When Sierpiński met Ulam

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For more details

All results are joint work with Assaf Rinot (BIU) from one of the following two papers.

[1] *Was Ulam right?*, Tanmay Inamdar and Assaf Rinot, submitted, available at `http://p.assafrinot.com/47`.

[2] *Relative club guessing*, Tanmay Inamdar and Assaf Rinot, in progress, will be available at `http://p.assafrinot.com/46`.

- By default, all results are from [1].

Conventions

Unless otherwise specified:

- κ an infinite regular cardinal, $\theta \leq \kappa$ a cardinal;
- $J^{\mathrm{bd}}[\kappa]$ is the ideal of *bounded* subsets of κ;
- NS_κ is the ideal of non-stationary subsets of κ;
- J an ideal on κ extending $J^{\mathrm{bd}}[\kappa]$;
- $E^\kappa_\xi := \{\delta < \kappa \mid \mathrm{cf}(\delta) = \xi\}$;
- $A \circledast B := \{(\alpha, \beta) \in A \times B \mid \alpha < \beta\}$;
- $\mathrm{Tr}(S) := \{\alpha \in E^\kappa_{>\aleph_0} \mid S \cap \alpha$ is stationary in $\alpha\}$;
- $\langle C_\delta \mid \delta \in S\rangle$ is a C-sequence if $C_\delta \subseteq \delta$ is a club for every $\delta \in S$;
- *subnormal* is a technical condition on ideals that covers both normal ideals and $J^{\mathrm{bd}}[\kappa]$;
- an *upper-regressive function* is one satisfying $c(\alpha, \beta) < \beta$ for ordinals $\alpha < \beta$ in its domain.

Motivating problem

Let $\xi < \kappa$ be infinite regular cardinals and $\langle C_\delta \mid \delta \in E^\kappa_\xi \rangle$ a C-sequence be such that

- for every $\delta \in E^\kappa_\xi$, $\mathrm{otp}(C_\delta) = \xi$;
- for every $D \subseteq \kappa$ a club there is $\delta \in E^\kappa_\xi$ such that $\sup(\mathrm{nacc}(C_\delta) \cap D) = \delta$.

Then, if $\theta \leq \xi$, are there $\langle h_\delta : C_\delta \to \theta \mid \delta \in E^\kappa_\xi \rangle$ such that for every club $D \subseteq \kappa$ there is $\delta \in E^\kappa_\xi$ such that for every $\tau < \theta$,

$$\sup(\mathrm{nacc}(C_\delta) \cap D \cap h_\delta^{-1}\{\tau\}) = \delta?$$

- Best results in Shelah572.
- Interest comes from strong colourings.
- For this and more, see [2].

Introducing onto and $\mathrm{unbounded}$

Definition

Let $c : [\kappa]^2 \to \theta$ be a colouring.

- c witnesses $\mathrm{onto}(J, \theta)$ if for every $B \in J^+$ there is an $\eta < \kappa$ such that $c[\{\eta\} \circledast B] = \theta$;
- c witnesses $\mathrm{unbounded}(J, \theta)$ if it is upper-regressive and for every $B \in J^+$ there is an $\eta < \kappa$ such that $\mathrm{otp}(c[\{\eta\} \circledast B]) = \theta$.

Another parameter has been suppressed, but an example is given when we discuss the onto mapping principle of Sierpiński.

Two classical results and one new result

Theorem (Jensen+Kunen ‘69)

Let κ be an uncountable cardinal.

1. *For κ regular, κ is ineffable iff $\mathrm{onto}(\mathsf{NS}_\kappa, 2)$ fails.*
2. *κ is almost ineffable iff $\mathrm{onto}(J^{\mathrm{bd}}[\kappa], 2)$ fails.*

Theorem

Let κ be an uncountable cardinal.
Then κ is weakly compact iff $\mathrm{onto}(J^{\mathrm{bd}}[\kappa], 3)$ fails.

