

Recent progress on the theory of higher forcing axioms

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Bibliography

Based on the following papers with Assaf Rinot:

- ▶ *Walks on uncountable ordinals and non-structure theorems for higher Aronszajn lines*, arXiv October 2024.
- ▶ *Partition relations for trees I: Incomparable trees*, RIMS proceedings 2025.
- ▶ *Partition relations for trees II: Entangled linear orders*, submitted August 2026.
- ▶ *What is a higher forcing axiom?*, survey in progress.

$\aleph_2$ is not $\aleph_1$

Trailer of a trailer

Theorem (Kunen, Rowbottom, Solovay (independently))

$\text{MA}_{\aleph_1}$ implies the product of any two $\aleph_1$ -cc posets is $\aleph_1$ -cc.

Theorem (Shelah, 1997)

There is an $\aleph_2$ -cc poset whose square is not $\aleph_2$ -cc.

Theorem (Moore, 2006)

PFA implies the class of special $\aleph_1$ -Aronszajn lines has a basis of size 2.

Theorem (Inamdar–Rinot, 2024)

The class of special $\aleph_2$ -Aronszajn lines is either empty, or any basis for it has size $2^{\aleph_2}$.

Recent interest in higher forcing axioms

Higher analog of the proper forcing axiom

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Fields Institute, October 2012

[Submitted on 4 Dec 2018]

Methods in Higher Forcing Axioms (Workshop Notes)

David Asperó, Asaf Karagila

Some higher forcing axioms

Monroe Eskew

University of Vienna

10th European Set Theory Conference
University of Helsinki
May 21, 2026

High and low forcing

January 11 to January 15, 2016
at the

American Institute of Mathematics, San Jose, California

organized by

Itay Neeman and Dima Sinapova

From $\aleph_2$ to infinity

May 15 to May 19, 2023
at the

American Institute of Mathematics, San Jose, California

organized by

James Cummings, Itay Neeman, and Dima Sinapova

In search of higher forcing axioms

Boban Velickovic

IMJ-PRG Université Paris Cité

IMPAN - Gdańsk Logic Colloquium
Gdańsk, November 2 2023

Two examples of higher forcing axioms

Theorem (Solovay, Tennenbaum, Martin, Rowbottom, 1969)

$\text{MA}_{\aleph_2}$ is consistent: It is consistent that for every **ccc** forcing $\mathbb{P}$ and any collection $\mathcal{D} = \langle D_\alpha \mid \alpha < \omega_2 \rangle$ of dense open subsets of $\mathbb{P}$, there is a generic set G which meets all the members of $\mathcal{D}$.

Theorem (Shelah, 1977)

$\text{GMA}_{\aleph_2}$ is consistent: It is consistent that for every forcing $\mathbb{P}$ which is **countably closed with glb, is well-met, has $\aleph_2$ -stationary-cc** and any collection $\mathcal{D} = \langle D_\alpha \mid \alpha < \omega_2 \rangle$ of dense open subsets of $\mathbb{P}$, there is a generic set G which meets all the members of $\mathcal{D}$.

- There are recent examples, the theme of this talk is why there are so few examples.

Two non-examples of higher forcing axioms

Theorem (Shelah, 1982)

CH refutes $\text{FA}_{\aleph_2}(\mathbb{P})$ for some $\mathbb{P}$ that is countably closed, $\aleph_2$ -stationary-cc.

- ▶ Note that without CH, there are no σ -closed separative $\aleph_2$ -cc posets.
- ▶ It's a beautiful proof, using uniformisation of ladder systems.

Theorem (Inamdar, 2013)

CH refutes $\text{FA}_{\aleph_2}(\mathbb{P})$ for some $\mathbb{P}$ that is countably-closed, precalibre $\aleph_2$, and well-met.

- ▶ It's a much simpler proof, using properties of the function ρ_2 from Todorćević's theory of walks on ordinals.

Why is it so hard to add an object of size $\aleph_4$?

1. You need to preserve $\aleph_1, \aleph_2, \aleph_3, \aleph_4$.
2. And if the object is interesting enough, you can't do this with $\aleph_1$ -cc, $\aleph_2$ -cc etc.
3. Maybe a fragment of GCH contradicts the existence of this object and you need to also add reals, subsets of ω_1 etc.
4. So you may need four different reasons why these cardinals are preserved.
5. And if you want to iterate such forcings to get a higher forcing axiom, the preservation theorems present an additional challenge.
6. And after all this effort, you need to additionally make sure that your class of forcings is not only non-empty, but actually interesting!
7. Maybe I convince you to instead work on infinite combinatorics and show such things are impossible...

