RAMSEY THEORY OVER PARTITIONS I: POSITIVE RAMSEY RELATIONS FROM FORCING AXIOMS

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Abstract. In this series of papers we advance Ramsey theory over partitions. In this part, a correspondence between anti-Ramsey properties of partitions and chain conditions of the natural forcing notions that homogenize colorings over them is uncovered. At the level of the first uncountable cardinal this gives rise to a duality theorem under Martin's Axiom: a function $p : [\omega_1]^2 \to \omega$ witnesses a weak negative Ramsey relation when p plays the role of a coloring if and only if a positive Ramsey relation holds over p when p plays the role of a partition.

The consistency of positive Ramsey relations over partitions does not stop at the first uncountable cardinal: it is established that at arbitrarily high uncountable cardinals these relations follow from forcing axioms without large cardinal strength. This result solves in particular two problems from [CKS21].

1. Introduction

1.1. **Ramsey relations.** For (finite or infinite) cardinals κ, λ and θ, the Ramsey relation $\kappa \to (\lambda)^2_\theta$ asserts that for every coloring $c : [\kappa]^2 \to \theta$ of unordered pairs of ordinals in κ by θ colors there is a c-homogeneous set of size λ, i.e., there exist $A \subseteq \kappa$ of size λ and $\tau \in \theta$ such that $c(\alpha, \beta) = \tau$ for any pair $\alpha < \beta$ of elements of A. In this notation, Ramsey's famous theorem [Ram30] is written as $\aleph_0 \to (\aleph_0)^2_2$.

If $\aleph_0$ is substituted in this relation by the next infinite cardinal, $\aleph_1$, then the relation fails, meaning that its negation $\aleph_1 \nrightarrow (\aleph_1)^2_2$ holds. In fact, Sierpiński [Sie33] proved that $2^\lambda \nrightarrow (\lambda^+)^2_2$ holds for every infinite cardinal λ and as $\lambda^+ \leq 2^\lambda$, by monotonicity, $\kappa \nrightarrow (\kappa)^2_2$ for every successor cardinal $\kappa = \lambda^+$.

Later on, work by Tarski, Erdős, Hanf and others (see [Kan09, Chapter 2]) has shown that if the relation $\kappa \to (\kappa)^2_2$ happens to hold for some cardinal $\kappa > \aleph_0$ then κ must be a weakly compact cardinal, a large cardinal which is inaccessible and satisfies the higher analog of König's lemma, that any tree of size κ all of whose levels have size $< \kappa$ admits a branch of size κ. As the existence of an inaccessible cardinal cannot be proved from the axiomatic system of set theory, ZFC,[1] this established that in ZFC alone it would be impossible to prove that any cardinal κ other than $\aleph_0$ satisfies the positive Ramsey relation $\kappa \to (\kappa)^2_2$.

A weaker Ramsey relation than the one above is the square brackets relation $\kappa \to [\lambda]^2_\theta$, which asserts that for every coloring $c : [\kappa]^2 \to \theta$ there exist $A \subseteq \kappa$ of size λ and $\tau \in \theta$ such that $c(\alpha, \beta) \neq \tau$ for any pair $\alpha < \beta$ of elements of A. Erdős,

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[1]Unless ZFC is inconsistent.

Hajnal and Rado [EHR65] proved $\lambda^+ \nrightarrow [\lambda^+]^2_{\lambda^+}$ (i.e. the negation of $\lambda^+ \rightarrow [\lambda^+]^2_{\lambda^+}$) for every infinite cardinal λ at which the Generalized Continuum Hypothesis (GCH) holds, that is, for every λ for which $2^\lambda = \lambda^+$.

Then, in [Tod87], Todorčević waived the hypothesis $2^\lambda = \lambda^+$ for infinite cardinals λ that are regular, establishing in particular that the negative Ramsey relation $\aleph_1 \nrightarrow [\aleph_1]^2_{\aleph_1}$ is a theorem of ZFC. Thus, the analog of Ramsey's theorem for $\aleph_0$ fails strongly on all successors of regular cardinals outright in ZFC: on every such λ^+ there is a coloring of pairs by λ^+ colors with the property that no color is omitted on the pairs from any subset $A \subseteq \lambda^+$ of full size.

In the positive direction, Shelah [She88] showed that Sierpiński's theorem is optimal in the sense that the consistency of a measurable cardinal is sufficient to get the consistency of $2^{\aleph_0} \rightarrow [\aleph_1]^2_3$. More works in the positive direction at the level of the first uncountable cardinal $\aleph_1$ include [Tod83, RT20], and the fact that the forcing axiom $\mathsf{MA}_{\aleph_1}$ (Martin's Axiom) implies that the product of any two *ccc* topological spaces is again *ccc*. The point is that the Ramsey relation $\kappa \rightarrow (\kappa)^2_2$ naturally reduces the question of κ-*cc* of the product of two spaces to the question of κ-*cc* of each factor, so $\mathsf{MA}_{\aleph_1}$ yields a consequence of $\aleph_1 \rightarrow (\aleph_1)^2_2$, even though the relation itself fails in ZFC.

However, this echo of a positive Ramsey relation on $\aleph_1$ turned out to be a very isolated case: from the *second* uncountable cardinal $\aleph_2$ on, for every successor cardinal κ, including successors of singular cardinals, there is a κ-cc space whose square violates the κ-cc (see [Rin14] for a comprehensive account of productivity of chain conditions and colorings).

In the positive direction, Mitchell and Silver [Mit73] proved that the consistency of a weakly compact cardinal is necessary and sufficient for the higher analog of König's lemma to hold at the level of $\aleph_2$.

In summary, the higher one goes on the scale of the Alephs, Ramsey theory steers away from Ramsey's original theorem towards strong negative relations. The consistency of positive relations on uncountable cardinals is rare and requires large cardinal strength.

1.2. **Ramsey relation over partitions.** Recently, a new, weaker type of Ramsey relations *over partitions* was discovered [CKS21]. Given a partition $p : [\kappa]^2 \rightarrow \mu$ of the unordered pairs from κ into μ cells, it is possible to relax the notion of homogeneity to *relative* homogeneity over p. Declare a set $A \subseteq \kappa$ as *homogeneous over* p for a coloring $c : [\kappa]^2 \rightarrow \theta$, or (p, c)-*homogeneous* for short, if all pairs from A which lie in any single p-cell are colored by one color which depends on the cell. More formally, for every $j < \mu$, the set $\{c(\alpha, \beta) \mid (\alpha, \beta) \in [A]^2 \ \&\ p(\alpha, \beta) = j\}$ has size no more than 1.

