\alpha)} C_\beta$ for every $\alpha < \kappa$.

Proposition

$\chi(\kappa) \leq \sup(\mathrm{Reg}(\kappa))$.

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Proof. Clearly, $\chi(\kappa) \leq \kappa$, so suppose $\sup(\mathrm{Reg}(\kappa)) < \kappa$.

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Then $\kappa = \lambda^+$ for $\lambda := \sup(\mathrm{Reg}(\kappa))$.

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Proof. Clearly, $\chi(\kappa) \leq \kappa$, so suppose $\sup(\mathrm{Reg}(\kappa)) < \kappa$.
Then $\kappa = \lambda^+$ for $\lambda := \sup(\mathrm{Reg}(\kappa))$. Let $\langle C_\beta \mid \beta < \kappa \rangle$ be arbitrary.
Then $\Delta := \bigcup_{\beta < \kappa} C_\beta$ is in $[\kappa]^\kappa$ and $|\Delta \cap \alpha| \leq \lambda$ for all $\alpha < \kappa$.

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Then $\Delta := \bigcup_{\beta < \kappa} C_\beta$ is in $[\kappa]^\kappa$ and $|\Delta \cap \alpha| \leq \lambda$ for all $\alpha < \kappa$.
Evidently, there is $b : \kappa \to [\kappa]^\lambda$ such that $\Delta \cap \alpha \subseteq \bigcup_{\beta \in b(\alpha)} C_\beta$ for all $\alpha < \kappa$.

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$\chi(\kappa) \leq \sup(\mathrm{Reg}(\kappa))$.

Proof. Clearly, $\chi(\kappa) \leq \kappa$, so suppose $\sup(\mathrm{Reg}(\kappa)) < \kappa$.
Then $\kappa = \lambda^+$ for $\lambda := \sup(\mathrm{Reg}(\kappa))$. Let $\langle C_\beta \mid \beta < \kappa \rangle$ be arbitrary.
Then $\Delta := \bigcup_{\beta < \kappa} C_\beta$ is in $[\kappa]^\kappa$ and $|\Delta \cap \alpha| \leq \lambda$ for all $\alpha < \kappa$.
Evidently, there is $b : \kappa \to [\kappa]^\lambda$ such that $\Delta \cap \alpha \subseteq \bigcup_{\beta \in b(\alpha)} C_\beta$ for all $\alpha < \kappa$.
So $\chi(\kappa) \leq \lambda$. □

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Recall: If κ is not weakly compact, then $\chi(\kappa)$ denotes the least cardinal $\chi \leq \kappa$ such that, for every C-sequence $\langle C_\beta \mid \beta < \kappa\rangle$, there exist $\Delta \in [\kappa]^\kappa$ and $b : \kappa \to [\kappa]^\chi$ with $\Delta \cap \alpha \subseteq \bigcup_{\beta \in b(\alpha)} C_\beta$ for every $\alpha < \kappa$.

Theorem (Todorcevic, 1987; see also [35])

$\mathrm{U}(\kappa, \omega, \chi(\kappa))$ holds.

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Proof. Fix a C-sequence $\langle C_\beta \mid \beta < \kappa\rangle$ witnessing the value of $\chi(\kappa)$.
In particular, $\sup(C_\beta) = \sup(\beta)$ for all $\beta < \kappa$.

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Proof. Fix a C-sequence $\langle C_\beta \mid \beta < \kappa\rangle$ witnessing the value of $\chi(\kappa)$.
Define $d : [\kappa]^2 \to \omega$ as follows.

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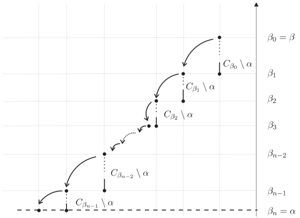
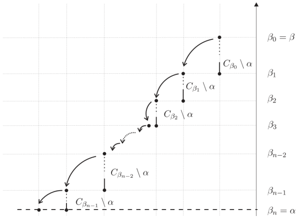
Proof. Fix a C-sequence $\langle C_\beta \mid \beta < \kappa\rangle$ witnessing the value of $\chi(\kappa)$. Define $d : [\kappa]^2 \to \omega$ as follows. Given $\alpha < \beta < \kappa$, recursively define:

The C-sequence number and yoU



Proof. Fix a C-sequence $\langle C_\beta \mid \beta < \kappa \rangle$ witnessing the value of $\chi(\kappa)$. Define $d : [\kappa]^2 \to \omega$ as follows. Given $\alpha < \beta < \kappa$, recursively define: $\beta_0 := \beta$.

The C-sequence number and yoU



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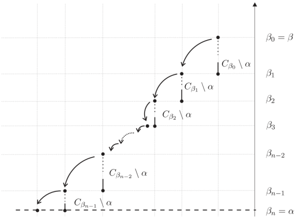
The C-sequence number and yoU



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Recall

U(κ,θ,χ) asserts there is a coloring $d:[\kappa]^2\rightarrow\theta$ s.t. for every $\chi'<\chi$, every family $\mathcal{A}\subseteq[\kappa]^{\chi'}$ consisting of κ-many pairwise disjoint sets, and every $i<\theta$, there is $\mathcal{B}\in[\mathcal{A}]^\kappa$ s.t. $\min(d[a\times b])>i$ for all $a<b$ from $\mathcal{B}$.

Proof. Fix a C-sequence $\langle C_\beta\mid\beta<\kappa\rangle$ witnessing the value of $\chi(\kappa)$. Define $d:[\kappa]^2\rightarrow\omega$ as follows. Given $\alpha<\beta<\kappa$, recursively define: $\beta_0:=\beta$. If $\beta_n>\alpha$, let $\beta_{n+1}:=\min(C_{\beta_n}\setminus\alpha)$. Otherwise, let $d(\alpha,\beta):=n$. We claim that d witnesses U$(\kappa,\omega,\chi(\kappa))$.

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Recall

$\mathrm{U}(\kappa,\theta,\chi)$ asserts there is a coloring $d:[\kappa]^2\rightarrow\theta$ s.t. for every $\chi'<\chi$, every family $\mathcal{A}\subseteq[\kappa]^{\chi'}$ consisting of κ-many pairwise disjoint sets, and every $i<\theta$, there is $\mathcal{B}\in[\mathcal{A}]^\kappa$ s.t. $\min(d[a\times b])>i$ for all $a<b$ from $\mathcal{B}$.

