COMPLICATED COLORINGS, REVISITED

ASSAF RINOT AND JING ZHANG

Abstract. In a paper from 1997, Shelah asked whether $\mathrm{Pr}_1(\lambda^+,\lambda^+,\lambda^+,\lambda)$ holds for every inaccessible cardinal λ. Here, we prove that an affirmative answer follows from $\square(\lambda^+)$. Furthermore, we establish that for every pair $\chi<\kappa$ of regular uncountable cardinals, $\square(\kappa)$ implies $\mathrm{Pr}_1(\kappa,\kappa,\kappa,\chi)$.

1. Introduction

The subject matter of this paper is the following two anti-Ramsey coloring principles:

Definition 1.1 (Shelah, [She88])**.** $\mathrm{Pr}_1(\kappa,\kappa,\theta,\chi)$ asserts the existence of a coloring $c:[\kappa]^2\rightarrow\theta$ such that for every $\sigma<\chi$, every pairwise disjoint subfamily $\mathcal{A}\subseteq[\kappa]^\sigma$ of size κ, and every $\tau<\theta$, there is $(a,b)\in[\mathcal{A}]^2$ such that $c[a\times b]=\{\tau\}$.

Definition 1.2 (Lambie-Hanson and Rinot, [LHR18])**.** $\mathrm{U}(\kappa,\mu,\theta,\chi)$ asserts the existence of a coloring $c:[\kappa]^2\rightarrow\theta$ such that for every $\sigma<\chi$, every pairwise disjoint subfamily $\mathcal{A}\subseteq[\kappa]^\sigma$ of size κ, and every $\tau<\theta$, there exists $\mathcal{B}\in[\mathcal{A}]^\mu$ such that, for every $(a,b)\in[\mathcal{B}]^2$, $\min(c[a\times b])\geq\tau$.[1]

For those who meet these definitions for the first time, it is beneficial to start by parsing the special case in which the 4^{th} parameter gets its smallest possible nontrivial value: The instance $\mathrm{Pr}_1(\kappa,\kappa,\theta,2)$ asserts the existence of a coloring $c:[\kappa]^2\rightarrow\theta$ such that c takes on every possible color on every large set, that is, for every $A\subseteq\kappa$ of size κ, the set $\{c(\alpha,\beta)\mid\alpha<\beta\text{ is a pair of elements of }A\}$ is equal to θ. Likewise, $\mathrm{U}(\kappa,2,\theta,2)$ asserts the existence of a coloring $c:[\kappa]^2\rightarrow\theta$ such that c takes on unboundedly many colors on every large set.

By increasing the 4^{th} parameter one gets "block-wise" version of the above principles. The importance of this line of study — especially in proving instances of $\mathrm{Pr}_1(\ldots)$ and $\mathrm{U}(\ldots)$ with a large value of the 4^{th} parameter — is explained in details in the introductions to [Rin14a, Rin14b, LHR18]. In what follows, we survey a few milestone results, depending on the identity of κ.

► At the level of the first uncountable cardinal $\kappa=\aleph_1$, the picture is complete: In his seminal paper [Tod87], Todorčević proved that $\mathrm{Pr}_1(\aleph_1,\aleph_1,\aleph_1,2)$ holds, improving upon a classic result of Sierpiński [Sie33] asserting that $\mathrm{Pr}_1(\aleph_1,\aleph_1,2,2)$ holds. In 1980, Galvin [Gal80] proved that $\mathrm{Pr}_1(\aleph_1,\aleph_1,\theta,\aleph_0)$ is independent of ZFC for any cardinal $\theta\in[2,\aleph_1]$. Finally, a few years ago, by pushing further ideas of Moore [Moo06], Peng and Wu [PW18] proved that $\mathrm{Pr}_1(\aleph_1,\aleph_1,\aleph_1,\chi)$ holds for every $\chi\in[2,\aleph_0)$. As for the other coloring principle, and in contrast with Galvin's result, by [LHR18], $\mathrm{U}(\aleph_1,\aleph_1,\theta,\aleph_0)$ holds for any cardinal $\theta\in[2,\aleph_1]$.

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[1]Note that $\mathrm{Pr}_1(\kappa,\kappa,\theta,\chi)$ implies $\mathrm{U}(\kappa,2,\theta,\chi)$. However, by [LHR21, Theorem 3.3], it does not imply $\mathrm{U}(\kappa,\kappa,\theta,\chi)$.