The standard positive Ramsey relation $\kappa \rightarrow (\lambda)^2_\theta$ can now be relaxed, for a partition p, to its "over p" version, $\kappa \rightarrow_p (\lambda)^2_\theta$, to mean that for every coloring $c : [\kappa]^2 \rightarrow \theta$ there is a set $A \subseteq \kappa$ of size λ which is *homogeneous over* p. Clearly, if A is homogeneous for c, it is also (p, c)-homogeneous for every partition p of the pairs from κ, so $\kappa \rightarrow_p (\lambda)^2_\theta$ follows from $\kappa \rightarrow (\lambda)^2_\theta$ for any partition p, but it is feasible that for some p, $\kappa \rightarrow_p (\lambda)^2_\theta$, even in situations where $\kappa \nrightarrow (\lambda)^2_\theta$.

When $A \subseteq \kappa$ is homogeneous over p, the p-cell $p(\alpha, \beta)$ of a pair from A determines its color $c(\alpha, \beta)$, so there is a function $\tau : \mu \rightarrow \theta$ such that $c(\alpha, \beta) = \tau(p(\alpha, \beta))$ for every $(\alpha, \beta) \in [A]^2$.

Similarly, when putting p in the weaker square brackets Ramsey relation $\kappa \to [\lambda]^2_\theta$, the relation $\kappa \to_p [\lambda]^2_\theta$ means that for every coloring $c : [\kappa]^2 \to \theta$ there is a set $A \subseteq \kappa$ of cardinality λ such that at least one color from θ is *omitted* by c in every p-cell intersected with $[A]^2$. In this case there will be a function $\tau : \mu \to \theta$ such that $c(\alpha, \beta) \neq \tau(p(\alpha, \beta))$ for every $(\alpha, \beta) \in [A]^2$.

In [CKS21] it was indeed shown that $\aleph_1 \to_p [\aleph_1]^2_{\aleph_1}$ is consistent in a *ccc* forcing extension for some generic partition $p : [\aleph_1]^2 \to \aleph_0$. In other words, although $\aleph_1 \nrightarrow [\aleph_1]^2_{\aleph_1}$ holds absolutely, by Todorčević's theorem, it is consistent that for some countable partition p the opposite, positive Ramsey relation $\aleph_1 \to_p [\aleph_1]^2_{\aleph_1}$ holds.

This consistency result led to many interesting questions. Primarily, whether $\mathsf{MA}_{\aleph_1}$ decides Ramsey theory over countable partitions on $\aleph_1$, and, of course, which way it decides it if it does. These problems were stated in [CKS21] and are solved as a corollary of the results in this paper.

1.3. **The results.** First, combinatorial properties of partitions are listed and it is shown that: (a) partitions which satisfy those properties exist in every model of ZFC; (b) MA implies that positive Ramsey relations hold over every partition with these properties. While the consistency results in [CKS21] rely on partitions that are generic, in the forcing sense, the partitions considered here will be definable from an ω_1 sequence of reals.

Theorem A. *If $\mathsf{MA}_{\aleph_1}(K)$ holds then for every partition $p : [\aleph_1]^2 \to \aleph_0$ with finite-to-one fibers, for every coloring $c : [\aleph_1]^2 \to \aleph_0$ there is decomposition $\aleph_1 = \biguplus_{i<\omega} X_i$ such that for all $i, j < \omega$,*

$$\{c(\alpha, \beta) \mid (\alpha, \beta) \in [X_i]^2 \ \&\ p(\alpha, \beta) = j\} \text{ is finite.}$$

The surprising point, though, is that this consistencey result on $\aleph_1$ is actually a special case of a more general theorem which applies to every successor of a regular cardinal via the Generalized Martin's Axiom (GMA). In other words, the consistency of positive Ramsey relations over partitions does not stop at $\aleph_1$, as is the case with classical positive Ramsey relation.

But even more is true. It is consistent that not only does every strong coloring omit at least one color in every p-cell of some set $X \subseteq \lambda^+$ of full size, but actually the positive rounded brackets Ramsey relation $\lambda^+ \to_p (\lambda^+)^2_\lambda$ holds over a suitable partition $p : [\lambda^+]^2 \to \lambda$ for an even stronger reason: λ^+ is the union of λ many (p, c)-homogeneous sets for every coloring $c : [\lambda^+]^2 \to \lambda$.

Theorem B. *If GMA_{λ^+} holds for a cardinal $\lambda = \lambda^{<\lambda}$ then there is a partition $p : [\lambda^+]^2 \to \lambda$ such that for every coloring $c : [\lambda^+]^2 \to \lambda$, there is a decomposition $\lambda^+ = \biguplus_{i<\lambda} X_i$ such that for all $i, j < \lambda$,*

$$|\{c(\alpha, \beta) \mid (\alpha, \beta) \in [X_i]^2 \ \&\ p(\alpha, \beta) = j\}| = 1.$$

The main new phenomenon which is revealed here about Ramsey theory over partitions is, then, that small partitions can consistently have positive Ramsey relations over them at uncountable cardinals at the presence of GMA. Since the consistency of GMA does not require consistency assumptions beyond that of ZFC, this stands in strong contrast to other characterizations of the Ramsey properties [Mit73].

Another interesting discovery is of a duality phenomenon in the presence of a mild forcing axiom (see Definition 3.1 below): a positive Ramsey relation over p, when viewed as a partition, is equivalent to p witnessing a certain negative Ramsey relation when viewed as a coloring:

Theorem C. *Suppose $\mathsf{MA}_{\aleph_1}(K)$ holds. Then for every function $p : [\aleph_1]^2 \to \aleph_0$, the following are equivalent:*

1. $\aleph_1 \to_p [\aleph_1]^2_{\aleph_0,\mathit{finite}}$;
2. *There exists $X \in [\aleph_1]^{\aleph_1}$ such that $p \restriction [X]^2$ witnesses $\mathrm{U}(\aleph_1, \aleph_1, \aleph_0, \aleph_0)$.*

Here $\aleph_1 \to_p [\aleph_1]^2_{\aleph_0,\text{finite}}$ asserts that for every coloring $c : [\aleph_1]^2 \to \aleph_0$ there exist an uncountable set $X \subseteq \aleph_1$ such that all $j < \omega$ the set $\{c(\alpha, \beta) \mid (\alpha, \beta) \in [X]^2 \ \&\ p(\alpha, \beta) = j\}$ is finite, or, almost all colors are omitted when restricting c to the pairs from X in any single p-cell. $\mathrm{U}(\aleph_1, \aleph_1, \aleph_0, \aleph_0)$ is a provable instance of the 4-parameter anti-Ramsey coloring principle $\mathrm{U}(\ldots)$ due to Lambie-Hanson and Rinot [LHR18]. Its definition is reproduced in Section 2.