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$\beta_0:=\beta$. If $\beta_n>\alpha$, let $\beta_{n+1}:=\min(C_{\beta_n}\setminus\alpha)$. Otherwise, let $d(\alpha,\beta):=n$.
We claim that d witnesses $\mathrm{U}(\kappa,\omega,\chi(\kappa))$.
We prove by induction on $i<\omega$ that for all $\chi'<\chi(\kappa)$ and $\mathcal{A}\subseteq[\kappa]^{\chi'}$ consisting of κ-many pairwise disjoint sets., there is $\mathcal{B}\in[\mathcal{A}]^\kappa$ such that $\min(d[a\times b])\geq i$ for all $a<b$ from $\mathcal{B}$.

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The C-sequence number and yoU

Recall: If κ is not weakly compact, then $\chi(\kappa)$ denotes the least cardinal $\chi \leq \kappa$ such that, for every C-sequence $\langle C_\beta \mid \beta < \kappa \rangle$, there exist $\Delta \in [\kappa]^\kappa$ and $b : \kappa \to [\kappa]^\chi$ with $\Delta \cap \alpha \subseteq \bigcup_{\beta \in b(\alpha)} C_\beta$ for every $\alpha < \kappa$.

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We prove d witnesses $\mathrm{U}(\kappa, \omega, \chi(\kappa))$ by induction on $i < \omega$.
Fix $\chi < \chi(\kappa)$ and $\mathcal{A} \subseteq [\kappa]^\chi$ consisting of κ-many pairwise disjoint sets.
Note that $S := \{\gamma \in E^\kappa_{>\chi} \mid \exists a \in \mathcal{A}[\gamma < a \ \& \ \gamma \notin \bigcup_{\beta \in a} C_\beta]\}$ is stationary.

The C-sequence number and yoU

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So $\beta_1 := \min(C_\beta \setminus \alpha) = \min(C_\beta \setminus \delta)$ and $d(\alpha, \beta) = d(\alpha, \beta_1) + 1 \geq i + 1$.$\square$

Analysis of the C-sequence number

Exercise

1. $\mathrm{cf}(\lambda) \leq \chi(\lambda^+) \leq \lambda$.

In particular, $\chi(\lambda^+) = \lambda$ whenever λ is regular.

Analysis of the C-sequence number

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1. $\mathrm{cf}(\lambda) \leq \chi(\lambda^+) \leq \lambda$.
2. If $\square(\kappa)$ holds, then $\chi(\kappa) = \sup(\mathrm{Reg}(\kappa))$.

In particular, it is consistent for $\chi(\kappa)$ to be a singular cardinal.

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Every stationary subset of $E^{\kappa}_{>\chi(\kappa)}$ reflects.

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Every stationary subset of $E^{\kappa}_{>\chi(\kappa)}$ reflects.

Proof. Suppose not. Fix $S \subseteq E^{\kappa}_{>\chi(\kappa)}$ stationary with $\mathrm{Tr}(S) = \emptyset$.

Analysis of the C-sequence number

Exercise

1. $\mathrm{cf}(\lambda) \leq \chi(\lambda^+) \leq \lambda$.
2. If $\square(\kappa)$ holds, then $\chi(\kappa) = \sup(\mathrm{Reg}(\kappa))$.
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Fix $A \in [E^{\kappa}_{>\chi(\kappa)}]^\kappa$ and $\epsilon < \kappa$ with $\sup(b(\alpha) \cap \alpha) = \epsilon$ for all $\alpha \in A$.

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Fix $A \in [E^{\kappa}_{>\chi(\kappa)}]^\kappa$ and $\epsilon < \kappa$ with $\sup(b(\alpha) \cap \alpha) = \epsilon$ for all $\alpha \in A$.
Fix $\delta \in \mathrm{acc}^+(\Delta \setminus \epsilon) \cap S$ and $\alpha \in A$ above δ.

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Fix $\delta \in \mathrm{acc}^+(\Delta \setminus \epsilon) \cap S$ and $\alpha \in A$ above δ. As $\mathrm{cf}(\delta) > |b(\alpha)|$, find $\beta \in b(\alpha)$ with $\sup(C_\beta \cap \delta) = \delta$.

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1. $\mathrm{cf}(\lambda) \leq \chi(\lambda^+) \leq \lambda$.
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Fix $\Delta \in [\kappa]^\kappa$ and $b : \kappa \to [\kappa]^{\chi(\kappa)}$ with $\Delta \cap \alpha \subseteq \bigcup_{\beta\in b(\alpha)} C_\beta$ for all $\alpha < \kappa$.
Fix $A \in [E^{\kappa}_{>\chi(\kappa)}]^\kappa$ and $\epsilon < \kappa$ with $\sup(b(\alpha) \cap \alpha) = \epsilon$ for all $\alpha \in A$.
Fix $\delta \in \mathrm{acc}^+(\Delta \setminus \epsilon) \cap S$ and $\alpha \in A$ above δ. As $\mathrm{cf}(\delta) > |b(\alpha)|$, find $\beta \in b(\alpha)$ with $\sup(C_\beta \cap \delta) = \delta$. Then $\beta \geq \alpha > \delta$ and $\delta \in \mathrm{acc}^+(C_\beta) \cap S$. $\square$

Analysis of the C-sequence number

Exercise

1. $\mathrm{cf}(\lambda) \leq \chi(\lambda^+) \leq \lambda$.
2. If $\square(\kappa)$ holds, then $\chi(\kappa) = \sup(\mathrm{Reg}(\kappa))$.
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Note that, under $V = L$, $\chi(\kappa) > 0 \implies \chi(\kappa) = \sup(\mathrm{Reg}(\kappa))$.

Lemma

Every stationary subset of $E^{\kappa}_{>\chi(\kappa)}$ reflects.

Corollary [35]

If κ is a successor, or if $\square(\kappa)$ holds, or if there is a non-reflecting stationary subset of κ, then there is a κ-Knaster poset $\mathbb{Q}$ for which $\mathbb{Q}^\omega$ is not κ-cc.