Weak-saturation

Theorem

Suppose that J is a subnormal κ-complete ideal such that $\mathrm{unbounded}(J, \theta)$ *holds. Then there is $d : [\kappa]^2 \to \theta$ such that for every element B of J^+ there is an $\eta < \kappa$ such that the set*

$$\{\tau < \theta \mid \{\beta \in B \mid d(\eta, \beta) = \tau\} \in J^+\}$$

has ordertype θ.
If in fact $\mathrm{onto}(J, \theta)$ *holds, then d can be found such that this set is all of θ.*

Corollary ([2])

If $\langle C_\delta \mid \delta \in E^\kappa_\xi \rangle$ guesses clubs and $\theta \leq \xi$ then we can partition it into θ-many pieces assuming

1. $\theta < \xi$ *and* $\mathrm{unbounded}(J^{\mathrm{bd}}[\xi], \theta)$ *holds;*
2. $\theta = \xi$ *and* $\mathrm{onto}(J^{\mathrm{bd}}[\xi], \xi)$ *holds.*

The onto mapping principle of Sierpiński

An example of the suppressed parameter.

Theorem (Sierpiński '34, Miller '14, Guzmán '17)

The following are equivalent:

1. $\mathrm{non}(\mathcal{M}) = \aleph_1$;
2. *there are functions* $\langle f_n : \aleph_1 \to \aleph_1 \mid n < \aleph_0 \rangle$ *such that for every cofinal* $B \subseteq \aleph_1$*, there is an* $n < \aleph_0$ *such that* $f_n[B] = \aleph_1$;
3. $\mathrm{onto}(\aleph_0, J^{\mathrm{bd}}[\aleph_1], \aleph_1)$;
4. *there are functions* $\langle f_n : \aleph_1 \to \aleph_1 \mid n < \aleph_0 \rangle$ *such that for every cofinal* $B \subseteq \aleph_1$*, for all but finitely many* $n < \aleph_0$ *we have* $f_n[B] = \aleph_1$;
5. $\mathrm{onto}([\aleph_0]^{\aleph_0}, J^{\mathrm{bd}}[\aleph_1], \aleph_1)$.

There's more, see Kojman+Rinot+Steprans: 'Sierpiński's onto mapping principle and partitions'.

Strongly amenable ideal I

Till otherwise stated, κ is assumed to be uncountable and regular.

Definition

Let $S \subseteq \kappa$. A C-sequence $\vec{C} = \langle C_\beta \mid \beta \in S \rangle$ is *strongly amenable in* κ if for every club D in κ, the set $\{\beta \in S \mid D \cap \beta \subseteq C_\beta\}$ is bounded in κ.

Theorem

The following are equivalent:

1. $\mathrm{unbounded}(J^{\mathrm{bd}}[\kappa], \kappa)$;
2. *there is a C-sequence $\vec{C} = \langle C_\beta \mid \beta \in \kappa \rangle$ which is strongly amenable in κ.*

Theorem

If κ is weakly compact then κ does not carry a strongly amenable C-sequence. In L *the converse is also true.*

Strongly amenable ideal II

Definition

$\mathsf{SA}_\kappa := \{S \subseteq \kappa \mid S \text{ carries a } C\text{-sequence strongly amenable in } \kappa\}$.

Theorem

SA_κ is a κ-complete ideal and the following sets are all subsets of SA_κ:

1. NS_κ;
2. $\{\{\beta < \kappa \mid \mathrm{cf}(\beta) < \beta\}\}$;
3. $\{\kappa \setminus \mathrm{Tr}(S) \mid S \in (\mathsf{NS}_\kappa)^+\}$;
4. $\{\kappa \setminus \mathrm{Tr}^\alpha(\kappa) \mid \alpha < \kappa\}$.

Proposition

Every stationary $S \subseteq \kappa$ contains a stationary subset S' such that $S' \in \mathsf{SA}_\kappa$.