2. Preliminaries

Throughout the paper, κ denotes a regular uncountable cardinal, χ, θ, μ denote cardinals $\leq \kappa$, and λ denotes an infinite cardinal $< \kappa$. For sets of ordinals a and b, we write $a < b$ if $\alpha < \beta$ for all $\alpha \in a$ and $\beta \in b$. For a set $\mathcal{A}$ which is either an ordinal or a collection of sets of ordinals, we interpret $[\mathcal{A}]^2$ as $\{(a, b) \in \mathcal{A} \times \mathcal{A} \mid a < b\}$. This is a convenient means to be able to write $c(\alpha, \beta)$ instead of $c(\{\alpha, \beta\})$. For an ordinal $\sigma > 2$ and a set of ordinals A, we write $[A]^\sigma$ for $\{B \subseteq A \mid \mathrm{otp}(B) = \sigma\}$. For a cardinal χ and a set $\mathcal{A}$, we write $[\mathcal{A}]^{<\chi} := \{\mathcal{B} \subseteq \mathcal{A} \mid |\mathcal{B}| < \chi\}$.

Definition 2.1 ([LHR18])**.** $\mathrm{U}(\kappa, \kappa, \mu, \chi)$ asserts the existence of a function $p : [\kappa]^2 \to \mu$ such that for every $\sigma < \chi$, every pairwise disjoint family $\mathcal{A} \subseteq [\kappa]^\sigma$ of size κ, for every $\delta < \mu$, there exists $\mathcal{B} \subseteq \mathcal{A}$ of size κ such that $\min(p[a \times b]) > \delta$ for all $(a, b) \in [\mathcal{B}]^2$.

By [LHR18, Corollary 4.12], for every pair $\mu \leq \lambda$ of infinite regular cardinals, $\mathrm{U}(\lambda^+, \lambda^+, \mu, \lambda)$ holds.

Definition 2.2. Let $p : [\kappa]^2 \to \mu$ be a partition. Then:

- p has *injective fibers* iff for all $\alpha < \alpha' < \beta$, $p(\alpha, \beta) \neq p(\alpha', \beta)$;
- p has *finite-to-one fibers* iff for all $\beta < \kappa$ and $j < \mu$, $\{\alpha < \beta \mid p(\alpha, \beta) = j\}$ is finite;
- p has *λ-almost-disjoint fibers* iff for all $\beta < \beta' < \kappa$:
$$|\{p(\alpha, \beta) \mid \alpha < \beta\} \cap \{p(\alpha, \beta') \mid \alpha < \beta\}| < \lambda;$$
- p has *λ-coherent fibers* iff for all $\beta < \beta' < \kappa$:
$$|\{\alpha < \beta \mid p(\alpha, \beta) \neq p(\alpha, \beta')\}| < \lambda;$$
- p has *λ-Cohen fibers* iff for every injection $g : a \to \mu$ with $a \in [\kappa]^{<\lambda}$, there are cofinally many $\beta < \kappa$ such that $g(\alpha) = p(\alpha, \beta)$ for all $\alpha \in a$.

Proposition 2.3. *Suppose that λ is an infinite regular cardinal.*

1. *There is a partition $p : [\lambda^+]^2 \to \lambda$ with injective and λ-coherent fibers;*
2. *For every cardinal χ such that $\lambda^{<\chi} = \lambda$, there is a partition $p : [\lambda^+]^2 \to \lambda$ with injective, λ-almost-disjoint and χ-Cohen fibers.*

Proof. (1) See, for instance, [Tod07, Lemma 6.25].

(2) Assuming $\lambda^{<\chi}=\lambda$, fix an enumeration $\langle g_\beta \mid \beta<\lambda^+\rangle$ of all injections g with $\mathrm{dom}(g)\in[\lambda^+]^{<\chi}$ and $\mathrm{Im}(g)\subseteq\lambda$ in which each such injection occurs cofinally often. For each $\beta<\lambda^+$, let $\gamma_\beta:=\sup(\mathrm{Im}(g_\beta))+1$.

Fix a sequence $\vec{Z}=\langle Z_\beta \mid \beta<\lambda^+\rangle$ of elements of $[\lambda]^\lambda$ such that, for all $\alpha<\beta<\lambda^+$, $|Z_\alpha\cap Z_\beta|<\lambda$. For all $\beta<\lambda^+$ and $\iota<\lambda$, let $Z_\beta(\iota)$ denote the unique $\zeta\in Z_\beta$ to satisfy $\mathrm{otp}(Z_\beta\cap\zeta)=\iota$. For every $\beta<\lambda^+$, fix an injection $i_\beta:\beta\rightarrow\lambda$. Then, define a partition $p:[\lambda^+]^2\rightarrow\lambda$ via:

$$p(\alpha,\beta):=\begin{cases} g_\beta(\alpha) & \text{if } \alpha\in\mathrm{dom}(g_\beta);\\ Z_\beta(\gamma_\beta+i_\beta(\alpha)) & \text{otherwise.}\end{cases}$$

A moment's reflection makes it clear that p has injective and χ-Cohen fibers.

To see that p is λ-almost-disjoint, fix an arbitrary pair $(\beta,\beta')\in[\lambda^+]^2$ and consider the set

$$A:=\{p(\alpha,\beta)\mid\alpha<\beta\}\cap\{p(\alpha,\beta')\mid\alpha<\beta\}.$$

Clearly, $|A|\leq|g_\beta|+|g_{\beta'}|+|Z_\beta\cap Z_{\beta'}|<\lambda$, as sought. $\square$

Definition 2.4. Let $p:[\kappa]^2\rightarrow\mu$ be a partition and $c:[\kappa]^2\rightarrow\theta$ be a coloring.

- A subset $A\subseteq\kappa$ is (p,c)-*homogeneous* iff there exists a function $\tau:\mu\rightarrow\theta$ such that $c(\alpha,\beta)=\tau(p(\alpha,\beta))$ for all $(\alpha,\beta)\in[A]^2$;
- The *p-cochromatic number of* c, denoted $z_p(c)$, is the least cardinal ζ such that κ may be covered by ζ many (p,c)–homogeneous sets.

3. Relations over partitions from forcing axioms

Our forcing convention is that $q\leq p$ means that q is a forcing condition which extends the forcing condition p. A notion of forcing $\mathbb{Q}$ has Knaster's Property (Property K) iff for every uncountable set A of conditions in $\mathbb{Q}$, there is an uncountable $B\subseteq A$ such that any two elements of B are compatible.

Definition 3.1. $\mathsf{MA}_{\aleph_1}(K)$ asserts that for every notion of forcing $\mathbb{Q}$ having Property K, for every sequence $\langle D_\beta\mid\beta<\omega_1\rangle$ of dense subsets of $\mathbb{Q}$, there is a filter G over $\mathbb{Q}$ that meets each of the D_β's.

