# STRONGEST TRANSFORMATIONS 

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#### Abstract

We continue our study of maps which transform high-dimensional complicated objects into squares of stationary sets. Previously, we proved that many such transformations exist in ZFC. Here we address the consistency of the strongest conceivable transformations.

Along the way, we obtain new results on Shelah's coloring principle $\operatorname{Pr}_{1}$ : For $\kappa$ inaccessible, we prove the consistency of $\operatorname{Pr}_{1}(\kappa, \kappa, \kappa, \kappa)$. For successors of regulars, we obtain a full lifting of Galvin's 1980 theorem. In contrast, the full lifting of Galvin's theorem to successors of singulars is shown to be inconsistent.


## 1. Introduction

Throughout the paper, $\kappa$ denotes a regular uncountable cardinal, and $\theta, \chi$ denote (possibly finite) cardinals $\leq \kappa$.

In an attempt to give a broader perspective on what Ramsey theory is about, Graham, Rothschild and Spencer quote in their book on the subject [GRS90] the following words of Burkill and Mirsky: ". . . every system of a certain class possesses a large subsystem with a higher degree of organization than the original system". Though this phenomenon extends and remains quite accurate at the level of the first infinite cardinal, it is no longer true for the uncountable. Already Sierpiński [Sie33] constructed "anti-Ramsey" colorings on the real line - colorings of pairs of reals by two colors that attain both colors on every uncountable subset. In a body of work from the 1960's, Erdős and his collaborators were able to get from instances of the Generalized Continuum Hypothesis (GCH) a plethora of anti-Ramsey colorings on all successors $\mu^{+}$of infinite cardinals $\mu$. For instance, they showed that if $2^{\mu}=\mu^{+}$, then $\mu^{+} \nrightarrow\left[\mu^{+}\right]_{\mu^{+}}^{2}$ holds, that is, there exists a coloring $c:\left[\mu^{+}\right]^{2} \rightarrow \mu^{+}$such that no color is omitted on the square $[A]^{2}$ of any subset $A \subseteq \mu^{+}$of full cardinality $\mu^{+}$. Later on, problems in set theoretic topology prompted Hajnal and Juhász [HJ74], Roitman [Roi78] and others to find colorings that can handle tasks of high dimensional nature. Most notably, in a paper from 1980, Galvin [Gal80] proved that the Continuum Hypothesis (CH) implies the existence of a coloring $c:\left[\aleph_{1}\right]^{2} \rightarrow 2$ with the property that for every positive dimension $n$, every uncountable pairwise disjoint subfamily $\mathcal{A} \subseteq\left[\aleph_{1}\right]^{n}$, and every color $\tau<2$, there are $a, b \in \mathcal{A}$ with $\max (a)<\min (b)$ such that $c[a \times b]=\{\tau\}$. Then, Shelah [She88] defined the coloring principle $\operatorname{Pr}_{1}(\kappa, \kappa, \theta, \chi)$ (see Definition 2.3) that simultaneously captures all of these strong colorings concepts, and wrote a series of papers dealing with their consistency (see the introduction to [Rin14a] for a survey). A few years ago, in [FR17], a study of additive Ramsey theory at the level of the uncountable demonstrated again the utility of strong colorings that solve problems of high dimension.

[^0]In [RZ21], the authors introduced the principle $\mathrm{P} \ell_{1}(\kappa, \theta, \chi)$ (see Definition 2.5) asserting the existence of maps which transform high-dimensional complicated objects into squares of stationary sets. They proved that $\mathrm{P} \ell_{1}(\kappa, \theta, \chi)$ is strictly stronger than Shelah's coloring principle $\operatorname{Pr}_{1}(\kappa, \kappa, \theta, \chi)$. However, and that was the main motivation for its introduction, the instance $\mathrm{P} \ell_{1}(\kappa, 1, \chi)$ implies that the high-dimensional coloring principle $\operatorname{Pr}_{1}(\kappa, \kappa, \theta, \chi)$ is no stronger than the classical negative partition relation $\kappa \nrightarrow[\kappa]_{\theta}^{2}$. In particular, $\mathrm{P} \ell_{1}(\kappa, 1, \omega)$ enables to reduce problems in additive Ramsey theory concerning groups of size $\kappa$ down to the classical partition calculus of the cardinal $\kappa$ studied by Erdős and his collaborators. Note that even the very weak instance $\mathrm{P} \ell_{1}(\kappa, 1,3)$ is quite powerful, as it allows the transformation of rectangles into squares, as in [Rin12].

Most of [RZ21] was devoted to providing sufficient conditions for instances of $\mathrm{P} \ell_{1}(\ldots)$ to hold. In particular, combining walks on ordinals with strong forms of the oscillation oracle $\mathrm{P} \ell_{6}(\ldots)$, it was shown that many instances of $\mathrm{P} \ell_{1}(\ldots)$ are theorems of ZFC.

The current paper is dedicated to studying the strongest instance of $\mathrm{P} \ell_{1}(\kappa, \theta, \chi)$, namely the case when $\theta:=\kappa$ and $\chi:=\sup (\operatorname{Reg}(\kappa))$, and a further strengthening of it that reads as follows (see Figure 1 on Page 3):

Definition 1.1. For a stationary subset $\Gamma \subseteq \kappa, \mathrm{P} \ell_{2}(\kappa, \Gamma, \chi)$ asserts the existence of a transformation $\mathbf{t}:[\kappa]^{2} \rightarrow[\kappa]^{2}$ satisfying the following:

- for every $(\alpha, \beta) \in[\kappa]^{2}$, if $\mathbf{t}(\alpha, \beta)=\left(\alpha^{*}, \beta^{*}\right)$, then $\alpha^{*} \leq \alpha<\beta^{*} \leq \beta$;
- for every $\sigma<\chi$ and every pairwise disjoint subfamily $\mathcal{A} \subseteq[\kappa]^{\sigma}$ of size $\kappa$, there exists a club $D \subseteq \kappa$, such that, for all $\left(\alpha^{*}, \beta^{*}\right) \in[\Gamma \cap D]^{2}$, there exists $(a, b) \in[\mathcal{A}]^{2}$ with $\mathbf{t}[a \times b]=\left\{\left(\alpha^{*}, \beta^{*}\right)\right\}$.

Our first main result concerns successors of regular cardinals:
Theorem A. For every infinite regular cardinal $\mu$, either of the following imply that $\mathrm{P} \ell_{2}\left(\mu^{+}, E_{\mu}^{\mu^{+}}, \mu\right)$ holds:
(1) ${ }^{\bullet}\left(\mu^{+}\right)$;
(2) $\left(\mu^{+}\right)^{\aleph_{0}}=\mu^{+}$and $\boldsymbol{Q}(S)$ holds for some nonreflecting stationary $S \subseteq \mu^{+}$.

Remark 1. ${ }^{\ominus}\left(\mu^{+}\right)$is the stick principle (see Definition 6.2) which is a weakening of the assertion that $2^{\mu}=\mu^{+}$. In Clause (2), we moreover get $\mathrm{P} \ell_{2}\left(\mu^{+}, \mu^{+}, \mu\right)$.

Theorem A sheds a new light on [She97, Question 2.3], in particular showing that, for every infinite regular cardinal $\mu, 2^{\mu}=\mu^{+}$implies $\operatorname{Pr}_{1}\left(\mu^{+}, \mu^{+}, \mu^{+}, \mu\right)$. As mentioned earlier, the case $\mu=\aleph_{0}$ was proved by Galvin back in 1980, but his original proof only generalizes to show that $2^{\mu}=\mu^{+}$implies $\operatorname{Pr}_{1}\left(\mu^{+}, \mu^{+}, \mu^{+}, \aleph_{0}\right)$.

Our second main result concerns (weakly) inaccessible cardinals:
Theorem B. For every inaccessible cardinal $\kappa$, any of the following imply that $\mathrm{P} \ell_{1}(\kappa, \kappa, \kappa)$ holds:
(1) $\square(\kappa)$ and $\diamond(S)$ for some stationary $S \subseteq \kappa$ that does not reflect at regulars;
(2) $\square(\kappa)$ and $\diamond^{*}(\kappa)$;
(3) $\boxtimes^{-}(\kappa)$ and $\diamond(\kappa)$;
(4) $\kappa$ is Mahlo, $\diamond(S)$ for some stationary $S \subseteq \kappa$ that does not reflect, and there exists a nonreflecting stationary subset of $\operatorname{Reg}(\kappa)$.

Remark 2. $\boxtimes^{-}(\kappa)$ is a simple instance of the Brodsky-Rinot proxy principle (see Definition 3.3). In Clauses (3) and (4), we moreover get $\mathrm{P} \ell_{2}(\kappa, \kappa, \kappa)$.

Our third main result concerns successors of singulars. Here, we uncover ZFC constraints on the extent of the combinatorial principles under discussion.

Theorem C. Let $\mu$ be a singular cardinal. Then:
(1) $\operatorname{Pr}_{1}\left(\mu^{+}, \mu^{+}, 2, \mu\right)$ fails;
(2) $\operatorname{Pr}_{1}\left(\mu^{+}, \mu^{+}, 2, \operatorname{cf}(\mu)^{+}\right)$fails, provided that $\mu$ is a (singular) limit of strongly compact cardinals;
(3) $\operatorname{Pr}_{6}\left(\mu^{+}, \mu^{+}, 2, \mu\right)$ fails;
(4) $\mathrm{P} \ell_{6}\left(\mu^{+}, \mu\right)$ fails.

As a corollary, we confirm that in Gödel's constructible universe, every regular uncountable cardinal admits the strongest conceivable transformation:

Theorem D. Assuming $V=L$, for every regular uncountable cardinal $\kappa$ and every regular cardinal $\chi \leq \chi(\kappa), \mathrm{P} \ell_{2}(\kappa, \kappa, \chi)$ holds. ${ }^{1}$


Figure 1. Illustration of Definition 1.1.
1.1. Organization of this paper. In Section 2, we define all the combinatorial principles discussed in this paper, prove Theorem C and also prove that $\mathrm{P} \ell_{6}(\mu, \mu)$ and $\operatorname{Pr}_{6}(\mu, \mu, 2, \mu)$ fail for any infinite cardinal $\mu$. These theorems should be understood as verifying positive partition relations.

[^1]In Section 3, we make some contributions to the theory of $C$-sequences. This will play a role in the proofs of Section 4.

In Section 4, we provide sufficient conditions for $\operatorname{Pr}_{1}(\kappa, \kappa, \kappa, \chi)$ to imply $\mathrm{P} \ell_{1}(\kappa, \kappa, \chi)$ or $\mathrm{P} \ell_{2}(\kappa, \Gamma, \chi)$.

In Section 5, we deal with inaccessible cardinals, in particular, proving Clauses (1)-(3) of Theorem B.

In Section 6, we first prove our generalization of Galvin's theorem and then use the results of Section 4 to obtain Clause (1) of Theorem A.

In Section 7, we derive transformations by walking along $C$-sequences witnessing an instance of the Brodsky-Rinot proxy principle. This is how Clause (2) of Theorem A, Clause (4) of Theorem B, and Theorem D are obtained.
1.2. Notation and conventions. By an inaccessible we mean a regular uncountable limit cardinal. Let $E_{\chi}^{\kappa}:=\{\alpha<\kappa \mid \operatorname{cf}(\alpha)=\chi\}$, and define $E_{\leq \chi}^{\kappa}, E_{<\chi}^{\kappa}$, $E_{\geq \chi}^{\kappa}, E_{>\chi}^{\kappa}, E_{\neq \chi}^{\kappa}$ analogously. The collection of all sets of hereditary cardinality less than $\kappa$ is denoted by $\mathcal{H}_{\kappa}$. The set of all infinite and regular cardinals below $\kappa$ is denoted by $\operatorname{Reg}(\kappa)$. The length of a finite sequence $\varrho$ is denoted by $\ell(\varrho)$. A stationary subset $S \subseteq \kappa$ is nonreflecting iff there exists no $\alpha \in E_{>\omega}^{\kappa}$ such that $S \cap \alpha$ is stationary in $\alpha$. For a set of ordinals $a$, we write $\operatorname{ssup}(a):=\sup \{\alpha+1 \mid$ $\alpha \in a\}, \operatorname{acc}^{+}(a):=\{\alpha<\operatorname{ssup}(a) \mid \sup (a \cap \alpha)=\alpha>0\}, \operatorname{acc}(a):=a \cap \operatorname{acc}^{+}(a)$, $\operatorname{nacc}(a):=a \backslash \operatorname{acc}(a)$, and $\operatorname{cl}(a):=a \cup \operatorname{acc}^{+}(a)$. For sets of ordinals, $a$ and $b$, we let $a \circledast b:=\{(\alpha, \beta) \in a \times b \mid \alpha<\beta\}$, and write $a<b$ to express that $a \times b$ coincides with $a \circledast b$. For any set $\mathcal{A}$, we write $[\mathcal{A}]^{\chi}:=\{\mathcal{B} \subseteq \mathcal{A}| | \mathcal{B} \mid=\chi\}$ and $[\mathcal{A}]^{<\chi}:=\{\mathcal{B} \subseteq \mathcal{A} \mid$ $|\mathcal{B}|<\chi\}$. This convention admits two refined exceptions:

- for an ordinal $\sigma$ and a set of ordinals $A$, we write $[A]^{\sigma}$ for $\{B \subseteq A \mid$ otp $(B)=\sigma\}$;
- for a set $\mathcal{A}$ which is either an ordinal or a collection of sets of ordinals, we interpret $[\mathcal{A}]^{2}$ as the collection of ordered pairs $\{(a, b) \in \mathcal{A} \times \mathcal{A} \mid a<b\}$.
In particular, $[\kappa]^{2}=\{(\alpha, \beta) \mid \alpha<\beta<\kappa\}$. Likewise, we let $[\kappa]^{3}:=\{(\alpha, \beta, \gamma) \in$ $\kappa \times \kappa \times \kappa \mid \alpha<\beta<\gamma<\kappa\}$.


## 2. Combinatorial principles and inconsistent instances of them

Convention 2.1. For any coloring $f:[\kappa]^{2} \rightarrow \theta$ and $\delta<\kappa$, while $(\delta, \delta) \notin[\kappa]^{2}$, we extend the definition of $f$, and agree to let $f(\delta, \delta):=0$.

Definition 2.2 ([EHR65, §18]). $\kappa \nrightarrow[\kappa]_{\theta}^{2}$ asserts the existence of a coloring $c$ : $[\kappa]^{2} \rightarrow \theta$ such that, for every $A \subseteq \kappa$ of size $\kappa, c^{\prime \prime}[A]^{2}=\theta$.

Definition 2.3 ([She88]). $\operatorname{Pr}_{1}(\kappa, \kappa, \theta, \chi)$ asserts the existence of a coloring $c$ : $[\kappa]^{2} \rightarrow \theta$ such that for every $\sigma<\chi$, every pairwise disjoint subfamily $\mathcal{A} \subseteq[\kappa]^{\sigma}$ of size $\kappa$, and every $\tau<\theta$, there is $(a, b) \in[\mathcal{A}]^{2}$ such that $c[a \times b]=\{\tau\}$.

Note that $\operatorname{Pr}_{1}(\kappa, \kappa, \theta, \chi)$ implies $\operatorname{Pr}_{1}\left(\kappa, \kappa, \theta^{\prime}, \chi^{\prime}\right)$ for all $\theta^{\prime} \leq \theta$ and $\chi^{\prime} \leq \chi$, and that the instance $\operatorname{Pr}_{1}(\kappa, \kappa, \theta, 2)$ coincides with the classical relation $\kappa \nrightarrow[\kappa]_{\theta}^{2}$. In particular, Ramsey's celebrated theorem asserts that $\operatorname{Pr}_{1}(\omega, \omega, 2,2)$ fails.
Definition $2.4([\mathrm{KRS} 21 \mathrm{a}]) . \operatorname{Pr}_{1}(\kappa, \mu \circledast \kappa / 1 \circledast \kappa, \theta, \chi)$ asserts the existence of a coloring $c:[\kappa]^{2} \rightarrow \theta$ satisfying that for every $\sigma<\chi$, and every pairwise disjoint subfamilies $\mathcal{A}, \mathcal{B}$ of $[\kappa]^{\sigma}$ with $|\mathcal{A}|=\mu$ and $|\mathcal{B}|=\kappa$, there is $a \in \mathcal{A}$ such that, for every $\tau<\theta$, there is $b \in \mathcal{B}$ with $a<b$ such that $c[a \times b]=\{\tau\}$.

Note that $\operatorname{Pr}_{1}(\kappa, \mu \circledast \kappa / 1 \circledast \kappa, \theta, \chi)$ implies $\operatorname{Pr}_{1}(\kappa, \kappa, \theta, \chi)$. By [KRS21b, Corollary 3.3], the former (unbalanced) principle is strictly stronger than the latter.

Definition 2.5 ([RZ21]). $\mathrm{P} \ell_{1}(\kappa, \theta, \chi)$ asserts the existence of a transformation $\mathbf{t}:[\kappa]^{2} \rightarrow[\kappa]^{3}$ satisfying the following:

- for every $(\alpha, \beta) \in[\kappa]^{2}$, if $\mathbf{t}(\alpha, \beta)=\left(\tau^{*}, \alpha^{*}, \beta^{*}\right)$, then $\tau^{*} \leq \alpha^{*} \leq \alpha<\beta^{*} \leq \beta$;
- for every $\sigma<\chi$ and every pairwise disjoint subfamily $\mathcal{A} \subseteq[\kappa]^{\sigma}$ of size $\kappa$, there exists a stationary $S \subseteq \kappa$ such that, for all $\left(\alpha^{*}, \beta^{*}\right) \in[S]^{2}$ and $\tau^{*}<\min \left\{\theta, \alpha^{*}\right\}$, there exists $(a, b) \in[\mathcal{A}]^{2}$ with $\mathbf{t}[a \times b]=\left\{\left(\tau^{*}, \alpha^{*}, \beta^{*}\right)\right\}$.
The motivation to study the preceding principle comes from the fact that it reduces strong coloring principles to weaker ones, as listed in the following result.

Fact 2.6 ([RZ21, Lemma 2.18 and Corollary 2.22]). Any of the following implies that $\operatorname{Pr}_{1}(\kappa, \kappa, \theta, \chi)$ holds:
(1) $\mathrm{P} \ell_{1}(\kappa, 1, \chi)$ and $\kappa \nrightarrow[\kappa]_{\theta}^{2}$;
(2) $\mathrm{P} \ell_{1}(\kappa, 1, \chi)$ and $\theta \leq \aleph_{0}$;
(3) $\mathrm{P} \ell_{1}(\kappa, \theta, \chi)$;
(4) $\mathrm{P} \ell_{1}(\kappa, \operatorname{cf}(\theta), \chi)$ and $\kappa \nrightarrow[\kappa]_{\eta}^{2}$ for all $\eta<\theta$;
(5) $\mathrm{P} \ell_{1}(\kappa, \nu, \chi)$ and there exists a $\nu^{+}-c c$ forcing extension satisfying $\kappa \nrightarrow[\kappa]_{\theta}^{2}$.

Definition 2.5 displays a minor asymmetry in the sense that the domain of the transformation is $[\kappa]^{2}$ whereas its range is $[\kappa]^{3}$. Definition 1.1 captures a scenario in which this problem may be mitigated. And indeed, the principle of Definition 1.1 is typically stronger than that of Definition 2.5:

Proposition 2.7. For every stationary $\Gamma \subseteq \kappa, \mathrm{P} \ell_{2}(\kappa, \Gamma, \chi)$ implies $\mathrm{P} \ell_{1}(\kappa, \kappa, \chi)$.
Proof. Suppose that $\mathrm{P} \ell_{2}(\kappa, \Gamma, \chi)$ holds for a given stationary $\Gamma \subseteq \kappa$. Fix a partition $\left\langle\Gamma_{p} \mid p \in[\kappa]^{2}\right\rangle$ of $\Gamma$ into stationary sets. Define a map $\pi: \kappa \rightarrow[\kappa]^{2}$ via:

$$
\pi(\alpha):= \begin{cases}p, & \text { if } \alpha \in \Gamma_{p} \& \alpha \geq \max (p) \\ (0,1), & \text { otherwise }\end{cases}
$$

Now, given a transformation $\mathbf{t}_{2}:[\kappa]^{2} \rightarrow[\kappa]^{2}$ witnessing that $\mathrm{P} \ell_{2}(\kappa, \Gamma, \chi)$ holds, we define a transformation $\mathbf{t}_{1}:[\kappa]^{2} \rightarrow[\kappa]^{3}$ by letting for every $(\alpha, \beta) \in[\kappa]^{2}$, $\mathbf{t}_{1}(\alpha, \beta):=\left(\tau_{1}, \alpha_{1}, \beta_{2}\right)$ provided that $\mathbf{t}_{2}(\alpha, \beta)=\left(\alpha_{2}, \beta_{2}\right)$ and $\pi\left(\alpha_{2}\right)=\left(\tau_{1}, \alpha_{1}\right)$.

To see that $\mathbf{t}_{1}$ witnesses $\mathrm{P} \ell_{1}(\kappa, \kappa, \chi)$, suppose that $\mathcal{A}$ is a $\kappa$-sized pairwise disjoint subfamily of $[\kappa]^{\sigma}$, for some fixed $\sigma<\chi$. By the choice of $\mathbf{t}_{2}$, let us fix a club $D \subseteq \kappa$, such that, for every $\left(\alpha_{2}, \beta_{2}\right) \in[\Gamma \cap D]^{2}$, there exists $(a, b) \in[\mathcal{A}]^{2}$ with $\mathbf{t}_{2}[a \times b]=\left\{\left(\alpha_{2}, \beta_{2}\right)\right\}$. Evidently, $E:=\left\{\gamma<\kappa \mid[\gamma]^{2} \subseteq \pi[\Gamma \cap D \cap \gamma]\right\}$ is a club, so that $S:=\Gamma \cap \operatorname{acc}(E)$ is stationary. Finally, given $\left(\alpha^{*}, \beta^{*}\right) \in[S]^{2}$, pick $\gamma \in E$ such that $\alpha^{*}<\gamma<\beta^{*}$. Now, given $\tau^{*}<\alpha^{*}$, pick some $\alpha_{2} \in \Gamma \cap D \cap \gamma$ such that $\pi\left(\alpha_{2}\right)=\left(\tau^{*}, \alpha^{*}\right)$, and then pick $(a, b) \in[\mathcal{A}]^{2}$ such that $\mathbf{t}_{2}[a \times b]=\left\{\left(\alpha_{2}, \beta^{*}\right)\right\}$. Clearly, $\mathbf{t}_{1}[a \times b]=\left\{\left(\tau^{*}, \alpha^{*}, \beta^{*}\right)\right\}$.
Definition 2.8 ([She97]). $\operatorname{Pr}_{6}(\kappa, \kappa, \theta, \chi)$ asserts the existence of a coloring $d$ : ${ }^{<\omega} \kappa \rightarrow \theta$ satisfying the following. For every $\tau<\theta$, and every sequence $\left\langle\left(u_{\alpha}, v_{\alpha}\right)\right|$ $\alpha \in E\rangle$ such that:
(1) $E$ is a club in $\kappa$;
(2) $u_{\alpha}$ and $v_{\alpha}$ are nonempty elements of $\left[\omega^{<\omega}\right]^{<\chi}$;
(3) $\alpha \in \operatorname{Im}(\varrho)$ for all $\varrho \in u_{\alpha}$;
(4) $\alpha \in \operatorname{Im}(\sigma)$ for all $\sigma \in v_{\alpha}$,
there exists $(\alpha, \beta) \in[E]^{2}$ such that $d\left(\varrho^{\circ} \sigma\right)=\tau$ for all $\varrho \in u_{\alpha}$ and $\sigma \in v_{\beta}$.
Definition 2.9 ([Rin14b]). $\mathrm{P} \ell_{6}(\kappa, \chi)$ asserts the existence of a map $d:<\omega \kappa \rightarrow \omega$ satisfying the following. For every sequence $\left\langle\left(u_{\alpha}, v_{\alpha}, \sigma_{\alpha}\right) \mid \alpha<\kappa\right\rangle$ and $\varphi: \kappa \rightarrow \kappa$ with
(1) $\varphi$ is eventually regressive. That is, $\varphi(\alpha)<\alpha$ for co-boundedly many $\alpha<\kappa$;
(2) $u_{\alpha}$ and $v_{\alpha}$ are nonempty elements of $\left[{ }^{<\omega} \kappa\right]^{<\chi}$;
(3) $\alpha \in \operatorname{Im}(\varrho)$ for all $\varrho \in u_{\alpha}$;
(4) $\sigma_{\alpha} \frown\langle\alpha\rangle \sqsubseteq \sigma$ for all $\sigma \in v_{\alpha}$,
there exists $(\alpha, \beta) \in[\kappa]^{2}$ with $\varphi(\alpha)=\varphi(\beta)$ such that $d\left(\varrho^{\wedge} \sigma\right)=\ell(\varrho)$ for all $\varrho \in u_{\alpha}$ and $\sigma \in v_{\beta}$.