▶ At the level of the second uncountable cardinal, $\kappa = \aleph_2$, a celebrated result of Shelah [She97] asserts that $\mathrm{Pr}_1(\aleph_2, \aleph_2, \aleph_2, \aleph_0)$ is a theorem of ZFC. Ever since, the following problem remained open:

Open problem (Shelah, [She97, She19])**.** (1) Does $\mathrm{Pr}_1(\aleph_2, \aleph_2, \aleph_2, \aleph_1)$ hold?
(2) Does $\mathrm{Pr}_1(\lambda^+, \lambda^+, \lambda^+, \lambda)$ hold for λ inaccessible?

In comparison, by [LHR18], $\mathrm{U}(\lambda^+, \lambda^+, \theta, \lambda)$ is a theorem of ZFC for every infinite regular cardinal λ and every cardinal $\theta \in [2, \lambda^+]$.

▶ At the level of $\kappa = \lambda^+$ for λ a singular cardinal, the main problem left open has to do with the 3rd parameter of $\mathrm{Pr}_1(\dots)$ rather than the 4th (see [She94a, ES05, ES09, Eis10, Eis13a, Eis13b]). This is a consequence of three findings. First, by the main result of [Rin12], for every singular cardinal λ and every cardinal $\theta \leq \lambda^+$, $\mathrm{Pr}_1(\lambda^+, \lambda^+, \theta, 2)$ implies $\mathrm{Pr}_1(\lambda^+, \lambda^+, \theta, \mathrm{cf}(\lambda))$. Second, by [RZ22, §2], if λ is the singular limit of strongly compact cardinals, then $\mathrm{Pr}_1(\lambda^+, \lambda^+, 2, (\mathrm{cf}(\lambda))^+)$ fails, meaning that the first result cannot be improved. Third, by [RZ22, §2], $\mathrm{Pr}_1(\lambda^+, \lambda^+, 2, \lambda)$ outright fails for every singular cardinal λ.

The situation with $\mathrm{U}(\dots)$ is slightly better. An analog of the first result may be found as [LHR18, Lemma 2.5 and Theorem 4.21(3)]. An analog of the second result may be found as [LHR18, Theorem 2.14]. In contrast, by [LHR18, Corollary 4.15], it is in fact consistent that $\mathrm{U}(\lambda^+, \lambda^+, \theta, \lambda)$ holds for every singular cardinal λ and every cardinal $\theta \in [2, \lambda^+]$.

▶ At the level of a Mahlo cardinal κ, by [She94b, Conclusion 4.8(2)], the existence of a stationary subset of $E^\kappa_{\geq\chi}$ that does not reflect at inaccessibles entails that $\mathrm{Pr}_1(\kappa, \kappa, \theta, \chi)$ holds for all $\theta < \kappa$. By [RZ22, §5], the existence of nonreflecting a stationary subset of $\mathrm{Reg}(\kappa)$ on which $\diamondsuit$ holds entails that $\mathrm{Pr}_1(\kappa, \kappa, \kappa, \kappa)$ holds.

The situation with $\mathrm{U}(\dots)$ is analogous: By [LHR18, Theorem 4.23], the existence of a stationary subset of $E^\kappa_{\geq\chi}$ that does not reflect at inaccessibles entails that $\mathrm{U}(\kappa, \kappa, \theta, \chi)$ holds for all $\theta < \kappa$. By [LHR22, Proposition 2.5], the existence of nonreflecting a stationary subset of $\mathrm{Reg}(\kappa)$ entails that $\mathrm{U}(\kappa, \kappa, \theta, \kappa)$ holds for all $\theta \leq \kappa$.

▶ At the level of an abstract regular cardinal $\kappa \geq \aleph_2$, we mention two key results. First, by [Rin14b], for every regular cardinal $\kappa \geq \aleph_2$ and every $\chi \in \mathrm{Reg}(\kappa)$ such that $\chi^+ < \kappa$, the existence of a nonreflecting stationary subset of $E^\kappa_{\geq\chi}$ entails that $\mathrm{Pr}_1(\kappa, \kappa, \kappa, \chi)$ holds (this is optimal, by [LHR21, Theorem 3.4], it is consistent that for some inaccessible cardinal κ, E^κ_χ admits a nonreflecting stationary set, and yet, $\mathrm{Pr}_1(\kappa, \kappa, \kappa, \chi^+)$ fails). Second, by [Rin14a], for every regular cardinal $\kappa \geq \aleph_2$ and every $\chi \in \mathrm{Reg}(\kappa)$ such that $\chi^+ < \kappa$, $\square(\kappa)$ entails that $\mathrm{Pr}_1(\kappa, \kappa, \kappa, \chi)$ holds.