A notion of forcing $\mathbb{Q}=(Q,\leq)$ is *well-met* iff every pair q_0,q_1 of compatible conditions has a greatest lower bound, i.e., an $r\leq q_0,q_1$ such that for any condition s, if $s\leq q_0,q_1$ then $s\leq r$. The notion of forcing $\mathbb{Q}$ is said to satisfy the λ^+-*stationary chain condition* (λ^+-stationary-cc, for short) iff for every sequence $\langle q_\delta\mid\delta<\lambda^+\rangle$ of conditions in $\mathbb{Q}$ there is a club $D\subseteq\lambda^+$ and a regressive map $h:D\cap E^{\lambda^+}_{\mathrm{cf}(\lambda)}\rightarrow\lambda^+$ such that for all $\gamma,\delta\in\mathrm{dom}(h)$, if $h(\gamma)=h(\delta)$ then q_γ and q_δ are compatible.

Definition 3.2 (Generalized Martin's Axiom)**.** GMA_{λ^+} asserts that for every notion of forcing $\mathbb{Q}=(Q,\leq)$ of size $<2^\lambda$ which satisfies the following conditions:

(a) $\mathbb{Q}$ is well-met;
(b) For all $\sigma<\lambda$, every $\leq$-decreasing sequence of conditions $\langle q_i\mid i<\sigma\rangle$ in $\mathbb{Q}$ admits a greatest lower bound;
(c) $\mathbb{Q}$ satisfies the λ^+-stationary-cc,

for every sequence $\langle D_\beta\mid\beta<\lambda^+\rangle$ of dense subsets of $\mathbb{Q}$ there is a filter G over $\mathbb{Q}$ that meets each of the D_β's.

By Fodor's lemma, any poset satisfying the ω_1-stationary-cc has Property K. Thus, $\mathsf{MA}_{\aleph_1} \implies \mathsf{MA}_{\aleph_1}(K) \implies \mathsf{GMA}_{\aleph_1}$.

Theorem 3.3 (Shelah, [She78])**.** *Suppose the* GCH *holds. Then for any prescribed regular cardinal λ there is a cofinality-preserving forcing extension in which $\lambda^{<\lambda} = \lambda$ and* GMA_{λ^+} *holds.*

Remark 3.4. The conjunction of $\lambda^{<\lambda} = \lambda$ and GMA_{λ^+} implies that $2^\lambda > \lambda^+$. Otherwise, fix an enumeration $\langle f_\beta \mid \beta < \lambda^+\rangle$ of ${}^\lambda\lambda$ and appeal to GMA_{λ^+} with $\mathbb{Q} := \mathrm{Add}(\lambda, 1)$ and $D_\beta := \{q \in {}^{<\lambda}\lambda \mid q \nsubseteq f_\beta\}$ for each $\beta < \lambda^+$.

Together with Proposition 2.3(2), the next result is Theorem B.

Theorem 3.5. *Suppose $\lambda^{<\lambda} = \lambda$ and GMA_{λ^+} holds. Let $p : [\lambda^+]^2 \to \lambda$ be any partition with injective and λ-almost-disjoint fibers.*

For every coloring $c : [\lambda^+]^2 \to \lambda$, there exists a decomposition $\lambda^+ = \biguplus_{i<\lambda} X_i$ such that for all $i < \lambda$:

- *X_i is (p, c)-homogeneous (recall Definition 2.4);*
- *if p has in addition λ-Cohen fibers, then $|X_i| = \lambda^+$ and $p[[X_i]^2] = \lambda$.*

In particular, $z_p(c) \le \lambda$ for every coloring $c : [\lambda^+]^2 \to \lambda$.

Proof. Fix an arbitrary coloring $c : [\lambda^+]^2 \to \lambda$. Define a notion of forcing $\mathbb{Q} = (Q, \supseteq)$, where Q consists of all functions $f : a \to \lambda$ such that:

(1) $a \in [\lambda^+]^{<\lambda}$;
(2) for all $i, j < \lambda$, the set
$$\Gamma^f_{i,j} := \{c(\alpha, \beta) \mid (\alpha, \beta) \in [a]^2,\ f(\alpha) = i = f(\beta),\ p(\alpha, \beta) = j\}$$
contains at most one element.

It will be shown that $\mathbb{Q}$ satisfies the requirement of Definition 3.2. We first dispose of an easy claim.

Claim 3.5.1. *Let $\mathcal{F}$ be a centered family of conditions of size $< \lambda$. Then $\bigcup \mathcal{F}$ is a condition.*

Proof. Suppose not. Denote $f := \bigcup \mathcal{F}$ and $a := \mathrm{dom}(f)$. As $a \in [\lambda^+]^{<\lambda}$ this must mean that there are $i, j < \lambda$ for which the set $\Gamma^f_{i,j}$ has more than one element. This means that we may pick $(\alpha_0, \beta_0), (\alpha_1, \beta_1) \in [a]^2$ such that:

- $f(\alpha_0) = f(\alpha_1) = i = f(\beta_0) = f(\beta_1)$;
- $p(\alpha_0, \beta_0) = j = p(\alpha_1, \beta_1)$;
- $c(\alpha_0, \beta_0) \neq c(\alpha_1, \beta_1)$.

Fix $f_0, f_1, f^0, f^1 \in \mathcal{F}$ such that $\alpha_0 \in \mathrm{dom}(f_0)$, $\alpha_1 \in \mathrm{dom}(f_1)$, $\beta_0 \in \mathrm{dom}(f^0)$ and $\beta_1 \in \mathrm{dom}(f^1)$. Since $\mathcal{F}$ is centered, there exists a condition q such that $f_0 \cup f_1 \cup f^0 \cup f^1 \subseteq q$. A moment's reflection makes it clear that $q(\alpha_0) = q(\alpha_1) = i = q(\beta_0) = q(\beta_1)$, so since $p(\alpha_0, \beta_0) = j = p(\alpha_1, \beta_1)$ and since q is a legitimate condition, this must mean that $c(\alpha_0, \beta_0) = c(\alpha_1, \beta_1)$. This is a contradiction. $\square$

Claim 3.5.2. *For every $\beta < \lambda^+$, $D_\beta := \{f \in Q \mid \beta \in \mathrm{dom}(f)\}$ is dense in $\mathbb{Q}$.*

Proof. Let $\beta < \lambda^+$. Given any condition f such that $\beta \notin \mathrm{dom}(f)$, we get that $f' := f \cup \{(\beta, \sup(\mathrm{Im}(f)) + 1)\}$ is an extension of f in D_β. $\square$

It thus follows that if G is a filter over $\mathbb{Q}$ that meets each of the D_β's, then $g := \bigcup G$ is a function from λ^+ to λ such that for every $i < \lambda$, if we let $X_i := \{\alpha < \lambda^+ \mid g(\alpha) = i\}$, then $|\{c(\alpha,\beta) \mid (\alpha,\beta) \in [X_i]^2 \ \&\ p(\alpha,\beta) = j\}| \leq 1$ for all $j < \lambda$. Thus, $z_p(c) \leq \lambda$ holds in the forcing extension by $\mathbb{Q}$.

