Proxy principles in combinatorial set theory

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Background — classical combinatorial principles

Jensen (1972):

- analyzed the fine structure of the constructible universe L;
- proved: In L, for every regular uncountable cardinal κ that is not weakly compact, there exists a κ-Souslin tree;
- formulated two new kinds of combinatorial principles:
  - prediction principles ($\diamondsuit$), and
  - coherence principles ($\square$);
- made the combinatorial properties of L accessible to generations of set theorists;
- enabled combinatorial constructions of complicated objects, leading to the settling of open problems in topology, measure theory, group theory, ...

Background — Square sequences

Fix a regular uncountable cardinal κ.

Definition

A coherent C-sequence over κ is a sequence $\langle C_\alpha \mid \alpha < \kappa \rangle$ such that, for every limit ordinal $\alpha < \kappa$:

- C_α is a club subset of α; and
- $C_\alpha \cap \bar{\alpha} = C_{\bar{\alpha}}$ for every $\bar{\alpha} \in \mathrm{acc}(C_\alpha)$.

Remark

For any set C of ordinals,
acc(C) stands for the set of its accumulation points:

$$\mathrm{acc}(C) = \{\beta \in C \mid \sup(C \cap \beta) = \beta > 0\}.$$

We can easily obtain a coherent C-sequence over κ by setting $C_\alpha = \alpha$ for every limit ordinal $\alpha < \kappa$.
But that's boring.

Background — Square sequences

More useful are principles asserting the existence of coherent C-sequences satisfying some non-triviality condition.
For example:

- For $\kappa = \lambda^+$, $\square_\lambda$ asserts the existence of a coherent C-sequence over λ^+, $\langle C_\alpha \mid \alpha < \lambda^+ \rangle$, such that $\mathrm{otp}(C_\alpha) \leq \lambda$ for every $\alpha < \lambda^+$. (Jensen)
- For a stationary set $E \subseteq \mathrm{acc}(\kappa)$, $\square(E)$ asserts the existence of a coherent C-sequence over κ, $\langle C_\alpha \mid \alpha < \kappa \rangle$, that avoids E, meaning that $\mathrm{acc}(C_\alpha) \cap E = \emptyset$ for every $\alpha < \kappa$. (Jensen)
- $\square(\kappa)$ asserts the existence of a coherent C-sequence over κ, $\langle C_\alpha \mid \alpha < \kappa \rangle$, that is unthreadable — meaning that there is no club $D \subseteq \kappa$ such that $D \cap \alpha = C_\alpha$ for every $\alpha \in \mathrm{acc}(D)$. (Todorčević)

Background — Square sequences

Many variants of both diamond and square have appeared — strengthening, weakening, or adapting each one as needed to solve various combinatorial problems.
Recall:

- Square principles are primarily concerned with coherence.
- Diamond principles are prediction principles, asserting that objects of size κ can be predicted by means of their initial segments.

The construction of complicated combinatorial objects such as κ-Souslin trees requires both prediction and coherence.
Strong combinations of square and diamond have also become useful, such as $\boxdiamond_\lambda$ (Gray) and its further strengthening $\boxdiamond_\lambda^+$ (Rinot & Schindler).

Limitations of the classical principles

- The classical constructions of κ-Souslin trees generally depend on the nature of κ: successor of regular, successor of singular, or inaccessible.
- Constructions include extensive bookkeeping, counters, timers, coding and decoding whose particular nature makes it difficult to transfer the process from one cardinal to another.
- Classical constructions of Souslin trees (following Jensen) require the $\diamondsuit$-sequence's predictions to occur in some non-reflecting stationary set E, which must then be avoided by the square sequence in order not to interfere with building higher levels of the tree.
- There are particular scenarios where the coherence requirements are transparent, such as for $\kappa = \aleph_1$ where $\square_{\aleph_0}$ holds trivially, and we thus find many classical constructions tailored to such cases alone.
- For inaccessible κ, there is a dearth of axiom-based constructions.

Limitations of the classical principles

All of these weaknesses are symptoms of a deeper problem:
The classical non-triviality conditions imposed on square-like sequences are somewhat arbitrary and do not correspond closely enough to the way in which the sequences are used in the desired constructions.

The solution: Square sequences with "hitting"

The core feature of the proxy principles is that the non-triviality of a square-like sequence is ensured by a hitting requirement — a weak form of prediction — that is tailored for the desired construction.

This tailoring enables uniform construction of κ-Souslin trees and other combinatorial objects, oblivious to the nature of κ.