Here, the situation with $\mathrm{U}(\dots)$ is again better. By [LHR18, Corollaries 4.12 and 4.15] and [LHR22, Theorems 4.4 and 4.13], the analogs of the two results are true even without requiring "$\chi^+ < \kappa$"!

After many years without progress on the above mentioned Open Problem, in the last few years, there have been a few breakthroughs. In an unpublished note from 2017, Todorčević proved that CH implies a weak form of $\mathrm{Pr}_1(\aleph_2, \aleph_2, \aleph_2, \aleph_1)$, strong enough to entail one of its intended applications (the existence of a σ-complete $\aleph_2$-cc partial order whose square does not satisfy that $\aleph_2$-cc). Next, in [RZ22, §6], the authors obtained a full lifting of Galvin's strong coloring theorem, proving that for every infinite regular cardinal λ, $\mathrm{Pr}_1(\lambda^+, \lambda^+, \lambda^+, \lambda)$ holds assuming the stick principle $\clubsuit$-stick(λ^+). In particular, an affirmative answer to (1) follows from $2^{\aleph_1} = \aleph_2$.

Then, very recently, in [She21], Shelah proved that for every regular uncountable cardinal λ, $\mathrm{Pr}_1(\lambda^+,\lambda^+,\lambda^+,\lambda)$ holds assuming the existence of a nonreflecting stationary subset of $E^{\lambda^+}_{<\lambda}$. So, by a standard fact from inner model theory, a negative answer to (1) implies that $\aleph_2$ is a Mahlo cardinal in Gödel's constructible universe.

The main result of this paper reads as follows:

Theorem A. *For every regular uncountable cardinal λ, if $\square(\lambda^+)$ holds, then so does $\mathrm{Pr}_1(\lambda^+,\lambda^+,\lambda^+,\lambda)$. In particular, a negative answer to (1) implies that $\aleph_2$ is a weakly compact cardinal in Gödel's constructible universe.*

Thanks to the preceding theorem, we can now waive the hypothesis "$\chi^+<\kappa$" from [Rin14a, Theorem B], altogether getting a clear picture:

Theorem A′. *For every pair $\chi<\kappa$ of regular uncountable cardinals, $\square(\kappa)$ implies $\mathrm{Pr}_1(\kappa,\kappa,\kappa,\chi)$.*

Now, let us say a few words about the proof. As made clear by the earlier discussion, in the case that $\kappa=\chi^+$, it is easier to prove $\mathrm{U}(\kappa,\kappa,\theta,\chi)$ than proving $\mathrm{Pr}_1(\kappa,\kappa,\theta,\chi)$. Therefore, we consider the following slight strengthening of $\mathrm{U}(\dots)$:

Definition 1.3. $\mathrm{U}_1(\kappa,\mu,\theta,\chi)$ asserts the existence of a coloring $c:[\kappa]^2\rightarrow\theta$ such that for every $\sigma<\chi$, every pairwise disjoint subfamily $\mathcal{A}\subseteq[\kappa]^\sigma$ of size κ, and every $\epsilon<\theta$, there exists $\mathcal{B}\in[\mathcal{A}]^\mu$ such that, for every $(a,b)\in[\mathcal{B}]^2$, there exists $\tau>\epsilon$ such that $c[a\times b]=\{\tau\}$.