Proposition 2.10. Let $\mu$ be a singular cardinal.
Then $\operatorname{Pr}_{1}\left(\mu^{+}, \mu^{+}, 2, \mu\right)$ and $\mathrm{P} \ell_{1}\left(\mu^{+}, 1, \mu\right)$ both fail.
Proof. As mentioned earlier, $\mathrm{P} \ell_{1}\left(\mu^{+}, 1, \mu\right)$ implies $\operatorname{Pr}_{1}\left(\mu^{+}, \mu^{+}, \aleph_{0}, \mu\right)$, so, towards a contradiction, let us suppose that $c:\left[\mu^{+}\right]^{2} \rightarrow 2$ witnesses $\operatorname{Pr}_{1}\left(\mu^{+}, \mu^{+}, 2, \mu\right)$.

Claim 2.10.1. There exists $A \in\left[\mu^{+}\right]^{\mu^{+}}$such that, for every $a \in[A]^{<\mu}$, there are cofinally many $\beta<\mu^{+}$with $c[a \times\{\beta\}]=\{0\}$.

Proof. Suppose not. Construct a sequence $\left\langle\left(A_{i}, a_{i}, B_{i}\right) \mid i<\mu^{+}\right\rangle$by recursion on $i<\mu^{+}$, as follows:

- Let $A_{0}:=\mu^{+}$. By the indirect assumption, we may find $a_{0} \in\left[\mu^{+}\right]^{<\mu}$ such that $B_{0}:=\left\{\beta<\mu^{+} \mid \beta>\sup \left(a_{0}\right) \& c\left[a_{0} \times\{\beta\}\right]=\{0\}\right\}$ is bounded in $\mu^{+}$.
- Suppose that $i<\mu^{+}$is nonzero, and that $\left\langle\left(A_{j}, a_{j}, B_{j}\right) \mid j<i\right\rangle$ has already been defined. Set $A_{i}:=\mu^{+} \backslash\left(\sup \left(\bigcup_{j<i} B_{j}\right)+1\right)$. By the indirect assumption, we may now find $a_{i} \in\left[A_{i}\right]^{<\mu}$ such that $B_{i}:=\left\{\beta<\mu^{+} \mid \beta>\sup \left(a_{i}\right) \& c\left[a_{i} \times\{\beta\}\right]=\{0\}\right\}$ is bounded in $\mu^{+}$.

This completes the description of the recursion. Fix $\sigma<\mu$ and $I \in\left[\mu^{+}\right]^{\mu^{+}}$such that $\operatorname{otp}\left(a_{i}\right)=\sigma$ for all $i \in I$. Then $\left\langle a_{i} \mid i \in I\right\rangle$ is a <-increasing sequence of elements of $\left[\mu^{+}\right]^{\sigma}$. Thus, by the choice of $c$, we may find $(j, i) \in[I]^{2}$ such that $c\left[a_{j} \times a_{i}\right]=\{0\}$. However, $a_{i} \subseteq A_{i}$, so that $a_{i} \cap B_{j}=\emptyset$. This is a contradiction.

Fix $A$ as in the claim. Without loss of generality, $\min (A) \geq \mu$. Let $\delta \in A$. Fix a decomposition $A \cap \delta=\biguplus_{i<\operatorname{cf}(\mu)} A_{\delta, i}$ such that $\left|A_{\delta, i}\right|<\mu$ for all $i<\operatorname{cf}(\mu)$, and then, for every $i<\operatorname{cf}(\mu)$, fix $\beta_{\delta, i}>\delta$ such that $c\left[A_{\delta, i} \times\left\{\beta_{\delta, i}\right\}\right] \subseteq\{0\}$. Denote $b_{\delta}:=\{\delta\} \cup\left\{\beta_{\delta, i} \mid i<\operatorname{cf}(\mu)\right\}$, so that $b_{\delta} \in\left[\mu^{+} \backslash \delta\right] \leq \operatorname{cf}(\mu)$.

Fix a sparse enough $\Delta \in[A]^{\mu^{+}}$such that, for every $(\gamma, \delta) \in[\Delta]^{2}, \sup \left(b_{\gamma}\right)<$ $\min \left(b_{\delta}\right)$ and $\operatorname{otp}\left(b_{\gamma}\right)=\operatorname{otp}\left(b_{\delta}\right)$. Now, by the choice of $c$, there must exist $(\gamma, \delta) \in$ $[\Delta]^{2}$ such that $c\left[b_{\gamma} \times b_{\delta}\right]=\{1\}$. Pick $i<\operatorname{cf}(\mu)$ such that $\gamma \in A_{\delta, i}$. Then $c\left(\gamma, \beta_{\delta, i}\right)=$ 0 , contradicting the fact that $\left(\gamma, \beta_{\delta, i}\right) \in b_{\gamma} \times b_{\delta}$.

The preceding proof makes it clear that the following holds, as well.
Proposition 2.11. For every infinite regular cardinal $\mu, \operatorname{Pr}_{1}\left(\mu^{+}, \mu^{+}, 2, \mu^{+}\right)$fails.

The next result is suggested by the proof of [LHR18, Theorem 2.14].
Proposition 2.12. Suppose that $\mu$ is a singular limit of strongly compact cardinals. Then $\operatorname{Pr}_{1}\left(\mu^{+}, \mu^{+}, 2, \operatorname{cf}(\mu)^{+}\right)$and $\mathrm{P} \ell_{1}\left(\mu^{+}, 1, \operatorname{cf}(\mu)^{+}\right)$both fail.

Proof. As mentioned earlier, $\mathrm{P} \ell_{1}\left(\mu^{+}, 1, \mathrm{cf}(\mu)^{+}\right)$implies $\operatorname{Pr}_{1}\left(\mu^{+}, \mu^{+}, \aleph_{0}, \operatorname{cf}(\mu)^{+}\right)$, so, towards a contradiction, let us suppose that $c:\left[\mu^{+}\right]^{2} \rightarrow 2$ witnesses $\operatorname{Pr}_{1}\left(\mu^{+}, \mu^{+}\right.$, $\left.2, \operatorname{cf}(\mu)^{+}\right)$.
Claim 2.12.1. Let $\theta<\mu$. There exists $X \in\left[\mu^{+}\right]^{\mu^{+}}$and $j<2$ such that, for every $x \in[X]^{\theta}$, there are cofinally many $\beta \in X$ with $c[x \times\{\beta\}]=\{j\}$.
Proof. For all $\alpha<\mu^{+}$and $j<2$, let

$$
B_{j}^{\alpha}:=\left\{\beta<\mu^{+} \mid \beta>\alpha \& c(\alpha, \beta)=j\right\}
$$

As there exists a strongly compact cardinal in between $\theta$ and $\mu$, let us fix a uniform, $\theta^{+}$-complete ultrafilter $U$ on $\mu^{+}$. Then, for each $\alpha<\mu^{+}$, find $j_{\alpha}<2$ such that $B_{j_{\alpha}}^{\alpha}$ is in $U$. Finally, find $j<2$ such that $X:=\left\{\alpha<\mu^{+} \mid j_{\alpha}=j\right\}$ is in $U$. Then $X$ is as sought.

Fix a strictly increasing sequence $\left\langle\mu_{i} \mid i<\operatorname{cf}(\mu)\right\rangle$ of cardinals converging to $\mu$, with $\mu_{0} \geq \operatorname{cf}(\mu)$. For each $i<\operatorname{cf}(\mu)$, let $X_{i}$ and $j_{i}$ be given by the above claim with $\mu_{i}$ playing the role of $\theta$. By thinning out, we may also assume the existence of $j<2$ such that $j_{i}=j$ for all $i<\operatorname{cf}(\mu)$.

For every $\delta<\mu^{+}$, fix a decomposition $\delta=\biguplus_{i<\operatorname{cf}(\mu)} \Gamma_{\delta, i}$ such that $\left|\Gamma_{\delta, i}\right| \leq \mu_{i}$ for all $i<\operatorname{cf}(\mu)$. We shall now construct a matrix $\left\langle\beta_{\delta, i} \mid \delta<\mu^{+}, i<\operatorname{cf}(\mu)\right\rangle$ in such a way that, for each $\delta<\mu^{+},\left\langle\beta_{\delta, i} \mid i<\operatorname{cf}(\mu)\right\rangle \in \prod_{i<\operatorname{cf}(\mu)} X_{i}$. The definition is by recursion on $\delta<\mu^{+}$:

- For $\delta=0$, let $\beta_{\delta, i}:=\min \left(X_{i}\right)$ for all $i<\operatorname{cf}(\mu)$.
- For $\delta>0$ such that $\left\langle\beta_{\gamma, i} \mid \gamma<\delta, i<\operatorname{cf}(\mu)\right\rangle$ has already been defined, since, for each $i<\operatorname{cf}(\mu), x_{\delta, i}:=\left\{\beta_{\gamma, i} \mid \gamma \in \Gamma_{\delta, i}\right\}$ is a subset of $X_{i}$ of size no more than $\mu_{i}$, we may pick $\beta_{\delta, i} \in X_{i}$ above $\sup \left\{\beta_{\gamma, \iota} \mid \gamma<\delta, \iota<\operatorname{cf}(\mu)\right\}$ such that $c\left[x_{\delta, i} \times\left\{\beta_{\delta, i}\right\}\right]=\{j\}$.

This completes the construction.
For each $\delta<\mu^{+}$, let $a_{\delta}:=\left\{\beta_{\delta, i} \mid i<\operatorname{cf}(\mu)\right\}$. Evidently, $\left\langle a_{\delta} \mid \delta<\mu^{+}\right\rangle$is $<$-increasing. Fix a stationary subset $S \subseteq \mu^{+}$on which the map $\delta \mapsto \operatorname{otp}\left(a_{\delta}\right)$ is constant. So, by the choice of the coloring $c$, we may pick $(\gamma, \delta) \in[S]^{2}$ such that $c\left[a_{\gamma} \times a_{\delta}\right]=\{1-j\}$. Find $i<\operatorname{cf}(\mu)$ such that $\gamma \in \Gamma_{\delta, i}$. Then $\beta_{\gamma, i} \in x_{\delta, i} \cap a_{\gamma}$, $\beta_{\delta, i} \in a_{\delta}$, and $c\left(\beta_{\gamma, i}, \beta_{\delta, i}\right)=j$. This is a contradiction.

By [Rin14b, Theorem 3.1], for every infinite regular cardinal $\mu$, if $\mathrm{P} \ell_{6}(\mu, \mu)$ holds then so does $\operatorname{Pr}_{1}\left(\mu^{+}, \mu^{+}, \mu^{+}, \mu\right)$, and for every infinite singular cardinal $\mu$, if $\mathrm{P} \ell_{6}\left(\mu^{+}\right.$, $\mu)$ holds, then so does $\operatorname{Pr}_{1}\left(\mu^{++}, \mu^{++}, \mu^{++}, \mu\right)$. The same conclusions may be drawn from $\operatorname{Pr}_{6}(\mu, \mu, \mu, \mu)$ and $\operatorname{Pr}_{6}\left(\mu^{+}, \mu^{+}, \mu^{+}, \mu\right)$, respectively. However, the upcoming series of results show that none of these instances are consistent.

Proposition 2.13. Let $\mu$ be a singular cardinal.
Then $\mathrm{P} \ell_{6}\left(\mu^{+}, \mu\right)$ and $\operatorname{Pr}_{6}\left(\mu^{+}, \mu^{+}, 2, \mu\right)$ both fail.
Proof. Towards a contradiction, suppose that $d:{ }^{<\omega} \mu^{+} \rightarrow \omega$ witnesses $\mathrm{P} \ell_{6}\left(\mu^{+}, \mu\right)$ (resp. $d:{ }^{<\omega} \mu^{+} \rightarrow 2$ witnesses $\operatorname{Pr}_{6}\left(\mu^{+}, \mu^{+}, 2, \mu\right)$ ).

Claim 2.13.1. Let $\beta<\mu^{+}$and $A \in[\beta]^{<\mu}$. There exists $\sigma \in{ }^{<\omega} \mu^{+}$such that $d(\langle\alpha, \beta\rangle \wedge \sigma) \neq 1$ for any $\alpha \in A$.
Proof. For all $\gamma<\mu^{+}$, let $u_{\gamma}:=\{\langle\alpha, \beta, \gamma\rangle \mid \alpha \in A\}$ and $v_{\gamma}:=\{\langle\gamma\rangle\}$.

- Assuming that $d:{ }^{<\omega} \mu^{+} \rightarrow \omega$ witnesses $\mathrm{P} \ell_{6}\left(\mu^{+}, \mu\right)$, we may now fix $(\gamma, \delta) \in$ $\left[\mu^{+}\right]^{2}$ such that $d\left(\eta^{\complement} \varrho\right)=\ell(\eta)$ for all $\eta \in u_{\gamma}$ and $\varrho \in v_{\delta}$. Let $\sigma:=\langle\gamma, \delta\rangle$. Then, for every $\alpha \in A,\langle\alpha, \beta, \gamma\rangle \in u_{\gamma}$ and $\langle\delta\rangle \in v_{\delta}$, so that $d(\langle\alpha, \beta\rangle \wedge \sigma)=d\left(\langle\alpha, \beta, \gamma\rangle^{\wedge}\langle\delta\rangle\right)=3$.

Assuming that $d:{ }^{<\omega} \mu^{+} \rightarrow 2$ witnesses $\operatorname{Pr}_{6}\left(\mu^{+}, \mu^{+}, 2, \mu\right)$, we may now fix $(\gamma, \delta) \in\left[\mu^{+}\right]^{2}$ such that $d\left(\eta^{\curvearrowleft} \varrho\right)=0$ for all $\eta \in u_{\gamma}$ and $\varrho \in v_{\delta}$. Let $\sigma:=\langle\gamma, \delta\rangle$. Then, for every $\alpha \in A,\langle\alpha, \beta, \gamma\rangle \in u_{\gamma}$ and $\langle\delta\rangle \in v_{\delta}$, so that $d(\langle\alpha, \beta\rangle \wedge \sigma)=0$.

Let $\beta<\mu^{+}$. Fix a decomposition $\beta=\biguplus_{i<\operatorname{cf}(\mu)} A_{\beta, i}$ such that $\left|A_{\beta, i}\right|<\mu$ for all $i<\operatorname{cf}(\mu)$, and then, for every $i<\operatorname{cf}(\mu)$, fix $\sigma_{\beta, i} \in<\omega \mu^{+}$such that $d\left(\langle\alpha, \beta\rangle \sigma_{\beta, i}\right) \neq$ 1 for any $\alpha \in A_{\beta, i}$. Let $u_{\beta}:=\{\langle\beta\rangle\}$ and $v_{\beta}:=\left\{\langle\beta\rangle{ }^{\wedge} \sigma_{\beta, i} \mid i<\operatorname{cf}(\mu)\right\}$.
$\rightarrow$ Assuming that $d:{ }^{<\omega} \mu^{+} \rightarrow \omega$ witnesses $\mathrm{P} \ell_{6}\left(\mu^{+}, \mu\right)$, we may now fix $(\alpha, \beta) \in$ $\left[\mu^{+}\right]^{2}$ such that $d\left(\eta^{`} \varrho\right)=\ell(\eta)$ for all $\eta \in u_{\alpha}$ and $\varrho \in v_{\beta}$. Find $i<\operatorname{cf}(\mu)$ such that $\alpha \in A_{\beta, i}$. As $\langle\alpha\rangle \in u_{\alpha}$ and $\langle\beta\rangle \wedge \sigma_{\beta, i} \in v_{\beta}$, this must mean that $d\left(\langle\alpha\rangle \vee\langle\beta\rangle \wedge \sigma_{\beta, i}\right)=$ 1 , contradicting the fact that $\alpha \in A_{\beta, i}$.

- Assuming that $d:{ }^{<\omega} \mu^{+} \rightarrow 2$ witnesses $\operatorname{Pr}_{6}\left(\mu^{+}, \mu^{+}, 2, \mu\right)$, we may now fix $(\alpha, \beta) \in\left[\mu^{+}\right]^{2}$ such that $d\left(\eta^{`} \varrho\right)=1$ for all $\eta \in u_{\alpha}$ and $\varrho \in v_{\beta}$. Find $i<\operatorname{cf}(\mu)$ such that $\alpha \in A_{\beta, i}$. As $\langle\alpha\rangle \in u_{\alpha}$ and $\langle\beta\rangle \cap \sigma_{\beta, i} \in v_{\beta}$, this must mean that $d\left(\langle\alpha\rangle^{\wedge}\langle\beta\rangle \wedge \sigma_{\beta, i}\right)=1$, contradicting the fact that $\alpha \in A_{\beta, i}$.

Proposition 2.14. Let $\mu$ be an infinite cardinal. Then $\mathrm{P} \ell_{6}(\mu, \mu)$ fails. Furthermore:
(a) For every map $d:{ }^{<\omega} \mu \rightarrow \omega$, there exist a cardinal $\nu<\mu$ and two sequences $\left\langle u_{\alpha} \mid \alpha<\mu\right\rangle$ and $\left\langle v_{\beta} \mid \beta<\mu\right\rangle$, with
(1) $u_{\alpha} \subseteq{ }^{<\omega} \mu,\left|u_{\alpha}\right|=\nu$, and, for all $\varrho \in u_{\alpha}, \alpha \in \operatorname{Im}(\varrho)$;
(2) $v_{\beta} \subseteq{ }^{<\omega} \mu,\left|v_{\beta}\right|=1$, and, for all $\sigma \in v_{\beta},\langle\beta\rangle \sqsubseteq \sigma$,
such that, for every $(\alpha, \beta) \in[\mu]^{2}$, there are $\varrho \in u_{\alpha}$ and $\sigma \in v_{\beta}$ with $d\left(\varrho^{\wedge} \sigma\right) \neq \ell(\varrho)$.
(b) For every map $d:{ }^{<\omega} \mu \rightarrow \omega$, there exist two sequences $\left\langle u_{\alpha} \mid \alpha<\mu\right\rangle$ and $\left\langle v_{\beta} \mid \beta<\mu\right\rangle$, with
(1) $u_{\alpha} \subseteq{ }^{<\omega} \mu,\left|u_{\alpha}\right|=1$, and, for all $\varrho \in u_{\alpha}, \alpha \in \operatorname{Im}(\varrho)$;
(2) $v_{\beta} \subseteq{ }^{<\omega} \mu,\left|v_{\beta}\right|=|\beta|$, and, for all $\sigma \in v_{\beta},\langle\beta\rangle \sqsubseteq \sigma$,
such that, for every $(\alpha, \beta) \in[\mu]^{2}$, there are $\varrho \in u_{\alpha}$ and $\sigma \in v_{\beta}$ with $d\left(\varrho^{\wedge} \sigma\right) \neq \ell(\varrho)$.

Proof. (a) Suppose not, and let $d$ be a counterexample.
Claim 2.14.1. Let $\beta<\mu$. There exists $\eta \in{ }^{<\omega} \mu$ such that, for all $\alpha<\beta$, $d(\langle\alpha, \beta\rangle \cap \eta) \neq 1$.
Proof. For every $\gamma<\mu$, let $u_{\gamma}:=\{\langle\alpha, \beta, \gamma\rangle \mid \alpha<\beta\}$. For every $\delta<\mu$, let $v_{\delta}:=$ $\{\langle\delta\rangle\}$. Now, by the choice of $d$ (using $\nu:=|\beta|$ ), there must exist $(\gamma, \delta) \in[\mu]^{2}$, such that $d\left(\varrho^{\frown} \sigma\right)=\ell(\varrho)$ for all $\varrho \in u_{\gamma}$ and $\sigma \in v_{\delta}$. So, for all $\alpha<\beta, d(\langle\alpha, \beta, \gamma, \delta\rangle)=3$. Set $\eta:=\langle\gamma, \delta\rangle$. Then, for all $\alpha<\beta, d\left(\langle\alpha, \beta\rangle^{\wedge} \eta\right)=3$.

For every $\alpha<\mu$, let $u_{\alpha}:=\{\langle\alpha\rangle\}$. For every $\beta<\mu$, pick $\eta_{\beta} \in{ }^{<\omega} \mu$ as in the claim, and let $v_{\beta}:=\left\{\langle\beta\rangle^{\wedge} \eta_{\beta}\right\}$. Now, by the choice of $d$ (using $\nu:=1$ ), there must exist $(\alpha, \beta) \in[\mu]^{2}$, such that $d\left(\varrho^{\wedge} \sigma\right)=\ell(\varrho)$ for all $\varrho \in u_{\alpha}$ and $\sigma \in v_{\beta}$. In particular, $d\left(\langle\alpha\rangle \wedge\langle\beta\rangle \wedge \eta_{\beta}\right)=1$, contradicting the choice of $\eta_{\beta}$.
(b) Left to the reader (but see the proof of Proposition 2.15(b)).

Proposition 2.15. Let $\mu$ be an infinite cardinal. Then $\operatorname{Pr}_{6}(\mu, \mu, 2, \mu)$ fails. Furthermore:
(a) For every map $d:{ }^{<\omega} \mu \rightarrow 2$, there exist a cardinal $\nu<\mu, i<2$, and two sequences $\left\langle u_{\alpha} \mid \alpha<\mu\right\rangle$ and $\left\langle v_{\beta} \mid \beta<\mu\right\rangle$, with
(1) $u_{\alpha} \subseteq{ }^{<\omega} \mu,\left|u_{\alpha}\right|=\nu$, and, for all $\varrho \in u_{\alpha}, \alpha \in \operatorname{Im}(\varrho)$;
(2) $v_{\beta} \subseteq{ }^{<\omega} \mu,\left|v_{\beta}\right|=1$, and, for all $\sigma \in v_{\beta}, \beta \in \operatorname{Im}(\sigma)$,
such that, for every $(\alpha, \beta) \in[\mu]^{2}$, there are $\varrho \in u_{\alpha}$ and $\sigma \in v_{\beta}$ with $d\left(\varrho^{\wedge} \sigma\right) \neq i$.
(b) For every map $d:<\omega \mu \rightarrow 2$, there exist $i<2$ and two sequences $\left\langle u_{\alpha}\right|$ $\alpha<\mu\rangle$ and $\left\langle v_{\beta} \mid \beta<\mu\right\rangle$, with
(1) $u_{\alpha} \subseteq{ }^{<\omega} \mu,\left|u_{\alpha}\right|=1$, and, for all $\varrho \in u_{\alpha}, \alpha \in \operatorname{Im}(\varrho)$;
(2) $v_{\beta} \subseteq{ }^{<\omega} \mu,\left|v_{\beta}\right|=|\beta|$, and, for all $\sigma \in v_{\beta}, \beta \in \operatorname{Im}(\sigma)$,
such that, for every $(\alpha, \beta) \in[\mu]^{2}$, there are $\varrho \in u_{\alpha}$ and $\sigma \in v_{\beta}$ with $d\left(\varrho^{\wedge} \sigma\right) \neq i$.

Proof. (a) Left to the reader (but see the proof of Proposition 2.14(a)).
(b) Suppose not, and let $d$ be a counterexample.

Claim 2.15.1. Let $(\alpha, \beta) \in[\mu]^{2}$. There exists $\eta \in{ }^{<\omega} \mu$ such that $d(\langle\alpha, \beta\rangle \wedge \eta) \neq 1$.
Proof. For every $\gamma<\mu$, let $u_{\gamma}:=\{\langle\alpha, \beta, \gamma\rangle\}$. For every nonzero $\delta<\mu$, let $v_{\delta}:=$ $\{\langle\delta\rangle\}$. By the choice of $d$, there must exist $(\gamma, \delta) \in[\mu]^{2}$, such that $d\left(\varrho^{\Upsilon} \sigma\right)=0$ for all $\varrho \in u_{\gamma}$ and $\sigma \in v_{\delta}$. Set $\eta:=\langle\gamma, \delta\rangle$. Then, $d\left(\langle\alpha, \beta\rangle^{\wedge} \eta\right)=0$.

For each $(\alpha, \beta) \in[\mu]^{2}$, let $\eta_{\alpha, \beta} \in{ }^{<\omega} \mu$ be given by the claim. For every $\alpha<\mu$, let $u_{\alpha}:=\{\langle\alpha\rangle\}$. For every $\beta<\mu$, let $v_{\beta}:=\left\{\langle\beta\rangle^{\wedge} \eta_{\alpha, \beta} \mid \alpha<\beta\right\}$. By the choice of $d$, pick $(\alpha, \beta) \in[\mu]^{2}$ such that $d\left(\varrho^{\complement} \sigma\right)=1$ for all $\varrho \in u_{\alpha}$ and $\sigma \in v_{\beta}$. Then $d\left(\langle\alpha\rangle^{\wedge}\langle\beta\rangle^{\wedge} \eta_{\alpha, \beta}\right)=1$, contradicting the choice of $\eta_{\alpha, \beta}$.

## 3. $C$-SEQUENCES

Definition 3.1. A $C$-sequence over $\kappa$ is sequence $\vec{C}=\left\langle C_{\alpha} \mid \alpha<\kappa\right\rangle$ such that, for all $\alpha<\kappa, C_{\alpha}$ is a closed subset of $\alpha$ with $\sup \left(C_{\alpha}\right)=\sup (\alpha)$.

Definition 3.2. A $C$-sequence $\left\langle C_{\alpha} \mid \alpha<\kappa\right\rangle$ is said to avoid a set $\Gamma$ iff $\operatorname{acc}\left(C_{\alpha}\right) \cap \Gamma=$ $\emptyset$ for all $\alpha<\kappa$.