The simplest instance of the proxy principle

Definition (B. & Rinot)

The proxy principle $\boxtimes^-(\kappa)$ asserts the existence of a sequence $\vec{C} = \langle C_\alpha \mid \alpha < \kappa \rangle$ such that:

- for every limit ordinal $\alpha < \kappa$, C_α is a club subset of α;
- for every limit ordinal $\alpha < \kappa$ and every $\bar{\alpha} \in \mathrm{acc}(C_\alpha)$, $C_\alpha \cap \bar{\alpha} = C_{\bar{\alpha}}$.
- for every cofinal $B \subseteq \kappa$, there exist stationarily many ordinals $\alpha < \kappa$ such that $\sup(\mathrm{nacc}(C_\alpha) \cap B) = \alpha$.

Remark

For any set C of ordinals, $\mathrm{nacc}(C)$ denotes the set of its non-accumulation points: $\mathrm{nacc}(C) = C \setminus \mathrm{acc}(C)$.

The "hitting" feature in the third bullet is our new non-triviality condition. It can be understood as a genericity feature of the coherent C-sequence.

More flexibility in the proxy principles

In order to maintain the flexibility to vary both the coherence and hitting features as needed to prove various desired results, we would like to be able to express something along the following lines:

- There exists a system $\vec{\mathcal{C}} = \langle \mathcal{C}_\alpha \mid \alpha < \kappa \rangle$ with each $\mathcal{C}_\alpha$ a nonempty collection of closed cofinal subsets of α;
- There is a prescribed bound for how many sets are there at each level, e.g., $|\mathcal{C}_\alpha| = 1$ for every $\alpha < \kappa$ (as in the simple instance above), or more generally, for some fixed cardinal μ, $|\mathcal{C}_\alpha| < \mu$ for every $\alpha < \kappa$;
- The elements of any level are compatible with the ones from below, i.e., there is a prescribed binary coherence relation $\mathcal{R}$ (such as $\sqsubseteq$) such that, for every $\alpha < \kappa$, every $C \in \mathcal{C}_\alpha$, and every $\bar{\alpha} \in \operatorname{acc}(C)$, there exists a $D \in \mathcal{C}_{\bar{\alpha}}$ with $D \mathrel{\mathcal{R}} C$;
- For some prescribed cardinal θ, every family $\mathcal{B} \subseteq [\kappa]^\kappa$ of size θ gets "hit" at some level α, i.e., each $C \in \mathcal{C}_\alpha$ meets each $B \in \mathcal{B}$. We may also want the α of interest to come from some prescribed set G, and we may want a meeting that is successful σ many times in a row for some prescribed σ.

The coherence relations $\mathcal{R}$

The basic coherence relation $\mathcal{R}$ is the end-extension relation, $\sqsubseteq$, indicating that $\vec{\mathcal{C}}$ is a coherent $\mathcal{C}$-sequence. A close examination of proxy-based constructions reveals that full coherence is not always necessary, and the $\sqsubseteq$ relation can be weakened in several ways, as follows.

First, considering some $C \in \bigcup_{\alpha<\kappa} \mathcal{C}_\alpha$ and some $\bar{\alpha} \in \mathrm{acc}(C)$, it may be that all we require is for some $D \in \mathcal{C}_{\bar{\alpha}}$ to agree with C at the final approach to $\bar{\alpha}$. If this is the case, then the construction will work just as well from a $\sqsubseteq^*$-coherent instance of the proxy principle, where $D \sqsubseteq^* C$ iff there is some $\varepsilon < \sup(D)$ such that $D \setminus \varepsilon \sqsubseteq C \setminus \varepsilon$.

The coherence relations $\mathcal{R}$

In another direction, some proxy-based constructions can be designed to require genuine coherence only for some of the clubs in $\vec{\mathcal{C}}$, or only at some of their accumulation points. Indeed, there are contexts in which, for some infinite cardinal χ, there is no need to require coherence for clubs of order-type $< \chi$, or possibly, there is no need to require coherence at accumulation points of cofinality $< \chi$. Thus, in such cases we may weaken $\sqsubseteq$ to either ${}_{\chi}{\sqsubseteq}$ or $\sqsubseteq_{\chi}$, where for a coherence relation $\mathcal{R}$:

- $D \ {}_{\chi}\mathcal{R} \ C$ iff $((D \ \mathcal{R} \ C)$ or $(\mathrm{cf}(\sup(D)) < \chi))$, and
- $D \ \mathcal{R}_{\chi} \ C$ iff $((D \ \mathcal{R} \ C)$ or $(\mathrm{otp}(C) < \chi$ and $\mathrm{nacc}(C)$ consists only of successor ordinals$))$."$\mathrm{nacc}(C)$ consists only of successor ordinals"

The significance of such a weakening is that unlike coherent square sequences that are typically refuted by reflection principles (such as large cardinals, strong forcing axioms, and simultaneous reflection of stationary sets), $\sqsubseteq_{\chi}$-coherent proxy principles are compatible with a gallery of reflection principles and provide an effective means of obtaining optimal incompactness results.