Shelah's proof from [She21] can be described as utilizing the hypothesis of his theorem twice: first to get $\mathrm{U}_1(\lambda^+,2,\lambda^+,\lambda)$, and then to derive $\mathrm{Pr}_1(\lambda^+,\lambda^+,\lambda^+,\lambda)$ from the latter.[2] Here, we shall follow a similar path, building on the progress made in [RZ21, §5] with respect to walking along well-chosen $\square(\kappa)$-sequences. We shall also present a couple of propositions translating $\mathrm{U}_1(\dots)$ to $\mathrm{Pr}_1(\dots)$ and vice versa, demonstrating that $\mathrm{U}_1(\kappa,\mu,\theta,\chi)$ is of interest also with $\theta<\kappa$. For instance, it will be proved that for every regular uncountable cardinal λ that is not greatly Mahlo (e.g., $\lambda=\aleph_1$), $\mathrm{U}_1(\lambda^+,2,\lambda,\lambda)$ iff $\mathrm{Pr}_1(\lambda^+,\lambda^+,\lambda^+,\lambda)$. Thus, the core contribution of this paper reads as follows.

Theorem B. *Suppose that $\chi\le\theta\le\kappa$ are infinite regular cardinals such that $\max\{\chi,\aleph_1\}<\kappa$. If $\square(\kappa)$ holds, then so does $\mathrm{U}_1(\kappa,2,\theta,\chi)$.*

1.1. **Organization of this paper.** In Section 2, we provide some necessary background on C-sequences and walks on ordinals. In response to a request of the referee, this section now offers many discussions around the relevant definitions, with the hope of helping the reader become comfortable with these definitions.

In Section 3, we prove Theorem B as Theorem 3.1.

In Section 4, we provide sufficient conditions for instances of U_1 to imply Pr_1, and then derive Theorem A as Corollary 4.7.

2. Preliminaries

In what follows, $\chi<\kappa$ denotes a pair of infinite regular cardinals. $\mathrm{Reg}(\kappa)$ stands for the set of all infinite and regular cardinals below κ. Let $E^\kappa_\chi:=\{\alpha<\kappa\mid \mathrm{cf}(\alpha)=\chi\}$, and define $E^\kappa_{\le\chi}$, $E^\kappa_{<\chi}$, $E^\kappa_{\ge\chi}$, $E^\kappa_{>\chi}$, $E^\kappa_{\neq\chi}$ analogously. A stationary

[2] We learned from Feldman that our principle U_1 is close to the principle Pr_4 from [She94b, Definition 4.3]. Specifically, $\mathrm{U}_1(\lambda^+,2,\lambda^+,\lambda)$ coincides with $\mathrm{Pr}_4(\lambda^+,\lambda^+,\lambda^+,\lambda)$.

subset $S \subseteq \kappa$ is *nonreflecting* (resp. *nonreflecting at inaccessibles*) iff there exists no $\alpha \in E^{\kappa}_{>\omega}$ (resp. α a regular limit uncountable cardinal) such that $S \cap \alpha$ is stationary in α. For a set of ordinals a, we write $\mathrm{ssup}(a) := \sup\{\alpha + 1 \mid \alpha \in a\}$, $\mathrm{acc}^+(a) := \{\alpha < \mathrm{ssup}(a) \mid \sup(a \cap \alpha) = \alpha > 0\}$, $\mathrm{acc}(a) := a \cap \mathrm{acc}^+(a)$ and $\mathrm{nacc}(a) := a \setminus \mathrm{acc}(a)$. For sets of ordinals that are not ordinals, a and b, we write $a < b$ to express that $\alpha < \beta$ for all $\alpha \in a$ and $\beta \in b$. For an ordinal σ and a set of ordinals A, we write $[A]^\sigma$ for $\{B \subseteq A \mid \mathrm{otp}(B) = \sigma\}$. In the special case that $\sigma = 2$ and $\mathcal{A}$ is either an ordinal or a collection of sets of ordinals, we interpret $[\mathcal{A}]^2$ as the collection of *ordered* pairs $\{(a, b) \in \mathcal{A} \times \mathcal{A} \mid a < b\}$. In particular, $[\kappa]^2 = \{(\alpha, \beta) \mid \alpha < \beta < \kappa\}$.

For the rest of this section, let us fix a *C-sequence* $\vec{C} = \langle C_\alpha \mid \alpha < \kappa\rangle$ over κ, i.e., for every $\alpha < \kappa$, C_α is a closed subset of α with $\sup(C_\alpha) = \sup(\alpha)$.

Definition 2.1 (Todorčević)**.** From $\vec{C}$, derive maps $\mathrm{Tr} : [\kappa]^2 \to {}^{\omega}\kappa$, $\rho_2 : [\kappa]^2 \to \omega$, $\mathrm{tr} : [\kappa]^2 \to {}^{<\omega}\kappa$ and $\lambda : [\kappa]^2 \to \kappa$, as follows. Let $(\alpha, \beta) \in [\kappa]^2$ be arbitrary.