In particular, there is an $\aleph_2$-Knaster poset $\mathbb{Q}$ such that $\mathbb{Q}^\omega$ is not $\aleph_2$-cc.

Increasing at the level of successors of singulars

Theorem [35]

If λ is a singular limit of supercompact cardinals, then $\chi(\lambda^+) = \mathrm{cf}(\lambda)$.

Theorem [35]

If λ is a singular limit of supercompact cardinals, and $\theta \in \mathrm{Reg}(\lambda) \setminus \mathrm{cf}(\lambda)$, then in some cofinality-preserving forcing extension, $\chi(\lambda^+) = \theta$.

Supercompact cardinals

Lemma. Suppose $\vec{C}$ is a C-sequence over κ.
If $\delta < \kappa$ is supercompact, then there is $A \in [\kappa]^\kappa$ such that for every $B \in [A]^{<\delta}$, there is $\beta < \kappa$ with $B \subseteq C_\beta$.

Proof. Let U be a normal, fine ultrafilter over $\mathcal{P}_\delta(\kappa)$, and let $j : V \to M \cong \mathrm{Ult}(V, U)$ be the corresponding ultrapower map.
Recall that $\mathrm{crit}(j) = \delta$, $j(\delta) > \kappa$, and ${}^\kappa M \subseteq M$.
Let $\langle D_\beta \mid \beta < j(\kappa)\rangle$ denote the enumeration of $j(\vec{C})$. Let $\gamma := \sup(j``\kappa)$, and let $A := \{\alpha < \kappa \mid j(\alpha) \in D_\gamma\}$. Since $j$ is continuous at ordinals of cofinality less than $\delta$, and since $D_\gamma$ is club in $\gamma$, it follows that $j``\kappa$ is $<\delta$-club in γ, and hence A is $<\delta$-club in κ. In particular, $|A| = \kappa$.
Let $\alpha \in A$ be arbitrary, and let $X_\alpha := \{x \in \mathcal{P}_\delta(\kappa) \mid \alpha \in C_{\sup(x)}\}$.
As $j(\alpha) \in D_\gamma$, we have $j``\kappa \in \{z \in \mathcal{P}_{j(\delta)}(j(\kappa)) \mid j(\alpha) \in D_{\sup(z)}\} = j(X_\alpha)$, and thus $X_\alpha \in U$.
Finally, for every $B \in [A]^{<\delta}$, use the δ-completeness of U to find $x \in \bigcap_{\alpha \in B} X_\alpha$, and note that $B \subseteq C_\beta$ for $\beta := \sup(x)$. □

Successors of singulars

Corollary [35]

If λ is a singular limit of supercompact cardinals, then $\chi(\lambda^+) = \mathrm{cf}(\lambda)$.

Proof. To see that $\chi(\lambda^+) \leq \mathrm{cf}(\lambda)$, fix a C-sequence $\vec{C} = \langle C_\beta \mid \beta < \lambda^+ \rangle$.
Fix an increasing sequence $\langle \lambda_i \mid i < \mathrm{cf}(\lambda) \rangle$ of supercompacts, $\nearrow \lambda$.
By the Lemma, for each $i < \mathrm{cf}(\lambda)$, let us pick $A_i \in [\lambda^+]^{\lambda^+}$ such that for every $B \in [A]^{<\lambda_i}$, for some $\beta < \lambda^+$, $B \subseteq C_\beta$.
Consider the club $\Delta := \bigcap_{i<\mathrm{cf}(\lambda)} \mathrm{acc}^+(A_i)$, and let $\alpha < \lambda^+$ be arbitrary.
We shall find $\langle \beta_i \mid i < \mathrm{cf}(\lambda) \rangle$ such that $\Delta \cap \alpha \subseteq \bigcup_{i<\mathrm{cf}(\lambda)} C_{\beta_i}$.
By increasing α, we may assume that $\mathrm{otp}(\Delta \cap \alpha) = \alpha$ and $\mathrm{cf}(\alpha) = \omega$.
Now, by definition of $\Delta \cap \alpha$, let us fix $\langle B_i \mid i < \mathrm{cf}(\lambda) \rangle$ such that

- for every $i < \mathrm{cf}(\lambda)$, $B_i \in [A_i]^{<\lambda_i}$ and $\sup(B_i) = \alpha$;
- $\Delta \cap \alpha = \bigcup_{i<\mathrm{cf}(\lambda)} \mathrm{acc}^+(B_i)$.

For each $i < \mathrm{cf}(\lambda)$, pick $\beta_i < \lambda^+$ such that $B_i \subseteq C_{\beta_i}$. As C_{β_i} is closed below α, we also have $\mathrm{acc}^+(B_i) \subseteq C_{\beta_i}$. So $\langle \beta_i \mid i < \mathrm{cf}(\lambda) \rangle$ is as sought. $\square$

Chromatic number of graphs - large gaps

Compactness and incompactness of chromatic number

Recall: A graph is a pair $G = (V, E)$, where $E \subseteq [V]^2$.
V is the set of *vertices* of G, and E is the set of *edges* of G.
The *chromatic number* of G, denoted $\mathrm{Chr}(G)$, is the least cardinal θ for which there exists a coloring $f : V \to \theta$ such that:

$$f(x) \neq f(y) \text{ for all } \{x, y\} \in E.$$

Theorem (Baumgartner, 1984)

It is consistent with GCH that there exists a graph of size and chromatic number $\aleph_2$ containing no subgraphs of chromatic number $\aleph_1$.

Theorem (Foreman-Laver, 1988)

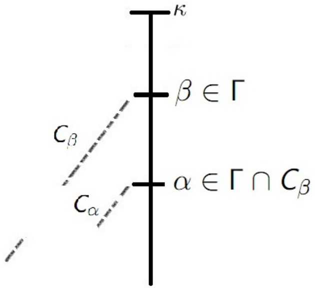
Assuming the consistency of a huge cardinal, it is consistent that GCH holds and any graph of size and chromatic number $\aleph_2$ contains a subgraph of size and chromatic number $\aleph_1$.