Another example

Theorem

1. ([Cohen, 1963](#)) You can add a club of ω which doesn't contain an infinite set from the ground model.
2. ([Baumgartner, 1984](#)) You can add a club of ω_1 which doesn't contain an infinite set from the ground model.
3. ([Friedman/Mitchell, ~2004](#)) You can add a club of ω_2 which doesn't contain an infinite set from the ground model.

Theorem

1. ([Folklore?](#)) It is consistent to have a forcing axiom for Cohen's poset.
 2. ([Asperó, 2014](#)) It is consistent to have a forcing axiom for Baumgartner's poset.
 3. ([Shelah, 1990s?](#)) There is no forcing axiom for the Friedman/Mitchell posets.
- Shelah used his ZFC club guessing theorem.

Independence: order vs chaos

- ▶ Forcing/forcing axioms and infinite combinatorics have a common goal, to delineate the boundary between provable and unprovable.
- ▶ But their approaches are dual.
- ▶ Forcing is taking the via negativa: “[...] is not provable”;
- ▶ Infinite combinatorics the opposite path: “[...] is provable”.
- ▶ At $\aleph_1$ the dividing line can be subtle:
 - ▶ Todorčević’ idea of forcing with models as side conditions was motivated by his 1982 solution of the S -space problem.
 - ▶ The dual L -space problem was solved by Moore only in 2005 (in ZFC) using oscillations and walks on ordinals.
- ▶ At $\aleph_2$ and beyond much remains to be done, but there is a shift in the balance.

What are the new tools available to infinite combinatorics beyond $\aleph_1$?

1. Shelah's club guessing theorem and many variations.
2. More generally, 'higher gap combinatorics'.
 - ▶ Use the old lemma: $\aleph_0 < \aleph_1 < \aleph_2$.
3. Todorćević's technique of walks on ordinals and new ideas there.
4. New Ulam-type matrices.
5. Strong colouring principles.
6. CH, or weak squares, or other assumptions which may be natural for the problem at hand.
7. Techniques for improving ladder systems/ C -sequences.
8. Shelah's PCF (possible cofinalities) theory.

An advertisement

Our paper [Walks on uncountable ordinals and non-structure theorems for higher Aronszajn lines](#) needed all of these tools. We laid the groundwork in our earlier papers:

- ▶ *A club guessing toolbox I*, Bull. Symb. Log., 59pp, 2024.
- ▶ *Was Ulam Right? I: Basic theory and subnormal ideals*, Topology Appl., 53pp, 2023.
- ▶ *Was Ulam Right? II: Small width and general ideals*, Algebra Universalis, 47pp, 2024.
- ▶ *Was Ulam Right? III: Indecomposable ideals*, J. Inst. Math. Jussieu, 33pp, 2026.

The paper also contains a detailed introduction with many examples.

A step back: what is so great about forcing axioms anyway?

What we talk about when we talk about forcing axioms

Forcing axioms at $\aleph_1$ get their prominence from a remarkable confluence of factors:

1. A **natural** formulation which captures our intuition.
 - ▶ Quoting Martin–Solovay: “Although this paper is about forcing, almost the whole paper can be read without any knowledge of forcing. ”
2. They are **useful** axioms.
3. Bonus: they have a number of equivalent formulations:
 - ▶ in terms of **saturation of the universe**;
 - ▶ in terms of **bootstrapping chain conditions**;
 - ▶ in terms of **Ramsey theory**,
 - ▶ in terms of **cardinal invariants of the continuum** etc.
4. Bonus: they suggest new ZFC theorems.

Each perspective, each result about forcing axioms at $\aleph_1$, suggests questions at $\aleph_2$, and in turn, a possibility of discovering new combinatorial phenomena.

Examples of forcing axioms

1. $\text{MA}_{\aleph_1}$,
2. $\text{MA}_{\aleph_2}$,
3. PFA,
4. SPFA,
5. $\text{GMA}_{\aleph_2}$,
6. ...

Template: For every partial order in the class Γ , for any sequence of κ -many dense sets etc. etc. etc.

Some equivalent forms: the saturation perspective

Theorem (Bagaria 1997, Stavi–Väänänen 2002 (independently))

For every ccc poset $\mathbb{P}$, the following are equivalent:

- $\text{FA}_\theta(\mathbb{P})$;
- $H(\theta^+) \preceq_{\Sigma_1} H(\theta^+)^{V^\mathbb{P}}$.