Strongly amenable ideal III

Theorem
If $\kappa \notin \mathsf{SA}_\kappa$ *then*

1. *for every* $\mu < \kappa$ *and* $\langle S_i \mid i < \mu \rangle$ *stationary subsets of* κ, $\mathrm{Reg}(\kappa) \cap \bigcap_{i<\mu} \mathrm{Tr}(S_i) \neq \emptyset$;
2. κ *is greatly Mahlo;*
3. $\square(\kappa, {<}\mu)$ *fails for all* $\mu < \kappa$
4. κ *is weakly compact in* L.

Theorem
Assuming the consistency of a weakly compact cardinal, it is consistent that

1. $\kappa = 2^{\aleph_0}$ *and* $\kappa \notin \mathsf{SA}_\kappa$;
2. κ *is strongly inaccessible, not weakly compact, and* $\kappa \notin \mathsf{SA}_\kappa$.

Amenable ideal I

Definition (Brodsky+Rinot)

Let $S \subseteq \kappa$. A C-sequence $\vec{C} = \langle C_\beta \mid \beta \in S \rangle$ is *amenable in* κ iff for every club D in κ, the set $\{\beta \in S \mid D \cap \beta \subseteq C_\beta\}$ is non-stationary in κ.

Theorem

The following are equivalent for $S \subseteq \kappa$ stationary:

1. *S carries an amenable C-sequence;*
2. $\mathrm{unbounded}(\mathsf{NS}_\kappa \restriction S, \kappa)$.

Definition

$\mathsf{A}_\kappa := \{S \subseteq \kappa \mid S \text{ carries a } C\text{-sequence amenable in } \kappa\}$.

Theorem

If $S \subseteq \kappa$ is ineffable then $S \notin \mathsf{A}_\kappa$. In L *the converse is also true.*

Amenable ideal II

Theorem

A_κ is a normal κ-complete ideal containing SA_κ and hence containing $\{\kappa \setminus \mathrm{Tr}^\alpha(\kappa) \mid \alpha < \kappa^+\}$.

Corollary

If $\kappa \notin \mathsf{A}_\kappa$ then

1. *for every sequence $\langle S_i \mid i < \kappa\rangle$ of stationary subsets of κ, there is $\delta < \kappa$ inaccessible such that $\delta \in \bigcap_{i<\delta} \mathrm{Tr}(S_i)$;*
2. *κ is greatly Mahlo;*
3. *$\square(\kappa, {<}\mu)$ fails for all $\mu < \kappa$*
4. *κ is weakly compact in L.*

Amenable ideal III

Conjecture

If $\kappa \notin \mathsf{A}_\kappa$ *then* κ *is ineffable in* L.

Theorem

Assuming the consistency of an ineffable cardinal, it is consistent that

1. $\kappa = 2^{\aleph_0}$ *and* $\kappa \notin \mathsf{A}_\kappa$;
2. κ *is strongly inaccessible, not weakly compact, and* $\kappa \notin \mathsf{A}_\kappa$.

Ulam matrices I

Definition (Ulam '30, Hajnal '69)

A matrix $\langle U_{\eta,\tau} \mid \eta < \tau < \kappa \rangle$ is a *triangular Ulam matrix* if

1. for every $\eta < \kappa$, $\{U_{\eta,\tau} \mid \eta < \tau < \kappa\}$ consists of pairwise disjoint subsets of κ;
2. the set $T := \{\tau < \kappa \mid |\kappa \setminus \bigcup_{\eta<\tau} U_{\eta,\tau}| < \kappa\}$ is stationary in κ. This set is called the *support*.

Ulam matrices II

Theorem (Hajnal '69, [1])

Let $T \subseteq \kappa$ be stationary. The following are equivalent:

1. $\mathrm{Tr}(T) \cap \mathrm{Reg}(\kappa)$ *is non-stationary;*
2. κ *carries a triangular Ulam matrix with support* T*;*
3. $\mathrm{unbounded}^*(J, \{T\})$ *holds for every normal* J*, which means: there is an upper-regressive colouring* $c : [\kappa]^2 \to \kappa$ *with the property that, for all* $B \in J^+$*, for every* $\tau \in T$*, there is an* $\eta < \tau$ *and a* $\beta \in B$ *such that* $c(\eta, \beta) = \tau$

In particular, $\mathrm{unbounded}(J^{\mathrm{bd}}[\kappa], \kappa)$ is a more applicable principle than Ulam matrices for obtaining non-weak-saturation results in a uniform manner.