The C-sequence graph

Definition [12]

Given a C-sequence $\vec{C} = \langle C_\alpha \mid \alpha \in \Gamma \rangle$, define a graph $G(\vec{C}) := (\Gamma, E)$ by:

$$E := \{\{\alpha, \beta\} \in [\Gamma]^2 \mid \alpha \in C_\beta, \min(C_\alpha) > \sup(C_\beta \cap \alpha) \geq \min(C_\beta)\}.$$



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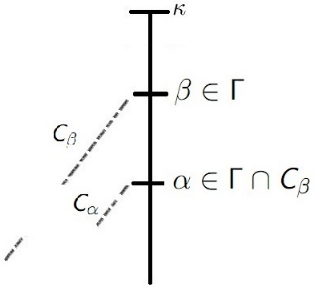
Exercise

Show that $G(\vec{C})$ is triangle free.
I.e., for all $\alpha < \beta < \gamma$, $\{\alpha, \beta, \gamma\}^2 \nsubseteq E$



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Recall

The type of a C-sequence $\vec{C} = \langle C_\alpha \mid \alpha \in \Gamma \rangle$ is the least ordinal ξ satisfying $\mathrm{otp}(C_\alpha) < \xi$ for all $\alpha \in \Gamma$.

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Exercise

For any cardinal θ, if $\mathrm{type}(\vec{C}) \leq \theta$, then $\mathrm{Chr}(G(\vec{C})) \leq \theta$.

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Definition

A C-sequence $\vec{C} = \langle C_\alpha \mid \alpha \in \Gamma \rangle$ is said to be coherent iff for every $\alpha \in \Gamma$ and $\bar{\alpha} \in \mathrm{acc}^+(C_\alpha)$, we have $\bar{\alpha} \in \Gamma$ and $C_{\bar{\alpha}} = C_\alpha \cap \bar{\alpha}$.

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Lemma [28]

If $\vec{C}$ is a coherent C-sequence over κ, then any small subgraph of $G(\vec{C})$ (i.e., of size $< \kappa$) is countably chromatic.

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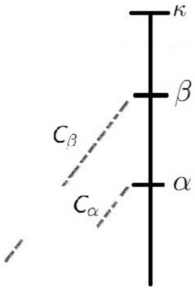
Large gaps for free [28]

Let $\vec{C}$ be a generic coherent C-sequence over κ.
Then $G(\vec{C})$ has chromatic number κ, but all of its small subgraphs are countably chromatic.

Initial segments are countably chromatic

Let $\vec{C} = \langle C_\alpha \mid \alpha \in \mathrm{acc}^+(\kappa)\rangle$ be a coherent C-sequence.
Recall that $E := \{\{\alpha, \beta\} \mid \alpha \in N_\beta\}$, where
$N_\beta := \{\alpha \in C_\beta \mid \min(C_\alpha) > \sup(C_\beta \cap \alpha) \geq \min(C_\beta)\}$.



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We shall show that for every $\gamma < \kappa$, there is a suitable coloring $f : \gamma \to \omega$:
(1) $f(\alpha) \neq f(\beta)$ for all $\{\alpha, \beta\} \in E$, and (2) $f[N_\delta]$ is finite for all $\delta < \kappa$.

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Observation: It suffices to verify that $f[N_\delta]$ is finite for all $\delta \in E^{\gamma+1}_\omega$.

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Observation: It suffices to verify that $f[N_\delta]$ is finite for all $\delta \in E^{\gamma+1}_\omega$.
Proof. If $\delta < \kappa$, and $f[N_\delta]$ is infinite, then there is $I \in [N_\delta \cap \gamma]^\omega$ on which f is injective. Put $\bar{\delta} := \sup(I)$. So $I \subseteq N_\delta \cap \bar{\delta} = N_{\bar{\delta}}$ and $f[N_{\bar{\delta}}]$ is infinite.
However, $\bar{\delta} \in E^{\gamma+1}_\omega$. □

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Suitable: $f(\alpha) \neq f(\beta)$ for all $\{\alpha, \beta\} \in E$, and $f[N_\delta]$ is finite for all $\delta < \kappa$.
Claim. For every $\bar{\gamma} < \gamma < \kappa$, suitable $\bar{f} : \bar{\gamma} \to \omega$ and $x \in [\omega]^\omega$, there is a suitable $f : \gamma \to \omega$ extending $\bar{f}$ with $f[\gamma \setminus \bar{\gamma}] \subseteq x$.

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Let $\bar f : \bar\gamma \to \omega$ be suitable with $\bar\gamma < \gamma$, and let $x \in [\omega]^\omega$ be arbitrary.
We need to find a suitable $f : \gamma \to \omega$ extending $\bar f$ such that $f[\gamma \setminus \bar\gamma] \subseteq x$.

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Claim. For every $\bar{\gamma} < \gamma < \kappa$, suitable $\bar{f} : \bar{\gamma} \to \omega$ and $x \in [\omega]^\omega$, there is a suitable $f : \gamma \to \omega$ extending $\bar{f}$ with $f[\gamma \setminus \bar{\gamma}] \subseteq x$.

Proof. By induction. Suppose $\gamma < \kappa$ and the claim holds for $\gamma' < \kappa$.
Let $\bar{f} : \bar{\gamma} \to \omega$ be suitable with $\bar{\gamma} < \gamma$, and let $x \in [\omega]^\omega$ be arbitrary.
We need to find a suitable $f : \gamma \to \omega$ extending $\bar{f}$ such that $f[\gamma \setminus \bar{\gamma}] \subseteq x$.

- If $\gamma = \gamma' + 1$, then $y := \bar{f}[N_{\gamma'}]$ is finite, and we may find $\xi \in x \setminus y$.

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- If $\gamma = \gamma' + 1$, then $y := \bar{f}[N_{\gamma'}]$ is finite, and we may find $\xi \in x \setminus y$.

Pick a suitable $f' : \gamma' \to \omega$ extending $\bar{f}$ with $f'[\gamma' \setminus \bar{\gamma}] \subseteq (x \setminus \{\xi\})$.