Note that a stationary subset $\Gamma$ of $\kappa$ is nonreflecting iff there exists a $C$-sequence over $\kappa$ that avoids it.

In this paper, we shall make use of two instances of the parameterized proxy principle from [BR17, BR21]. The first instance reads as follows (see [BR17, Definition 1.3]):

Definition 3.3. $\boxtimes^{-}(\kappa)$ asserts the existence of a $C$-sequence $\left\langle C_{\alpha} \mid \alpha<\kappa\right\rangle$ such that:

- for all $\alpha<\kappa$ and $\delta \in \operatorname{acc}\left(C_{\alpha}\right), C_{\alpha} \cap \delta=C_{\delta}$;
- for every cofinal $B \subseteq \kappa$, there exist stationarily many $\alpha<\kappa$ such that $\sup \left(\operatorname{nacc}\left(C_{\alpha}\right) \cap B\right)=\alpha$.

The second instance reads as follows (see [BR21, Definition 4.10, Theorem 4.15(iii) and Convention 4.18]):
Definition 3.4. For a stationary subset $S \subseteq \kappa, \mathrm{P}^{-}\left(\kappa, \kappa^{+}, \sqsubseteq^{*}, 1,\{S\}, 2\right)$ asserts the existence of a $C$-sequence $\left\langle C_{\alpha} \mid \alpha<\kappa\right\rangle$ and a stationary subset $\Delta \subseteq S$ such that:

- for all $\alpha<\kappa$ and $\delta \in \operatorname{acc}\left(C_{\alpha}\right) \cap \Delta, \sup \left(\left(C_{\alpha} \cap \delta\right) \triangle C_{\delta}\right)<\delta$;
- for every cofinal $B \subseteq \kappa$, there exist stationarily many $\alpha \in \Delta$ such that:

$$
\sup \left\{\varepsilon \in B \cap \alpha \mid \min \left(C_{\alpha} \backslash(\varepsilon+1)\right) \in B\right\}=\alpha
$$

3.1. Walks on ordinals. For the rest of this subsection, let us fix a $C$-sequence $\vec{C}=\left\langle C_{\alpha} \mid \alpha<\kappa\right\rangle$ over $\kappa$. The next definition is due to Todorcevic; see [Tod07] for a comprehensive treatment.
Definition 3.5 (Todorcevic). From $\vec{C}$, derive maps $\operatorname{Tr}:[\kappa]^{2} \rightarrow{ }^{\omega} \kappa, \rho_{2}:[\kappa]^{2} \rightarrow \omega$, $\operatorname{tr}:[\kappa]^{2} \rightarrow{ }^{<\omega} \kappa$ and $\lambda:[\kappa]^{2} \rightarrow \kappa$, as follows. Let $(\alpha, \beta) \in[\kappa]^{2}$ be arbitrary.

- $\operatorname{Tr}(\alpha, \beta): \omega \rightarrow \kappa$ is defined by recursion on $n<\omega$ :
$\operatorname{Tr}(\alpha, \beta)(n):= \begin{cases}\beta, & n=0 ; \\ \min \left(C_{\operatorname{Tr}(\alpha, \beta)(n-1)} \backslash \alpha\right), & n>0 \& \operatorname{Tr}(\alpha, \beta)(n-1)>\alpha ; \\ \alpha, & \text { otherwise. }\end{cases}$
- $\rho_{2}(\alpha, \beta):=\min \{n<\omega \mid \operatorname{Tr}(\alpha, \beta)(n)=\alpha\}$;
- $\operatorname{tr}(\alpha, \beta):=\operatorname{Tr}(\alpha, \beta) \upharpoonright \rho_{2}(\alpha, \beta)$;
- $\lambda(\alpha, \beta):=\max \left\{\sup \left(C_{\operatorname{Tr}(\alpha, \beta)(i)} \cap \alpha\right) \mid i<\rho_{2}(\alpha, \beta)\right\}$.

The next three facts are quite elementary. See [RZ21, §2.1] for details.
Fact 3.6. Whenever $0<\delta<\beta<\kappa$, if $\delta \notin \bigcup_{\alpha<\kappa} \operatorname{acc}\left(C_{\alpha}\right)$, then $\lambda(\delta, \beta)<\delta$.
Fact 3.7. Whenever $\lambda(\delta, \beta)<\alpha<\delta<\beta<\kappa$, $\operatorname{tr}(\alpha, \beta)=\operatorname{tr}(\delta, \beta)^{\wedge} \operatorname{tr}(\alpha, \delta)$.
Fact 3.8. Whenever $\alpha<\delta<\beta<\kappa$ with $\delta \in \operatorname{Im}(\operatorname{tr}(\alpha, \beta))$,

$$
\lambda(\alpha, \beta)=\max \{\lambda(\delta, \beta), \lambda(\alpha, \delta)\}
$$

Definition 3.9 ([RZ21, Definition 2.10]). For every $(\alpha, \beta) \in[\kappa]^{2}$, we define an ordinal д $_{\alpha, \beta} \in[\alpha, \beta]$ via:

$$
\check{\delta}_{\alpha, \beta}:= \begin{cases}\alpha, & \text { if } \lambda(\alpha, \beta)<\alpha \\ \min (\operatorname{Im}(\operatorname{tr}(\alpha, \beta))), & \text { otherwise }\end{cases}
$$

Fact 3.10 ([RZ21, Lemma 2.11]). Let $(\alpha, \beta) \in[\kappa]^{2}$ with $\alpha>0$. Then
(1) $\lambda\left(\partial_{\alpha, \beta}, \beta\right)<\alpha ;^{2}$
(2) If ס $_{\alpha, \beta} \neq \alpha$, then $\alpha \in \operatorname{acc}\left(C_{\check{\partial}_{\alpha, \beta}}\right)$;
(3) $\operatorname{tr}\left(\mathrm{\partial}_{\alpha, \beta}, \beta\right) \sqsubseteq \operatorname{tr}(\alpha, \beta)$.

For the purpose of this paper, we also introduce the following ad-hoc notation.
Definition 3.11. For every ordinal $\eta<\kappa$ and a pair $(\alpha, \beta) \in[\kappa]^{2}$, we let

$$
\eta_{\alpha, \beta}:=\min \left\{n<\omega \mid \eta \in C_{\operatorname{Tr}(\alpha, \beta)(n)} \text { or } n=\rho_{2}(\alpha, \beta)\right\}+1 .
$$

We conclude this subsection by proving a useful lemma.
Lemma 3.12. For ordinals $\eta<\alpha<\delta<\beta<\kappa$, if $\lambda(\delta, \beta)=\eta$ and $\rho_{2}(\delta, \beta)=\eta_{\delta, \beta}$, then $\operatorname{tr}(\alpha, \beta)\left(\eta_{\alpha, \beta}\right)=\delta$.
Proof. Under the above hypothesis, Fact 3.7 entails that $\operatorname{tr}(\alpha, \beta)=\operatorname{tr}(\delta, \beta)^{\wedge} \operatorname{tr}(\alpha, \delta)$. As $\eta_{\delta, \beta}=\rho_{2}(\delta, \beta)<\rho_{2}(\delta, \beta)+1$, it altogether follows that

$$
\begin{aligned}
\eta_{\alpha, \beta} & =\min \left\{n<\omega \mid \eta \in C_{\operatorname{Tr}(\alpha, \beta)(n)} \text { or } n=\rho_{2}(\alpha, \beta)\right\}+1 \\
& =\min \left\{n<\omega \mid \eta \in C_{\operatorname{Tr}(\delta, \beta)(n)}\right\}+1 \\
& =\rho_{2}(\delta, \beta)
\end{aligned}
$$

so that $\operatorname{tr}(\alpha, \beta)\left(\eta_{\alpha, \beta}\right)=\delta$.

[^2]
### 3.2. Cardinal characteristics of $C$-sequences.

Definition 3.13. For a $C$-sequence $\vec{C}=\left\langle C_{\alpha} \mid \alpha<\kappa\right\rangle$ and a subset $\Gamma \subseteq \kappa$ :

- $\chi_{1}(\vec{C})$ is the supremum of $\sigma+1$ over all $\sigma<\kappa$ satisfying the following. For every pairwise disjoint subfamily $\mathcal{B} \subseteq[\kappa]^{\sigma}$ of size $\kappa$, there are a stationary set $\Delta \subseteq \kappa$ and an ordinal $\eta<\kappa$ such that, for every $\delta \in \Delta$, there exist $\kappa$ many $b \in \mathcal{B}$ such that, for every $\beta \in b, \lambda(\delta, \beta)=\eta$ and $\rho_{2}(\delta, \beta)=\eta_{\delta, \beta}$.
- $\chi_{2}(\vec{C}, \Gamma)$ is the supremum of $\sigma+1$ over all $\sigma<\kappa$ satisfying the following. For every pairwise disjoint subfamily $\mathcal{B} \subseteq[\kappa]^{\sigma}$ of size $\kappa$, there are club many $\delta \in \Gamma$ for which there exist an ordinal $\eta<\delta$ and $\kappa$ many $b \in \mathcal{B}$ such that, for every $\beta \in b, \lambda(\delta, \beta)=\eta$ and $\rho_{2}(\delta, \beta)=\eta_{\delta, \beta}$.

Note that if $\Gamma$ is stationary, then $\chi_{2}(\vec{C}, \Gamma) \leq \chi_{1}(\vec{C})$.
Lemma 3.14. Suppose that $\Gamma \subseteq \kappa$ is a nonreflecting stationary set, and $\kappa \geq \aleph_{2}$. Then there exists a $C$-sequence $\vec{C}$ that avoids $\Gamma$ such that

$$
\chi_{1}(\vec{C}) \geq \sup \left\{\sigma<\kappa \mid \Gamma \cap E_{>\sigma}^{\kappa} \text { is stationary }\right\}
$$

Proof. As $\Gamma$ is nonreflecting, we commence by fixing a $C$-sequence $\vec{C}=\left\langle C_{\alpha} \mid \alpha<\kappa\right\rangle$ that avoids $\Gamma$. It is clear that $\vec{C}$ is amenable in the sense of [BR19a, Definition 1.3]. Let $\Lambda:=\left\{\sigma<\kappa \mid \Gamma \cap E_{>\sigma}^{\kappa}\right.$ is stationary $\}$. It is clear that $\Lambda$ is some limit ordinal with $\omega \leq \Lambda \leq \kappa$. For each $\sigma \in \Lambda, \Omega^{\sigma}:=\Gamma \cap E_{>\sigma}^{\kappa}$ is stationary. Following the terminology of [BR19a, Definition 1.8], by [BR19a, Lemma 1.15], we may now fix a conservative postprocessing function $\Phi: \mathcal{K}(\kappa) \rightarrow \mathcal{K}(\kappa)$, a cofinal subset $\Sigma \subseteq \Lambda$ and an injection $h: \Sigma \rightarrow \kappa$ such that $\left\{\alpha \in \Omega^{\sigma} \mid \min \left(\Phi\left(C_{\alpha}\right)\right)=h(\sigma)\right\}$ is stationary for all $\sigma \in \Sigma$. In simple words, this means that there exists a $C$-sequence $\vec{D}=\left\langle D_{\alpha}\right|$ $\alpha<\kappa\rangle$ such that:

- $D_{\alpha} \subseteq C_{\alpha}$ for all $\alpha<\kappa$;
- $\Gamma^{\sigma}:=\left\{\alpha \in \Gamma \cap E_{>\sigma}^{\kappa} \mid \min \left(D_{\alpha}\right)=h(\sigma)\right\}$ is stationary for all $\sigma \in \Sigma$.

Note that since $h$ is injective, $\left\langle\Gamma^{\sigma} \mid \sigma \in \Sigma\right\rangle$ consists of pairwise disjoint stationary sets. For each $\sigma \in \Sigma$, since $\vec{D} \upharpoonright \Gamma^{\sigma}$ is an amenable $C$-sequence, it follows from [BR19a, Lemma 2.2 and Fact 2.4(2)] that there exists a $C$-sequence $\left\langle C_{\alpha}^{\bullet} \mid \alpha \in \Gamma^{\sigma}\right\rangle$ such that:

- $\operatorname{acc}\left(C_{\alpha}^{\bullet}\right) \subseteq \operatorname{acc}\left(D_{\alpha}\right)$ for all $\alpha \in \Gamma^{\sigma} ;$
- For every club $D \subseteq \kappa$, there exists $\alpha \in \Gamma^{\sigma}$ with $\sup \left(\operatorname{nacc}\left(C_{\alpha}^{\bullet}\right) \cap D\right)=\alpha$.

For every $\alpha \in \kappa \backslash \bigcup_{\sigma \in \Sigma} \Gamma^{\sigma}$, let $C_{\alpha}^{\bullet}:=D_{\alpha}$. Recalling that $\Sigma$ is a cofinal subset of $\Lambda$, we altogether infer that:

- $\operatorname{acc}\left(C_{\alpha}^{\bullet}\right) \cap \Gamma=\emptyset$ for all $\alpha<\kappa$;
- For every club $D \subseteq \kappa$ and every $\sigma<\kappa$ such that $\Gamma \cap E_{>\sigma}^{\kappa}$ is stationary, there exists $\alpha \in \Gamma \cap E_{>\sigma}^{\kappa}$ with $\sup \left(\operatorname{nacc}\left(C_{\alpha}^{\bullet}\right) \cap D\right)=\alpha$.
We now walk along $\vec{C}^{\bullet}:=\left\langle C_{\alpha}^{\bullet} \mid \alpha<\kappa\right\rangle$, and verify that it is as sought. For this, suppose that $\sigma<\kappa$ is such that $\Gamma \cap E_{>\sigma}^{\kappa}$ is stationary, and that we are given a pairwise disjoint subfamily $\mathcal{B} \subseteq[\kappa]^{\sigma}$ of size $\kappa$. It suffices to prove that for every club $D \subseteq \kappa$, there exists $\delta \in D$, such that, for $\kappa$ many $b \in \mathcal{B}$, for some $\eta<\delta$, for every $\beta \in b, \lambda(\delta, \beta)=\eta$ and $\rho_{2}(\delta, \beta)=\eta_{\delta, \beta}$.

Thus, let $D$ be an arbitrary club in $\kappa$. Without loss of generality, $D \subseteq \operatorname{acc}(\kappa)$. Pick $\gamma \in \Gamma \cap E_{>\sigma}^{\kappa}$ with $\sup \left(\operatorname{nacc}\left(C_{\gamma}^{\bullet}\right) \cap D\right)=\gamma$.

Claim 3.14.1. Let $b \in \mathcal{B}$ with $\min (b)>\gamma$. There are $\delta \in D \cap \gamma$ and $\eta<\delta$ such that, for every $\beta \in b, \lambda(\delta, \beta)=\eta$ and $\rho_{2}(\delta, \beta)=\eta_{\delta, \beta}$.

Proof. Set $\epsilon:=\sup \{\lambda(\gamma, \beta) \mid \beta \in b\}$. As $|b|<\operatorname{cf}(\gamma)$ and $\gamma \in \Gamma$, it follows from Fact 3.6 that $\epsilon<\gamma$. Now pick $\delta \in \operatorname{nacc}\left(C_{\gamma}^{\bullet}\right) \cap D$ for which $\eta:=\sup \left(C_{\gamma}^{\bullet} \cap \delta\right)$ is bigger than $\epsilon$. For every $\beta \in b, \lambda(\gamma, \beta) \leq \epsilon<\delta<\gamma<\beta$, so by Facts 3.7 and $3.8, \lambda(\delta, \beta)=\max \{\lambda(\gamma, \beta), \lambda(\delta, \gamma)\}$. As $\lambda(\gamma, \beta) \leq \epsilon<\eta=\lambda(\delta, \gamma)$, it follows that $\lambda(\delta, \beta)=\eta$ and $\rho_{2}(\delta, \beta)=\eta_{\delta, \beta}$.

As $|D \cap \gamma|<\kappa=|\mathcal{B}|$, there must be $\delta \in D \cap \gamma$ and $\eta<\delta$ such that, for $\kappa$ many $b \in \mathcal{B}$, for every $\beta \in b, \lambda(\delta, \beta)=\eta$ and $\rho_{2}(\delta, \beta)=\eta_{\delta, \beta}$.

It follows that for every regular uncountable cardinal $\mu$, there exists a $C$-sequence $\vec{C}$ over $\mu^{+}$with $\chi_{1}(\vec{C}) \geq \mu$. As made clear by the proof of [Rin12, Lemma 2.4], for every singular cardinal $\mu$ of uncountable cofinality, there exists a $C$-sequence $\vec{C}$ over $\mu^{+}$with $\chi_{1}(\vec{C}) \geq \operatorname{cf}(\mu)$. This raises the following question:

Question 3.15. Suppose that $\mu$ is an infinite cardinal of countable cofinality. Must there exist a $C$-sequence $\vec{C}$ over $\mu^{+}$with $\chi_{1}(\vec{C}) \geq \omega$ ?

Lemma 3.16. (1) If $\square(\kappa)$ holds and $\kappa \geq \aleph_{2}$, then there exists a $C$-sequence $\vec{C}$ over $\kappa$ such that $\chi_{1}(\vec{C})=\sup (\operatorname{Reg}(\kappa))$;
(2) If $\vec{C}$ is $a \boxtimes^{-}(\kappa)$-sequence, then $\chi_{2}(\vec{C}, \kappa)=\sup (\operatorname{Reg}(\kappa))$.

Proof. In Case (1), since $\square(\kappa)$ holds and $\kappa \geq \aleph_{2}$, by [Rin17, Proposition 3.5], we may fix a $C$-sequence $\vec{C}=\left\langle C_{\alpha} \mid \alpha<\kappa\right\rangle$ satisfying the following two items:
(※) for every $\alpha<\kappa$ and $\delta \in \operatorname{acc}\left(C_{\alpha}\right), C_{\delta}=C_{\alpha} \cap \delta$;
(】) for every club $D \subseteq \kappa$, there exists $\gamma>0$ with $\sup \left(\operatorname{nacc}\left(C_{\gamma}\right) \cap D\right)=\gamma$.
In Case (2), just let $\vec{C}=\left\langle C_{\alpha} \mid \alpha<\kappa\right\rangle$ be a $\boxtimes^{-}(\kappa)$-sequence.
Let $X_{1}$ denote an arbitrary club in $\kappa$, and let $X_{2}$ denote an arbitrary stationary set in $\kappa$. Let $n \in\{1,2\}$. To verify Clause ( $n$ ), we shall prove that given $\sigma<$ $\sup (\operatorname{Reg}(\kappa))$ and a pairwise disjoint subfamily $\mathcal{B} \subseteq[\kappa]^{\sigma}$ of size $\kappa$, there exists $\delta \in X_{n}$, such that, for $\kappa$ many $b \in \mathcal{B}$, for some $\eta<\delta$, for every $\beta \in b, \lambda(\delta, \beta)=\eta$ and $\rho_{2}(\delta, \beta)=\eta_{\delta, \beta}$.

Without loss of generality, $X_{n} \subseteq \operatorname{acc}(\kappa)$. For each $\tau \in E_{>\sigma}^{\kappa}$, fix $b_{\tau} \in \mathcal{B}$ with $\min \left(b_{\tau}\right)>\tau$. Define a function $f: E_{>\sigma}^{\kappa} \rightarrow \kappa$ via

$$
f(\tau):=\sup \left\{\lambda\left(\check{\partial}_{\tau, \beta}, \beta\right) \mid \beta \in b_{\tau}\right\}
$$

As $\left|b_{\tau}\right|<\operatorname{cf}(\tau)$, Fact $3.10(1)$ entails that $f$ is regressive. So, fix a stationary $T \subseteq E_{>\sigma}^{\kappa}$ such that $f \upharpoonright T$ is constant with value, say, $\zeta$. Now, if $n=1$, then using Clause ( $\beth)$, we may pick a nonzero ordinal $\gamma<\kappa$ with $\sup \left(\operatorname{nacc}\left(C_{\gamma}\right) \cap\left(X_{1} \backslash \zeta\right)\right)=\gamma$, and if $n=2$, then we may pick a nonzero ordinal $\gamma<\kappa$ with $\sup \left(\operatorname{nacc}\left(C_{\gamma}\right) \cap\left(X_{2} \backslash\right.\right.$ $\zeta))=\gamma$.

Claim 3.16.1. Let $\tau \in T$ above $\gamma$. There are $\delta \in X_{n} \cap \gamma$ and $\eta<\delta$ such that, for every $\beta \in b_{\tau}, \lambda(\delta, \beta)=\eta$ and $\rho_{2}(\delta, \beta)=\eta_{\delta, \beta}$.

Proof. By Fact 3.10(1), the following ordinal is smaller than $\gamma$ :

$$
\zeta^{\prime}:= \begin{cases}0, & \text { if } \gamma \in \operatorname{acc}\left(C_{\tau}\right) \\ \sup \left(C_{\tau} \cap \gamma\right), & \text { if } \gamma \in \operatorname{nacc}\left(C_{\tau}\right) \\ \lambda\left(\partial_{\gamma, \tau}, \tau\right), & \text { if } \gamma \notin C_{\tau}\end{cases}
$$

Thus, we may pick a large enough $\delta \in \operatorname{nacc}\left(C_{\gamma}\right) \cap X_{n}$ such that $\sup \left(C_{\gamma} \cap \delta\right)>$ $\max \left\{\zeta, \zeta^{\prime}\right\}$. Denote $\eta:=\sup \left(C_{\gamma} \cap \delta\right)$, so that $\eta<\delta$. Let $\beta \in b_{\tau}$ be arbitrary. We have

$$
\lambda\left(\check{\partial}_{\tau, \beta}, \beta\right) \leq f(\tau) \leq \max \left\{\zeta, \zeta^{\prime}\right\}<\eta<\eta+1<\delta<\gamma<\tau<\beta
$$

We shall show that $\lambda(\delta, \beta)=\eta$ and $\rho_{2}(\delta, \beta)=\eta_{\delta, \beta}$.
By Fact 3.7 and the inequality presented above, $\operatorname{tr}(\delta, \beta)=\operatorname{tr}\left(\check{\partial}_{\tau, \beta}, \beta\right)^{\wedge} \operatorname{tr}\left(\delta, \check{\partial}_{\tau, \beta}\right)$. So, by Fact 3.8, $\lambda(\delta, \beta)=\max \left\{\lambda\left(\check{\partial}_{\tau, \beta}, \beta\right), \lambda\left(\delta, \partial_{\tau, \beta}\right)\right\}$. Now, there are three cases to consider:

- If $\gamma \in \operatorname{acc}\left(C_{\tau}\right)$, then $C_{\gamma}=C_{\tau} \cap \gamma=\left(C_{\check{\partial}_{\tau, \beta}} \cap \tau\right) \cap \gamma$. So, since $\delta \in C_{\gamma}$, $\operatorname{tr}(\delta, \beta)=\operatorname{tr}\left(\check{\partial}_{\tau, \beta}, \beta\right)^{\wedge}\left\langle\partial_{\tau, \beta}\right\rangle$ and $\lambda\left(\delta, \check{\partial}_{\tau, \beta}\right)=\sup \left(C_{\gamma} \cap \delta\right)=\eta>\zeta \geq \lambda\left(\partial_{\tau, \beta}, \beta\right)$, hence the conclusion follows.
- If $\gamma \in \operatorname{nacc}\left(C_{\tau}\right)$, then, since $\delta \in C_{\gamma}, \operatorname{tr}(\delta, \beta)=\operatorname{tr}\left(\partial_{\tau, \beta}, \beta\right) \wedge\left\langle\partial_{\tau, \beta}, \gamma\right\rangle$, so that $\lambda(\delta, \beta)=\max \left\{\lambda\left(\partial_{\tau, \beta}, \beta\right), \sup \left(C_{\check{\partial}_{\tau, \beta}} \cap \delta\right), \sup \left(C_{\gamma} \cap \delta\right)\right\}=\max \left\{\lambda\left(\partial_{\tau, \beta}, \beta\right), \zeta^{\prime}, \eta\right\}$, and the conclusion follows.
- If $\gamma \notin C_{\tau}$, then $\partial_{\gamma, \tau} \neq \tau$. Consequently, $\lambda\left(\partial_{\gamma, \tau}, \tau\right)=\zeta^{\prime}<\delta<\gamma \leq \partial_{\gamma, \tau}<\tau$, and so, by Fact 3.7, $\operatorname{tr}(\delta, \tau)=\operatorname{tr}\left(\check{\partial}_{\gamma, \tau}, \tau\right)^{\wedge} \operatorname{tr}\left(\delta, \partial_{\gamma, \tau}\right)$. Thus, by Fact 3.8,

$$
\lambda(\delta, \tau)=\max \left\{\lambda\left(\partial_{\gamma, \tau}, \tau\right), \lambda\left(\delta, \partial_{\gamma, \tau}\right)\right\}=\max \left\{\zeta^{\prime}, \lambda\left(\delta, \partial_{\gamma, \tau}\right)\right\}
$$

By Clause ( $\aleph$ ) together with Fact $3.10(2), \lambda\left(\delta, \partial_{\tau, \beta}\right)=\lambda(\delta, \tau)$. As $\delta \in C_{\gamma}=$ $C_{\check{\partial}_{\gamma, \tau}} \cap \gamma$, we get that $\lambda\left(\delta, \check{\partial}_{\gamma, \tau}\right)=\sup \left(C_{\gamma} \cap \delta\right)=\eta$. Altogether, $\lambda(\delta, \beta)=$ $\max \left\{\lambda\left(\partial_{\tau, \beta}, \beta\right), \zeta^{\prime}, \eta\right\}$. But, $\eta>\max \left\{\zeta, \zeta^{\prime}\right\} \geq\left\{\lambda\left(\partial_{\tau, \beta}, \beta\right), \zeta^{\prime}\right\}$, and the conclusion follows.