Claim 3.5.3. *If p has λ-Cohen fibers, then, for all $\epsilon < \lambda^+$ and $i, j < \lambda$:*
1. *$D^\epsilon := \{f \in Q \mid \exists\beta \in \mathrm{dom}(f)\,(\beta \geq \epsilon \ \&\ f(\beta) = i)\}$ is dense;*
2. *$D_{i,j} := \{f \in Q \mid \exists(\alpha,\beta) \in [\mathrm{dom}(f)]^2, f(\alpha) = i = f(\beta) \ \&\ p(\alpha,\beta) = j\}$ is dense.*

Proof. Suppose that p has λ-Cohen fibers.

(1) Given $\epsilon < \lambda^+$ and a condition $f : a \to \lambda$, fix a large enough $\eta < \lambda^+$ such that $\{p(\alpha,\beta) \mid (\alpha,\beta) \in [a]^2\} \subseteq \eta$, and then define an injection $g : a \to \lambda$ via

$$g(\alpha) := \eta + \mathrm{otp}(a \cap \alpha).$$

Now, as p has λ-Cohen fibers, we may find some $\beta < \lambda^+$ with $a \cup \epsilon \subseteq \beta$ such that $p(\alpha,\beta) = g(\alpha)$ for all $\alpha \in a$. It is clear that $f' := f \cup \{(\beta, i)\}$ is an extension of f lying in D^ϵ.

(2) Given $i, j < \lambda^+$ and a condition $f : a \to \lambda$, we do the following. First, by Clause (1), we may assume the existence of some $\alpha^* \in a$ such that $f(\alpha^*) = i$. Of course, if $f \in D_{i,j}$, then we are done, thus, hereafter, assume that $f \notin D_{i,j}$.

Fix a large enough $\eta \in \lambda^+ \setminus (j+1)$ such that $\{p(\alpha,\beta) \mid (\alpha,\beta) \in [a]^2\} \subseteq \eta$. Define an injection $g : a \to \lambda$ via

$$g(\alpha) := \begin{cases} j & \text{if } \alpha = \alpha^*; \\ \eta + \mathrm{otp}(a \cap \alpha) & \text{otherwise.} \end{cases}$$

Now, as p has λ-Cohen fibers, we may find some $\beta < \lambda^+$ with $a \subseteq \beta$ such that $p(\alpha,\beta) = g(\alpha)$ for all $\alpha \in a$. As $f \notin D_{i,j}$, it immediately follows that $f' := f \cup \{(\beta, i)\}$ is a legitimate condition lying in $D_{i,j}$. So we are done. $\square$

Clearly, $\mathbb{Q}$ has size no more than $|[\lambda^+]^{<\lambda}| = \lambda^+ < 2^\lambda$. In addition, by Claim 3.5.1, Clauses (a) and (b) of Definition 3.2 hold true. Thus, to complete the proof, we are left with addressing Clause (c) of Definition 3.2. To this end, assume we are given a sequence $\langle f_\delta \mid \delta < \lambda^+\rangle$ of conditions in $\mathbb{Q}$; we need to find a club $D \subseteq \lambda^+$ and a regressive map $h : D \cap E^{\lambda^+}_\lambda \to \lambda^+$ such that for all $\gamma, \delta \in D \cap E^{\lambda^+}_\lambda$, if $h(\gamma) = h(\delta)$ then f_γ and f_δ are compatible.

Consider the club $C := \{\delta < \lambda^+ \mid \forall\gamma < \delta\,(\sup(\mathrm{dom}(f_\gamma)) < \delta)\}$. Fix an injective enumeration $\langle(\psi_\tau, \varphi_\tau, \xi_\tau, \mu_\tau, \epsilon_\tau) \mid \tau < \lambda^+\rangle$ of $[\lambda^3]^{<\lambda} \times [\lambda^+ \times \lambda]^{<\lambda} \times \lambda \times \lambda \times \lambda^+$, and then consider the subclub:

$$D := \{\delta \in C \mid \{(\psi_\tau, \varphi_\tau, \xi_\tau, \mu_\tau, \epsilon_\tau) \mid \tau < \delta\} = [\lambda^3]^{<\lambda} \times [\delta \times \lambda]^{<\lambda} \times \lambda \times \lambda \times \delta\}.$$

We define the function $h : D \cap E^{\lambda^+}_\lambda \to \lambda^+$ as follows. Given $\delta \in D \cap E^{\lambda^+}_\lambda$, let $h(\delta) := \tau$ for the least $\tau < \delta$ which satisfies all of the following:

(a) $\big\{(i, j, c(\alpha,\beta)) \mid i, j < \lambda, (\alpha,\beta) \in [\mathrm{dom}(f_\delta)]^2, f_\delta(\alpha) = i = f_\delta(\beta), p(\alpha,\beta) = j\big\} = \psi_\tau$;

(b) $f_\delta \restriction \delta = \varphi_\tau$;

(c) $\bigcup\{\{p(\alpha,\beta) \mid \alpha < \beta\} \cap \{p(\alpha,\beta') \mid \alpha < \beta\} \mid (\beta,\beta') \in [\mathrm{dom}(f_\delta)]^2\} \subseteq \xi_\tau$;

(d) $\{p(\alpha,\beta) \mid (\alpha,\beta) \in [\mathrm{dom}(f_\delta)]^2\} \subseteq \mu_\tau$;

(e) $\{\alpha < \delta \mid \exists\beta \in \mathrm{dom}(f_\delta) \setminus \delta\ [p(\alpha,\beta) \leq \max\{\xi_\tau, \mu_\tau\}]\} \subseteq \epsilon_\tau$.

Claim 3.5.4. *h is well-defined.*

Proof. Let $\delta \in D \cap E^{\lambda^+}_\lambda$. First we make note of the following:
- The corresponding set of Clause (a) is a subset of λ^3 of size $< \lambda$;
- The corresponding set of Clause (b) is a subset of $\delta \times \lambda$ of size $< \lambda$;
- As $|\operatorname{dom}(f_\delta)| < \lambda$, the fact that p has λ-almost-disjoint fibers ensures that an ordinal $\xi_\tau < \lambda$ as in Clause (c) exists;
- As $|\operatorname{dom}(f_\delta)| < \lambda$, an ordinal $\mu_\tau < \lambda$ as in Clause (d) does exist;
- As $|\operatorname{dom}(f_\delta)| < \lambda = \operatorname{cf}(\delta)$, the fact that p has injective fibers ensures that an ordinal $\epsilon_\tau < \lambda$ as in Clause (e) exists.