Initial segments are countably chromatic

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Recall that $E := \{\{\alpha, \beta\} \mid \alpha \in N_\beta\}$, where
$N_\beta := \{\alpha \in C_\beta \mid \min(C_\alpha) > \sup(C_\beta \cap \alpha) \geq \min(C_\beta)\}$.
Suitable: $f(\alpha) \neq f(\beta)$ for all $\{\alpha, \beta\} \in E$, and $f[N_\delta]$ is finite for all $\delta < \kappa$.
Claim. For every $\bar{\gamma} < \gamma < \kappa$, suitable $\bar{f} : \bar{\gamma} \to \omega$ and $x \in [\omega]^\omega$, there is a suitable $f : \gamma \to \omega$ extending $\bar{f}$ with $f[\gamma \setminus \bar{\gamma}] \subseteq x$.
Proof. By induction. Suppose $\gamma < \kappa$ and the claim holds for $\gamma' < \kappa$.
Let $\bar{f} : \bar{\gamma} \to \omega$ be suitable with $\bar{\gamma} < \gamma$, and let $x \in [\omega]^\omega$ be arbitrary.
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- If $\gamma = \gamma' + 1$, then $y := \bar{f}[N_{\gamma'}]$ is finite, and we may find $\xi \in x \setminus y$.
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Claim. For every $\bar\gamma < \gamma < \kappa$, suitable $\bar f : \bar\gamma \to \omega$ and $x \in [\omega]^\omega$, there is a suitable $f : \gamma \to \omega$ extending $\bar f$ with $f[\gamma \setminus \bar\gamma] \subseteq x$.

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Also, $f[N_\delta] = f'[N_\delta]$ is finite for all $\delta \in E_\omega^{\gamma+1} = E_\omega^{\gamma'+1}$.
Finally, if $\{\alpha,\beta\} \in E$ and $f(\alpha) = f(\beta)$, then by $\mathrm{dom}(f) = \mathrm{dom}(f') \cup \{\gamma'\}$ we may assume that $\beta = \gamma'$ and $\alpha \in N_{\gamma'}$. So, $f(\beta) = \xi$.

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    - If $\alpha < \bar\gamma$, then $f(\alpha) = \bar f(\alpha) \in y$. So, $f(\alpha) \neq \xi$.
    - If $\bar\gamma \leq \alpha < \gamma'$, then $f(\alpha) \in f'[\gamma' \setminus \bar\gamma]$. So, $f(\alpha) \neq \xi$.

Initial segments are countably chromatic (cont.)

Let $\bar{f} : \bar{\gamma} \rightarrow \omega$ be suitable with $\bar{\gamma} < \gamma$, and let $x \in [\omega]^\omega$ be arbitrary.
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Evidently, if we succeed, then f_γ would be as sought.

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As $g \restriction \eta' = f_{\eta'}$ and $f_{\eta'}^{-1}\{\xi\} \setminus \bar{\gamma} = N_\gamma \setminus \bar{\gamma}$, we have $\bar{f} = f_{\bar{\gamma}} \subseteq f_{\eta'} = g \restriction \eta' \subseteq f_\eta$.

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For $\delta < \kappa$, $f_\eta[N_\delta] \subseteq g[N_\delta] \cup \{\xi\}$ is finite. Also, $f_\eta[\eta \setminus \bar{\gamma}] \subseteq g[\eta \setminus \bar{\gamma}] \cup \{\xi\} \subseteq x$.

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Finally, if $\{\alpha, \beta\} \in E$ and $f_\eta(\alpha) = f_\eta(\beta)$, then the latter equals ξ.

Initial segments are countably chromatic (cont.)

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For $\delta < \kappa$, $f_\eta[N_\delta] \subseteq g[N_\delta] \cup \{\xi\}$ is finite. Also, $f_\eta[\eta \setminus \bar{\gamma}] \subseteq g[\eta \setminus \bar{\gamma}] \cup \{\xi\} \subseteq x$.
Finally, if $\{\alpha, \beta\} \in E$ and $f_\eta(\alpha) = f_\eta(\beta)$, then the latter equals ξ.
Say $\alpha < \beta$. Then $\alpha \in N_\beta$ and $\beta \geq \bar{\gamma}$.

Initial segments are countably chromatic (cont.)

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As $g \restriction \eta' = f_{\eta'}$ and $f_{\eta'}^{-1}\{\xi\} \setminus \bar{\gamma} = N_\gamma \setminus \bar{\gamma}$, we have $\bar{f} = f_{\bar{\gamma}} \subseteq f_{\eta'} = g \restriction \eta' \subseteq f_\eta$.
For $\delta < \kappa$, $f_\eta[N_\delta] \subseteq g[N_\delta] \cup \{\xi\}$ is finite. Also, $f_\eta[\eta \setminus \bar{\gamma}] \subseteq g[\eta \setminus \bar{\gamma}] \cup \{\xi\} \subseteq x$.
Finally, if $\{\alpha, \beta\} \in E$ and $f_\eta(\alpha) = f_\eta(\beta)$, then the latter equals ξ.
Say $\alpha < \beta$. Then $\alpha \in N_\beta$ and $\beta \geq \bar{\gamma}$. So, $\beta \in N_\gamma \setminus \bar{\gamma}$.

Initial segments are countably chromatic (cont.)

Let $\bar{f} : \bar{\gamma} \to \omega$ be suitable with $\bar{\gamma} < \gamma$, and let $x \in [\omega]^\omega$ be arbitrary.
We need to find a suitable $f : \gamma \to \omega$ extending $\bar{f}$ such that $f[\gamma \setminus \bar{\gamma}] \subseteq x$.

▶ If $\gamma \in \mathrm{acc}^+(\kappa)$, then by extending $\bar{f}$ (using the induction hypothesis), we may assume that $\bar{\gamma} \in C_\gamma$. Put $y := \bar{f}[N_{\bar{\gamma}}]$ and pick $\xi \in x \setminus y$.
Construct a chain of suitable colorings $\{f_\eta : \eta \to \omega \mid \eta \in (C_\gamma \setminus \bar{\gamma}) \cup \{\gamma\}\}$ such that: (i) $\bar{f} \subseteq f_\eta$, (ii) $f_\eta[\eta \setminus \bar{\gamma}] \subseteq x$, (iii) $f_\eta^{-1}\{\xi\} \setminus \bar{\gamma} = N_\gamma \setminus \bar{\gamma}$.

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Say $\alpha < \beta$. Then $\alpha \in N_\beta$ and $\beta \geq \bar{\gamma}$. So, $\beta \in N_\gamma \setminus \bar{\gamma}$. As $\xi \notin y$, $\beta \neq \bar{\gamma}$.