As $\left|X_{n} \cap \gamma\right|<\kappa=|T|$, there must be $\delta \in X_{n} \cap \gamma$ and $\eta<\delta$ such that, for $\kappa$ many $b \in \mathcal{B}$, for some $\eta<\delta$, for every $\beta \in b, \lambda(\delta, \beta)=\eta$ and $\rho_{2}(\delta, \beta)=\eta_{\delta, \beta}$.

Lemma 3.17. Suppose that $S \subseteq \kappa$ is a stationary set and $\vec{C}$ is a witness to $\mathrm{P}^{-}\left(\kappa, \kappa^{+}, \sqsubseteq^{*}, 1,\{S\}, 2\right)$. For every cardinal $\chi$ such that $S \cap E_{<\chi}^{\kappa}$ is nonstationary, $\chi_{2}(\vec{C}, \kappa) \geq \chi$.
Proof. Write $\vec{C}$ as $\left\langle C_{\alpha} \mid \alpha<\kappa\right\rangle$. Fix a stationary subset $\Delta \subseteq S$ such that:
(1) for all $\alpha<\kappa$ and $\delta \in \operatorname{acc}\left(C_{\alpha}\right) \cap \Delta$, $\sup \left(\left(C_{\alpha} \cap \delta\right) \triangle C_{\delta}\right)<\delta$;
(2) for every cofinal $B \subseteq \kappa$, there exist stationarily many $\alpha \in \Delta$ such that:

$$
\sup \left\{\varepsilon \in B \cap \alpha \mid \min \left(C_{\alpha} \backslash(\varepsilon+1)\right) \in B\right\}=\alpha
$$

Now, suppose that $\chi$ is cardinal such that $S \cap E_{<\chi}^{\kappa}$ is nonstationary. Let $\sigma<\chi$, let $\mathcal{B} \subseteq[\kappa]^{\sigma}$ be a pairwise disjoint family of size $\kappa$, and let $\Gamma$ be an arbitrary stationary subset of $\kappa$; we shall show that there exist $\gamma \in \Gamma$ and $\mathcal{B}^{\prime} \in[\mathcal{B}]^{\kappa}$ such that, for every $b \in \mathcal{B}^{\prime}$, there exists $\eta<\gamma$, such that, for every $\beta \in b, \lambda(\gamma, \beta)=\eta$ and $\rho_{2}(\gamma, \beta)=\eta_{\gamma, \beta}$.

Using Clause (2), fix $\delta \in \Delta \cap E_{>\sigma}^{\kappa}$ such that $\sup \left(\operatorname{nacc}\left(C_{\delta}\right) \cap \Gamma\right)=\delta$, and then set $\mathcal{B}^{\prime}:=\{b \in \mathcal{B} \mid \min (b)>\delta\}$.

Now, let $b \in \mathcal{B}^{\prime}$ be arbitrary. By Fact $3.10(1)$ and as $\operatorname{cf}(\delta)>\sigma=\operatorname{otp}(b)$, $\Lambda:=\sup _{\beta \in b} \lambda\left(\mathrm{\partial}_{\delta, \beta}, \beta\right)$ is $<\delta$. Using Clause (1) and Fact 3.10(2),

$$
\Lambda^{\prime}:=\sup _{\beta \in b} \min \left\{\epsilon \in C_{\delta} \backslash \Lambda \mid C_{\delta} \cap[\epsilon, \delta)=C_{\check{\partial}_{\delta, \beta}} \cap[\epsilon, \delta)\right\}
$$

is $<\delta$, as well. Fix a large enough $\gamma \in \operatorname{nacc}\left(C_{\delta}\right) \cap \Gamma$ for which $\eta:=\sup \left(C_{\delta} \cap \gamma\right)$ is $>\Lambda^{\prime}$. Note that $\gamma \in \Gamma \cap \delta$ and $\eta<\gamma$. Let $\beta \in b$ be arbitrary. We have:

$$
\lambda\left(\check{\partial}_{\delta, \beta}, \beta\right) \leq \Lambda<\gamma<\delta \leq \mathscr{\partial}_{\delta, \beta} \leq \beta,
$$

so, by Facts 3.7 and $3.8, \lambda(\gamma, \beta)=\max \left\{\lambda\left(\check{\partial}_{\delta, \beta}, \beta\right), \lambda\left(\gamma, \check{\partial}_{\delta, \beta}\right)\right\}$. Fix $\epsilon \in C_{\delta} \cap[\Lambda, \gamma)$ such that $C_{\delta} \cap[\epsilon, \delta)=C_{\tilde{\partial}_{\delta, \beta}} \cap[\epsilon, \delta)$. As $\gamma>\epsilon$, we have that $\gamma \in \operatorname{nacc}\left(C_{ஓ_{\delta, \beta}}\right)$ and $\lambda\left(\gamma, \check{\partial}_{\delta, \beta}\right)=\sup \left(C_{\check{\partial}_{\delta, \beta}} \cap \gamma\right) \geq \epsilon \geq \Lambda \geq \lambda\left(\check{\partial}_{\delta, \beta}, \beta\right)$. Altogether, $\lambda(\gamma, \beta)=$ $\sup \left(C_{\check{\delta}_{\delta, \beta}} \cap \gamma\right)=\sup \left(C_{\delta} \cap \gamma\right)=\eta$.
Lemma 3.18. For a cardinal $\chi$, assume either of the following:
(1) $\mathrm{P} \ell_{1}(\kappa, 1, \chi)$ holds, or
(2) There exists a $C$-sequence $\vec{C}$ over $\kappa$ with $\chi_{1}(\vec{C}) \geq \chi$.

Then there exists a coloring $d_{1}:[\kappa]^{2} \rightarrow \kappa$ good for $\chi$ in the following sense. For every $\sigma<\chi$ and every pairwise disjoint subfamily $\mathcal{B} \subseteq[\kappa]^{\sigma}$ of size $\kappa$, there are $a$ stationary set $\Delta \subseteq \kappa$ and an ordinal $\eta<\kappa$ such that, for every $\delta \in \Delta$, there exists $b \in \mathcal{B}$ with $\min (b)>\max \{\eta, \delta\}$ satisfying $d_{1}[\{\eta\} \times b]=\{\delta\}$.
Proof. (1) Suppose that $\mathbf{t}:[\kappa]^{2} \rightarrow[\kappa]^{3}$ witnesses $\mathrm{P} \ell_{1}(\kappa, 1, \chi)$. Define $d_{1}:[\kappa]^{2} \rightarrow \kappa$ by letting $d_{1}(\eta, \beta):=\beta^{*}$ whenever $\mathbf{t}(\eta, \beta)=\left(\tau, \alpha^{*}, \beta^{*}\right)$. Now given $\sigma<\chi$ and a pairwise disjoint subfamily $\mathcal{B} \subseteq[\kappa]^{\sigma}$ of size $\kappa$, we do the following. For each $\gamma<\kappa$, pick $b_{\gamma} \in \mathcal{B}$ such that $\min \left(b_{\gamma}\right)>\gamma$. Consider the club $C:=\{\gamma<\kappa \mid$ $\left.\forall \bar{\gamma}<\gamma\left(\sup \left(b_{\bar{\gamma}}\right)<\gamma\right)\right\}$. Clearly, $\mathcal{A}:=\left\{b_{\gamma} \mid \gamma \in C\right\}$ is a pairwise disjoint subfamily of $[\kappa]^{\sigma}$ of size $\kappa$. So, since $\mathbf{t}$ witnesses $\mathrm{P} \ell_{1}(\kappa, 1, \chi)$, the two clauses of Definition 2.5 tell us that we may fix a stationary subset $S \subseteq C$ such that, for every $\left(\alpha^{*}, \beta^{*}\right) \in[S]^{2}$, there exist $a, b \in \mathcal{A}$ with

$$
\alpha^{*} \leq \min (a) \leq \sup (a)<\beta^{*} \leq \min (b)
$$

such that $\mathbf{t}[a \times b]=\left\{\left(0, \alpha^{*}, \beta^{*}\right)\right\}$; pick $\eta \in a$, and pick $\gamma \in C$ such that $b=b_{\gamma}$. Since $\beta^{*} \in C$, if $\gamma<\beta^{*}$, then $\min (b) \leq \sup (b)<\beta^{*}$, contradicting the fact that $\beta^{*} \leq \min (b)$. So $\gamma \geq \beta^{*}$ and hence $\min (b)=\min \left(b_{\gamma}\right)>\gamma \geq \beta^{*}$. Altogether, for every $\beta \in b_{\gamma}$,

$$
\alpha^{*} \leq \eta<\beta^{*} \leq \gamma<\beta
$$

and $d_{1}(\eta, \beta)=\beta^{*}$. It follows that we may define a regressive map $f: S \rightarrow \kappa$ via

$$
f\left(\beta^{*}\right):=\min \left\{\eta<\beta^{*} \mid \exists b \in \mathcal{B}\left(\min (b)>\beta^{*} \& d_{1}[\{\eta\} \times b]=\left\{\beta^{*}\right\}\right)\right\}
$$

Fix a stationary subset $\Delta \subseteq S$ on which $f$ is constant with value, say, $\eta$. Then, for every $\delta \in \Delta$, there exists $b \in \mathcal{B}$ with $\min (b)>\max \{\eta, \delta\}$ satisfying $d_{1}[\{\eta\} \times b]=\{\delta\}$.
(2) Walk along such a $\vec{C}$. Now, pick any coloring $d_{1}:[\kappa]^{2} \rightarrow \kappa$ such that, for every $\eta, \beta<\kappa$ with $\eta+1<\beta, d_{1}(\eta, \beta)=\operatorname{Tr}(\eta+1, \beta)\left(\eta_{\eta+1, \beta}\right)$. By Lemma 3.12, $d_{1}$ is as sought.

We conclude this section with an improvement of [RZ21, Lemma 2.16].
Definition 3.19 ([LHR21]). For a $C$-sequence $\vec{C}=\left\langle C_{\alpha} \mid \alpha<\kappa\right\rangle, \chi(\vec{C})$ is the least cardinal $\chi \leq \kappa$ such that there exists $\Delta \in[\kappa]^{\kappa}$ with the property that, for every $\epsilon<\kappa$, for some $a \in[\kappa]^{\chi}, \Delta \cap \epsilon \subseteq \bigcup_{\alpha \in a} C_{\alpha}$.

If $\kappa$ is weakly compact, then $\chi(\kappa)$ is defined to be $0 ;{ }^{3}$ otherwise, $\chi(\kappa)$ is the supremum of $\chi(\vec{C})$ over all $C$-sequences $\vec{C}=\left\langle C_{\alpha} \mid \alpha<\kappa\right\rangle$.

Lemma 3.20. Suppose that $\kappa$ is an inaccessible cardinal.
For any cardinal $\chi$ such that there exists a coloring $d_{1}$ good for $\chi$ in the sense of Lemma 3.18, $\chi(\kappa) \geq \chi$. In particular, if $\mathrm{P} \ell_{1}(\kappa, 1, \chi)$ holds, then $\chi(\kappa) \geq \chi$.

[^3]Proof. Suppose $d_{1}:[\kappa]^{2} \rightarrow \kappa$ is a coloring good for $\chi$. By a straightforward modification, we may assume that $d_{1}(\eta, \beta)<\beta$ for all $\eta<\beta<\kappa$. Denote $\Sigma:=$ $\{\alpha<\kappa \mid \operatorname{cf}(\alpha)<\alpha\}$. Now, define a $C$-sequence $\vec{C}=\left\langle C_{\alpha} \mid \alpha<\kappa\right\rangle$, as follows.

- Set $C_{0}:=\emptyset$ and $C_{\omega}:=\omega$;
- For every $\alpha \in \Sigma$, let $C_{\alpha}$ be a closed subset of $\alpha$ with $\sup \left(C_{\alpha}\right)=\sup (\alpha)$ and $\min \left(C_{\alpha}\right) \geq \operatorname{cf}(\alpha)=\operatorname{otp}\left(C_{\alpha}\right) ;$
- For every regular uncountable cardinal $\alpha<\kappa$, set $C_{\alpha}:=\{\gamma<\alpha \mid \forall \eta<$ $\left.\gamma\left[d_{1}(\eta, \alpha)<\gamma\right]\right\}$.

Note that it follows that $C_{\beta+1}=\{\beta\}$ for all $\beta<\kappa$. Now, towards a contradiction, suppose that $\chi(\kappa)<\chi$. In particular, $\chi(\kappa)<\kappa$ so that, by [LHR21, Lemma 2.21(1)], $\kappa$ is a Mahlo cardinal.

Claim 3.20.1. There exist $A \in[\kappa]^{\kappa}, \sigma<\chi$ and a pairwise disjoint subfamily $\mathcal{B} \subseteq[\operatorname{Reg}(\kappa)]^{\sigma}$ of size $\kappa$ with the property that, for every $\epsilon<\kappa$, for some $b \in \mathcal{B}$, $A \cap \epsilon \subseteq \bigcup_{\alpha \in b} C_{\alpha}$.

Proof. Set $\sigma:=\chi(\vec{C})$, so that $\sigma \leq \chi(\kappa)<\chi$. Fix $\Delta \in[\kappa]^{\kappa}$ and a sequence $\left\langle a_{\epsilon}\right|$ $\epsilon<\kappa\rangle$ of sets in $[\kappa]^{\sigma}$ with the property that, for every $\epsilon<\kappa, \Delta \cap \epsilon \subseteq \bigcup_{\alpha \in a_{\epsilon}} C_{\alpha}$. Define a function $f_{0}: \operatorname{Reg}(\kappa) \backslash(\sigma+1) \rightarrow \kappa$ via:

$$
f_{0}(\epsilon):=\sup \left(\left(a_{\epsilon} \cap \epsilon\right) \cup \bigcup\left\{\operatorname{otp}\left(C_{\alpha} \cap \epsilon\right) \mid \alpha \in a_{\epsilon} \cap \Sigma\right\}\right)
$$

Note that $f_{0}$ is regressive, because otherwise for some $\alpha \in a_{\epsilon} \cap \Sigma$, otp $\left(C_{\alpha} \cap \epsilon\right)=$ $\epsilon>0$, and in particular $\epsilon>\min \left(C_{\alpha}\right) \geq \operatorname{otp}\left(C_{\alpha}\right) \geq \operatorname{otp}\left(C_{\alpha} \cap \epsilon\right)=\epsilon$.

Fix a stationary subset $S_{0} \subseteq \operatorname{dom}\left(f_{0}\right)$ on which $f_{0}$ is constant with value, say, $\tau_{0}$. Define a function $f_{1}: S_{0} \backslash\left(\tau_{0}+1\right) \rightarrow \kappa$ via:

$$
f_{1}(\epsilon):=\sup \left(\bigcup\left\{\sup \left(C_{\alpha} \cap \epsilon\right) \mid \alpha \in a_{\epsilon} \cap \Sigma\right\}\right)
$$

Note that $f_{1}$ is regressive, because otherwise for some $\alpha \in a_{\epsilon} \cap \Sigma, \sup \left(C_{\alpha} \cap \epsilon\right)=\epsilon$, and in particular $\tau_{0}=f_{0}(\epsilon) \geq \operatorname{otp}\left(C_{\alpha} \cap \epsilon\right) \geq \operatorname{cf}(\epsilon)=\epsilon>\tau_{0}$.

Fix a stationary subset $S_{1} \subseteq \operatorname{dom}\left(f_{1}\right)$ on which $f_{1}$ is constant with value, say, $\tau_{1}$. Set $A:=\Delta \backslash\left(\tau_{0}+\tau_{1}+1\right)$, and for every $\epsilon \in S_{1}$, set $b_{\epsilon}:=\left(a_{\epsilon} \cap \operatorname{Reg}(\kappa)\right) \backslash \epsilon$.

As $\min \left(b_{\epsilon}\right) \geq \epsilon$ for all $\epsilon \in S_{1}$, we may find $S_{2} \in\left[S_{1}\right]^{\kappa}$ such that $\mathcal{B}:=\left\{b_{\epsilon} \mid \epsilon \in S_{2}\right\}$ is a pairwise disjoint family (of size $\kappa$ ). Now, to see that $A, \sigma$ and $\mathcal{B}$ are as sought, it suffices to show that for every $\epsilon \in S_{2}$, we have $A \cap \epsilon \subseteq \bigcup_{\alpha \in b_{\epsilon}} C_{\alpha}$.

Let $\epsilon \in S_{2}$ and $\delta \in A \cap \epsilon$ be arbitrary. In particular, $\delta \in \Delta \cap \epsilon$, so we may fix $\alpha \in a_{\epsilon}$ such that $\delta \in C_{\alpha}$.

- If $\alpha \in a_{\epsilon} \cap \epsilon$, then $\alpha \leq f_{0}(\epsilon)=\tau_{0}<\min (A) \leq \delta$, contradicting the fact that $\delta \in C_{\alpha} \subseteq \alpha$
- If $\alpha \in a_{\epsilon} \cap \Sigma$, then $\delta \leq f_{1}(\epsilon) \leq \tau_{1}<\min (A) \leq \delta$ which is a contradiction.

So, $\alpha \in b_{\epsilon}$. Altogether, $A \cap \epsilon \subseteq \bigcup_{\alpha \in b_{\epsilon}} C_{\alpha}$, as sought.
Let $A, \sigma$ and $\mathcal{B}$ be given by the claim. By throwing away at most one set from $\mathcal{B}$, we may assume that $\omega \notin b$ for all $b \in \mathcal{B}$. By thinning out even further we may assume the existence of a club $D \subseteq \kappa$ such that, for all $\delta \in D$ and $b \in \mathcal{B}$, if $\min (b)>\delta$, then $A \cap \delta \subseteq \bigcup_{\alpha \in b} C_{\alpha}$.

Now, by the choice of the coloring $d_{1}$, we may fix a stationary subset $\Delta \subseteq \kappa$ and an ordinal $\eta<\kappa$ such that, for every $\delta \in \Delta$, there exists $b \in \mathcal{B}$ with $\min (b)>$ $\max \{\eta, \delta\}$ satisfying $d_{1}[\{\eta\} \times b]=\{\delta\}$. Without loss of generality, we may assume that $\min (\Delta)>\eta$.

Fix $\delta \in D \cap \operatorname{acc}^{+}(A) \cap \Delta$, and then fix $b \in \mathcal{B}$ with $\min (b)>\delta$ satisfying $d_{1}[\{\eta\} \times b]=\{\delta\}$.

As $\delta \in D, A \cap \delta \subseteq \bigcup_{\alpha \in b} C_{\alpha}$. As $\delta \in \operatorname{acc}^{+}(A)$, we may fix $\alpha \in b$ and $\gamma \in C_{\alpha}$ with $\eta<\gamma<\delta$. As $\alpha \in b$, it is a regular uncountable cardinal, so it follows from the definition of $C_{\alpha}$ that $d_{1}(\eta, \alpha)<\gamma<\delta$, contradicting the fact that $d_{1}(\eta, \alpha)=\delta$.

Question 3.21. Does $\mathrm{P} \ell_{1}(\kappa, 1, \chi(\kappa))$ hold for every inaccessible cardinal $\kappa$ ?

## 4. From colorings to transformations

For the sake of this section, we introduce the following ad-hoc principle.
Definition 4.1. $\operatorname{Pr}_{1}^{+}(\kappa, \theta, \chi)$ asserts the existence of a coloring $o:[\kappa]^{2} \rightarrow \theta$ satisfying:
(1) For all nonzero $\alpha<\beta<\kappa$, o( $\alpha, \beta)<\alpha$;
(2) For all $\zeta<\theta, \sigma<\chi$, and every pairwise disjoint subfamily $\mathcal{A} \subseteq[\kappa]^{\sigma}$ of size $\kappa$, there is $\gamma<\kappa$ such that, for every $b \in[\kappa \backslash \gamma]^{\sigma}$, for some $a \in \mathcal{A} \cap \mathcal{P}(\gamma)$, $o[a \times b]=\{\zeta\}$.

It turns out that the above variation is not much stronger than the original. In particular, the following lemma shows that if $\chi$ is an infinite cardinal, then $\operatorname{Pr}_{1}(\kappa, \kappa, \theta, \chi)$ implies $\operatorname{Pr}_{1}^{+}(\kappa, \theta, \chi)$.
Lemma 4.2. Suppose that $\operatorname{Pr}_{1}\left(\kappa, \kappa, \theta, \chi_{2}\right)$ holds for the cardinal $\chi_{2}:=\chi+\chi$. Then so does $\operatorname{Pr}_{1}^{+}(\kappa, \theta, \chi)$.
Proof. Let $c$ be a witness to $\operatorname{Pr}_{1}\left(\kappa, \kappa, \theta, \chi_{2}\right)$. Define a coloring $o:[\kappa]^{2} \rightarrow \theta$ by letting $o(\alpha, \beta):=c(\alpha, \beta)$ whenever $c(\alpha, \beta)<\alpha<\beta<\kappa$, and $o(\alpha, \beta):=0$, otherwise. Towards a contradiction, suppose that $o$ is not as sought. Let $\zeta, \sigma$ and $\mathcal{A}$ form together a counterexample. This means that, for each $\gamma<\kappa$, we may fix $b_{\gamma} \in[\kappa \backslash \gamma]^{\sigma}$ such that, for all $a \in \mathcal{A} \cap \mathcal{P}(\gamma), o\left[a \times b_{\gamma}\right] \neq\{\zeta\}$. Also, fix $a_{\gamma} \in \mathcal{A}$ with $\min \left(a_{\gamma}\right)>\gamma$, and then set $x_{\gamma}:=a_{\gamma} \cup b_{\gamma}$.

Fix a club $C \subseteq \kappa$ such that, for all $\gamma \in C,\left(\bigcup_{\gamma^{\prime}<\gamma} x_{\gamma^{\prime}}\right) \subseteq \gamma$. In particular, $\left\langle x_{\gamma}\right|$ $\gamma \in C \backslash \zeta\rangle$ is a <-increasing sequence of elements of $[\kappa] \leq \sigma+\sigma \subseteq[\kappa]^{<\chi_{2}}$. Thus, by the choice of $c$, we may find $\left(\gamma^{\prime}, \gamma\right) \in[C \backslash \zeta]^{2}$ such that $c\left[x_{\gamma^{\prime}} \times x_{\gamma}\right]=\{\zeta\}$. As $\min \left(a_{\gamma^{\prime}}\right)>\gamma^{\prime} \geq \zeta$, it thus follows that $o\left[a_{\gamma^{\prime}} \times b_{\gamma}\right]=\{\zeta\}$, contradicting the fact that $a_{\gamma^{\prime}} \in \mathcal{A} \cap \mathcal{P}(\gamma)$ and the choice of $b_{\gamma}$.

Theorem 4.3. Suppose that $\mu$ is an infinite regular cardinal, $\chi \leq \mu$, and $\Gamma \subseteq \mu^{+}$ is a nonreflecting stationary set.

If $\operatorname{Pr}_{1}^{+}\left(\mu^{+}, \mu, \chi\right)$ holds, then so does $\mathrm{P} \ell_{2}\left(\mu^{+}, \Gamma, \chi\right)$.
Proof. Suppose that $\operatorname{Pr}_{1}^{+}\left(\mu^{+}, \mu, \chi\right)$ holds, as witnessed by $o:\left[\mu^{+}\right]^{2} \rightarrow \mu$. Fix a $C$-sequence $\vec{C}=\left\langle C_{\alpha} \mid \alpha<\mu^{+}\right\rangle$such that, for all $\alpha<\mu^{+}$, otp $\left(C_{\alpha}\right)=\operatorname{cf}(\alpha)$ and $C_{\alpha} \cap \Gamma=\emptyset$. We shall walk along $\vec{C}$. Fix a bijection $\pi: \mu \leftrightarrow \mu \times \mu$. For every $\beta<\mu^{+}$, fix a surjection $\psi_{\beta}: \mu \rightarrow \beta+1$. Fix an almost disjoint family $\left\{Z_{\epsilon} \mid\right.$ $\left.\epsilon<\mu^{+}\right\} \subseteq[\mu]^{\mu}$. For every ordinal $\xi<\mu$ and a pair $(\alpha, \beta) \in\left[\mu^{+}\right]^{2}$, we let

$$
\xi^{\alpha, \beta}:=\min \left\{n<\omega \mid \xi \in Z_{\operatorname{Tr}(\alpha, \beta)(n)} \text { or } n=\rho_{2}(\alpha, \beta)+1\right\}
$$

Define $\mathbf{t}:\left[\mu^{+}\right]^{2} \rightarrow\left[\mu^{+}\right]^{2}$, as follows. Let $\mathbf{t}(\alpha, \beta):=\left(\alpha^{*}, \beta^{*}\right)$ provided that the following hold:

$$
\text { - }(\tau, \xi):=\pi(o(\alpha, \beta)) ;
$$

- $\beta^{*}:=\operatorname{Tr}(\alpha, \beta)\left(\xi^{\alpha, \beta}\right)$ is $>\alpha$;
- $\alpha^{*}:=\psi_{\beta^{*}}(\tau)$ is $<\alpha$.