Pumping-up I

Theorem

Let $\theta < \kappa$. The following all imply $\mathrm{unbounded}(J^{\mathrm{bd}}[\kappa], \theta)$*:*

1. $\kappa \nrightarrow [\mathsf{Stat}(\kappa)]^2_\theta$;
2. *there is a κ-Souslin tree;*
3. $\mathsf{cf}(\theta) = \theta$ *and there is a tree $\mathcal{T}$ of height θ with at least κ-many branches such that each level has size less than κ;*
4. $\mathrm{unbounded}(J^{\mathrm{bd}}[\kappa], \theta^+)$ *if $\theta^+ < \kappa$.*

Proposition (probably Erdős+Hajnal)

Let $\theta < \kappa$. Then $\kappa \nrightarrow [\kappa; \kappa]^2_\theta$ is equivalent to a strong form of $\mathrm{onto}(J^{\mathrm{bd}}[\kappa], \theta)$.

Pumping-up II

Theorem

Let $\theta \leq \chi < \kappa$ and J be subnormal. Then $\mathrm{unbounded}(J, \chi)$ implies $\mathrm{onto}(J, \theta)$ if any of the following occurs:

1. $\mathrm{cf}(\theta) = \theta < \chi$;
2. $\mathcal{C}(\theta, \chi) < \kappa$, *which means: there is a subfamily $\mathcal{X}$ of $[\chi]^\theta$ of size less than κ such that every club $C \subseteq \chi$ contains some $X \in \mathcal{X}$;*
3. $\theta = \chi$ *and* $2^\theta < \kappa$.

Theorem

Let θ be regular. If $\mathrm{onto}(J^{\mathrm{bd}}[\theta], \theta)$ holds and $\mathfrak{b}_\theta = \theta^+$ then $\mathrm{onto}(J^{\mathrm{bd}}[\theta^+], \theta^+)$ holds as well.

ZFC conclusions

Theorem

The following all hold

1. $\mathrm{unbounded}(J^{\mathrm{bd}}[\aleph_0], \aleph_0)$;
2. $\mathrm{unbounded}(J^{\mathrm{bd}}[\kappa], n)$ *for* $\aleph_0 < \kappa \leq 2^{\aleph_0}$ *and* $0 < n < \aleph_0$ *and* $\mathrm{cf}(\kappa) \leq \kappa$;
3. $\mathrm{unbounded}(J^{\mathrm{bd}}[\theta^+], \theta)$ *for* θ *a singular cardinal;*
4. $\mathrm{unbounded}(J^{\mathrm{bd}}[\mathfrak{d}_\theta], \theta)$ *for* θ *regular;*
5. $\mathrm{unbounded}(J^{\mathrm{bd}}[\mathfrak{b}_\theta], \theta)$ *for* θ *regular.*

Some more results for singulars

Theorem

Let κ be singular.

1. $\mathrm{unbounded}(J^{\mathrm{bd}}[\kappa], \theta)$ *holds iff* $\theta \leq \mathsf{cf}(\kappa)$*;*
2. $\mathrm{onto}(J^{\mathrm{bd}}[\kappa], \theta)$ *holds for every regular* $\theta < \mathsf{cf}(\kappa)$*.*
3. $\mathrm{onto}(J^{\mathrm{bd}}[\kappa], \theta)$ *holds for every singular* θ *such that* $\theta^+ < \mathsf{cf}(\kappa)$*.*

onto with maximal colours

Theorem (Guzmán '17)

$\mathsf{non}(\mathcal{M}) = \aleph_1$ *implies* $\mathrm{onto}(J^{\mathrm{bd}}[\aleph_1], \aleph_1)$.