Initial segments are countably chromatic (cont.)

Let $\bar{f} : \bar{\gamma} \to \omega$ be suitable with $\bar{\gamma} < \gamma$, and let $x \in [\omega]^\omega$ be arbitrary.
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Initial segments are countably chromatic (cont.)

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Initial segments are countably chromatic (cont.)

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Then $\alpha \in N_\gamma \setminus \bar{\gamma}$. However, $G(\vec{C})$ triangle-free!!

Initial segments are countably chromatic (cont.)

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So $c_\eta[N_\delta] = c_\eta[N_\gamma] = \bar{c}[N_\delta] \cup c_\eta[N_\gamma \setminus \bar{\gamma}] = \bar{c}[N_\delta] \cup \{\xi\}$ is finite. □

Chromatic number of the C-sequence graph

A typical feature of a generic coherent C-sequence

For every sequence $\langle A_i \mid i < \theta \rangle$ of cofinal subsets of κ, with $\theta < \kappa$, there is $\gamma \in \mathrm{acc}^+(\kappa)$ such that for all $i < \theta$:

- $\min(A_i) \leq \min(C_\gamma)$;
- there is $\alpha \in C_\gamma \cap A_i$ such that $\min(C_\gamma \setminus (\alpha + 1)) \in A_i$.

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Claim. Suppose for every sequence $\langle A_i \mid i < \theta\rangle$ of cofinal subsets of κ, with $\theta < \kappa$, there is $\gamma \in \Gamma$ such that for all $i < \theta$:

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Then $\mathrm{Chr}(G(\vec{C})) = \kappa$.

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Proof. Suppose $\mathrm{Chr}(G(\vec{C})) = \theta < \kappa$, as witnessed by $f : \Gamma \to \theta$.

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Proof. Suppose $\mathrm{Chr}(G(\vec{C})) = \theta < \kappa$, as witnessed by $f : \Gamma \to \theta$.
Let I be the set of colors $i < \theta$ such that $\sup\{\min(C_\beta) \mid \beta \in f^{-1}\{i\}\} = \kappa$.

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Let I be the set of colors $i < \theta$ such that $\sup\{\min(C_\beta) \mid \beta \in f^{-1}\{i\}\} = \kappa$.
For $i \in I$, define $g_i : \kappa \to \kappa$ by $g_i(\eta) := \min\{\beta \in f^{-1}\{i\} \mid \min(C_\beta) > \eta\}$.

Chromatic number of the C-sequence graph

Claim. Suppose for every sequence $\langle A_i \mid i < \theta \rangle$ of cofinal subsets of κ, with $\theta < \kappa$, there is $\gamma \in \Gamma$ such that for all $i < \theta$:

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Then $\mathrm{Chr}(G(\vec{C})) = \kappa$.

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(otherwise, $\min(C_\gamma) < \min(D) \leq \min(A_i) \leq \min(C_\gamma)$.)

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Then $f(\beta) = i$ and $\min(C_\beta) > \eta > \delta > \alpha$, where $\alpha = \sup(C_\gamma \cap \beta)$.

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Then $f(\beta) = i$ and $\min(C_\beta) > \eta > \delta > \alpha$, where $\alpha = \sup(C_\gamma \cap \beta)$.
So $\beta \in N_\gamma$, contradicting the fact that $f(\gamma) = i$. □

Large gaps above a strongly-compact cardinal

Theorem (de Bruijn-Erdős, 1951)

If G is a graph, $\theta < \delta$, δ strongly-compact, and all $(< \delta)$-sized subgraphs of G have chromatic number $\leq \theta$, then $\mathrm{Chr}(G) \leq \theta$.

Suppose $\delta < \kappa$ is a Laver-indestructible supercompact cardinal.
Force with $\mathbb{P}$ consisting of conditions $p := \langle C_\alpha \mid \alpha \in \gamma + 1 \rangle$ such that:

- $\gamma < \kappa$;
- for all $\alpha \leq \gamma$, C_α is a closed subset of α with $\sup(C_\alpha) = \sup(\alpha)$;
- for all $\alpha \leq \gamma$ and $\bar{\alpha} \in \mathrm{acc}^+(C_\alpha)$, if $\mathrm{otp}(C_\alpha) \geq \delta$, then $C_{\bar{\alpha}} = C_\alpha \cap \bar{\alpha}$.

$\mathbb{P}$ is $<\delta$-directed closed, hence, in $V^{\mathbb{P}}$, δ remains supercompact.
Let $\vec{C}$ be the generic C-sequence.
Then $G(\vec{C})$ has size and chromatic number κ,
all of whose small subgraphs have chromatic number $\leq \delta$.

The distributivity number of a C-sequence

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$\mathfrak{h}(\vec{C})$ is the least cardinal $\theta \leq \kappa$ such that for some sequence $\langle A_i \mid i < \theta\rangle$ of cofinal subsets of κ, for every $\gamma \in \Gamma$, there is $i < \theta$ for which one of the following fails:

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Recall: $\mathrm{nacc}(x) := x \setminus \mathrm{acc}^+(x)$.

The 2-distributivity number of a C-sequence

$\mathfrak{h}_2(\vec{C})$ is the least cardinal $\theta \leq \kappa$ such that for some sequence $\langle A_i \mid i < \theta \rangle$ of cofinal subsets of κ, for every $\gamma \in \Gamma$, there is $i < \theta$ for which one of the following fails:

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The argument we gave shows: $\mathrm{Chr}(G(\vec{C})) \geq \mathfrak{h}_2(\vec{C})$.

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The argument we gave shows: $\mathrm{Chr}(G(\vec{C})) \geq \mathfrak{h}_2(\vec{C})$.

Theorem [28]

Suppose that $\diamondsuit(\kappa)$ holds, and let $\theta \in \mathrm{Reg}(\kappa)$.
Then $\exists$ postprocessing function $\Phi : \mathcal{K}(\kappa) \to \mathcal{K}(\kappa)$ satisfying the following.
For every $\vec{C}$: if $\mathfrak{h}(\vec{C} \restriction E^\kappa_\theta) > 1$, then $\mathfrak{h}_2(\vec{C}^\Phi) > \theta$.