So, since $\delta \in D$, a $\tau < \delta$ for which Clause (a)–(e) are satisfied does exist. □

To see that h is as sought, fix a pair $\gamma < \delta$ of ordinals in $D \cap E^{\lambda^+}_\lambda$ such that $h(\gamma) = h(\delta)$, say, both are equal to τ. As $\delta \in C$, Clause (b) implies that $f := f_\gamma \cup f_\delta$ is a function. To see that $f \in Q$, it suffices to verify Clause (2) above with $a := \operatorname{dom}(f)$. Towards a contradiction, suppose that there $i, j < \lambda$ and $(\alpha_0, \beta_0), (\alpha_1, \beta_1) \in [a]^2$ such that:
- $f(\alpha_0) = f(\alpha_1) = i = f(\beta_0) = f(\beta_1)$;
- $p(\alpha_0, \beta_0) = j = p(\alpha_1, \beta_1)$;
- $c(\alpha_0, \beta_0) \neq c(\alpha_1, \beta_1)$.

Denote $a_\gamma := \operatorname{dom}(f_\gamma)$ and $a_\delta := \operatorname{dom}(f_\delta)$. As $h(\gamma) = \tau = h(\delta)$, Clause (a) implies that it cannot be the case that $\{(\alpha_0, \beta_0), (\alpha_1, \beta_1)\} \subseteq [a_\gamma]^2 \cup [a_\delta]^2$. So, without loss of generality, assume that $(\alpha_0, \beta_0) \notin [a_\gamma]^2 \cup [a_\delta]^2$. By Clause (b), in particular, $a_\gamma \cap \gamma = a_\delta \cap \delta$, and so, since $(\alpha_0, \beta_0) \notin [a_\gamma]^2 \cup [a_\delta]^2$, it must be the case that $\alpha_0 \geq \gamma$ and $\beta_0 \geq \delta$. If $\alpha_0 \geq \delta$, then since $\delta \in C$, we would get that $(\alpha_0, \beta_0) \in [a_\delta]^2$, which is not the case. Altogether, $\gamma \leq \alpha_0 < \delta \leq \beta_0$. In particular, $\alpha_0 \in (a_\gamma \setminus \gamma)$ and $\beta_0 \in (a_\delta \setminus \delta)$.

By Clause (e), $\epsilon_\tau < \gamma$, and hence $\alpha_0 > \epsilon_\tau$. It thus follows from Clause (e) that $p(\alpha_0, \beta_0) > \max\{\xi_\tau, \mu_\tau\}$. So $p(\alpha_1, \beta_1) = j = p(\alpha_0, \beta_0) > \mu_\tau$. Recalling Clause (d), this means that $(\alpha_1, \beta_1) \notin [a_\gamma]^2 \cup [a_\delta]^2$. Hence, the same analysis we had for (α_0, β_0) is valid also for (α_1, β_1). In particular, $\gamma \leq \alpha_1 < \delta \leq \beta_1$ (so that $\{\beta_0, \beta_1\} \subseteq a_\delta \setminus \delta$) and $p(\alpha_1, \beta) > \xi_\tau$. By Clause (c) for γ, $\xi_\tau < \gamma < \alpha_0, \alpha_1$ and then by Clause (c) for δ we infer that if $\beta_0 \neq \beta_1$, then $p(\alpha_0, \beta_0) \neq p(\alpha_1, \beta_1)$. It thus follows that $\beta_0 = \beta_1$. As p has has injective fibers it follows that $\alpha_0 = \alpha_1$, contradicting the fact that $c(\alpha_0, \beta_0) \neq c(\alpha_1, \beta_1)$. □

The next result answers two questions from [CKS21]: Question 48 in the negative and Question 49 in the affirmative. Recall that by Proposition 2.3(2), the set of partitions $p : [\omega_1]^2 \to \omega$ with injective, $\aleph_1$-almost-disjoint and $\aleph_0$-Cohen fibers is not vacuous.

Corollary 3.6. *Suppose $\mathsf{MA}_{\aleph_1}(K)$ holds. Then for every partition $p : [\omega_1]^2 \to \omega$ with injective, $\aleph_1$-almost-disjoint and $\aleph_0$-Cohen fibers, for every coloring $c : [\omega_1]^2 \to \omega$ there exists a decomposition $\aleph_1 = \biguplus_{i<\omega} X_i$ such that, for every $i < \omega$:*
- *X_i is uncountable;*
- *X_i is p-omnichromatic, i.e., $p[[X_i]^2] = \omega$;*
- *X_i is (p, c)-homogeneous, i.e., for every $j < \omega$,*

$$c \restriction \{(\alpha, \beta) \in [X_i]^2 \mid p(\alpha, \beta) = j\} \text{ is constant.}$$

Proof. By Theorem 3.5. □

It may be interesting to point out that the middle item in Corollary 3.6 is not required for establishing $z_p(c) \leq \aleph_0$, but nevertheless holds. It can be interpreted as a silent witness for p being, from the viewpoint of a coloring, anti-Ramsey.

The upcoming theorem applies to a broader set of partitions than the one in 3.6 in return for allowing a finite number of colors rather than a single color. It also implies Theorem A.

Theorem 3.7. *Suppose* $\mathsf{MA}_{\aleph_1}(K)$ *holds. Then for every partition p which witnesses* $\mathrm{U}(\omega_1, \omega_1, \omega, \omega)$*, and every coloring* $c : [\omega_1]^2 \to \omega$*, there is a decomposition* $\omega_1 = \biguplus_{i<\omega} X_i$ *such that for all* $i, j < \omega$*,*

$$\{c(\alpha, \beta) \mid (\alpha, \beta) \in [X_i]^2 \ \&\ p(\alpha, \beta) = j\} \text{ is finite.}$$

Proof. Define a notion of forcing $\mathbb{Q}$ consisting of conditions $q = (m_q, f_q, g_q)$ as follows:

(1) $m_q < \omega$;
(2) $f_q : m_q \times m_q \to \omega$ is a function;
(3) $g_q : a_q \to m_q$ is a function, with $a_q \in [\omega_1]^{<\aleph_0}$;
(4) for all $(\alpha, \beta) \in [a_q]^2$, $p(\alpha, \beta) < m_q$;
(5) for all $(\alpha, \beta) \in [a_q]^2$, if $g_q(\alpha) = g_q(\beta)$, then $c(\alpha, \beta) < f_q(g_q(\alpha), p(\alpha, \beta))$.

A condition q extends a condition $\bar{q}$ iff $m_q \geq m_{\bar{q}}$, $f_q \supseteq f_{\bar{q}}$ and $g_q \supseteq g_{\bar{q}}$.