- $\mathrm{Tr}(\alpha, \beta) : \omega \to \kappa$ is defined by recursion on $n < \omega$:

$$\mathrm{Tr}(\alpha,\beta)(n) := \begin{cases} \beta, & n = 0 \\ \min(C_{\mathrm{Tr}(\alpha,\beta)(n-1)} \setminus \alpha), & n > 0 \ \&\ \mathrm{Tr}(\alpha,\beta)(n-1) > \alpha \\ \alpha, & \text{otherwise} \end{cases}$$

- $\rho_2(\alpha, \beta) := \min\{l < \omega \mid \mathrm{Tr}(\alpha, \beta)(l) = \alpha\}$;
- $\mathrm{tr}(\alpha, \beta) := \mathrm{Tr}(\alpha, \beta) \restriction \rho_2(\alpha, \beta)$;
- $\lambda(\alpha, \beta) := \max\{\sup(C_{\mathrm{Tr}(\alpha,\beta)(n)} \cap \alpha) \mid n < \rho_2(\alpha, \beta)\}$.

The above definition comes from the theory of *walks on ordinals* [Tod07]. Given a pair of ordinals $\alpha < \beta$ below κ, one would like to *walk* from β down to α. This is done by recursion, letting $\beta_0 := \beta$, and $\beta_{n+1} := \min(C_{\beta_n} \setminus \alpha)$, thus, obtaining an ordinal β_{n+1} such that $\alpha \leq \beta_{n+1} \leq \beta_n$. Since the ordinals are well-founded, there must exist some integer k such that $\beta_{k+1} = \alpha$, so that, the walk is $\beta = \beta_0 > \beta_1 > \cdots > \beta_{k+1} = \alpha$. The functions of Definition 2.1 record various aspects of this walk. The walk itself is recorded by $\mathrm{Tr}(\alpha, \beta)$, since, for every $n \leq k$, we have that $\mathrm{Tr}(\alpha, \beta) = \beta_n$, and for every $n > k$, we have that $\mathrm{Tr}(\alpha, \beta) = \alpha$. The length of the walk is recorded by $\rho_2(\alpha, \beta)$. Now, since $\mathrm{Tr}(\alpha, \beta)$ is eventually constant with value α, its nontrivial part is those ordinals greater than α, i.e., $\beta_0 > \beta_1 > \cdots > \beta_k$; this is recorded by $\mathrm{tr}(\alpha, \beta)$. Next, notice that for every $i < k$, since $\beta_{i+1} = \min(C_{\beta_i} \setminus \alpha)$ is still bigger than α, the fact that C_{β_i} is a closed subset of β_i implies that $\sup(C_{\beta_i} \cap \alpha)$ is smaller than α. In the special case that also $\sup(C_{\beta_k} \cap \alpha)$ is smaller than α, altogether $\lambda(\alpha, \beta)$ is an ordinal smaller than α, and then a very useful concatenation phenomenon is taking place. Indeed, in this case, for every ordinal ϵ lying in-between $\lambda(\alpha, \beta)$ and α, the walk from β to ϵ is the outcome of first walking from β to α and then walking from α to ϵ. See, e.g., [Rin14b, Claim 3.1.2] for a proof of the following elementary fact.

Fact 2.2. *Whenever $\lambda(\alpha, \beta) < \epsilon < \alpha < \beta < \kappa$, $\mathrm{tr}(\epsilon, \beta) = \mathrm{tr}(\alpha, \beta)^\frown \mathrm{tr}(\epsilon, \alpha)$.*

In [Rin14a], a natural variation of $\lambda(\cdot, \cdot)$ was considered. It is called $\lambda_2(\cdot, \cdot)$ and it has the property that $\lambda_2(\alpha, \beta) < \alpha$ whenever $0 < \alpha < \beta < \kappa$.