Compactness at the service of incompactness

Theorem [24]

Assume GCH. Suppose that $\kappa = \lambda^+$ and $\theta \in \mathrm{Reg}(\lambda)$.
If $\square(\kappa)$ holds, then there is a coherent $\langle C_\alpha \mid \alpha \in \mathrm{acc}^+(\kappa)\rangle$ such that for every cofinal $A \subseteq \kappa$, there is $\gamma \in E^\kappa_\theta$ with $\sup(\mathrm{nacc}(C_\gamma) \cap A) = \gamma$.

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So GCH $+\square(\lambda^+)$ yields for each $\theta < \lambda$ a graph of size λ^+ and chromatic number $> \theta$, all of whose small subgraphs are countably chromatic.

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So GCH $+\square(\lambda^+)$ yields for each $\theta < \lambda$ a graph of size λ^+ and chromatic number $> \theta$, all of whose small subgraphs are countably chromatic.
By taking the disjoint union of these graphs, we get:

Corollary

GCH $+\square(\lambda^+)$ yields a graph of size λ^+ and chromatic number $\geq \lambda$, all of whose small subgraphs are countably chromatic.

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Exercise

Suppose that $\kappa = \theta^{++}$ and for every stationary $S \subseteq E^\kappa_\theta$, $\mathrm{Tr}(S) \neq \emptyset$.
Suppose that $\langle C_\alpha \mid \alpha \in \mathrm{acc}^+(\kappa)\rangle$ is a coherent C-sequence such that for every cofinal $A \subseteq \kappa$, there is $\gamma \in E^\kappa_\theta$ with $\sup(\mathrm{nacc}(C_\gamma) \cap A) = \gamma$.
Then for every cofinal $A \subseteq \kappa$, $\exists \gamma \in E^\kappa_{\theta^+}$ with $\sup(\mathrm{nacc}(C_\gamma) \cap A) = \gamma$.

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So GCH $+\square(\aleph_2)+$all stationary subsets of $E^{\omega_2}_\omega$ reflect yields a graph of size and chromatic # $\aleph_2$ all of whose small subgraphs are countably chromatic.

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Recall: $\mathrm{GCH} + \square(\lambda^+)$ yields a graph of size λ^+ and chromatic number $\geq \lambda$, all of whose small subgraphs are countably chromatic.

Question

Assume $\mathrm{GCH} + \square(\lambda^+)$ for λ singular.
Must there exist a graph of size and chromatic number λ^+ all of whose small subgraphs are countably chromatic?

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Question

Assume GCH $+\square(\lambda^+)$ for λ singular.
Must there exist a graph of size and chromatic number λ^+ all of whose small subgraphs are countably chromatic?

Recall that if $\mathrm{otp}(C_\alpha) < \lambda$ for all α, then $\mathrm{Chr}(G(\vec{C})) \leq \lambda$.

More on the type of a C-sequence

Definition

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Exercise

If $\kappa = \lambda^+$ with λ singular, then $\exists$ pp function $\Phi : \mathcal{K}(\kappa) \to \mathcal{K}(\kappa)$ satisfying: For every $\vec{C}$ over κ with type$(\vec{C}) = \lambda + 1$, type$(\vec{C}^\Phi) = \lambda$.

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So, for λ singular, $\square_\lambda$ may be witnessed by $\vec{C}$ with $\mathrm{type}(\vec{C}) = \lambda$.

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So, for λ singular, $\Box_\lambda$ may be witnessed by $\vec{C}$ with type$(\vec{C}) = \lambda$.
Unfortunately, postprocessing functions won't help, as type$(\vec{C}^\Phi) \leq$ type$(\vec{C})$.

To increase type$(\vec{C})$ from λ to $\lambda + 1$ (or even to λ^+) in a coherent way, we had to devise another method.
Here is a sample result in this vein...

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So, for λ singular, $\square_\lambda$ may be witnessed by $\vec{C}$ with $\mathrm{type}(\vec{C}) = \lambda$.

Theorem [19]

For every singular cardinal λ, the following are equivalent:

- $\square_\lambda$ holds and $2^\lambda = \lambda^+$;
- There is a coherent C-sequence $\vec{C}$ over λ^+ with $\mathrm{type}(\vec{C}) = \lambda + 1$, satisfying: for every sequence $\langle A_i \mid i < \lambda \rangle$ of cofinal subsets of λ^+, there is $\alpha \in \mathrm{acc}^+(\lambda^+)$ with $C_\alpha(i+1) \in A_i$ for all $i < \lambda$.

Blowing up the type of a C-sequence

For an indecomposable ordinal $\Lambda < \kappa$, write

$$I(\Lambda) := \min\left\{\sup\{\varepsilon + 1 \mid \varepsilon \in \mathrm{Im}(\vec{\Lambda})\} \,\middle|\, \vec{\Lambda} \in {}^{\mathrm{cf}(\Lambda)}\Lambda \;\&\; \sum \vec{\Lambda} = \Lambda\right\}.$$

For example:

- If $\Lambda < \kappa$ is a regular cardinal, then $I(\Lambda) = 2$;
- If $\Lambda < \kappa$ is a singular cardinal, then $I(\Lambda) = \Lambda$;
- If $\Lambda = \lambda \cdot \eta$ (ordinal multiplication), with $\eta = \mathrm{cf}(\eta) \leq \lambda < \kappa$, then $I(\Lambda) = \lambda + 1$.