Otherwise, just let $\mathbf{t}(\alpha, \beta):=(\alpha, \beta)$.
To verify that $\mathbf{t}$ witnesses $\mathrm{P} \ell_{2}\left(\mu^{+}, \Gamma, \chi\right)$, suppose that we are given $\sigma<\chi$ and a pairwise disjoint subfamily $\mathcal{A} \subseteq\left[\mu^{+}\right]^{\sigma}$ of size $\mu^{+}$. Fix a sequence $\vec{x}=\left\langle x_{\delta} \mid \delta<\mu^{+}\right\rangle$ such that, for all $\delta<\mu^{+}, x_{\delta} \in \mathcal{A}$ with $\min \left(x_{\delta}\right)>\delta$.
Claim 4.3.1. There exists $\eta<\mu^{+}$and a stationary $\Delta \subseteq E_{\mu}^{\mu^{+}}$such that, for every $\delta \in \Delta$ and $\beta \in x_{\delta}, \lambda(\delta, \beta) \leq \eta$.

Proof. Let $\delta \in E_{\mu}^{\mu^{+}}$be arbitrary. For every $\alpha<\mu^{+}$, otp $\left(C_{\alpha}\right)=\operatorname{cf}(\alpha) \leq \mu=\operatorname{cf}(\delta)$, and hence $\delta \notin \operatorname{acc}\left(C_{\alpha}\right)$. So, as $\left|x_{\delta}\right|<\chi \leq \mu$, Fact 3.6 entails the existence of a large enough $\eta<\delta$ such that $\lambda(\delta, \beta) \leq \eta$ for all $\beta \in x_{\delta}$. Now, appeal to Fodor's lemma.

Let $\eta$ and $\Delta$ be given by the claim. Let $D \subseteq \mu^{+}$be the club of all $\delta<\mu^{+}$for which there exists an elementary submodel $\mathcal{M}_{\delta} \prec \mathcal{H}_{\mu^{++}}$with $\mathcal{M}_{\delta} \cap \mu^{+}=\delta$ such that $\left\{\left\langle x_{\gamma} \mid \gamma \in \Delta\right\rangle, o, \pi, \eta\right\} \in \mathcal{M}_{\delta}$.
Claim 4.3.2. Let $\left(\alpha^{*}, \beta^{*}\right) \in[D \cap \Gamma]^{2}$. Then there exists $(a, b) \in[\mathcal{A}]^{2}$ such that $\mathbf{t}[a \times b]=\left\{\left(\alpha^{*}, \beta^{*}\right)\right\}$.
Proof. Fix $\delta^{*} \in \Delta$ above $\beta^{*}$. Pick $\xi \in Z_{\beta^{*}} \backslash \bigcup\left\{Z_{\operatorname{tr}\left(\beta^{*}, \beta\right)(n)} \mid \beta \in x_{\delta^{*}}, n<\rho_{2}\left(\beta^{*}, \beta\right)\right\}$, and $\tau<\mu$ such that $\psi_{\beta^{*}}(\tau)=\alpha^{*}$. Let $\zeta:=\pi^{-1}(\tau, \xi)$. As $\beta^{*} \in \Gamma$, it follows from Fact 3.6 that $\lambda\left(\beta^{*}, \delta^{*}\right)<\beta^{*}$. Set $\eta^{*}:=\max \left\{\eta, \lambda\left(\beta^{*}, \delta^{*}\right)\right\}$. Let $\mathcal{A}^{\prime}:=\left\{x_{\delta} \mid\right.$ $\left.\delta \in \Delta \backslash \eta^{*}\right\}$. As $\mathcal{A}^{\prime}$ and $o$ are in $\mathcal{M}_{\beta^{*}}$ there exists some $\gamma<\beta^{*}$ such that, for every $b \in\left[\mu^{+} \backslash \gamma\right]^{\sigma}$, for some $a \in \mathcal{A}^{\prime} \cap \mathcal{P}(\gamma), o[a \times b]=\{\zeta\}$. In particular, we may find $\delta \in \mathcal{M}_{\beta^{*}} \cap \Delta \backslash \eta^{*}$ such that $o\left[x_{\delta} \times x_{\delta^{*}}\right]=\{\zeta\}$. Denote $a:=x_{\delta}$ and $b:=x_{\delta^{*}}$, so that $(a, b) \in[\mathcal{A}]^{2}$. Let $(\alpha, \beta) \in a \times b$ be arbitrary. Evidently,

$$
\max \left\{\lambda\left(\delta^{*}, \beta\right), \lambda\left(\beta^{*}, \delta^{*}\right)\right\} \leq \eta^{*} \leq \delta<\alpha<\beta^{*}<\delta^{*}<\beta
$$

So, Fact 3.7 implies that $\operatorname{tr}(\alpha, \beta)=\operatorname{tr}\left(\delta^{*}, \beta\right)^{\wedge} \operatorname{tr}\left(\beta^{*}, \delta^{*}\right)^{\wedge} \operatorname{tr}\left(\alpha, \beta^{*}\right)$. Now, by the choice of $\xi$, we have $\operatorname{Tr}(\alpha, \beta)\left(\xi^{\alpha, \beta}\right)=\beta^{*}$. So, as $(\tau, \xi)=\pi(o(\alpha, \beta))$ and $\psi_{\beta^{*}}(\tau)=\alpha^{*}$, it follows that $(\alpha, \beta)=\left(\alpha^{*}, \beta^{*}\right)$, as sought.

This completes the proof.
Corollary 4.4. (1) For every integer $n \geq 2, \mathrm{P} \ell_{2}\left(\aleph_{1}, \aleph_{1}, n\right)$ holds;
(2) If $\operatorname{Pr}_{1}\left(\aleph_{1}, \aleph_{1}, \aleph_{0}, \aleph_{0}\right)$ holds, then so does $\mathrm{P} \ell_{2}\left(\aleph_{1}, \aleph_{1}, \aleph_{0}\right)$.

Proof. (1) By Lemma 4.2, Theorem 4.3, and the fact that, by Peng and Wu [PW18, Theorem 2], $\operatorname{Pr}_{1}\left(\aleph_{1}, \aleph_{1}, \aleph_{1}, n+n\right)$ holds for every positive integer $n$.
(2) By Lemma 4.2, $\operatorname{Pr}_{1}\left(\aleph_{1}, \aleph_{1}, \aleph_{0}, \aleph_{0}\right)$ implies $\operatorname{Pr}_{1}^{+}\left(\aleph_{1}, \aleph_{1}, \aleph_{0}\right)$. In addition, $\Gamma:=$ $\aleph_{1}$ is a stationary set that does not reflect (to see this coheres with Definition 3.2, note that by taking a $C$-sequence $\left\langle C_{\alpha} \mid \alpha<\aleph_{1}\right\rangle$ such that otp $\left(C_{\alpha}\right)=\operatorname{cf}(\alpha)$ for all $\alpha<\omega_{1}$, one gets that $\operatorname{acc}\left(C_{\alpha}\right)=\emptyset$ for all $\left.\alpha<\omega_{1}\right)$. Now appeal to Theorem 4.3.

We are now in conditions to push Theorem 4.3 from the special case of $\kappa$ being a successor of a regular cardinal to the general case.

Theorem 4.5. Suppose that $\chi \leq \kappa$, and $\Gamma \subseteq \kappa$ is a nonreflecting stationary set such that $\Gamma \cap E_{>\sigma}^{\kappa}$ stationary for every $\sigma<\chi$.

If $\operatorname{Pr}_{1}^{+}(\kappa, \kappa, \chi)$ holds, then so does $\mathrm{P} \ell_{2}(\kappa, \Gamma, \chi)$.

Proof. By Corollary 4.4, we may assume that $\kappa \geq \aleph_{2}$. So, by Lemma 3.14, we may fix a $C$-sequence $\vec{C}=\left\langle C_{\alpha} \mid \alpha<\kappa\right\rangle$ that avoids $\Gamma$ and satisfying $\chi_{1}(\vec{C}) \geq \chi$. Fix a coloring $o:[\kappa]^{2} \rightarrow \kappa$ witnessing $\operatorname{Pr}_{1}^{+}(\kappa, \kappa, \chi)$. Fix a bijection $\pi: \kappa \leftrightarrow \omega \times \kappa \times \kappa$. Define $\mathbf{t}:[\kappa]^{2} \rightarrow[\kappa]^{2}$, as follows. Let $\mathbf{t}(\alpha, \beta):=\left(\alpha^{*}, \beta^{*}\right)$ provided that the following hold:

- $(n, \tau, \eta):=\pi(o(\alpha, \beta)) ;$
- $\beta^{*}:=\operatorname{Tr}(\alpha, \beta)\left(\eta_{\alpha, \beta}+n\right)$ is $>\alpha$;
- $\alpha^{*}:=\tau$ is $<\alpha$.

Otherwise, just let $\mathbf{t}(\alpha, \beta):=(\alpha, \beta)$.
To verify that $\mathbf{t}$ witnesses $\mathrm{P} \ell_{2}(\kappa, \Gamma, \chi)$, suppose that we are given $\sigma<\chi$ and a pairwise disjoint subfamily $\mathcal{A} \subseteq[\kappa]^{\sigma}$ of size $\kappa$. As $\sigma<\chi_{1}(\vec{C})$, we may fix $\eta<\kappa$ and a sequence $\left\langle x_{\delta} \mid \delta \in \Delta\right\rangle$ such that $\Delta$ is a stationary subset of $\kappa$, and, for every $\delta \in \Delta, x_{\delta} \in \mathcal{A}$ with $\min \left(x_{\delta}\right)>\delta$, and, for every $\beta \in x_{\delta}, \lambda(\delta, \beta)=\eta$ and $\rho_{2}(\delta, \beta)=\eta_{\delta, \beta}$.

Let $D \subseteq \kappa$ be the club of all $\delta<\kappa$ for which there exists an elementary submodel $\mathcal{M}_{\delta} \prec \mathcal{H}_{\kappa^{+}}$with $\mathcal{M}_{\delta} \cap \kappa=\delta$ such that $\left\{\left\langle x_{\gamma} \mid \gamma \in \Delta\right\rangle, o, \pi, \eta\right\} \in \mathcal{M}_{\delta}$.
Claim 4.5.1. Let $\left(\alpha^{*}, \beta^{*}\right) \in[D \cap \Gamma]^{2}$. Then there exists $(a, b) \in[\mathcal{A}]^{2}$ such that $\mathbf{t}[a \times b]=\left\{\left(\alpha^{*}, \beta^{*}\right)\right\}$.
Proof. Fix $\delta^{*} \in \Delta$ above $\beta^{*}$. Let $n:=\rho_{2}\left(\beta^{*}, \delta^{*}\right)$. Let $\zeta:=\pi^{-1}\left(n, \alpha^{*}, \eta\right)$. As $\beta^{*} \in \Gamma$, it follows from Fact 3.6 that $\lambda\left(\beta^{*}, \delta^{*}\right)<\beta^{*}$. Set $\eta^{*}:=\max \left\{\eta, \lambda\left(\beta^{*}, \delta^{*}\right)\right\}$. Let $\mathcal{A}^{\prime}:=\left\{x_{\delta} \mid \delta \in \Delta \backslash \eta^{*}\right\}$. As $\mathcal{A}^{\prime}$ and $o$ are in $\mathcal{M}_{\beta^{*}}$ there exists some $\gamma<\beta^{*}$ such that, for every $b \in[\kappa \backslash \gamma]^{\sigma}$, for some $a \in \mathcal{A}^{\prime} \cap \mathcal{P}(\gamma)$, $o[a \times b]=\{\zeta\}$. In particular, we may find $\delta \in \mathcal{M}_{\beta^{*}} \cap \Delta \backslash \eta^{*}$ such that $o\left[x_{\delta} \times x_{\delta^{*}}\right]=\{\zeta\}$. Denote $a:=x_{\delta}$ and $b:=x_{\delta^{*}}$, so that $(a, b) \in[\mathcal{A}]^{2}$. Let $(\alpha, \beta) \in a \times b$ be arbitrary. Evidently,

$$
\max \left\{\lambda\left(\delta^{*}, \beta\right), \lambda\left(\beta^{*}, \delta^{*}\right)\right\} \leq \eta^{*} \leq \delta<\alpha<\beta^{*}<\delta^{*}<\beta
$$

So, Fact 3.7 implies that $\operatorname{tr}(\alpha, \beta)=\operatorname{tr}\left(\delta^{*}, \beta\right)^{\wedge} \operatorname{tr}\left(\beta^{*}, \delta^{*}\right)^{\wedge} \operatorname{tr}\left(\alpha, \beta^{*}\right)$. Now, by the choice of $\eta$, we have $\operatorname{Tr}(\alpha, \beta)\left(\eta_{\alpha, \beta}\right)=\delta^{*}$ and $\operatorname{Tr}(\alpha, \beta)\left(\eta_{\alpha, \beta}+n\right)=\beta^{*}$. So, as $\left(n, \alpha^{*}, \eta\right)=\pi(o(\alpha, \beta))$, it altogether follows that $\mathbf{t}(\alpha, \beta)=\left(\alpha^{*}, \beta^{*}\right)$, as sought.

This completes the proof.
Corollary 4.6. Suppose that $\chi<\chi^{+}<\kappa$ are infinite regular cardinals, and $\Gamma \subseteq \kappa$ is a nonreflecting stationary set such that $\Gamma \cap E_{\geq \chi}^{\kappa}$ stationary.

Then $\mathrm{P} \ell_{2}(\kappa, \Gamma, \chi)$ holds.
Proof. By the main result of [Rin14b], the hypothesis implies that $\operatorname{Pr}_{1}(\kappa, \kappa, \kappa, \chi)$ holds. So by Lemma 4.2, $\operatorname{Pr}_{1}^{+}(\kappa, \kappa, \chi)$ holds, as well. Now, appeal to Theorem 4.5.

Theorem 4.7. Suppose that $\operatorname{Pr}_{1}^{+}(\kappa, \kappa, \chi)$ holds for some cardinal $\chi \leq \kappa$, and that $\Gamma \subseteq \kappa$ is a stationary set.
(1) If there is a C-sequence $\vec{C}$ over $\kappa$ with $\chi_{1}(\vec{C}) \geq \chi$, then $\mathrm{P} \ell_{1}(\kappa, \kappa, \chi)$ holds;
(2) If there is a $C$-sequence $\vec{C}$ over $\kappa$ with $\chi_{2}(\vec{C}, \Gamma) \geq \chi$, then $\mathrm{P} \ell_{2}(\kappa, \Gamma, \chi)$ holds.

Proof. Fix a coloring $o:[\kappa]^{2} \rightarrow \kappa$ witnessing $\operatorname{Pr}_{1}^{+}(\kappa, \kappa, \chi)$.
(1) Suppose that $\vec{C}=\left\langle C_{\alpha} \mid \alpha<\kappa\right\rangle$ is a $C$-sequence over $\kappa$ with $\chi_{1}(\vec{C}) \geq \chi$, and let us conduct walks along $\vec{C}$. Fix a bijection $\pi: \kappa \leftrightarrow \kappa \times \kappa \times \kappa$. Define a
transformation $\mathbf{t}:[\kappa]^{2} \rightarrow[\kappa]^{3}$ by letting $\mathbf{t}(\alpha, \beta):=\left(\tau^{*}, \alpha^{*}, \beta^{*}\right)$ provided that the following hold:

- $\left(\tau^{*}, \alpha^{*}, \eta\right):=\pi(o(\alpha, \beta))$;
- $\beta^{*}:=\operatorname{Tr}(\alpha, \beta)\left(\eta_{\alpha, \beta}\right)$ is $>\alpha$;
- $\tau^{*}<\alpha^{*}<\alpha$.

Otherwise, just let $\mathbf{t}(\alpha, \beta):=(0, \alpha, \beta)$.
To verify that $\mathbf{t}$ is as sought, suppose that we are given $\sigma<\chi$ and a pairwise disjoint subfamily $\mathcal{A} \subseteq[\kappa]^{\sigma}$ of size $\kappa$. As $\chi_{1}(\vec{C}) \geq \chi>\sigma$, we may fix a stationary $\Delta \subseteq \kappa$ such that, for every $\delta \in \Delta$, there exist an ordinal $\eta_{\delta}<\delta$ and a subfamily $\mathcal{A}_{\delta} \in[\mathcal{A}]^{\kappa}$ such that, for all $b \in \mathcal{A}_{\delta}$ and $\beta \in b, \lambda(\delta, \beta)=\eta_{\delta}$ and $\rho_{2}(\delta, \beta)=\left(\eta_{\delta}\right)_{\delta, \beta}$. Let $D$ be the club of all $\delta<\kappa$ for which there exists an elementary submodel $\mathcal{M}_{\delta} \prec \mathcal{H}_{\kappa^{+}}$with $\mathcal{M}_{\delta} \cap \kappa=\delta$ such that $\left\{\left\langle\mathcal{A}_{\gamma} \mid \gamma \in \Delta\right\rangle, o, \pi\right\} \in \mathcal{M}_{\delta}$.
Claim 4.7.1. Let $\left(\alpha^{*}, \beta^{*}\right) \in[\Delta \cap D]^{2}$ and $\tau^{*}<\alpha$.
Then there exists a pair $(a, b) \in[\mathcal{A}]^{2}$ such that $\mathbf{t}[a \times b]=\left\{\left(\tau^{*}, \alpha^{*}, \beta^{*}\right)\right\}$.
Proof. Denote $\eta:=\eta_{\beta^{*}}$. Evidently, $\left\{a \in \mathcal{A}_{\alpha^{*}} \mid \min (a)>\eta\right\}$ and $\zeta:=\pi^{-1}\left(\tau^{*}, \alpha^{*}, \eta\right)$ are in $\mathcal{M}_{\beta^{*}}$. As $o \in \mathcal{M}_{\beta^{*}}$, it follows that there exists $\gamma<\beta^{*}$ such that, for every $b \in[\kappa \backslash \gamma]^{\sigma}$, for some $a \in \mathcal{A}_{\alpha^{*}} \cap \mathcal{P}(\gamma)$ with $\min (a)>\eta, o[a \times b]=\{\zeta\}$.

Fix an arbitrary $b \in \mathcal{A}_{\beta^{*}}$ with $\min (b)>\beta^{*}$, and then pick $a \in \mathcal{A}_{\alpha^{*}} \cap \mathcal{P}(\gamma)$ with $\min (a)>\eta$ such that $o[a \times b]=\{\zeta\}$. Now, let $(\alpha, \beta) \in a \times b$, and we shall show that $\mathbf{t}(\alpha, \beta)=\left(\alpha^{*}, \beta^{*}\right)$. All of the following hold:

- $\eta<\alpha<\gamma<\beta^{*}<\beta$,
- $\lambda\left(\beta^{*}, \beta\right)=\eta$, and
- $\rho_{2}\left(\beta^{*}, \beta\right)=\eta_{\beta^{*}, \beta}$.

So, by Lemma 3.12 (using $\left.\delta:=\beta^{*}\right), \operatorname{tr}(\alpha, \beta)\left(\eta_{\alpha, \beta}\right)=\beta^{*}$. Recalling that $\pi(o(\alpha, \beta))=$ $\left(\tau^{*}, \alpha^{*}, \eta\right)$, we infer from the definition of $\mathbf{t}$ that $\mathbf{t}(\alpha, \beta)=\left(\tau^{*}, \alpha^{*}, \beta^{*}\right)$, as sought.
(2) Suppose that $\vec{C}=\left\langle C_{\alpha} \mid \alpha<\kappa\right\rangle$ is a $C$-sequence over $\kappa$ with $\chi_{2}(\vec{C}, \Gamma) \geq \chi$, and let us walk along $\vec{C}$. Fix a bijection $\pi: \kappa \leftrightarrow \kappa \times \kappa$. Define a transformation $\mathbf{t}:[\kappa]^{2} \rightarrow[\kappa]^{2}$ by letting $\mathbf{t}(\alpha, \beta):=\left(\alpha^{*}, \beta^{*}\right)$ provided that the following hold:

- $\left(\alpha^{*}, \eta\right):=\pi(o(\alpha, \beta))$;
- $\beta^{*}:=\operatorname{Tr}(\alpha, \beta)\left(\eta_{\alpha, \beta}\right)$ is $>\alpha$;
- $\alpha^{*}<\alpha$.

Otherwise, just let $\mathbf{t}(\alpha, \beta):=(\alpha, \beta)$.
To verify that $\mathbf{t}$ is as sought, suppose that we are given $\sigma<\chi$ and a pairwise disjoint subfamily $\mathcal{A} \subseteq[\kappa]^{\sigma}$ of size $\kappa$. As $\chi_{2}(\vec{C}, \Gamma) \geq \chi>\sigma$, we may fix $\Delta \subseteq \Gamma$ for which $\Gamma \backslash \Delta$ is nonstationary such that, for every $\delta \in \Delta$, there exist an ordinal $\eta_{\delta}<\delta$ and a subfamily $\mathcal{A}_{\delta} \in[\mathcal{A}]^{\kappa}$ such that, for all $b \in \mathcal{A}_{\delta}$ and $\beta \in b, \lambda(\delta, \beta)=\eta_{\delta}$ and $\rho_{2}(\delta, \beta)=\left(\eta_{\delta}\right)_{\delta, \beta}$. Let $D$ be the club of all $\delta<\kappa$ for which there exists an elementary submodel $\mathcal{M}_{\delta} \prec \mathcal{H}_{\kappa^{+}}$with $\mathcal{M}_{\delta} \cap \kappa=\delta$ such that $\left\{\left\langle\mathcal{A}_{\gamma} \mid \gamma \in \Delta\right\rangle, o, \pi\right\} \in$ $\mathcal{M}_{\delta}$.

Claim 4.7.2. Let $\left(\alpha^{*}, \beta^{*}\right) \in[\Delta \cap D]^{2}$. Then there exists $(a, b) \in[\mathcal{A}]^{2}$ such that $\mathbf{t}[a \times b]=\left\{\left(\alpha^{*}, \beta^{*}\right)\right\}$.
Proof. Denote $\eta:=\eta_{\beta^{*}}$. As $\left\{a \in \mathcal{A}_{\alpha^{*}} \mid \min (a)>\eta\right\}, \zeta:=\pi^{-1}\left(\alpha^{*}, \eta\right)$ and $o$ are all in $\mathcal{M}_{\beta^{*}}$, there exists $\gamma<\beta^{*}$ such that, for every $b \in[\kappa \backslash \gamma]^{\sigma}$, for some $a \in \mathcal{A}_{\alpha^{*}} \cap \mathcal{P}(\gamma)$ with $\min (a)>\eta, o[a \times b]=\{\zeta\}$. Fix $b \in \mathcal{A}_{\beta^{*}}$ with $\min (b)>\beta^{*}$, and then pick
$a \in \mathcal{A}_{\alpha^{*}} \cap \mathcal{P}(\gamma)$ with $\min (a)>\eta$ such that $o[a \times b]=\{\zeta\}$. Now, let $(\alpha, \beta) \in a \times b$, and we shall show that $\mathbf{t}(\alpha, \beta)=\left(\alpha^{*}, \beta^{*}\right)$. All of the following hold:

- $\eta<\alpha<\gamma<\beta^{*}<\beta$,
- $\lambda\left(\beta^{*}, \beta\right)=\eta$, and
- $\rho_{2}\left(\beta^{*}, \beta\right)=\eta_{\beta^{*}, \beta}$.

So, $\operatorname{tr}(\alpha, \beta)\left(\eta_{\alpha, \beta}\right)=\beta^{*}$. Recalling that $\pi(o(\alpha, \beta))=\left(\alpha^{*}, \eta\right)$, we infer from the definition of $\mathbf{t}$ that $\mathbf{t}(\alpha, \beta)=\left(\alpha^{*}, \beta^{*}\right)$, as sought.

This completes the proof.
Corollary 4.8. Suppose that $\chi \leq \kappa$ is an infinite cardinal such that $\operatorname{Pr}_{1}(\kappa, \kappa, \kappa, \chi)$ holds.
(1) If $\square(\kappa)$ holds, then so does $\mathrm{P} \ell_{1}(\kappa, \kappa, \chi)$;
(2) If $\boxtimes^{-}(\kappa)$ holds, then so does $\mathrm{P} \ell_{2}(\kappa, \kappa, \chi)$.