So, MA_θ holds iff $H(\theta^+)$ is a Σ_1 -elementary substructure of the $H(\theta^+)$ of any ccc forcing extension of the universe.

- ▶ The universe is similar to the generic extension.

Some equivalent forms: chain conditions

Recall (Kunen, Rowbottom, Solovay (independently))

$\text{MA}_{\aleph_1}$ *implies the product of any two ccc posets is ccc.*

Theorem (Todorćević–Veličković, 1987)

1. MA holds iff every ccc poset of size less than $2^{\aleph_0}$ is σ -centered;
2. $\text{MA}_{\aleph_1}$ holds iff every ccc poset has precalibre $\aleph_1$.
3. $\text{MA}_{\aleph_1}$ iff $\omega_1 \xrightarrow{\text{ccc}} (\omega_1)^{<\omega}$.

Theorem (Peng, 2025)

$\text{MA}_{\aleph_1}$ holds iff every ccc poset has property K_3 .

The historical origin of forcing axioms

MA was first extracted from the proof of the consistency of Suslin's Hypothesis.

Theorem (Solovay–Tennenbaum, 1969)

$\text{SH}_{\aleph_1}$ is consistent: Every complete ccc dense linear ordered without endpoints is isomorphic to $(\mathbb{R}, <)$.

PFA was inspired by the proof of the following:

Theorem (Baumgartner, 1973)

Consistently every two $\aleph_1$ -dense sets of reals without endpoints are order-isomorphic.

Both these results extend Cantor's characterisation of $\mathbb{Q}$.

Forcing axioms to classify uncountable structures

$V = L$ is also a very decisive axiom. But forcing axioms prioritise **order** over **chaos** at the boundary of the countable and the uncountable. Many examples:

- ▶ (Abraham–Shelah, 1985) PFA implies that every two $\aleph_1$ -Aronszajn trees are club-isomorphic.
- ▶ (Todorćević, 1985) PFA implies there are five Tukey types of size $\aleph_1$.
- ▶ (Foreman–Magidor–Shelah, 1988) SPFA implies that $NS_{\aleph_1}$ is saturated.
- ▶ (Todorćević, 1996) PFA gives a classification of transitive relations on $\aleph_1$.
- ▶ (Moore, 2006) PFA implies that there is a 5-element basis for the uncountable linear orders.
- ▶ (Moore, 2009) PFA implies that there is a universal $\aleph_1$ -Aronszajn line.
- ▶ (Farah, 2011) PFA implies that all automorphisms of the Calkin algebra are inner.

More uses of forcing axioms

The following are all consequences of PFA.

- ▶ All automorphisms of $\mathcal{P}(\omega)/\text{Fin}$ are trivial;
- ▶ Uniformization over ω_1 ;
- ▶ Every compact space of countable tightness is sequential;
- ▶ Every Hajnal–Maté graph is countably chromatic;
- ▶ Every Aronszajn tree is special;
- ▶ P-ideal dichotomy;
- ▶ Open Graph Axiom.

Summarising, forcing axioms are useful axioms, and they provide a well-behaved structure theory for classes of objects of size $\aleph_1$.

Summary

Forcing axioms at $\aleph_1$ are useful axioms, they are decisive axioms, but furthermore they often decide questions *the right way*, providing a well-behaved structure theory for classes of objects of size $\aleph_1$.

We can separate and analyse the ‘unreasonable effectiveness of forcing axioms at $\aleph_1$ ’ into the following perspectives:

- ▶ **natural perspective**: diagonalising families of dense sets;
- ▶ **richness perspective**: the universe is saturated with respect to forcing, or close to the generic extension;
- ▶ **chain condition perspective**: a Ramsey-theoretic perspective focusing on one technical aspect;
- ▶ **historical/utilitarian perspective**: focusing on their historical role in solving longstanding open problems and in classifying uncountable structures.

Higher forcing axioms : some limitations

Some 'obvious' higher forcing axioms are inconsistent

Recall (Shelah, 1982)

CH refutes $\text{FA}_{\aleph_2}(\mathbb{P})$ for some $\mathbb{P}$ that is countably closed, $\aleph_2$ -stationary-cc.

Recall (Inamdar, 2013)

CH refutes $\text{FA}_{\aleph_2}(\mathbb{P})$ for some $\mathbb{P}$ that is countably-closed, precalibre $\aleph_2$, and well-met.

Theorem (Shelah, 2004)

$\text{FA}_{\aleph_2}(\Gamma)$ fails for the class Γ of posets which add no ω_1 -sequences of ordinals and preserve stationary subsets of $\aleph_2$.