Blowing up the type of a C-sequence

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Theorem [29]

Suppose that $\diamondsuit(\kappa)$ holds and $\Lambda < \kappa$ is indecomposable.
Suppose $\vec{C}$ is a C-sequence over κ such that for every cofinal $A \subseteq \kappa$ and every $\Lambda' < I(\Lambda)$, $\exists \alpha \in \operatorname{acc}^+(\kappa)$ with $\Lambda' \leq \operatorname{otp}(C_\alpha) < \Lambda$ and $\operatorname{nacc}(C_\alpha) \subseteq A$.
Then there is a C-sequence $\vec{D}$ over κ with:

1. $\operatorname{width}(\vec{D}) \leq \operatorname{width}(\vec{C})$;
2. $\operatorname{type}(\vec{D}) \leq \max\{\operatorname{type}(\vec{C}), \Lambda + 1\}$;
3. for every cofinal $A \subseteq \kappa$, there is a cofinal $B \subseteq \kappa$, for which

$$\left\{\alpha \in \operatorname{acc}^+(\kappa) \,\middle|\, \begin{array}{c} \operatorname{nacc}(C_\alpha) \subseteq B, \\ \operatorname{otp}(C_\alpha) = \operatorname{cf}(\Lambda) \end{array}\right\} \subseteq \left\{\alpha < \kappa \,\middle|\, \begin{array}{c} \operatorname{nacc}(D_\alpha) \subseteq A, \\ \operatorname{otp}(D_\alpha) = \Lambda \end{array}\right\}.$$

No need to force

Corollary [29]

Suppose $\Box(\lambda^+)$ holds for a given singular cardinal λ.
Assuming GCH, we can cook up a $\Box(\lambda^+)$-sequence $\vec{C}$ with $\mathfrak{h}_2(\vec{C}) = \lambda^+$.

In particular, $G(\vec{C})$ forms a graph of size and chromatic number λ^+ all of whose small subgraphs are countably chromatic.

Distributive Aronszajn trees

Definition

$\text{width}(\vec{C}, \Omega)$ is the least cardinal μ satisfying $|\mathcal{G}_\beta(\vec{C})| < \mu$ for all $\beta \in \Omega$, where $\mathcal{G}_\beta(\vec{C}) := \{C_\alpha \cap \gamma \mid \alpha \in \Gamma,\ sup(C_\alpha \cap \beta) = \beta\}$.

Of course, $\text{width}(\vec{C})$ is nothing but $\text{width}(\vec{C}, \kappa)$.

Theorem [29]

Assume $\diamondsuit(\kappa)$, and $\vec{C}$ is a C-sequence over κ with $\text{width}(\vec{C}) \leq \kappa$.
Suppose for every club $D \subseteq \kappa$, there is $\beta \in D$ with $\sup(\text{nacc}(g) \cap D) = \beta$ for all $g \in \mathcal{G}_\beta(\vec{C})$ (aka, "wide club-guessing").
Then there is a corresponding κ-Aronszajn tree $\mathcal{T}(\vec{C})$ which is θ-distributive for every cardinal $\theta < \mathfrak{h}(\vec{C})$.

In particular, if $\mathfrak{h}(\vec{C}) = \kappa$, then forcing with $\mathcal{T}(\vec{C})$ does not collapse κ while adding a chain of size κ, which must mean that $\mathcal{T}(\vec{C})$ is non-special.

Diagonal distributivity

Set $\mathfrak{h}(\vec{C}) := \kappa + 1$ iff for any sequence $\langle A_i \mid i < \kappa \rangle$ of cofinal subsets of κ, there is a (limit, nonzero) $\gamma \in \Gamma$, such that for all $i < \gamma$, the two hold:

- $\min(A_i) \leq \min(C_\gamma)$;
- there is $\alpha \in \mathrm{nacc}(C_\gamma) \cap A_i$.

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- $\min(A_i) \leq \min(C_\gamma)$;
- there is $\alpha \in \mathrm{nacc}(C_\gamma) \cap A_i$.

Theorem [22]

Suppose $\diamondsuit(\kappa)$ holds, and $\vec{C}$ is a C-sequence over κ with $\mathrm{width}(\vec{C}) \leq \kappa$.

- If $\mathrm{width}(\vec{C}) = 2$ and $\mathfrak{h}(\vec{C}) = \kappa + 1$,
  then there is a coherent κ-Souslin tree.

Coherent: For all $s, t \in \mathcal{T}$, $\{\alpha \in \mathrm{dom}(s) \cap \mathrm{dom}(t) \mid s(\alpha) \neq t(\alpha)\}$ is finite.

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- $\min(A_i) \leq \min(C_\gamma)$;
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Theorem [22],[32]

Suppose $\diamondsuit(\kappa)$ holds, and $\vec{C}$ is a C-sequence over κ with $\mathrm{width}(\vec{C}) \leq \kappa$.

- If $\mathrm{width}(\vec{C}) = 2$ and $\mathfrak{h}(\vec{C}) = \kappa + 1$, then there is a coherent κ-Souslin tree.
- If there is $\Omega \subseteq \kappa$ with $\mathrm{width}(\vec{C}, \Omega) = 2$ and $\mathfrak{h}(\vec{C} \restriction \Omega) = \kappa + 1$, then there is a free κ-Souslin tree.

Free: For all pairwise distinct $t_0, \dots, t_n \in \mathcal{T}$ with $\mathrm{ht}(t_0) = \cdots = \mathrm{ht}(t_n)$, the product of the upper cones $t_0{}^{\uparrow} \otimes \cdots \otimes t_n{}^{\uparrow}$ is again κ-Souslin.

Diagonal distributivity

Set $\mathfrak{h}(\vec{C}) := \kappa + 1$ iff for any sequence $\langle A_i \mid i < \kappa \rangle$ of cofinal subsets of κ, there is a (limit, nonzero) $\gamma \in \Gamma$, such that for all $i < \gamma$, the two hold:

- $\min(A_i) \leq \min(C_\gamma)$;
- there is $\alpha \in \mathrm{nacc}(C_\gamma) \cap A_i$.

More notions of forcing add a Souslin tree [26]

Suppose $\lambda^{<\lambda} = \lambda$ is a regular uncountable cardinal and $2^\lambda = \lambda^+$.
Suppose $\mathbb{P}$ is a λ^+-cc notion of forcing of size $\leq \lambda^+$ and:

- $\mathbb{P}$ forces that $\mathrm{cf}(\lambda) < |\lambda|$ (e.g., Prikry), or
- $\mathbb{P}$ preserves the regularity of λ, and is not ${}^\lambda\lambda$-bounding (e.g., Hechler).

Then, in $V^{\mathbb{P}}$, there is a $\vec{C}$ over λ^+ and $\Omega \subseteq \lambda^+$ such that:

- $\mathrm{width}(\vec{C}) \leq \lambda^+$;
- $\mathrm{type}(\vec{C}) = \lambda + 1$;
- $\mathrm{width}(\vec{C} \restriction \Omega) = 2$ and $\mathfrak{h}(\vec{C} \restriction \Omega) = \lambda^+ + 1$.