The coherence relations $\mathcal{R}$

Note that the extreme case $\mathcal{R} = {}_{\kappa}{\sqsubseteq}$ amounts to saying that no coherence is needed at all, and we call it the trivial coherence relation. In this case, every $\mathcal{C}_\alpha$ may be shrunk to a singleton, yielding a proxy sequence with $\mu = 2$.

In yet another direction, there are circumstances in which it is helpful to indicate that $\vec{\mathcal{C}}$ avoids a particular class of ordinals Ω, meaning that $\mathrm{acc}(C) \cap \Omega = \emptyset$ for every $C \in \bigcup_{\alpha<\kappa} \mathcal{C}_\alpha$. This requirement is indicated by prepending Ω as a superscript to the coherence relation $\mathcal{R}$, thereby strengthening it to ${}^{\Omega}\mathcal{R}$. In general, if $\vec{\mathcal{C}}$ is a ${}^{\Omega}{\sqsubseteq}$-coherent proxy sequence, then for any $\alpha \in \Omega$, one is free to shrink $\mathcal{C}_\alpha$ to a singleton, and this has important ramifications.

The parameterized version of the proxy principles

Definition (B. & Rinot)

The proxy principle $\mathrm{P}^-_\xi(\kappa, \mu, \mathcal{R}, \theta, \mathcal{S}, \nu, \sigma)$ asserts the existence of a sequence $\vec{\mathcal{C}} = \langle \mathcal{C}_\alpha \mid \alpha < \kappa \rangle$ such that the following three requirements are satisfied:

1. for every $\alpha < \kappa$, $\mathcal{C}_\alpha$ is a nonempty collection of less than μ many closed subsets C of α with $\sup(C) = \sup(\alpha)$ and $\mathrm{otp}(C) \leq \xi$;
2. for all $\alpha < \kappa$, $C \in \mathcal{C}_\alpha$ and $\bar{\alpha} \in \mathrm{acc}(C)$, there is a $D \in \mathcal{C}_{\bar{\alpha}}$ such that $D \mathrel{\mathcal{R}} C$;
3. for every sequence $\langle B_i \mid i < \theta \rangle$ of cofinal subsets of κ, for every $S \in \mathcal{S}$, there exist stationarily many $\alpha \in S$ for which:
   - $|\mathcal{C}_\alpha| < \nu$, and
   - for all $C \in \mathcal{C}_\alpha$ and $i < \min\{\alpha, \theta\}$:

$$\sup\{\beta \in C \mid \mathrm{succ}_\sigma(C \setminus \beta) \subseteq B_i\} = \alpha. \qquad (\star)$$

Recent proxy-based constructions of Souslin trees

Theorem (Yadai)

Assuming $\mathrm{P}(\kappa, 2, \sqsubseteq, \kappa)$*, if there exists a* κ*-Kurepa tree, then there exists a* κ*-Aronszajn tree* $\mathbf{T}$ *admitting* κ^+*-many* κ*-Souslin subtrees such that the product of any finitely many of them is again Souslin.*

This result is optimal in the sense that the tree $\mathbf{T}$ itself cannot be κ-Souslin.
We obtain a proxy-based construction of a Souslin tree whose square is special. More generally:

Theorem (Yadai)

For an infinite cardinal λ*, assuming* $\mathrm{P}_\lambda(\lambda^+, 2, \sqsubseteq, \lambda^+)$*, for every positive integer n, there exists a* λ^+*-Souslin tree* $\mathbf{T}$ *satisfying the following:*

- *all n-derived trees of* $\mathbf{T}$ *are Souslin;*
- *the* $(n+1)$*-power of* $\mathbf{T}$ *is special.*

Additional applications of the proxy principles

Constructions aside from Souslin trees, using proxy principles together with $\diamondsuit$:

- distributive Aronszajn trees, as well as special trees with a non-special projection [B. & Rinot];
- a large pairwise far family of Aronszajn trees [Krueger];
- minimal non-σ-scattered linear orders [Shalev].

Constructions using proxy principles alone, without $\diamondsuit$ or any arithmetic hypothesis:

- a highly chromatic graph all of whose smaller subgraphs are countably chromatic [Lambie-Hanson & Rinot];
- Ulam-type matrices [Inamdar & Rinot];
- a Dowker space whose square is still Dowker [Rinot & Shalev];
- a C-sequence suitable for conducting walks on ordinals, proving the consistency of an extreme instance of Saharon Shelah's strong colouring principle [Rinot & Zhang].

Conclusion and riddle

Recall that one of the limitations of classical square sequences is that they are not as useful for constructions on inaccessible cardinals.

What do the proxy principles have in common with Prikry forcing?

Conclusion and riddle

Recall that one of the limitations of classical square sequences is that they are not as useful for constructions on inaccessible cardinals.

What do the proxy principles have in common with Prikry forcing?

They both make inaccessible cardinals accessible.