Definition 2.3 ([Rin14a, Definition 2.8])**.** Define $\lambda_2 : [\kappa]^2 \to \kappa$ via

$$\lambda_2(\alpha, \beta) := \sup(\alpha \cap \{\sup(C_\delta \cap \alpha) \mid \delta \in \mathrm{Im}(\mathrm{tr}(\alpha, \beta))\}).$$

With λ_2, one gets Fact 2.2 as Case (1) of two possible cases:

Fact 2.4 ([LHR18, Lemma 4.7]). *Suppose that $\lambda_2(\alpha,\beta)<\epsilon<\alpha<\beta<\kappa$. Then $\mathrm{tr}(\epsilon,\beta)$ end-extends $\mathrm{tr}(\alpha,\beta)$, and one of the following cases holds:*

(1) *$\alpha\in\mathrm{Im}(\mathrm{tr}(\epsilon,\beta))$; or*
(2) *$\alpha\in\mathrm{acc}(C_\delta)$ for $\delta:=\min(\mathrm{Im}(\mathrm{tr}(\alpha,\beta)))$.*

Given an ordinal $\eta<\kappa$ and a walk of the form $\beta=\beta_0>\beta_1>\cdots>\beta_{k+1}=\alpha$, it is quite natural to ask whether η shows up as an element of C_{β_n} for some $n\le k$. If it is the case, then we would like to record the least such n, and otherwise, we would like to record that it is not the case. This is achieved, as follows.

Convention 2.5 ([RZ21]). For every ordinal $\eta<\kappa$ and a pair $(\alpha,\beta)\in[\kappa]^2$, let

$$\eta_{\alpha,\beta}:=\min\{n<\omega\mid \eta\in C_{\mathrm{Tr}(\alpha,\beta)(n)}\text{ or }n=\rho_2(\alpha,\beta)\}+1.$$

By now, it should be clear that the definition of the walk from β to α and all the corresponding charachteristic functions (such as ρ_2 and λ_2) are highly sensitive to the initial choice of the sequence $\vec{C}$. This motivates the introduction of measures that calibrate various aspects of $\vec{C}$. Motivated by [Tod07, Theorem 6.3.2], in [LHR21, Definition 1.7(1)], the measure $\chi(\cdot)$ was introduced, and it was shown that by walking along $\vec{C}$, the outcome ρ_2 witnesses $\mathrm{U}(\kappa,\kappa,\omega,\chi(\vec{C}))$ (see [LHR21, Lemma 5.8]). Then, in [RZ22, Definition 3.13], two more measures were defined. The two measures are χ_1,χ_2, and it is the case that $\chi_2(\vec{C},\kappa)\le\chi_1(\vec{C})\le\chi(\vec{C})$. For our purpose, it suffices to define χ_1. Its definition makes use of Convention 2.5, as follows.

Definition 2.6 ([RZ22, §3]). $\chi_1(\vec{C})$ stands for the supremum of $\sigma+1$ over all $\sigma<\kappa$ satisfying the following condition:

For every pairwise disjoint subfamily $\mathcal{A}\subseteq[\kappa]^\sigma$ of size κ, there are a stationary set $\Delta\subseteq\kappa$ and an ordinal $\eta<\kappa$ such that, for every $\delta\in\Delta$, there exist κ many $b\in\mathcal{A}$ such that, for every $\beta\in b$, $\lambda(\delta,\beta)=\eta$ and $\rho_2(\delta,\beta)=\eta_{\delta,\beta}$.

This means that for every ordinal $\sigma<\chi_1(\vec{C})$ and every pairwise disjoint family consisting of κ many blocks, each being a subset of κ of ordertype σ, there are stationarily many $\delta<\kappa$ for which there are κ many blocks b in the given family that are 'good for δ' in the following sense:

(a) $\beta\mapsto\lambda(\delta,\beta)$ is a constant map over b, whose sole value is smaller than δ;
(b) for any $\beta\in b$, if $\beta=\beta_0>\beta_1>\cdots>\beta_{k+1}=\delta$ denotes the walk from β to δ, then k is the least (equivalently, unique) $n\le k$ to satisfy $\lambda(\delta,\beta)=\sup(C_{\beta_n}\cap\delta)$.

By pressing down, we can find a stationary set $\Delta\subseteq\kappa$ and some ordinal $\eta<\kappa$ such that, in Clause (a), the sole value is η for every $\delta\in\Delta$. Recalling Fact 2.2, this tells us that for every $\delta\in\Delta$ and every ϵ lying in-between η and δ, for every β in a block good for δ, the walk from β to ϵ passes through δ, i.e., there exists some $n<\omega$ such that $\mathrm{Tr}(\epsilon,\beta)(n)=\delta$. Then, Clause (b) tells us that, in fact, $n=\eta_{\delta,\beta}$.