Theorem (Larson '07)

It is consistent that $\mathrm{onto}(\mathsf{NS}_{\aleph_1}, \aleph_1)$, *and hence* $\mathrm{onto}(J^{\mathrm{bd}}[\aleph_1], \aleph_1)$ *as well, fails.*

Theorem

1. *For* κ *a successor cardinal,* $\stick(\kappa)$ *implies* $\mathrm{onto}(J^{\mathrm{bd}}[\kappa], \kappa)$.
2. $\diamondsuit^*(\kappa)$ *implies* $\mathrm{onto}(\mathsf{NS}_\kappa, \kappa)$.
3. *If* $\diamondsuit(S)$ *holds for some* $S \subseteq \kappa$ *stationary not reflecting at regulars then* $\mathrm{onto}(J^{\mathrm{bd}}[\kappa], \kappa)$ *holds.*

Weakly compact cardinals

Theorem

The following are equivalent for κ uncountable:

1. *κ is not weakly compact;*
2. $\mathrm{unbounded}(J^{\mathrm{bd}}[\kappa], \aleph_0)$ *holds.*

Theorem

The following are equivalent for $\kappa \geq 2^{\aleph_0}$:

1. *κ is not weakly compact;*
2. $\mathrm{onto}(J^{\mathrm{bd}}[\kappa], \aleph_0)$ *holds.*

Ineffable cardinals

Theorem

The following are equivalent for κ uncountable regular:

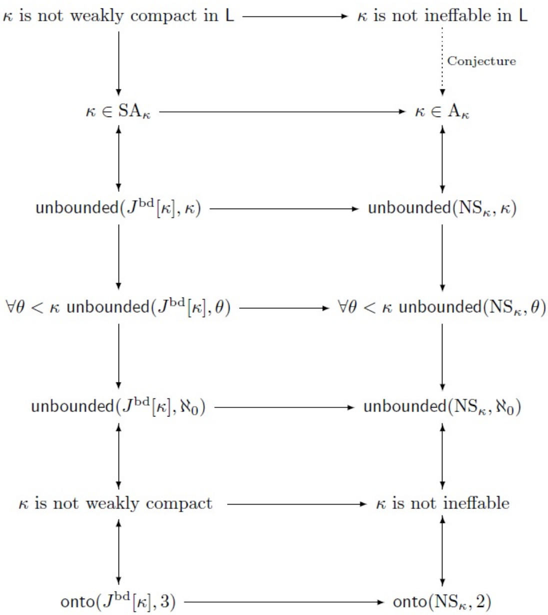
1. κ *is not ineffable;*
2. $\mathrm{unbounded}(\mathsf{NS}_\kappa, \aleph_0)$ *holds.*

Theorem

The following are equivalent for regular $\kappa \geq 2^{\aleph_0}$:

1. κ *is not ineffable;*
2. $\mathrm{onto}(\mathsf{NS}_\kappa, \aleph_0)$ *holds.*

The end?



κ is not weakly compact in L
κ is not ineffable in L
Conjecture
$\kappa \in \mathrm{SA}_\kappa$
$\kappa \in \mathrm{A}_\kappa$
unbounded$(J^{\mathrm{bd}}[\kappa], \kappa)$
unbounded$(\mathrm{NS}_\kappa, \kappa)$
$\forall \theta < \kappa$ unbounded$(J^{\mathrm{bd}}[\kappa], \theta)$
$\forall \theta < \kappa$ unbounded$(\mathrm{NS}_\kappa, \theta)$
unbounded$(J^{\mathrm{bd}}[\kappa], \aleph_0)$
unbounded$(\mathrm{NS}_\kappa, \aleph_0)$
κ is not weakly compact
κ is not ineffable
onto$(J^{\mathrm{bd}}[\kappa], 3)$
onto$(\mathrm{NS}_\kappa, 2)$