Theorem (Asperó–Tananimit, 2022)

$\text{FA}_{\aleph_2}(\Gamma)$ fails for the class Γ of posets which preserve stationary subsets of $\aleph_1$ and have the $\aleph_2$ -cc.

The limitations of GMA

- ▶ $\text{GMA}_{\aleph_2}$ does have some structural consequences, e.g. see Tall, 1994.
- ▶ $\text{GMA}_{\aleph_2}$ has no consistency strength beyond ZFC.
- ▶ In particular, $\text{GMA}_{\aleph_2}$ is consistent with the existence of an $\aleph_2$ -Suslin tree.
- ▶ It is not even known if $\text{GMA}_{\aleph_2}$ is consistent with $\text{SH}_{\aleph_2}$.
- ▶ There are even higher forcing axioms which **imply** that there is an $\aleph_2$ -Suslin tree: Shelah–Stanley 1982, Lambie-Hanson–Rinot 2018.

Limitations on higher forcing axioms: bootstrapping chain conditions

Recall (Kunen, Rowbottom, Solovay (independently))

$\text{MA}_{\aleph_1}$ *implies the product of any two ccc posets is ccc.*

Recall (Shelah, 1997)

There is a well-met, $\aleph_2$ -cc poset whose square is not $\aleph_2$ -cc.

Theorem (Todorćević, 2017)

CH yields a well-met, countably closed, $\aleph_2$ -cc poset whose square is not $\aleph_2$ -cc.

Higher forcing axioms: some good news

Old good news

Recall (Solovay–Tennenbaum, 1969)

$MA_{\aleph_1}$ *implies* $SH_{\aleph_1}$.

Theorem (Laver–Shelah, 1978)

Assuming the consistency of a weakly compact cardinal, $SH_{\aleph_2}$ is consistent with CH.

Recall (Abraham–Shelah, 1985)

PFA implies that every two $\aleph_1$ -Aronszajn trees are club-isomorphic.

Theorem (Krueger, 2018)

Assuming the consistency of an ineffable cardinal, it is consistent that CH holds and any two normal σ -closed $\aleph_2$ -Aronszajn trees are club-isomorphic.

Newer good news I

Kuzeljević–Shalev–Todorčević recently considered a structural dichotomy for κ -lists, $\text{LD}(\kappa)$.

Theorem (Kuzeljević–Shalev–Todorčević, 202 ∞)

Assuming the consistency of a weakly compact cardinal, it is consistent that $\diamond(\aleph_1)$ and $\text{LD}(\aleph_2)$ both hold.

- ▶ The principle $\text{LD}(\aleph_1)$ follows from $\text{MA}_{\aleph_1}$.
- ▶ At a successor cardinal, $\text{LD}(\lambda^+)$ has several consequences:
 1. Every λ^+ -Aronszajn tree is special;
 2. There are no λ^+ -Souslin lower semi-lattices;
 3. Every λ^+ -tower in $(\mathcal{P}(\lambda), \subseteq^*)$ is Hausdorff.

Newer good news II

Definition (Hajnal–Maté, 1975)

A Hajnal–Maté graph over ω_1 is a graph of the form (ω_1, E) where for every $\beta < \omega_1$, the set $\{\alpha < \beta \mid \alpha E \beta\}$ is either finite or an ω -cofinal subset of β .

Recall (Hajnal–Maté, 1975)

$\text{MA}_{\aleph_1}$ implies that every Hajnal–Maté graph over ω_1 has chromatic number $\leq \omega$.

Definition

A Hajnal–Maté graph over ω_2 is a graph of the form (ω_2, E) where for every $\beta < \omega_2$, the set $\{\alpha < \beta \mid \alpha E \beta\}$ is either countable or an ω_1 -cofinal subset of β .

Theorem (Feldman, 202 ∞)

Assuming the consistency of a Mahlo cardinal and an inaccessible cardinal above it, it is consistent that every Hajnal–Maté graph over ω_2 has chromatic number $\leq \omega_1$.