It is not hard to see that $\chi_1(\vec{C})$ cannot exceed $\sup(\mathrm{Reg}(\kappa))$. In the proof of [RZ22, Lemma 3.16], a sufficient condition for $\vec{C}$ to attain its maximal possible χ_1-value was given. The condition is a combination of $\vec{C}$ being *coherent* and satisfying a modest form of *club-guessing*. To be more specific:

Fact 2.7 ([RZ22, §3]). *If the following hold:*

($\aleph$) *for all $\alpha < \kappa$ and $\delta \in \mathrm{acc}(C_\alpha)$, $C_\delta = C_\alpha \cap \delta$, and*
($\beth$) *for every club $D \subseteq \kappa$, there exists $\gamma > 0$ with $\sup(\mathrm{nacc}(C_\gamma) \cap D) = \gamma$,*

then $\chi_1(\vec{C}) = \sup(\mathrm{Reg}(\kappa))$.

The existence of C-sequences $\vec{C}$ over an inaccessible κ satisfying $\chi_1(\vec{C}) = \kappa$ were used in [RZ22, §5] to prove the consistency of $\mathrm{Pr}_1(\kappa, \kappa, \kappa, \kappa)$.

Definition 2.8 (Todorčević, [Tod87]). For a cardinal $\mu \leq \kappa$, $\square(\kappa, {<}\mu)$ asserts the existence of a sequence $\vec{\mathcal{C}} = \langle \mathcal{C}_\alpha \mid \alpha < \kappa \rangle$ such that

1. for every $\alpha < \kappa$, $\mathcal{C}_\alpha$ is nonempty collection of less than μ many closed subsets C of α with $\sup(C) = \sup(\alpha)$;
2. for all $\alpha < \kappa$, $C \in \mathcal{C}_\alpha$ and $\delta \in \mathrm{acc}(C)$, $C \cap \delta \in \mathcal{C}_\delta$;
3. there exists no club C in κ such that $C \cap \alpha \in \mathcal{C}_\alpha$ for all $\alpha \in \mathrm{acc}(C)$.

The special case of $\square(\kappa, {<}\mu)$ with $\mu = 2$ is denoted by $\square(\kappa)$. Equivalently, $\square(\kappa)$ asserts the existence of a coherent C-sequence $\vec{C} = \langle C_\alpha \mid \alpha < \kappa \rangle$ with the property that for every club C in κ, there exists some $\alpha \in \mathrm{acc}(C)$ such that $C \cap \alpha \neq C_\alpha$.

Fact 2.9 (Hayut and Lambie-Hanson, [HLH17, Lemma 2.4]). *Clause (3) of Definition 2.8 is preserved in any κ-cc forcing extension, provided that $\mu < \kappa$.*

3. Theorem B

Theorem 3.1. *Suppose that $\chi \leq \theta \leq \kappa$ are infinite regular cardinals such that $\max\{\chi, \aleph_1\} < \kappa$. If $\square(\kappa)$ holds, then so does $\mathrm{U}_1(\kappa, 2, \theta, \chi)$.*

Proof. Suppose that $\square(\kappa)$ holds. Then, by [RZ21, Lemma 5.1], we may fix a C-sequence $\vec{C} = \langle C_\alpha \mid \alpha < \kappa \rangle$ satisfying the following:

1. $C_{\alpha+1} = \{0, \alpha\}$ for every $\alpha < \kappa$;
2. for every club $D \subseteq \kappa$, there exists $\gamma > 0$ with $\sup(\mathrm{nacc}(C_\gamma) \cap D) = \gamma$;
3. for every $\alpha \in \mathrm{acc}(\kappa)$ and $\bar{\alpha} \in \mathrm{acc}(C_\alpha)$, $C_{\bar{\alpha}} = C_\alpha \cap \bar{\alpha}$;
4. for every $i < \kappa$, $\{\alpha < \kappa \mid \min(C_\alpha) = i\}$ is stationary.