Proof. As $\operatorname{Pr}_{1}(\kappa, \kappa, 2, \chi)$ in particular holds, Propositions 2.10 and 2.11 tell us that $\chi \leq \sup (\operatorname{Reg}(\kappa))$. So if $\kappa=\aleph_{1}$, then $\chi \leq \aleph_{0}$, and then the conclusion follows from Corollary 4.4. Thus, we may assume that $\kappa \geq \aleph_{2}$.
(1) If $\square(\kappa)$ holds, then by Lemma $3.16(1)$, we may find a $C$-sequence $\vec{C}$ over $\kappa$ such that $\chi_{1}(\vec{C})=\sup (\operatorname{Reg}(\kappa))$. Now, appeal to Theorem 4.7.
(2) If $\boxtimes^{-}(\kappa)$ holds, then by Lemma $3.16(2)$, we may find a $C$-sequence $\vec{C}$ over $\kappa$ such that $\chi_{2}(\vec{C}, \kappa)=\sup (\operatorname{Reg}(\kappa))$. Now, appeal to Theorem 4.7.

Remark 4.9. By [RZ21, Proposition 2.19(1)], it is consistent that for an inaccessible cardinal $\kappa, \operatorname{Pr}_{1}(\kappa, \kappa, \kappa, \omega)$ holds, but $\mathrm{P} \ell_{1}(\kappa, 1, \omega)$ fails.

## 5. INACCESSIBLE CARDINALS

Fact 5.1 ([IR22, Lemma 8.4 and Corollary 8.6]). Assume either of the following:

- $\diamond(S)$ holds for some stationary $S \subseteq \kappa$ that does not reflect at regulars;
- $\diamond^{*}(\kappa)$ holds.

Then there exists a coloring $d_{0}:[\kappa]^{2} \rightarrow \kappa$ satisfying that, for every stationary $\Delta \subseteq \kappa$, there exists $\tau<\kappa$ such that $d_{0}[\{\tau\} \circledast \Delta]=\kappa$.

Theorem 5.2. Suppose that $\kappa=\kappa^{<\kappa}$ is an inaccessible cardinal and there are $a$ map $d_{0}:[\kappa]^{2} \rightarrow \kappa$ as in Fact 5.1 and a map $d_{1}:[\kappa]^{2} \rightarrow \kappa$ good for $\chi$ in the sense of Lemma 3.18. Then $\operatorname{Pr}_{1}(\kappa, \kappa \circledast \kappa / 1 \circledast \kappa, \kappa, \chi)$ holds.

Proof. Fix $d_{0}$ and $d_{1}$ as above. Fix a bijection $\pi: \kappa \leftrightarrow \kappa \times \kappa$. Fix a surjection $\psi: \kappa \rightarrow \kappa$ such that the preimage of any singleton is cofinal in $\kappa$. Next, define an auxiliary coloring $e:[\kappa]^{2} \rightarrow \kappa$ as follows. Given $j<\beta<\kappa$, set $(\tau, \eta):=\pi(\psi(j))$ and then let $e(j, \beta):=\psi\left(d_{0}\left(\tau, d_{1}(\eta, \beta)\right)\right)$ provided that $\eta<\beta$ and $\tau<d_{1}(\eta, \beta)$. Otherwise, just let $e(j, \beta):=0$.

Claim 5.2.1. For every $\sigma<\chi$ and every pairwise disjoint subfamily $\mathcal{B} \subseteq[\kappa]^{\sigma}$ of size $\kappa$, there are cofinally many $j<\kappa$ such that, for every $\gamma<\kappa$, there are $\kappa$ many $b \in \mathcal{B}$ with $e[\{j\} \times b]=\{\gamma\}$.
Proof. Let $\mathcal{B} \subseteq[\kappa]^{\sigma}$ be as above. As $d_{1}$ is good for $\chi$, we may fix a stationary set $\Delta \subseteq \kappa$ and an ordinal $\eta<\kappa$ such that for every $\delta \in \Delta$, there exists $b \in \mathcal{B}$ with $\min (b)>\max \{\eta, \delta\}$ satisfying $d_{1}[\{\eta\} \times b]=\{\delta\}$. Now, by the choice of $d_{0}$, find
$\tau<\kappa$ such that $d_{0}[\{\tau\} \circledast \Delta]=\kappa$. Evidently, $J:=\left\{j<\kappa \mid \psi(j)=\pi^{-1}(\tau, \eta)\right\}$ is cofinal in $\kappa$. Let $j \in J$ be arbitrary.

Now, given $\gamma<\kappa$, as $d_{0}[\{\tau\} \circledast \Delta]=\kappa$, the following set has size $\kappa$ :

$$
\Delta^{\prime}:=\left\{\delta \in \Delta \backslash(\tau+1) \mid \psi\left(d_{0}(\tau, \delta)\right)=\gamma\right\}
$$

Recalling the choice of $\eta$, it follows that the following set has size $\kappa$, as well:

$$
\mathcal{B}^{\prime}:=\left\{b \in \mathcal{B} \mid \exists \delta \in \Delta^{\prime}\left(\min (b)>\max \{\eta, \delta, j\} \& d_{1}[\{\eta\} \times b]=\{\delta\}\right)\right\}
$$

Let $b \in \mathcal{B}^{\prime}$ be arbitrary. Let $\delta \in \Delta^{\prime}$ be a witness for $b$ being in $\mathcal{B}^{\prime}$. For every $\beta \in b, e(j, \beta)=\psi\left(d_{0}\left(\tau, d_{1}(\eta, \beta)\right)\right)=\psi\left(d_{0}(\tau, \delta)\right)=\gamma$.

Fix a strictly increasing sequence $\left\langle\kappa_{j} \mid j<\kappa\right\rangle$ of infinite cardinals below $\kappa$, such that, for all $j<\kappa,\left(\sup _{i<j} \kappa_{i}\right)<\kappa_{j}$. For every $j<\kappa$, let $\Phi_{j}:=\bigcup\left\{{ }^{x} \kappa \mid\right.$ $\left.x \subseteq \kappa,|x|=\kappa_{j}\right\}$. As $\kappa^{<\kappa}=\kappa,\left|\Phi_{j}\right|=\kappa$, so we may fix an injective enumeration $\left\langle\phi_{j}^{\gamma} \mid \gamma<\kappa\right\rangle$ of $\Phi_{j}$. Now, define a coloring $c:[\kappa]^{2} \rightarrow \kappa$ by letting for all $\alpha<\beta<\kappa$ :

$$
c(\alpha, \beta):= \begin{cases}0 & \text { if } \alpha \notin \bigcup_{i<\kappa} \operatorname{dom}\left(\phi_{i}^{e(i, \beta)}\right) \\ \phi_{j}^{e(j, \beta)}(\alpha) & \text { if } j=\min \left\{i<\kappa \mid \alpha \in \operatorname{dom}\left(\phi_{i}^{e(i, \beta)}\right)\right\} .\end{cases}
$$

To see that $c$ is as sought, fix $\sigma<\chi$ and pairwise disjoint subfamilies $\mathcal{A}, \mathcal{B}$ of $[\kappa]^{\sigma}$ of size $\kappa$. By Claim 5.2.1, fix $j<\kappa$ with $\kappa_{j}>\sigma$ such that, for every $\gamma<\kappa$, there are $\kappa$ many $b \in \mathcal{B}$ with $e[\{j\} \times b]=\{\gamma\}$. Let $\left\langle a_{\iota} \mid \iota<\kappa_{j}\right\rangle$ be an injective sequence consisting of elements of $\mathcal{A}$.
Claim 5.2.2. There exists $\iota<\kappa_{j}$ such that, for every $\delta<\kappa$, there is $b \in \mathcal{B}$ with $a_{\iota}<b$ such that $c\left[a_{\iota} \times b\right]=\{\delta\}$.
Proof. Suppose not. Then, for every $\iota<\kappa_{j}$, we may find some $\delta_{\iota}<\kappa$ such that, for all $b \in \mathcal{B}$ with $a_{\iota}<b, c\left[a_{\iota} \times b\right] \neq\left\{\delta_{\iota}\right\}$. Since $\mathcal{A}$ is a pairwise disjoint family, we may define a function $\phi: \bigcup\left\{a_{\iota} \mid \iota<\kappa_{j}\right\} \rightarrow \kappa$ by letting $\phi(\alpha):=\delta_{\iota}$ iff $\alpha \in a_{\iota}$. As $\kappa_{j}>\sigma$, we infer that $\phi \in \Phi_{j}$, so we may fix $\gamma<\kappa$ such that $\phi=\phi_{j}^{\gamma}$. Now, by the choice of $j$, let us pick $b \in \mathcal{B}$ with $\operatorname{dom}(\phi)<b$ such that $e[\{j\} \times b]=\{\gamma\}$.

For every $i<j$ and $\beta \in b$, let $x_{i}^{\beta}:=\operatorname{dom}\left(\phi_{i}^{e(i, \beta)}\right)$, so that $\left|x_{i}^{\beta}\right|=\kappa_{i}$. Next, set $x:=\bigcup\left\{x_{i}^{\beta} \mid i<j, \beta \in b\right\}$, so that $|x|<\kappa_{j}$. In particular, we may fix $\iota<\kappa_{j}$ such that $a_{\iota} \cap x=\emptyset$. Now, let $(\alpha, \beta) \in a_{\iota} \times b$ be arbitrary. As $e(j, \beta)=\gamma$, we infer that $\phi_{j}^{e(j, \beta)}=\phi$. In particular, $\alpha \in a_{\iota} \subseteq \operatorname{dom}\left(\phi_{j}^{e(j, \beta)}\right)$. Recalling that $\alpha \notin x$, it follows that $\min \left\{i<\kappa \mid \alpha \in \operatorname{dom}\left(\phi_{i}^{e(i, \beta)}\right)\right\}=j$, and hence

$$
c(\alpha, \beta)=\phi_{j}^{e(j, \beta)}(\alpha)=\phi(\alpha)=\delta_{\iota} .
$$

Altogether, $c\left[a_{\iota} \times b\right]=\left\{\delta_{\iota}\right\}$, contradicting the choice of $\delta_{\iota}$.
This completes the proof.
Theorem 5.3. Suppose that $\kappa$ is an inaccessible cardinal and $\boxtimes^{-}(\kappa)$ and $\diamond(\kappa)$ both hold. Then $\operatorname{Pr}_{1}(\kappa, \kappa \circledast \kappa / 1 \circledast \kappa, \kappa, \kappa)$ holds, as well.

Proof. As $\diamond(\kappa)$ holds, let us fix a sequence $\left\langle f_{\delta} \mid \delta<\kappa\right\rangle$ such that, for every $\delta<\kappa, f_{\delta}$ is a function from to $\delta$ to $\delta$, and, for every function $f: \kappa \rightarrow \kappa$, the set $G(f):=\left\{\delta<\kappa|f| \delta=f_{\delta}\right\}$ is stationary. Let $\vec{C}$ be a $\boxtimes^{-}(\kappa)$-sequence, and we shall walk along $\vec{C}$. Now, pick any coloring $d:[\kappa]^{2} \rightarrow \kappa$ such that, for all $\eta, \beta<\kappa$ with $\eta+1<\beta, d(\eta, \beta)=f_{\operatorname{Tr}(\eta+1, \beta)\left(\eta_{\eta+1, \beta)}\right)}(\eta)$. Fix a regressive surjection $\psi: \kappa \rightarrow \kappa$
such that the preimage of any singleton is cofinal in $\kappa$. Define an auxiliary coloring $e:[\kappa]^{2} \rightarrow \kappa$ via $e(j, \beta):=d(\psi(j), \beta)$.

Claim 5.3.1. For every $\sigma<\kappa$ and every pairwise disjoint subfamily $\mathcal{B} \subseteq[\kappa]^{\sigma}$ of size $\kappa$, there are cofinally many $j<\kappa$ such that, for every $\gamma<\kappa$, there are $\kappa$ many $b \in \mathcal{B}$ with $e[\{j\} \times b]=\{\gamma\}$.

Proof. Let $\sigma<\kappa$ and let $\mathcal{B}$ be a pairwise disjoint subfamily of $[\kappa]^{\sigma}$ of size $\kappa$. It suffices to prove that there exists $\eta<\kappa$ such that, for every $\gamma<\kappa$, there are $\kappa$ many $b \in \mathcal{B}$ with $d[\{\eta\} \times b]=\{\gamma\}$. Towards a contradiction, suppose that this is not the case, and fix a function $f: \kappa \rightarrow \kappa$ such that, for every $\eta<\kappa$,

$$
\mathcal{B}_{\eta}:=\{b \in \mathcal{B} \mid d[\{\eta\} \times b]=\{f(\eta)\}\}
$$

has size $<\kappa$.
By Lemma $3.16(2), \chi_{2}(\vec{C}, \kappa)=\kappa>\sigma$, so since $G(f)$ is stationary, we may fix $\delta \in G(f), \eta<\delta$ and $b \in \mathcal{B} \backslash \mathcal{B}_{\eta}$ such that, for every $\beta \in b, \lambda(\delta, \beta)=\eta$ and $\rho_{2}(\delta, \beta)=\eta_{\delta, \beta}$.

For each $\beta \in b$, by appealing to Lemma 3.12 with $\alpha:=\eta+1$, we get that

$$
d(\eta, \beta)=f_{\operatorname{Tr}(\eta+1, \beta)\left(\eta_{\eta+1, \beta)}\right.}(\eta)=f_{\delta}(\eta)=f(\eta)
$$

So $b \in \mathcal{B}_{\eta}$, contradicting its choice.
Now continue as in the proof of Theorem 5.2, with the preceding playing the role of Claim 5.2.1.

Remark 5.4. The preceding proof actually shows that if $\mathrm{P}^{\bullet}(\kappa, 2, \sqsubseteq, 1,\{\kappa\}, 2)$ holds (see [BR21, Definition 5.9 and Proposition 5.10]), then so does $\operatorname{Pr}_{1}(\kappa, \kappa \circledast \kappa / 1 \circledast \kappa, \kappa, \kappa)$.

The next corollary yields Clauses (1)-(3) of Theorem B. It in particular provides sufficient conditions for $\operatorname{Pr}_{1}(\kappa, \kappa, \kappa, \chi)$ to hold for the extreme case of $\chi:=\kappa .^{4}$
Corollary 5.5. Suppose that $\kappa$ is an inaccessible cardinal.
(1) If $\square(\kappa)$ and $\diamond(S)$ both hold for some stationary $S \subseteq \kappa$ that does not reflect at regulars, then so does $\mathrm{P} \ell_{1}(\kappa, \kappa, \kappa)$;
(2) If $\square(\kappa)$ and $\diamond^{*}(\kappa)$ both hold, then so does $\mathrm{P} \ell_{1}(\kappa, \kappa, \kappa)$;
(3) If $\boxtimes^{-}(\kappa)$ and $\diamond(\kappa)$ both hold, then so does $\mathrm{P} \ell_{2}(\kappa, \kappa, \kappa)$.

Proof. To prove Clauses (1) and (2), let $d_{0}$ be given by Fact 5.1. Next, let $d_{1}$ be given by Lemma 3.18(2), using Lemma 3.16(1). Now, appeal to Theorem 5.2, and finally appeal to Corollary 4.8.
(3) By Theorem 5.3 and Corollary 4.8.

## 6. SUCCESSORS OF REGULAR CARDINALS

Lemma 6.1. Suppose that $\mu$ is an infinite regular cardinal. Then there exists a sequence $\vec{f}=\left\langle f_{j} \mid j<\mu\right\rangle$ of functions from $\mu^{+}$to $\mu^{+}$such that, for every pairwise disjoint subfamily $\mathcal{B} \subseteq\left[\mu^{+}\right]^{<\mu}$ of size $\mu^{+}$, for every $\gamma<\mu^{+}$, there exist $j<\mu$ and $b \in \mathcal{B}$ such that $f_{j}[b]=\{\gamma\}$.

[^4]Proof. Fix a surjection $g: \mu^{+} \rightarrow \mu^{+}$such that the preimage of any singleton is cofinal in $\mu^{+}$. As $\mu$ is regular, using [Tod07, Lemma 6.25], we may fix a function $p:\left[\mu^{+}\right]^{2} \rightarrow \mu$ having injective and $\mu$-coherent fibers; the latter means that $\mid\{\alpha<\beta \mid$ $\left.p(\alpha, \beta) \neq p\left(\alpha, \beta^{\prime}\right)\right\} \mid<\mu$ for all $\beta<\beta^{\prime}<\mu^{+}$. Now, for every $j<\mu$, define a function $f_{j}: \mu^{+} \rightarrow \mu^{+}$via:

$$
f_{j}(\beta):= \begin{cases}0, & \text { if } j \notin\{p(\alpha, \beta) \mid \alpha<\beta\} \\ g(\alpha) & \text { if } p(\alpha, \beta)=j\end{cases}
$$

Let $\mathcal{B}$ be a pairwise disjoint subfamily of $\left[\mu^{+}\right]^{<\mu}$ of size $\mu^{+}$, and let $\gamma<\mu^{+}$be a prescribed color. Find $\delta \in E_{\mu}^{\mu^{+}}$such that $A:=\{\alpha<\delta \mid g(\alpha)=\gamma\}$ is cofinal in $\delta$. Pick $b \in \mathcal{B}$ with $\min (b)>\delta$. As $p$ is $\mu$-coherent and $|b|<\mu$, we may find some $\Delta \in[\delta]^{<\mu}$ such that for all $\beta, \beta^{\prime} \in b$ and $\alpha \in \delta \backslash \Delta, p(\alpha, \beta)=p\left(\alpha, \beta^{\prime}\right)$. Now, pick $\alpha \in A \backslash \Delta$. Let $j$ denote the unique element of the singleton $\{p(\alpha, \beta) \mid \beta \in b\}$. Then, for all $\beta \in b, f_{j}(\beta)=g(\alpha)=\gamma$, as sought.
Definition 6.2 ([BGKT78]). ${ }^{\bullet}\left(\mu^{+}\right)$asserts the existence of a sequence $\left\langle X_{\gamma}\right| \gamma<$ $\left.\mu^{+}\right\rangle$such that, for every $X \in\left[\mu^{+}\right]^{\mu^{+}}$, there exists $\gamma<\mu^{+}$such that $X_{\gamma} \in[X]^{\mu}$.

Theorem 6.3. Suppose that $\mu$ is a regular uncountable cardinal, and $\boldsymbol{\varphi}\left(\mu^{+}\right)$holds. Then $\operatorname{Pr}_{1}\left(\mu^{+}, \mu^{+} \circledast \mu^{+} / 1 \circledast \mu^{+}, \mu^{+}, \mu\right)$ holds.

If $2^{\mu}=\mu^{+}$, then moreover $\operatorname{Pr}_{1}\left(\mu^{+}, \mu \circledast \mu^{+} / 1 \circledast \mu^{+}, \mu^{+}, \mu\right)$ holds.
Proof. For an ordinal $\eta$, let $\Phi_{\eta}$ denote the collection of all sequences $\left\langle\left(a_{\iota}, \delta_{\iota}\right) \mid \iota<\eta\right\rangle$ such that $\left\langle a_{\iota} \mid \iota<\eta\right\rangle$ is a sequence of pairwise disjoint elements of $\left[\mu^{+}\right]^{<\mu} \backslash\{\emptyset\}$, and $\left\langle\delta_{\iota}\right| \iota\langle\eta\rangle$ is a sequence of elements of $\mu^{+}$. Define an ordering $\unlhd$ of $\bigcup_{\eta<\mu^{+}} \Phi_{\eta}$ by letting

$$
\left\langle\left(a_{\iota}, \delta_{\iota}\right) \mid \iota<\eta\right\rangle \unlhd\left\langle\left(a_{\iota}^{\prime}, \delta_{\iota}^{\prime}\right) \mid \iota<\eta^{\prime}\right\rangle
$$

iff for every $\iota<\eta$, there exists $\iota^{\prime}<\eta^{\prime}$ such that $a_{\iota} \supseteq a_{\iota^{\prime}}$ and $\delta_{\iota}=\delta_{\iota^{\prime}}$.
Claim 6.3.1. There exists a sequence $\left\langle\phi_{\gamma} \mid \gamma<\mu^{+}\right\rangle$of elements of $\Phi_{\mu}$ such that for every $\phi \in \Phi_{\mu^{+}}$, there exists $\gamma<\mu^{+}$with $\phi_{\gamma} \unlhd \phi$.
Proof. For every $\beta<\mu^{+}$, fix a surjection $\psi_{\beta}: \mu \rightarrow \beta+1$. Then let $\mathcal{A}:=\left\{\psi_{\beta}[\epsilon] \backslash \alpha \mid\right.$ $\left.\epsilon<\mu, \alpha<\beta<\mu^{+}\right\}$. Evidently, $|\mathcal{A}|=\mu^{+}$, so, as ${ }^{\bullet}\left(\mu^{+}\right)$holds, we may fix a sequence $\left\langle X_{\gamma} \mid \gamma<\mu^{+}\right\rangle$with the property that, for every $X \in\left[\mathcal{A} \times \mu^{+}\right]^{\mu^{+}}$, there exists $\gamma<\mu^{+}$such that $X_{\gamma} \in[X]^{\mu}$. Now, pick a sequence $\left\langle\phi_{\gamma} \mid \gamma<\mu^{+}\right\rangle$of elements of $\Phi_{\mu}$ with the property that, for every $\gamma<\mu^{+}$, if there exists $\phi \in \Phi_{\mu}$ such that $\operatorname{Im}(\phi) \subseteq X_{\gamma}$, then $\phi_{\gamma}$ is such a $\phi$.

To see that $\left\langle\phi_{\gamma} \mid \gamma<\mu^{+}\right\rangle$is as sought, let $\left\langle\left(a_{\iota}, \delta_{\iota}\right) \mid \iota<\mu^{+}\right\rangle$be an arbitrary element of $\Phi_{\mu^{+}}$. For every $\iota<\mu^{+}$, let $\alpha_{\iota}:=\min \left(a_{\iota}\right), \beta_{\iota}:=\operatorname{ssup}\left(a_{\iota}\right)$, and $\epsilon_{\iota}:=$ $\operatorname{ssup}\left(\psi_{\beta_{\iota}}^{-1}\left[a_{\iota}\right]\right)$. Clearly, $\overline{a_{\iota}}:=\psi_{\beta_{\iota}}\left[\epsilon_{\iota}\right] \backslash \alpha_{\iota}$ is an element of $\mathcal{A}$ satisfying $\overline{a_{\iota}} \supseteq a_{\iota}$ and $\min \left(\overline{a_{\iota}}\right)=\min \left(a_{\iota}\right)$. Recalling that $\left\langle a_{\iota} \mid \iota<\mu^{+}\right\rangle$is a sequence of pairwise disjoint elements of $\left[\mu^{+}\right]^{<\mu} \backslash\{\emptyset\}$, it follows that we may fix a sparse enough $I \in\left[\mu^{+}\right]^{\mu^{+}}$ such that $\left\langle\overline{a_{\iota}} \mid \iota \in I\right\rangle$ is a <-increasing sequence of elements of $\left[\mu^{+}\right]^{<\mu} \backslash\{\emptyset\}$. Consequently, $X:=\left\{\left(\overline{a_{\iota}}, \delta_{\iota}\right) \mid \iota \in I\right\}$ is in $\left[\mathcal{A} \times \mu^{+}\right]^{\mu^{+}}$. Now, pick $\gamma<\mu^{+}$such that $X_{\gamma} \in[X]^{\mu}$. Evidently, $\phi_{\gamma} \unlhd \phi$.

If $2^{\mu}=\mu^{+}$, then we fix an injective enumeration $\left\langle\phi_{\gamma} \mid \gamma<\mu^{+}\right\rangle$of $\Phi_{\mu}$. Otherwise, we let $\left\langle\phi_{\gamma} \mid \gamma<\mu^{+}\right\rangle$be given by the preceding claim. Let $\vec{f}$ be given by Lemma 6.1.

Fix a surjection $\psi: \mu \rightarrow \mu$ such that the preimage of any singleton is stationary. For every $\beta<\mu^{+}$and $j<\mu$, write $\left\langle\left(a_{\iota}^{j, \beta}, \delta_{\iota}^{j, \beta}\right) \mid \iota<\mu\right\rangle$ for $\phi_{f_{\psi(j)}(\beta)}$.

Let $\beta<\mu^{+}$. We now recursively construct a strictly increasing sequence $\left\langle\iota^{j, \beta}\right|$ $j<\mu\rangle$ of ordinals below $\mu$. Suppose that $j<\mu$ and that $\left\langle\iota^{i, \beta} \mid i<j\right\rangle$ has already been defined. If there exists $\iota<\mu$ such that:

- $a_{\iota}^{j, \beta} \subseteq \beta \backslash \bigcup_{i<j} a_{\iota^{i, \beta}}^{i, \beta}$, and
- $\iota \geq \sup _{i<j}\left(\iota^{i, \beta}+1\right)$,
then let $\iota^{j, \beta}$ denote the least such $\iota$. Otherwise, just let $\iota^{j, \beta}:=\sup _{i<j}\left(\iota^{i, \beta}+1\right)$.
Finally, we define a coloring $c:\left[\mu^{+}\right]^{2} \rightarrow \mu^{+}$by letting for all $\alpha<\beta<\mu^{+}$:

$$
c(\alpha, \beta):= \begin{cases}0 & \text { if } \alpha \notin \bigcup_{i<\mu} a_{\iota^{i, \beta}}^{i, \beta} ; \\ \delta_{\iota^{j}, \beta}^{j, \beta} & \text { if } j=\min \left\{i<\mu \mid \alpha \in a_{\iota^{i, \beta}}^{i, \beta}\right\}\end{cases}
$$

Assuming $2^{\mu}=\mu^{+}$, to see that $c$ witnesses $\operatorname{Pr}_{1}\left(\mu^{+}, \mu \circledast \mu^{+} / 1 \circledast \mu^{+}, \mu^{+}, \mu\right)$, fix $\sigma<\mu$ and pairwise disjoint subfamilies $\mathcal{A}, \mathcal{B}$ of $\left[\mu^{+}\right]^{\sigma}$ such that $|\mathcal{A}|=\mu$ and $|\mathcal{B}|=\mu^{+}$. Assuming $2^{\mu}>\mu^{+}$, to see that $c$ witnesses $\operatorname{Pr}_{1}\left(\mu^{+}, \mu^{+} \circledast \mu^{+} / 1 \circledast \mu^{+}, \mu^{+}, \mu\right)$, fix $\sigma<\mu$ and pairwise disjoint subfamilies $\mathcal{A}, \mathcal{B}$ of $\left[\mu^{+}\right]^{\sigma}$ such that $|\mathcal{A}|=|\mathcal{B}|=\mu^{+}$.