Linear orders as test problems for higher forcing
axioms

Forcing axioms and uncountable linear orders

Forcing axioms were particularly successful at giving a coherent structure theory for linear orders at $\aleph_1$. Many examples:

1. (Solovay–Tennenbaum 1969) $\text{MA}_{\aleph_1}$ implies $\text{SH}_{\aleph_1}$;
2. (Baumgartner, 1973) PFA implies that any two $\aleph_1$ -sized suborders of $\mathbb{R}$ are bi-embeddable.
3. (Todorćević, 2007) $\text{MA}_{\aleph_1}$ implies that every $\aleph_1$ -Countryman line is minimal.
4. (Moore, 2006) PFA implies there is a 5-element basis for the uncountable linear orders.
5. (Moore, 2009) PFA implies there is a universal $\aleph_1$ -Aronszajn line.
6. (Martinez–Ranero 2011) PFA implies that $\aleph_1$ -Aronszajn lines are wqo.
7. (Dasilva-Barbosa 2021) PFA implies the $\aleph_1$ -Aronszajn lines have finite 1-dimensional Ramsey degree.

Analogue of Moore's Basis Theorem

Question (Moore, 2016)

Is it consistent that every $\aleph_2$ -Aronszajn line contains an $\aleph_2$ -Countryman line?

- ▶ Proving the analogous statement at $\aleph_1$ from PFA was the key step in Moore's proof of the Basis Theorem.

Theorem (Inamdar–Rinot 2024)

If there is an $\aleph_2$ -Aronszajn line, then there is an $\aleph_2$ -Aronszajn line not containing an $\aleph_2$ -Countryman line.

- ▶ “If”: (Mitchell–Silver, 1973) Assuming the consistency of a weakly compact cardinal, it is consistent that there are no $\aleph_2$ -Aronszajn lines.

Chaos for special Aronszajn lines: $\aleph_2$ and beyond

Theorem (Inamdar–Rinot, 2024)

Suppose $\kappa \geq \aleph_2$ is a successor of regular. The following are equivalent:

- 1. The class of special κ -Aronszajn lines is nonempty;*
- 2. The class of special κ -Aronszajn lines is not well-founded;*
- 3. Any basis for the class of special κ -Aronszajn lines has size 2^κ ;*
- 4. Total chaos: There is a lexicographically ordered special κ -Aronszajn tree $\mathbf{T}$ such that $\mathbf{T} \overset{\wedge}{\not\rightarrow} [\kappa]_\kappa^n$ holds for every positive integer n .*

- ▶ We have similar results for strongly inaccessible and also successors of singulars and inaccessible but the latter two not outright in ZFC.
- ▶ $\mathbf{T} \overset{\wedge}{\not\rightarrow} [\kappa]_\kappa^n$ is a partition relation for trees which is of interest for Aronszajn trees as well as weak Kurepa trees.

Another application of the partition relation for trees

Further applications of the partition relation for trees are given in a new series of papers. For example:

Theorem (Inamdar–Rinot, 2026)

CH implies that there is a family of $2^{\aleph_1}$ -many pairwise-incomparable $\aleph_1$ -coherent $\aleph_1$ -complete $\aleph_2$ -Aronszajn trees.

The above should be compared to the following result.

Theorem (Todorčević, 2007)

$\text{MA}_{\aleph_1}$ implies that all $\aleph_0$ -coherent $\aleph_1$ -Aronszajn trees are comparable.

Some more applications of partition relations for trees

Theorem (Inamdar–Rinot, 2026)

If λ is strong inaccessible not weakly compact, then for every pair $\mu < \theta$ of nonzero cardinals below λ , there is a λ -complete poset $\mathbb{P}$ such that $\mathbb{P}^\mu$ has the λ^+ -cc, but $\mathbb{P}^\theta$ does not.

Theorem (Inamdar–Rinot, 2026)

If $\text{cov}(\mathcal{M}) = \text{cof}(\mathcal{M}) = \kappa$, then for every natural number n , there is a (κ, n) -entangled linear order which is not $(\kappa, n+1)$ -entangled.

Theorem (Inamdar–Rinot, 2026)

If $\lambda > \beth_\omega$ is a cardinal such that $\lambda^{<\lambda} = \lambda$, then for some regular $\kappa \in [\lambda^+, 2^\lambda]$, there is a $(\kappa, <\lambda)$ -entangled linear order of size κ .

- Under PFA, no linear order of size $\aleph_1$ is $(\aleph_1, 3)$ -entangled.

Some conclusions

1. At $\aleph_1$, the two extreme possibilities for classes of structures are **order** (from forcing axioms say) and **chaos** (from $V = L$ say).
2. As the Mitchell model shows, at $\aleph_2$ and beyond there is a third possibility: **trivial**.
3. In the case of special $\aleph_2$ -Aronszajn lines, we reduce this **trivial**, **order**, **chaos** trichotomy to a dichotomy between the two extremes: **trivial** and **chaos**.
4. Such results on the test problems undercuts *any* possible forcing axiom, not just some specific forcing axiom.

The end.