Claim 3.7.1. *For every* $\beta < \omega_1$*,* $D_\beta := \{q \mid \beta \in a_q\}$ *is dense in* $\mathbb{Q}$

Proof. Let $\beta < \omega_1$. Given any condition (m, f, g) in $\mathbb{Q}$ such that $\beta \notin \mathrm{dom}(g)$, let

$$m' := \max\{m, p(\alpha, \beta) \mid \alpha \in \mathrm{dom}(g)\}$$

and define a condition $q = (m_q, f_q, g_q)$, by letting $m_q := m' + 1$, letting $f_q : m_q \times m_q \to \omega$ be an arbitrary function extending f, and letting $g_q := g \cup \{(\beta, m')\}$. $\square$

It thus follows that if G is a filter over $\mathbb{Q}$ such that $G \cap D_\beta \neq \emptyset$ for all $\beta < \omega_1$, then by letting $f := \bigcup\{f_q \mid q \in G\}$ and $g := \bigcup\{g_q \mid q \in G\}$, we get that $\mathrm{dom}(f) = \omega \times \omega$, $\mathrm{dom}(g) = \omega_1$, and for every $i, j < \omega$, if we let $X_i := \{\beta < \omega_1 \mid g(\beta) = i\}$, then $\{c(\alpha, \beta) \mid (\alpha, \beta) \in [X_i]^2 \ \&\ p(\alpha, \beta) = j\} \subseteq f(i, j)$.

Now, let us verify that $\mathbb{Q}$ has Property K. To this end, suppose that A is an uncountable family of conditions in $\mathbb{Q}$. By the pigeonhole principle, we may assume the existence of an integer m and a function f such that, for all $q \in A$, $m_q = m$ and $f_q = f$. By the Δ-system lemma, it may also be assumed that $\{a_q \mid q \in A\}$ forms a Δ-system with some root r. For any $q \in A$, denote $a'_q := a_q \setminus r$. Since the a_q's are subsets of ω_1, it can be assumed that r forms an initial segment of a_q for each q and that, if $q \neq \bar{q}$, then either $\max(a'_q) < \min(a'_{\bar{q}})$ or $\max(a'_{\bar{q}}) < \min(a'_q)$. By shrinking further, we may assume that $q \mapsto g_q \restriction r$ is constant over A. Next, by the choice of the partition p, we fix an uncountable $B \subseteq A$ with the property that for all $\bar{q}, q \in B$, if $\max(a'_{\bar{q}}) < \min(a'_q)$, then $\min(p[a'_{\bar{q}} \times a'_q]) > m$.

To see that $\{a_q \mid q \in B\}$ is directed, fix two conditions $\bar{q} \neq q$ in B. Without loss of generality, we may assume that $\max(a'_{\bar{q}}) < \min(a'_q)$.

Set $a^* := a_{\bar{q}} \cup a_q$, $g^* := g_{\bar{q}} \cup g_q$ and $m^* := \max(p[a^*]^2) + 1$.

Claim 3.7.2. *For every* $(\alpha, \beta) \in [a^*]^2 \setminus ([a_{\bar{q}}]^2 \cup [a_q]^2)$*,* $(\alpha, \beta) \in a'_{\bar{q}} \times a'_q$*.*

Proof. Let $(\alpha, \beta) \in [a^*]^2 \setminus ([a_{\bar{q}}]^2 \cup [a_q]^2)$. As $r = a_{\bar{q}} \cap a_q$, we infer that $\{\alpha, \beta\} \cap r = \emptyset$. So $\{\alpha, \beta\} \cap a'_{\bar{q}}$ and $\{\alpha, \beta\} \cap a'_q$ are singletons. Since $\alpha < \beta$ and $\max(a'_{\bar{q}}) < \min(a'_q)$, it altogether follows that $(\alpha, \beta) \in a'_{\bar{q}} \times a'_q$. $\square$

Fix any function $f^* : m^* \times m^* \to \omega$ extending f by letting, for all $(i,j) \in (m^* \times m^*) \setminus (m \times m)$,

$$f^*(i,j) := \max\{c(\alpha,\beta) \mid (\alpha,\beta) \in [a^*]^2, g(\alpha) = i = g(\beta) \ \&\ p(\alpha,\beta) = j)\} + 1.$$

Looking at Clauses (1)–(5) above, it is clear that for $q^* := (m^*, f^*, g^*)$ to be a condition in $\mathbb{Q}$ it suffices to verify the following claim.

Claim 3.7.3. *Let $(\alpha,\beta) \in [a^*]^2$ with $g^*(\alpha) = g^*(\beta)$. Then*

$$c(\alpha,\beta) < f^*(g^*(\alpha), p(\alpha,\beta)).$$

Proof. Denote $i := g^*(\alpha)$ and $j := p(\alpha,\beta)$. We shall show that $c(\alpha,\beta) < f^*(i,j)$.

Of course, if $(\alpha,\beta) \in [a_{\bar q}]^2$, then $g_{\bar q}(\alpha) = i < m$, $p(\alpha,\beta) < m$ and $c(\alpha,\beta) = f_{\bar q}(i,j) = f^*(i,j)$. Likewise, if $(\alpha,\beta) \in [a_q]^2$, then $c(\alpha,\beta) = f_q(i,j) = f^*(i,j)$.

Next, assume that $(\alpha,\beta) \notin [a_{\bar q}]^2 \cup [a_q]^2$. So, by Claim 3.7.2, $(\alpha,\beta) \in a'_{\bar q} \times a'_q$. As $j \geq \min(p[a'_{\bar q} \times a'_q]) > m > \max(p[[a_{\bar q}]^2 \cup [a_q]^2])$, we infer that $(i,j) \in (m^* \times m^*) \setminus (m \times m)$ and hence the definition of $f^*(i,j)$ makes it clear that $c(\alpha,\beta) < f^*(i,j)$, as sought. □

So q^* is a legitimate condition witnessing that $\bar q$ and q are compatible. Thus, we have demonstrated that $\mathbb{Q}$ indeed satisfies Property K. □

We now present two ZFC results which show that the preceding is optimal. To see how the first result connects to Theorem 3.7 note that any partition $p : [\omega_1]^2 \to \omega$ with injective (or just finite-to-one) fibers witnesses $\mathrm{U}(\omega_1, \omega_1, \omega, \omega)$.[2]

Theorem 3.8. *There exist a partition $p : [\omega_1]^2 \to \omega$ with injective fibers and a coloring $c : [\omega_1]^2 \to \omega$ such that, for every $k < \omega$, and every $X \subseteq \omega_1$ with $\mathrm{otp}(X) = \omega + k$, there exists $j < \omega$ such that $|\{c(\alpha,\beta) \mid (\alpha,\beta) \in [X]^2 \ \&\ p(\alpha,\beta) = j\}| \geq k$.*

Proof. By Proposition 2.3, let us fix a partition $p : [\omega_1]^2 \to \omega$ with injective and ω-coherent fibers, and a coloring $c : [\omega_1]^2 \to \omega$ with injective and ω-almost-disjoint fibers. Now, given $k < \omega$ and an increasing sequence $\langle \xi_n \mid n < \omega + k\rangle$ of countable ordinals, we do the following. For each $i < k$, denote $\beta_i := \xi_{\omega+i}$.