Towards a contradiction, suppose that, for every $a \in \mathcal{A}$, there exists some $\delta(a)<$ $\mu^{+}$such that, for every $b \in \mathcal{B}$ with $a<b, c[a \times b] \neq\{\delta(a)\}$. Fix $\phi \in \Phi_{|\mathcal{A}|}$ such that $\operatorname{Im}(\phi)=\{(a, \delta(a)) \mid a \in \mathcal{A}\}$.

- If $2^{\mu}=\mu^{+}$, then $\phi \in \Phi_{\mu}$, and we may fix $\gamma<\mu^{+}$such that $\phi=\phi_{\gamma}$. In particular, $\phi_{\gamma} \unlhd \phi$.
- If $2^{\mu}>\mu^{+}$, then $\phi \in \Phi_{\mu^{+}}$, so we may fix $\gamma<\mu^{+}$with $\phi_{\gamma} \unlhd \phi$.

Write $\phi_{\gamma}$ as $\left\langle\left(a_{\iota}, \delta_{\iota}\right) \mid \iota<\mu\right\rangle$. Set $\epsilon:=\operatorname{ssup}\left(\bigcup_{\iota<\mu} a_{\iota}\right)$, and then fix a bijection $\pi: \epsilon \leftrightarrow \mu$.

Claim 6.3.2. (1) $D:=\left\{j<\mu \mid\left\{\iota<\mu \mid \pi\left[a_{\iota}\right] \cap j \neq \emptyset\right\} \subseteq j\right\}$ is a club in $\mu$;
(2) For every $\beta<\mu^{+}, C_{\beta}:=\left\{j<\mu \mid \pi\left[\bigcup_{i<j} a_{\iota^{i, \beta}}^{i, \beta}\right] \subseteq j=\sup _{i<j}\left(\iota^{i, \beta}+1\right)\right\}$ is a club in $\mu$.

Proof. (1) Define a function $g_{0}: \mu \rightarrow \mu$ via $g_{0}(i):=\sup \left\{\iota<\mu \mid i \in \pi\left[a_{\iota}\right]\right\}$. As the elements of $\left\langle a_{\iota} \mid \iota<\mu\right\rangle$ are pairwise disjoint, $g_{0}$ is well-defined. Clearly, $D$ coincides with the club $\left\{j<\mu \mid g_{0}[j] \subseteq j\right\}$.
(2) Let $\beta<\mu^{+}$. It is clear that $C_{\beta}$ is closed. To see it is unbounded, define two functions $g_{1}, g_{2}: \mu \rightarrow \mu$ via $g_{1}(i):=\sup \left(\pi\left[a_{\iota^{i, \beta}}^{i, \beta}\right]\right)$ and $g_{2}(i):=\iota^{i, \beta}+1$. As the elements of $\left\langle\pi\left[a_{\iota}\right] \mid \iota<\mu\right\rangle$ are elements of $[\mu]^{<\mu}, g_{1}$ is well-defined. Recalling that the sequence $\left\langle\iota^{i, \beta} \mid i<\mu\right\rangle$ is strictly increasing, it follows that $C_{\beta}$ covers the intersection of the clubs $\left\{j<\mu \mid g_{1}[j] \subseteq j\right\}$ and $\left\{j<\mu \mid g_{2}[j] \subseteq j\right\}$.

Next, by the choice of $\vec{f}$, fix $j^{*}<\mu$ and $b \in \mathcal{B}$ with $\min (b)>\epsilon$ such that $f_{j^{*}}[b]=\{\gamma\}$. Then, by the choice of $\psi$, pick $j \in D \cap \bigcap_{\beta \in b} C_{\beta}$ such that $\psi(j)=j^{*}$. Consequently, for all $\beta \in b, \phi_{f_{\psi(j)}(\beta)}=\phi_{\gamma}$, meaning that

$$
\left\langle\left(a_{\iota}^{j, \beta}, \delta_{\iota}^{j, \beta}\right) \mid \iota<\mu\right\rangle=\left\langle\left(a_{\iota}, \delta_{\iota}\right) \mid \iota<\mu\right\rangle,
$$

and in particular, $\left(\bigcup_{\iota<\mu} a_{\iota}^{j, \beta}\right) \subseteq \epsilon=\operatorname{dom}(\pi)$.
Now, let $\beta \in b$; we have:
(1) $\left\{\iota<\mu \mid \pi\left[a_{\iota}\right] \cap j \neq \emptyset\right\} \subseteq j$;
(2) $\pi\left[\bigcup_{i<j} a_{\iota^{i, \beta}}^{i, \beta}\right] \subseteq j$;
(3) $\sup _{i<j}\left(\iota^{i, \beta}+1\right)=j$.

By Clause (1), $\pi\left[a_{j}\right] \cap j=\emptyset$. Together with Clause (2), it thus follows that $a_{j} \subseteq \epsilon \backslash \bigcup_{i<j} a_{\iota^{i, \beta}}^{i, \beta} \subseteq \beta \backslash \bigcup_{i<j} a_{\iota^{i, \beta}}^{i, \beta}$. So, by Clause (3) and the definition of $\iota^{j, \beta}$, we infer that $\iota^{j, \beta}=j$. Altogether, for all $\alpha \in a_{j}, \min \left\{i<\mu \mid \alpha \in a_{\iota^{i, \beta}}^{i, \beta}\right\}=j$, and hence $c(\alpha, \beta)=\delta_{j}^{j, \beta}=\delta_{j}$.

Finally, since $\phi_{\gamma} \unlhd \phi$ and $\operatorname{Im}(\phi)=\{(a, \delta(a)) \mid a \in \mathcal{A}\}$, we may find some $a \in \mathcal{A}$ such that $a_{j} \supseteq a$ and $\delta_{j}=\delta(a)$. Then, $c[a \times b] \subseteq c\left[a_{j} \times b\right]=\left\{\delta_{j}\right\}=\{\delta(a)\}$. This is a contradiction.
Corollary 6.4. Suppose that $\mu$ is an infinite regular cardinal and ${ }^{\bullet}\left(\mu^{+}\right)$holds. Then $\mathrm{P} \ell_{2}\left(\mu^{+}, E_{\mu}^{\mu^{+}}, \mu\right)$ holds, as well.

Proof. If $\mu=\aleph_{0}$, then a straightforward adjustment of Galvin's proof from [Gal80] shows that $\dagger\left(\aleph_{1}\right)$ implies $\operatorname{Pr}_{1}\left(\aleph_{1}, \aleph_{1}, \aleph_{1}, \aleph_{0}\right)$. Now, appeal to Corollary 4.4.

- If $\mu>\aleph_{0}$, then by Theorem 6.3, in particular, $\operatorname{Pr}_{1}\left(\mu^{+}, \mu^{+}, \mu^{+}, \mu\right)$ holds. Now appeal to Lemma 4.2 and Theorem 4.3.


## 7. From a proxy Principle

As recommended by anonymous referee X , we point out that Clause (2) of the upcoming theorem in particular implies that for every $\partial<\kappa$ and $\delta \in \operatorname{acc}\left(C_{\check{\delta}}\right) \cap \Delta$, there exists $\epsilon \in C_{\delta}$ such that $C_{\check{\delta}} \cap[\epsilon, \delta)=C_{\delta} \cap[\epsilon, \delta)$, and Clause (3) in particular implies that $\Delta$ is a stationary set.

Theorem 7.1. Suppose that $\chi \leq \kappa, \Delta \subseteq \kappa$, and $\left\langle h_{\delta}: C_{\delta} \rightarrow \kappa \mid \delta<\kappa\right\rangle$ is a sequence satisfying the following:
(1) $\vec{C}:=\left\langle C_{\delta} \mid \delta<\kappa\right\rangle$ is a $C$-sequence;
(2) For every $\delta<\kappa$ and $\delta \in \operatorname{acc}\left(C_{\check{\delta}}\right) \cap \Delta$, there exists $\epsilon \in C_{\delta}$ such that $h_{\text {Ø}} \upharpoonright[\epsilon, \delta)=h_{\delta} \upharpoonright[\epsilon, \delta) ;$
(3) For every $\sigma<\chi$ and every club $D \subseteq \kappa$, there exists $\delta \in \Delta \cap E_{>\sigma}^{\kappa}$ such that $\sup \left(\operatorname{nacc}\left(C_{\delta}\right) \cap D\right)=\delta ;$
(4) For every $\sigma<\chi$, every pairwise disjoint subfamily $\mathcal{A} \subseteq[\kappa]^{\sigma}$ of size $\kappa$ and every $\tau<\kappa$, there exists $\delta \in \Delta \cap \operatorname{acc}(\kappa)$ such that

$$
\sup \left\{\min (x) \mid x \in \mathcal{A} \cap \mathcal{P}\left(C_{\delta}\right) \& h_{\delta}[x]=\{\tau\}\right\}=\delta
$$

Then $\mathrm{P} \ell_{1}(\kappa, \kappa, \chi)$ holds.
Proof. We may assume that $C_{\delta+1}=\{\delta\}$ for every $\delta<\kappa$. We shall now walk along $\vec{C}$. Fix a bijection $\pi: \kappa \leftrightarrow \kappa \times \kappa$. Define a transformation $\mathbf{t}:[\kappa]^{2} \rightarrow[\kappa]^{3}$, letting $\mathbf{t}(\alpha, \beta):=(\tau, \gamma, \delta)$ provided that the following conditions are met:

- $(\eta, \tau):=\pi\left(h_{\min (\operatorname{Im}(\operatorname{tr}(\alpha, \beta)))}(\alpha)\right)$ and $\max \{\eta+1, \tau\}<\gamma$,
- $\delta=\operatorname{Tr}(\alpha, \beta)\left(\eta_{\alpha, \beta}\right)$,
- $\gamma=\operatorname{Tr}(\eta+1, \alpha)\left(\eta_{\eta+1, \alpha}\right)$.

Otherwise, let $\mathbf{t}(\alpha, \beta):=(0, \alpha, \beta)$.
We verify that this works. Given $\sigma<\chi$ and a pairwise disjoint subfamily $\mathcal{A} \subseteq[\kappa]^{\sigma}$ of size $\kappa$, we shall find a stationary subset $S \subseteq \kappa$ witnessing the definition of $\mathrm{P} \ell_{1}(\kappa, \kappa, \chi)$.
Claim 7.1.1. There exist a stationary $\Gamma \subseteq \kappa$, a sequence $\left\langle\mathcal{A}_{\gamma} \mid \gamma \in \Gamma\right\rangle$ and an ordinal $\eta<\kappa$ such that, for all $\gamma \in \Gamma, \mathcal{A}_{\gamma} \in[\mathcal{A}]^{\kappa}$ and, for all $x \in \mathcal{A}_{\gamma}$ and $\alpha \in x$ :

- $\lambda(\gamma, \alpha)=\eta<\gamma<\alpha$;
- $\eta_{\gamma, \alpha}=\rho_{2}(\gamma, \alpha)$.

Proof. The proof uses Clauses (2) and (3) and is almost identical to that of Lemma 3.17.

Let $\left\langle\mathcal{A}_{\gamma} \mid \gamma \in \Gamma\right\rangle$ and $\eta$ be given by the preceding claim. By Clause (4), for every $\tau<\kappa$ and $\gamma \in \Gamma$, we may let $\zeta_{\tau, \gamma}$ denote the least $\zeta \in \Delta \cap \operatorname{acc}(\kappa)$ which satisfies:

$$
\sup \left\{\min (x) \mid x \in \mathcal{A}_{\gamma} \cap \mathcal{P}\left(C_{\zeta}\right) \& h_{\zeta}[x]=\left\{\pi^{-1}(\eta, \tau)\right\}\right\}=\zeta
$$

Fix a club $E \subseteq \operatorname{acc}(\kappa)$ with the property that, for every $(\tau, \gamma, \delta) \in \kappa \circledast \Gamma \circledast E$, $\zeta_{\tau, \gamma}<\delta$. We claim that $S:=\Gamma \cap E$ is as sought. To see this, let $(\tau, \gamma, \delta) \in \kappa \circledast S \circledast S$ be arbitrary.

Let $\zeta:=\zeta_{\tau, \gamma}$, so that $\zeta<\delta$. Using Clause (2) and Fact 3.10(2), fix $\epsilon \in C_{\zeta}$ such that $h_{\boldsymbol{\partial}_{\zeta, \delta}} \upharpoonright[\epsilon, \zeta)=h_{\zeta} \upharpoonright[\epsilon, \zeta)$. Using Fact $3.10(1)$, pick $a \in \mathcal{A}_{\gamma} \cap \mathcal{P}\left(C_{\zeta}\right)$ with $\min (a)>\max \left\{\lambda\left(\widetilde{\partial}_{\zeta, \delta}, \delta\right), \epsilon\right\}$ such that $\left(\pi \circ h_{\zeta}\right)[a]=\{(\eta, \tau)\}$. Pick $b \in \mathcal{A}_{\delta}$ arbitrarily.

Claim 7.1.2. Let $(\alpha, \beta) \in a \times b$. Then $\mathbf{t}(\alpha, \beta)=(\tau, \gamma, \delta)$.
Proof. As $b \in \mathcal{A}_{\delta}$ and $\beta \in b, \lambda(\delta, \beta)=\eta<\gamma<\alpha<\zeta<\delta<\beta$. So, by Fact 3.7,

$$
\operatorname{tr}(\alpha, \beta)=\operatorname{tr}(\delta, \beta)^{\wedge} \operatorname{tr}(\alpha, \delta)
$$

It thus follows from $\eta_{\delta, \beta}=\rho_{2}(\delta, \beta)$ that $\operatorname{Tr}(\alpha, \beta)\left(\eta_{\alpha, \beta}\right)=\delta$.
Next, since $\lambda\left(\mathrm{O}_{\zeta, \delta}, \delta\right)<\min (a) \leq \alpha<\zeta<\delta$, we have

$$
\operatorname{tr}(\alpha, \delta)=\operatorname{tr}\left(\check{\mathrm{O}}_{\zeta, \delta}, \delta\right)^{\wedge} \operatorname{tr}\left(\alpha, \check{\partial}_{\zeta, \delta}\right)
$$

As $\alpha \in C_{\zeta} \cap[\epsilon, \zeta)$ and $\pi\left(h_{\zeta}(\alpha)\right)=(\eta, \tau)$, we infer that $\alpha \in C_{\check{\partial}_{\zeta, \delta}}$ and $\pi\left(h_{\widetilde{ð}_{\zeta, \delta}}(\alpha)\right)=$ $(\eta, \tau)$. Altogether, $\min (\operatorname{Im}(\operatorname{tr}(\alpha, \beta)))=\min (\operatorname{Im}(\alpha, \delta)))=\partial_{\zeta, \delta}$, and

$$
\pi\left(h_{\min (\operatorname{Im}(\operatorname{tr}(\alpha, \beta)))}(\alpha)\right)=(\eta, \tau)
$$

Finally, since $\lambda(\gamma, \alpha)=\eta<\eta+1<\gamma<\alpha, \operatorname{tr}(\eta+1, \alpha)=\operatorname{tr}(\gamma, \alpha)^{\wedge} \operatorname{tr}(\eta+1, \gamma)$, and as $\eta_{\gamma, \alpha}=\rho_{2}(\gamma, \alpha)$, we infer that $\operatorname{Tr}(\eta+1, \alpha)\left(\eta_{\eta+1, \alpha}\right)=\gamma$.

This completes the proof.
When reading the statement of the next theorem, the reader may want to recall Definition 3.4.

Theorem 7.2. Suppose that $\kappa=\mu^{+}$for some infinite regular cardinal $\mu$ and $\mathrm{P}^{-}\left(\kappa, \kappa^{+}, \sqsubseteq^{*}, 1,\left\{E_{\mu}^{\kappa}\right\}, 2\right)$ holds. Then $\mathrm{P} \ell_{2}(\kappa, \kappa, \mu)$ holds, as well.

Proof. We shall establish that $\mathrm{P} \ell_{1}(\kappa, \kappa, \mu)$ holds, using Theorem 7.1. The pumping up to $\mathrm{P} \ell_{2}(\kappa, \kappa, \mu)$ uses Theorem $4.7(2)$ as follows. First, by Lemma 3.17, $\mathrm{P}^{-}\left(\kappa, \kappa^{+}, \sqsubseteq^{*}, 1,\left\{E_{\mu}^{\kappa}\right\}, 2\right)$ gives rise to a $C$-sequence $\vec{C}$ such that $\chi_{2}(\vec{C}, \kappa) \geq \mu$. Second, by [RZ21, Lemma 2.18(2)], $\mathrm{P} \ell_{1}(\kappa, \kappa, \mu)$ implies $\operatorname{Pr}_{1}(\kappa, \kappa, \kappa, \mu)$, and by Lemma 4.2, $\operatorname{Pr}_{1}(\kappa, \kappa, \kappa, \mu)$ implies $\operatorname{Pr}_{1}^{+}(\kappa, \kappa, \mu)$.

For every nonzero $\delta<\kappa$, fix a surjection $\psi_{\delta}: \mu \rightarrow \delta$. Then, let

- $\mathcal{X}:=[\kappa]^{<\omega} \cup\left\{\psi_{\delta}[\eta] \backslash \alpha \mid \alpha<\delta<\kappa, \eta<\mu\right\}$, and
- $\mathcal{F}:=\{\operatorname{cl}(x) \times\{j\} \mid x \in \mathcal{X}, j<\kappa\}$.

Fix an enumeration (possibly, with repetitions) $\left\langle f_{\gamma} \mid \gamma<\kappa\right\rangle$ of $\mathcal{F}$ such that, for every $\gamma<\kappa$, $\operatorname{dom}\left(f_{\gamma}\right) \subseteq \gamma$. For every $(\beta, \gamma) \in[\kappa]^{2}$, set $f_{\gamma}^{\beta}:=f_{\gamma} \upharpoonright(\beta, \gamma)$, so that $f_{\gamma}^{\beta}$ is a constant function, and $\operatorname{dom}\left(f_{\gamma}^{\beta}\right)$ is a closed set of ordinals of order-type $<\mu$.

Let $\vec{D}=\left\langle D_{\delta} \mid \delta<\kappa\right\rangle$ and $\Delta \subseteq E_{\mu}^{\kappa}$ be witnesses to $\mathrm{P}^{-}\left(\kappa, \kappa^{+}, \sqsubseteq^{*}, 1,\left\{E_{\mu}^{\kappa}\right\}, 2\right)$. We now construct a sequence $\left\langle h_{\delta}: C_{\delta} \rightarrow \kappa \mid \delta<\kappa\right\rangle$ that satisfies the requirements of Theorem 7.1, with $\chi:=\mu$.

For each $\delta \in E_{<\mu}^{\kappa}$, fix a closed subset $C_{\delta} \subseteq \delta$ with $\sup \left(C_{\delta}\right)=\sup (\delta)$ and $\operatorname{otp}\left(C_{\delta}\right)=\operatorname{cf}(\delta)$, and then let $h_{\delta}:=C_{\delta} \times\{0\}$. Next, for each $\delta \in E_{\mu}^{\kappa}$, let

$$
h_{\delta}:=\left(D_{\delta} \times\{0\}\right) \cup \bigcup\left\{f_{\gamma}^{\beta} \mid \beta \in D_{\delta}, \gamma=\min \left(D_{\delta} \backslash(\beta+1)\right)\right\}
$$

so that $C_{\delta}:=\operatorname{dom}\left(h_{\delta}\right)$ is a club in $\delta$ and $\operatorname{acc}\left(C_{\delta}\right) \cap E_{\mu}^{\kappa}=\operatorname{acc}\left(D_{\delta}\right) \cap E_{\mu}^{\kappa}$.
Claim 7.2.1. Suppose $\delta<\kappa$ and $\bar{\delta} \in \operatorname{acc}\left(C_{\delta}\right) \cap \Delta$. Then there exists $\epsilon \in C_{\bar{\delta}}$ such that $h_{\delta} \upharpoonright[\epsilon, \bar{\delta})=h_{\bar{\delta}} \upharpoonright[\epsilon, \bar{\delta})$.

Proof. As $\bar{\delta} \in \operatorname{acc}\left(C_{\delta}\right) \cap \Delta \subseteq E_{\mu}^{\kappa}$, it follows that otp $\left(C_{\delta}\right)>\mu$, so that $\operatorname{cf}(\delta)=\mu$. As the domain of any function $f_{\gamma}^{\beta}$ is a set of ordinals of size $<\mu$, every pair of successive ordinals in $C_{\delta}$ have less than $\mu$ many elements in between them lying in $D_{\delta}$. Consequently, $\bar{\delta} \in \operatorname{acc}\left(D_{\delta}\right)$. In particular, $\varepsilon:=\sup \left(\left(D_{\delta} \cap \bar{\delta}\right) \triangle D_{\bar{\delta}}\right)$ is $<\bar{\delta}$. Let $\epsilon:=\min \left(D_{\bar{\delta}} \backslash(\varepsilon+1)\right)$. Then $D_{\delta} \cap[\epsilon, \bar{\delta})=D_{\bar{\delta}} \cap[\epsilon, \bar{\delta})$, and it follows that $h_{\delta} \upharpoonright[\epsilon, \bar{\delta})=h_{\bar{\delta}} \upharpoonright[\epsilon, \bar{\delta})$.

Claim 7.2.2. Let $A \subseteq \kappa$ be cofinal. Then there exists $\delta \in \Delta$ such that $\sup \left(\operatorname{nacc}\left(C_{\delta}\right) \cap\right.$ $A)=\delta$.

Proof. As $\{\{\alpha\} \mid \alpha \in A\} \subseteq[\kappa]^{<\omega} \subseteq \mathcal{X}$, we infer that $\{\{(\alpha, 0)\} \mid \alpha \in A\} \subseteq \mathcal{F}$. So, we may recursively construct a sequence $\left\langle\left(\alpha_{i}, \beta_{i}\right) \mid i<\kappa\right\rangle$ such that, for all $\left(i, i^{\prime}\right) \in[\kappa]^{2}$ :

- $\alpha_{i} \in A$;
- $f_{\beta_{i}}=\left\{\left(\alpha_{i}, 0\right)\right\}$;
- $\beta_{i}<\alpha_{i^{\prime}}$.

Set $B:=\left\{\beta_{i} \mid i<\kappa\right\}$. Now, by the choice of $\vec{D}$, we may fix $\delta \in \Delta$ for which the following set is cofinal in $\delta$ :

$$
\Gamma:=\left\{\gamma \in \operatorname{nacc}\left(D_{\delta}\right) \cap B \mid \exists \varepsilon \in B\left(\sup \left(D_{\delta} \cap \gamma\right) \leq \varepsilon<\gamma\right)\right\}
$$

Now, given $\gamma \in \Gamma$, fix $\varepsilon_{\gamma} \in B$ such that $\sup \left(D_{\delta} \cap \gamma\right) \leq \varepsilon_{\gamma}<\gamma$. Find $\left(i, i^{\prime}\right) \in[\kappa]^{2}$ such that $\varepsilon_{\gamma}=\beta_{i}$ and $\gamma=\beta_{i^{\prime}}$. Set $\beta:=\sup \left(D_{\delta} \cap \gamma\right)$. Then $\beta \leq \beta_{i}<\alpha_{i^{\prime}}$ and $\beta_{i^{\prime}}=\gamma$, so that $f_{\gamma}^{\beta}=f_{\gamma}=\left\{\left(\alpha_{i^{\prime}}, 0\right)\right\}$ and $C_{\delta} \cap(\beta, \gamma)=\left\{\alpha_{i^{\prime}}\right\}$. Thus, we have established that, for every $\gamma \in \Gamma, \operatorname{nacc}\left(C_{\delta}\right) \cap A \backslash \varepsilon_{\gamma}$ is nonempty. Consequently, $\sup \left(\operatorname{nacc}\left(C_{\delta}\right) \cap A\right)=\delta$.