- As p has ω-coherent fibers, $a := \bigcup_{i<i'<k}\{\alpha < \beta_0 \mid p(\alpha,\beta_i) \neq p(\alpha,\beta_{i'})\}$ is finite;
- As c has ω-almost-disjoint fibers, $T := \bigcup_{i<i'<k}\{c(\alpha,\beta_i) \mid \alpha < \beta_0\} \cap \{c(\alpha,\beta_{i'}) \mid \alpha < \beta_0\}$ is finite;
- As c has injective fibers, $a' := \bigcup_{i<k}\{\alpha < \beta_0 \mid c(\alpha,\beta_i) \in T\}$ is finite.

Now, pick $n < \omega$ such that $\xi_n \notin a \cup a'$, and set $\alpha := \xi_n$. Let $j := p(\alpha,\beta_0)$.

- As $\alpha \in \beta_0 \setminus a$, we infer that $p(\alpha,\beta_i) = j$ for all $i < k$;
- As $\alpha \in \beta_0 \setminus a'$, we infer that $c(\alpha,\beta_i) \neq c(\alpha,\beta_{i'})$ for all $i < i' < k$.

So $|\{c(\alpha,\beta) \mid (\alpha,\beta) \in [X]^2 \ \&\ p(\alpha,\beta) = j\}| \geq k$. □

Theorem 3.9. *For every partition $p : [\omega_1]^2 \to \omega$ and every uncountable $X \subseteq \omega_1$ such that $p \restriction [X]^2$ does not witness $\mathrm{U}(\omega_1, \omega_1, \omega, \omega)$, for every coloring $c : [\omega_1]^2 \to \omega$ with finite-to-one fibers, there exists $j < \omega$ such that*

$$\{c(\alpha,\beta) \mid (\alpha,\beta) \in [X]^2, p(\alpha,\beta) = j\} \text{ is infinite.}$$

[2] For every regular uncountable cardinal κ that is not greatly Mahlo, there is a partition $p : [\kappa]^2 \to \omega$ with nowhere bounded-to-one fibers that nevertheless satisfy $\mathrm{U}(\kappa,\kappa,\omega,\omega)$; see [LHR21, Lemma 2.12(3), Lemma 2.2(3) and the proof Lemma 5.8].

Proof. Suppose p and X are as above. Fix $n, k < \omega$ and an uncountable pairwise disjoint family $\mathcal{A} \subseteq [X]^k$, such that for every uncountable $\mathcal{B} \subseteq \mathcal{A}$ there is a pair $(a, b) \in [\mathcal{B}]^2$ such that $p[a \times b] \cap n \neq \emptyset$. By the Dushhnik-Miller theorem, then, there exists a $<$-increasing sequence $\langle a_i \mid i < \omega + 1\rangle$ of elements of $\mathcal{A}$ such that $p[a_i \times a_{i'}] \cap n \neq \emptyset$ for all $i < i' < \omega + 1$. It follows that there exist $I \in [\omega]^\omega$, $\beta \in a_\omega$, and $\langle \alpha_i \mid i \in I\rangle \in \prod_{i\in I} a_i$ such that $i \mapsto p(\alpha_i, \beta)$ is constant over I with some value $j < n$. Then, for every coloring $c : [\omega_1]^2 \to \omega$ with finite-to-one fibers, the set $\{c(\alpha, \beta) \mid (\alpha, \beta) \in [X]^2, p(\alpha, \beta) = j\}$ is infinite. □

For a partition $p : [\kappa]^2 \to \mu$, denote by $\kappa \to_p [\kappa]^2_{\theta,<\theta'}$ the assertion that for every coloring $c : [\kappa]^2 \to \theta$, there is $X \subseteq \kappa$ of size κ such that, for any cell $j < \mu$,

$$|\{c(\alpha, \beta) \mid (\alpha, \beta) \in [X]^2 \ \&\ p(\alpha, \beta) = j\}| < \theta'.$$

The next result is Theorem C.

Corollary 3.10. *Suppose $\mathsf{MA}_{\aleph_1}(K)$ holds. Then for every partition $p : [\omega_1]^2 \to \omega$, the following are equivalent:*

(1) $\omega_1 \to_p [\omega_1]^2_{\omega,\mathit{finite}}$;
(2) *There exists $X \in [\omega_1]^{\aleph_1}$ such that $p \restriction [X]^2$ witnesses* $\mathrm{U}(\omega_1, \omega_1, \omega, \omega)$.

Proof. (1) $\implies$ (2): Fix any coloring $c : [\omega_1]^2 \to \omega$ with finite-to-one fibers. Assuming that $\omega_1 \to_p [\omega_1]^2_{\omega,\text{finite}}$ holds, let us now fix $X \in [\omega_1]^{\aleph_1}$ that witnesses the instance $\omega_1 \to_p [\omega_1]^2_{\omega,\text{finite}}$ for the coloring c. This means that $\{c(\alpha, \beta) \mid (\alpha, \beta) \in [X]^2 \ \&\ p(\alpha, \beta) = j\}$ is finite for every $j < \omega$. So, by Theorem 3.9, $p \restriction [X]^2$ must witness $\mathrm{U}(\omega_1, \omega_1, \omega, \omega)$.

(2) $\implies$ (1): Fix $X \in [\omega_1]^{\aleph_1}$ such that $p \restriction [X]^2$ witnesses $\mathrm{U}(\omega_1, \omega_1, \omega, \omega)$. Then, by Theorem 3.7, for every coloring $c : [\omega_1]^2 \to \omega$, there is a decomposition $X = \biguplus_{i<\omega} X_i$ such that, for all $i, j < \omega$, $\{c(\alpha, \beta) \mid (\alpha, \beta) \in [X_i]^2 \ \&\ p(\alpha, \beta) = j\}$ is finite. Fix $i < \omega$ such that X_i is uncountable. Then X_i witnesses the instance $\omega_1 \to_p [\omega_1]^2_{\omega,\text{finite}}$ for the coloring c. □

The same proof yields:

Corollary 3.11. *Assuming $\mathsf{MA}_{\aleph_1}(K)$, for every partition $p : [\omega_1]^2 \to \omega$, the following are equivalent:*

(1) *There is a decomposition $\omega_1 = \biguplus_{i<\omega} X_i$ such that, for all $i, j < \omega$,*

$$\{c(\alpha, \beta) \mid (\alpha, \beta) \in [X_i]^2 \ \&\ p(\alpha, \beta) = j\} \text{ is finite;}$$

(2) *There is a decomposition $\omega_1 = \biguplus_{i<\omega} X_i$ such that, for all $i < \omega$, $p \restriction [X_i]^2$ witnesses* $\mathrm{U}(\omega_1, \omega_1, \omega, \omega)$. □

For completeness, we mention that by [CKS21, Corollary 29] it is consistent with $\mathsf{MA}_{\omega_1}(\sigma\text{-linked})$ that $\omega_1 \nrightarrow_p [\omega_1]^2_\omega$ (in fact, $\mathrm{Pr}_0(\omega_1, \omega_1, \omega_1, \omega)_p$) holds for any partition $p : [\omega_1]^2 \to \omega$.

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