Claim 7.2.3. Suppose $\mathcal{A} \subseteq[\kappa]^{<\mu}$ is a family consisting of $\kappa$ many pairwise disjoint sets, and $\tau<\kappa$. Then there exists $\delta \in \Delta$ such that

$$
\sup \left\{\min (x) \mid x \in \mathcal{A} \cap \mathcal{P}\left(C_{\delta}\right) \& h_{\delta}[x]=\{\tau\}\right\}=\delta
$$

Proof. For every $x \in \mathcal{A}$, we may find a large enough $\delta<\kappa$ such that $x \in[\delta]^{<\mu}$, and so, by regularity of $\mu$, we may find $\eta<\mu$ such that $x \subseteq \psi_{\delta}[\eta]$ so that $x^{\prime}:=$ $\psi_{\delta}[\eta] \backslash \min (x)$ is an element of $\mathcal{X}$ satisfying $x \subseteq x^{\prime}$ and $\min (x)=\min \left(x^{\prime}\right)$. It follows that we may recursively construct a sequence $\left\langle\left(x_{i}, \beta_{i}\right) \mid i<\kappa\right\rangle$ such that, for all $\left(i, i^{\prime}\right) \in[\kappa]^{2}$ :

- $x_{i} \in \mathcal{A}$;
- $\left(x_{i} \times\{\tau\}\right) \subseteq f_{\beta_{i}} ;$
- $\beta_{i}<\min \left(\operatorname{dom}\left(f_{\beta_{i^{\prime}}}\right)\right)$.

Set $B:=\left\{\beta_{i} \mid i<\kappa\right\}$.
Now, by the choice of $\vec{D}$, we may fix $\delta \in \Delta \cap \operatorname{acc}(\kappa)$ for which the following set is cofinal in $\delta$ :

$$
\Gamma:=\left\{\gamma \in \operatorname{nacc}\left(D_{\delta}\right) \cap B \mid \exists \varepsilon \in B\left(\sup \left(D_{\delta} \cap \gamma\right) \leq \varepsilon<\gamma\right)\right\}
$$

Let $\gamma \in \Gamma$. Fix $\varepsilon_{\gamma} \in B$ such that $\sup \left(D_{\delta} \cap \gamma\right) \leq \varepsilon_{\gamma}<\gamma$. Find $\left(i, i^{\prime}\right) \in[\kappa]^{2}$ such that $\varepsilon_{\gamma}=\beta_{i}$ and $\gamma=\beta_{i^{\prime}}$. Set $\beta:=\sup \left(D_{\delta} \cap \gamma\right)$. Then $\beta \leq \beta_{i}<\min \left(\operatorname{dom}\left(f_{\beta_{i^{\prime}}}\right)\right)$ and $\beta_{i^{\prime}}=\gamma$, so that $h_{\delta} \upharpoonright(\beta, \gamma)=f_{\gamma}^{\beta}=f_{\gamma} \supseteq\left(x_{i^{\prime}} \times\{\tau\}\right)$. As $x_{i^{\prime}} \in \mathcal{A}$ with $\varepsilon_{\gamma}<\min \left(x_{i^{\prime}}\right)$ and $\sup \left\{\varepsilon_{\gamma} \mid \gamma \in \Gamma\right\}=\delta$, we are done.

Now, we are in a position to appeal to Theorem 7.1.
The next corollary yields Clause (2) of Theorem A.
Corollary 7.3. For every infinite regular cardinal $\mu$, either of the following imply that $\mathrm{P} \ell_{2}\left(\mu^{+}, \mu^{+}, \mu\right)$ holds:
(1) $\boldsymbol{\varrho}\left(E_{\mu}^{\mu^{+}}\right)$holds;
(2) $\left(\mu^{+}\right)^{\aleph_{0}}=\mu^{+}$and $\boldsymbol{\varrho}(S)$ holds for some nonreflecting stationary $S \subseteq \mu^{+}$.

Proof. By Theorem 7.2, it suffices to prove that $\mathrm{P}^{-}\left(\mu^{+}, \mu^{++}, \sqsubseteq^{*}, 1,\left\{E_{\mu}^{\mu^{+}}\right\}, 2\right)$ holds. It is clear that the hypothesis of Clause (1) implies this instance. By [BR21, Lemma 4.20], also the hypothesis of Clause (2) implies this instance.
Corollary 7.4. For every infinite regular cardinal $\mu$, $V^{\operatorname{Add}(\mu, 1)} \models \mathrm{P} \ell_{2}\left(\mu^{+}, \mu^{+}, \mu\right) .{ }^{5}$
Proof. As $\operatorname{Add}(\mu, 1)$ is equivalent to $\operatorname{Add}(\mu, 2)$, we may assume that we are forcing over a model of $\mu^{<\mu}=\mu$. Now, by the same proof of [Rin15, Theorem 2.3], while ignoring any aspect of coherence (as it is not needed here; just ensuring that all the clubs have order-type at most $\mu$ is enough $), V^{\operatorname{Add}(\mu, 1)} \models \mathrm{P}^{-}\left(\kappa, \kappa^{+}, \sqsubseteq^{*}, 1,\left\{E_{\mu}^{\kappa}\right\}, 2\right)$ for $\kappa:=\mu^{+}$. Upon a request of anonymous referee Z, we now sketch that proof.

Work in $V$. Let $\mathbb{P}:=\operatorname{Add}(\mu, 1)$, so that a condition in $\mathbb{P}$ is an element $p$ of $<\mu \mu$. In particular, $\mathbb{P}$ has size $\mu^{<\mu}=\mu$ and it preserves the cardinals structure.

Let $\left\langle C_{\alpha} \mid \alpha<\kappa\right\rangle$ be any $C$-sequence such that otp $\left(C_{\alpha}\right)=\operatorname{cf}(\alpha)$ for every $\alpha<\kappa$. For every $\alpha<\kappa$, let $\pi_{\alpha}: \operatorname{otp}\left(C_{\alpha}\right) \rightarrow C_{\alpha}$ denote the inverse collapse, and let $\psi_{\alpha}: \mu \backslash\{0\} \rightarrow \alpha$ be an arbitrary surjection. For every function $p \in \leq \mu \mu$, and every $\alpha<\kappa$, denote

$$
1_{\alpha}^{p}:=\left\{j \in \operatorname{dom}(p) \cap \operatorname{dom}\left(\pi_{\alpha}\right) \mid p(j) \neq 0\right\}
$$

For every $j \in 1_{\alpha}^{p}$, we write $p_{\alpha}(j):=\psi_{\pi_{\alpha}(j)}(p(j))$. Note that $p_{\alpha}(j)<\pi_{\alpha}(j)$. Then, put

$$
C_{\alpha}^{p}:=\left\{\pi_{\alpha}(j) \mid j \in \operatorname{acc}^{+}\left(1_{\alpha}^{p}\right)\right\} \cup\left\{\max \left\{p_{\alpha}(j), \pi_{\alpha}\left(\sup \left(1_{\alpha}^{p} \cap j\right)\right)\right\} \mid j \in \operatorname{nacc}\left(1_{\alpha}^{p}\right)\right\}
$$

Notice that if $q$ extends $p$, then $C_{\alpha}^{q}$ is an end-extension of $C_{\alpha}^{p}$.
Next, let $G \subseteq \mathbb{P}$ be generic over $V$, and work in $V[G]$. Put $g:=\bigcup G$. Then, define for every $\alpha<\kappa$ :

$$
D_{\alpha}:= \begin{cases}C_{\alpha}^{g}, & \text { if } \sup \left(C_{\alpha}^{g}\right)=\alpha \\ C_{\alpha}, & \text { otherwise }\end{cases}
$$

[^5]Denote $\Delta:=E_{\mu}^{\kappa}$. Since $g$ is Cohen, the set $\{j<\mu \mid g(j) \neq 0\}$ is cofinal in $\mu$, and hence $D_{\alpha}=C_{\alpha}^{g}$ for every $\alpha \in \Delta$. Also note that, for every $\alpha<\kappa$, $\operatorname{otp}\left(D_{\alpha}\right)=\operatorname{cf}(\alpha) \leq \mu$, and hence $\operatorname{acc}\left(D_{\alpha}\right) \cap \Delta=\emptyset$
Claim 7.4.1. Let $A \subseteq \kappa$ be cofinal. Then there exists a club $E$ in $\kappa$ such that, for every $\alpha \in E \cap \Delta$,

$$
\sup \left\{\varepsilon \in A \cap \alpha \mid \min \left(D_{\alpha} \backslash(\varepsilon+1)\right) \in A\right\}=\alpha
$$

Proof. As $\mathbb{P}$ has size $\mu, A$ contains a ground model cofinal subset of $\kappa$. So, without loss of generality, we may assume that $A$ lies in $V$. Consider the club $D:=\operatorname{acc}^{+}(A)$. By [Rin15, Claim 2.3.1], for every $i<\mu$, there exists a club $E_{i} \subseteq \kappa$ such that for every $\alpha \in E_{i} \cap \Delta$, the following set is dense in $\mathbb{P}$ :
$\mathbb{D}(D, A, i, \alpha):=\left\{q \in{ }^{<\mu} \mu \mid\left\{\gamma \in \operatorname{nacc}\left(C_{\alpha}^{q}\right) \cap A \backslash \pi_{\alpha}(i) \mid\left(\sup \left(C_{\alpha}^{q} \cap \gamma\right), \gamma\right) \cap D \neq \emptyset\right\} \neq \emptyset\right\}$.
Set $E:=\bigcap_{i<\mu} E_{i}$. Then for all $\alpha \in E \cap \Delta$ and $i<\mu$, we may find a condition $q_{i} \in G \cap \mathbb{D}(D, A, i, \alpha)$. So $q_{i} \sqsubseteq g, C_{\alpha}^{q_{i}} \sqsubseteq C_{\alpha}^{g}=D_{\alpha}$, and

$$
\sup \left\{\gamma \in \operatorname{nacc}\left(C_{\alpha}^{q_{i}}\right) \cap A \mid\left(\sup \left(C_{\alpha}^{q_{i}} \cap \gamma\right), \gamma\right) \cap D \neq \emptyset\right\} \geq \pi_{\alpha}(i)
$$

In particular, the following set is cofinal in $\alpha$ :

$$
\Gamma:=\left\{\gamma \in \operatorname{nacc}\left(D_{\alpha}\right) \cap A \mid\left(\sup \left(D_{\alpha} \cap \gamma\right), \gamma\right) \cap D \neq \emptyset\right\}
$$

For each $\gamma \in \Gamma$, pick $\delta_{\gamma} \in D$ such that $\sup \left(D_{\alpha} \cap \gamma\right)<\delta_{\gamma}<\gamma$. Recalling that $D=\operatorname{acc}^{+}(A)$, it follows that we may pick $\varepsilon_{\gamma} \in A$ such that $\sup \left(D_{\alpha} \cap \gamma\right)<\varepsilon_{\gamma}<\delta_{\gamma}$. It is clear that $\sup \left\{\varepsilon_{\gamma} \mid \gamma \in \Gamma\right\}=\alpha$. So, since

$$
\left\{\varepsilon_{\gamma} \mid \gamma \in \Gamma\right\} \subseteq\left\{\varepsilon \in A \cap \alpha \mid \min \left(D_{\alpha} \backslash(\varepsilon+1)\right) \in A\right\}
$$

the latter is a cofinal subset of $\alpha$, as sought.
We have established that $\left\langle D_{\alpha} \mid \alpha<\kappa\right\rangle$ witnesses $\mathrm{P}^{-}\left(\kappa, \kappa^{+}, \sqsubseteq^{*}, 1,\left\{E_{\mu}^{\kappa}\right\}, 2\right)$, so we may now appeal to Theorem 7.2.

Corollary 7.5. Suppose that $\mu$ is a regular uncountable cardinal satisfying $2^{\mu}=$ $\mu^{+}$, and $\mathbb{P}$ is a $\mu^{+}$-cc notion of forcing of size $\leq \mu^{+}$that preserves the regularity of $\mu$ but does not satisfy the ${ }^{\mu} \mu$-bounding property. Then $V^{\mathbb{P}}=\mathrm{P} \ell_{2}\left(\mu^{+}, \mu^{+}, \mu\right)$.

Proof. By [BR19b, Theorem 3.4], if we also assume that $\mu^{<\mu}=\mu$, then, in $V^{\mathbb{P}}$, a strong from of the proxy principle denoted $\mathrm{P}^{*}\left(E_{\mu}^{\mu^{+}}, \mu\right)$ holds. By Remark (iv) that follows Definition 3.3 of [BR19b], $\mathrm{P}^{*}\left(E_{\mu}^{\mu^{+}}, \mu\right)$ stands for $\mathrm{P}^{-}\left(\mu^{+}, \infty, \sqsubseteq, 1, \mathrm{NS}_{\mu^{+}}^{+} \upharpoonright E_{\mu}^{\mu^{+}}\right.$, $\left.2,<\infty, \mathcal{E}_{\mu}\right)$. In particular, in $V^{\mathbb{P}}, \mathrm{P}^{-}\left(\mu^{+}, \mu^{+}, \sqsubseteq, 1,\left\{E_{\mu}^{\mu^{+}}\right\}, 2,<\infty, \mathcal{E}_{\mu}\right)$ holds.

By waiving the hypothesis " $\mu^{<\mu}=\mu$ ", the only thing that breaks down is [BR19b, Claim 3.4.3], meaning that $\mathrm{P}^{-}\left(\mu^{+}, \mu^{++}, \sqsubseteq, \mu^{+},\left\{E_{\mu}^{\mu^{+}}\right\}, 2,<\infty, \mathcal{E}_{\mu}\right)$ holds in $V^{\mathbb{P}}$, instead (i.e., the second parameter gets enlarged from $\mu^{+}$to $\mu^{++}$). In particular, $V^{\mathbb{P}} \models \mathrm{P}^{-}\left(\kappa, \kappa^{+}, \sqsubseteq^{*}, 1,\left\{E_{\mu}^{\kappa}\right\}, 2\right)$ holds for $\kappa:=\mu^{+}$, and we may appeal to Theorem 7.2.

Theorem 7.6. Suppose that $\kappa=\kappa^{<\kappa}$ is a Mahlo cardinal and $\mathrm{P}^{-}\left(\kappa, \kappa^{+}, \sqsubseteq^{*}, 1\right.$, $\{\operatorname{Reg}(\kappa)\}, 2)$ holds. Then $\mathrm{P} \ell_{2}(\kappa, \kappa, \kappa)$ holds, as well.
Proof. By an explanation almost identical to that from the beginning of the proof of Theorem 7.2, it suffices to establish that $\mathrm{P} \ell_{1}(\kappa, \kappa, \kappa)$ holds.

Let $\mathcal{F}:=\bigcup\left\{\operatorname{cl}(x) \times\{j\} \mid x \in[\kappa]^{<\kappa}, j<\kappa\right\}$. Fix an enumeration (possibly, with repetitions) $\left\langle f_{\gamma} \mid \gamma<\kappa\right\rangle$ of $\mathcal{F}$ such that, for every $\gamma<\kappa$, $\operatorname{dom}\left(f_{\gamma}\right) \subseteq \gamma$. For every
$(\beta, \gamma) \in[\kappa]^{2}$, set $f_{\gamma}^{\beta}:=f_{\gamma} \upharpoonright(\beta, \gamma)$, so that $f_{\gamma}^{\beta}$ is a constant function, and $\operatorname{dom}\left(f_{\gamma}^{\beta}\right)$ is a closed set of ordinals of order-type $<\kappa$.

Let $\vec{D}=\left\langle D_{\delta} \mid \delta<\kappa\right\rangle$ and $\Delta \subseteq \operatorname{Reg}(\kappa)$ be witnesses to the fact $\mathrm{P}^{-}\left(\kappa, \kappa^{+}, \sqsubseteq^{*}, 1\right.$, $\{\operatorname{Reg}(\kappa)\}, 2)$ holds. We now construct a sequence $\left\langle h_{\delta}: C_{\delta} \rightarrow \kappa \mid \delta<\kappa\right\rangle$ that satisfies the requirements of Theorem 7.1, with $\chi:=\kappa$.

For each $\delta \in \operatorname{acc}(\kappa)$, let

$$
h_{\delta}:=\left(D_{\delta} \times\{0\}\right) \cup \bigcup\left\{f_{\gamma}^{\beta} \mid \beta \in D_{\delta}, \gamma=\min \left(D_{\delta} \backslash(\beta+1)\right), \operatorname{otp}\left(\operatorname{dom}\left(f_{\gamma}^{\beta}\right)\right)<\beta\right\}
$$

so that $C_{\delta}:=\operatorname{dom}\left(h_{\delta}\right)$ is a club in $\delta$ and $\operatorname{acc}\left(C_{\delta}\right) \cap \operatorname{Reg}(\kappa)=\operatorname{acc}\left(D_{\delta}\right) \cap \operatorname{Reg}(\kappa)$.
Claim 7.6.1. Suppose $\delta<\kappa$ and $\bar{\delta} \in \operatorname{acc}\left(C_{\delta}\right) \cap \Delta$. Then there exists $\epsilon \in C_{\bar{\delta}}$ such that $h_{\delta} \upharpoonright[\epsilon, \bar{\delta})=h_{\bar{\delta}} \upharpoonright[\epsilon, \bar{\delta})$.

Proof. By the choice of $\vec{D}, \varepsilon:=\sup \left(\left(D_{\delta} \cap \bar{\delta}\right) \triangle D_{\bar{\delta}}\right)$ is $<\bar{\delta}$. Let $\epsilon:=\min \left(D_{\bar{\delta}} \backslash(\varepsilon+1)\right)$. Then $D_{\delta} \cap[\epsilon, \bar{\delta})=D_{\bar{\delta}} \cap[\epsilon, \bar{\delta})$, and it follows that $h_{\delta} \upharpoonright[\epsilon, \bar{\delta})=h_{\bar{\delta}} \upharpoonright[\epsilon, \bar{\delta})$.

As made clear by the proof of Claim 7.2.2, for every cofinal $A \subseteq \kappa$, there exists $\delta \in \Delta$ such that $\sup \left(\operatorname{nacc}\left(C_{\delta}\right) \cap A\right)=\delta$.

Claim 7.6.2. Suppose $\sigma<\kappa$ and $\mathcal{A} \subseteq[\kappa]^{\sigma}$ is a family consisting of $\kappa$ many pairwise disjoint sets, and $\tau<\kappa$. Then there exists $\delta \in \Delta$ such that

$$
\sup \left\{\min (x) \mid x \in \mathcal{A} \cap \mathcal{P}\left(C_{\delta}\right) \& h_{\delta}[x]=\{\tau\}\right\}=\delta
$$

Proof. Recursively construct a sequence $\left\langle\left(x_{i}, \beta_{i}\right) \mid i<\kappa\right\rangle$ such that, for all $\left(i, i^{\prime}\right) \in$ $[\kappa]^{2}$ :

- $x_{i} \in \mathcal{A}$;
- $f_{\beta_{i}}$ is the constant function from $x_{i}$ to $\tau$;
- $\beta_{i}<\min \left(x_{i^{\prime}}\right)$.

Set $B:=\left\{\beta_{i} \mid i<\kappa\right\}$. By the choice of $\vec{D}$, we may fix $\delta \in \Delta \cap \operatorname{acc}(\kappa \backslash \sigma)$ for which the following set is cofinal in $\delta$ :

$$
\Gamma:=\left\{\gamma \in \operatorname{nacc}\left(D_{\delta}\right) \cap B \mid \exists \varepsilon \in B\left(\sup \left(D_{\delta} \cap \gamma\right) \leq \varepsilon<\gamma\right)\right\}
$$

Let $\gamma \in \Gamma$ with $\sup \left(D_{\delta} \cap \gamma\right) \geq \sigma$. Fix $\varepsilon_{\gamma} \in B$ such that $\sup \left(D_{\delta} \cap \gamma\right) \leq \varepsilon_{\gamma}<\gamma$. Find $\left(i, i^{\prime}\right) \in[k]^{2}$ such that $\varepsilon_{\gamma}=\beta_{i}$ and $\gamma=\beta_{i^{\prime}}$. Set $\beta:=\sup \left(D_{\delta} \cap \gamma\right)$. Then

$$
\sigma \leq \beta \leq \beta_{i}<\min \left(x_{i^{\prime}}\right)=\min \left(\operatorname{dom}\left(f_{\beta_{i^{\prime}}}\right)\right)
$$

and $\beta_{i^{\prime}}=\gamma$, so that $h_{\delta} \upharpoonright(\beta, \gamma)=f_{\gamma}^{\beta}=f_{\gamma}=\left(x_{i^{\prime}} \times\{\tau\}\right)$. As $x_{i^{\prime}} \in \mathcal{A}$ with $\varepsilon_{\gamma}<\min \left(x_{i^{\prime}}\right)$ and $\sup \left\{\varepsilon_{\gamma} \mid \gamma \in \Gamma \& \sup \left(D_{\delta} \cap \gamma\right) \geq \sigma\right\}=\delta$, we are done.

Now, we are in a position to appeal to Theorem 7.1.
Remark 7.7. The proof of the preceding makes it clear that the conclusion of the theorem remains valid also after relaxing the arithmetic hypothesis of $\kappa=\kappa^{<\kappa}$ down to $\sup \left\{\sigma \in \operatorname{Reg}(\kappa) \mid \operatorname{cf}\left([\kappa]^{\sigma}, \subseteq\right)=\kappa\right\}=\kappa$.

The next corollary yields Clause (4) of Theorem B.
Corollary 7.8. Suppose that $\kappa$ is a Mahlo cardinal, and there exists a nonreflecting stationary $E \subseteq \kappa$ such that $\diamond(E)$ holds. If $\square(E)$ holds or if there exists a nonreflecting stationary subset of $\operatorname{Reg}(\kappa)$, then $\mathrm{P} \ell_{2}(\kappa, \kappa, \kappa)$ holds.

Proof. Recall that $\diamond(E)$ implies $\kappa^{<\kappa}=\kappa$. So, by Theorem 7.6, it suffices to prove that $\mathrm{P}^{-}\left(\kappa, \kappa^{+}, \sqsubseteq^{*}, 1,\{S\}, 2\right)$ holds for some stationary subset $S$ of $\operatorname{Reg}(\kappa)$.

- If $\square(E)$ holds, then by [BR21, Corollary 4.19(2)], $\mathrm{P}^{-}\left(\kappa, 2, \sqsubseteq^{*}, 1,\{S\}, 2\right)$ holds for every stationary $S \subseteq \kappa$.
- Suppose that $S$ is a nonreflecting stationary subset of $\operatorname{Reg}(\kappa)$. By [BR21, Corollary 4.27], if in addition $\kappa$ is a strong limit, then $\mathrm{P}^{-}\left(\kappa, \kappa, \sqsubseteq^{*}, 1,\{S\}, 2\right)$ holds. The same proof shows that, in the general case, $\mathrm{P}^{-}\left(\kappa, \kappa^{+}, \sqsubseteq^{*}, 1,\{S\}, 2\right)$ holds.

We can now derive Theorem D.
Corollary 7.9. Suppose that $V=L$. For every regular uncountable cardinal $\kappa$ and every regular cardinal $\chi \leq \chi(\kappa), \mathrm{P} \ell_{2}(\kappa, \kappa, \chi)$ holds.

Proof. There are four cases to consider:

- If $\kappa=\mu^{+}$for $\mu$ regular, then by [LHR21, Lemma 2.2(5)], $\chi(\kappa)=\mu$, and by [Jen72], $\diamond\left(E_{\mu}^{\mu^{+}}\right)$holds, so by Corollary $7.3(1), \mathrm{P} \ell_{2}(\kappa, \kappa, \mu)$ holds.
- If $\kappa=\mu^{+}$for $\mu$ singular, then by [LHR21, Lemma 2.2(5)], $\chi(\kappa)=\mu$, and by [Jen72], for every regular $\chi \leq \mu$, there exists a nonreflecting stationary subset of $E_{\chi}^{\kappa}$, so by the main result of [Rin14b], $\operatorname{Pr}_{1}(\kappa, \kappa, \kappa, \chi)$ holds. In addition, by [BR17, Corollary $1.10(5)], \boxtimes^{-}(\kappa)$ holds, so by Corollary 4.8(2), $\mathrm{P} \ell_{2}(\kappa, \kappa, \chi)$ holds.
- If $\kappa$ is inaccessible which is not weakly compact, then by [LHR21, Lemma 2.2(5)], $\chi(\kappa)=\kappa$, and by [Jen72], there exists a nonreflecting stationary subset $E \subseteq \kappa$ such $\diamond(E)$ and $\square(E)$ both hold. So, by Corollary $7.8, \mathrm{P} \ell_{2}(\kappa, \kappa, \kappa)$ holds.
$\checkmark$ If $\kappa$ is weakly compact, then $\chi(\kappa)=0$, so there is nothing to prove here.


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[^1]:    ${ }^{1}$ The definition of the cardinal characteristic $\chi(\kappa)$ is reproduced in Definition 3.19 below.

[^2]:    ${ }^{2}$ Recall Convention 2.1.

[^3]:    ${ }^{3} \chi(\kappa)$ should be understood as a measure of how far $\kappa$ is from being weakly compact. By [Tod07, Theorem 6.3.5], if $\kappa$ is weakly compact, then $\chi(\vec{C})=1$ for every $C$-sequence $\vec{C}$ over $\kappa$.

[^4]:    ${ }^{4}$ Recall that by Propositions 2.10 and 2.11 , this only makes sense for $\kappa$ a limit cardinal.

[^5]:    ${ }^{5}$ Here, $\mu^{+}$stands for the successor of $\mu$ in the generic extension.