Note that, by Fact 2.7, $\chi_1(\vec{C}) = \sup(\mathrm{Reg}(\kappa))$. If $\theta < \kappa$, then let $\mu := \theta$; otherwise, let $\mu := \chi$. Derive a coloring $h : \kappa \to \mu$ via

$$h(\alpha) := \begin{cases} \min(C_\alpha), & \text{if } \min(C_\alpha) < \mu; \\ 0, & \text{otherwise.} \end{cases}$$

We shall walk along $\vec{C}$. Derive a function $\mathrm{tr}_h : [\kappa]^2 \to {}^{<\omega}\kappa$ via

$$\mathrm{tr}_h(\alpha, \beta) := \langle h(\mathrm{Tr}(\alpha, \beta)(i)) \mid i < \rho_2(\alpha, \beta) \rangle.$$

Then, define a coloring $d : [\kappa]^2 \to \mu$ via

$$d(\alpha, \beta) := \max(\mathrm{Im}(\mathrm{tr}_h(\alpha, \beta))).$$

Simply put, for every pair $\alpha < \beta$ of ordinals in κ, if $\beta = \beta_0 > \cdots > \beta_{k+1} = \alpha$ denotes the walk from β to α, and there exists some $n \leq k$ such that $\min(C_{\beta_n}) < \mu$, then $d(\alpha, \beta)$ is equal to the largest possible value of $\min(C_{\beta_n})$ over all such n's.

Claim 3.1.1. *Suppose that α, β, γ are ordinals, and $\lambda_2(\gamma, \beta) < \alpha < \gamma < \beta < \kappa$.*

Then $\mathrm{Im}(\mathrm{tr}_h(\alpha, \beta)) = \mathrm{Im}(\mathrm{tr}_h(\alpha, \gamma)) \cup \mathrm{Im}(\mathrm{tr}_h(\gamma, \beta))$. In particular, $d(\alpha, \beta) = \max\{d(\alpha, \gamma), d(\gamma, \beta)\}$.

Proof. By Fact 2.4, one of the following cases holds:

- $\gamma \in \mathrm{Im}(\mathrm{tr}(\alpha,\beta))$. In this case, $\mathrm{tr}(\alpha,\beta) = \mathrm{tr}(\gamma,\beta)^\frown \mathrm{tr}(\alpha,\gamma)$, so we done.
- $\gamma \in \mathrm{acc}(C_\delta)$ for $\delta := \min(\mathrm{Im}(\mathrm{tr}(\gamma,\beta)))$. In this case, $\mathrm{tr}(\alpha,\beta) = \mathrm{tr}(\delta,\beta)^\frown \mathrm{tr}(\alpha,\delta)$, so that $\mathrm{Im}(\mathrm{tr}_h(\alpha,\beta)) = \mathrm{Im}(\mathrm{tr}_h(\alpha,\delta)) \cup \mathrm{Im}(\mathrm{tr}_h(\delta,\beta))$. Since $\gamma \in \mathrm{acc}(C_\delta)$, Clause (3) above and the definition of the function h together imply that $\mathrm{tr}_h(\alpha,\delta) = \mathrm{tr}_h(\alpha,\gamma)$. In addition, $\mathrm{tr}(\gamma,\beta) = \mathrm{tr}(\delta,\beta)^\frown\langle\delta\rangle$, so that $\mathrm{Im}(\mathrm{tr}_h(\gamma,\beta)) = \mathrm{Im}(\mathrm{tr}_h(\delta,\beta)) \cup \{h(\delta)\}$. Since $h(\delta) \in \mathrm{Im}(\mathrm{tr}_h(\alpha,\delta))$, altogether,

$$\mathrm{Im}(\mathrm{tr}_h(\alpha,\gamma)) \cup \mathrm{Im}(\mathrm{tr}_h(\gamma,\beta)) = \mathrm{Im}(\mathrm{tr}_h(\alpha,\delta)) \cup \mathrm{Im}(\mathrm{tr}_h(\delta,\beta)). \qquad \square$$

We are now ready to define the sought coloring c. If $\mu = \theta$, then let $c := d$, and otherwise define $c : [\kappa]^2 \to \theta$ via

$$c(\alpha,\beta) := \max\{\xi \in \mathrm{Im}(\mathrm{tr}(\alpha,\beta)) \mid h(\xi) = d(\alpha,\beta)\}.$$

To see that c witnesses $\mathrm{U}_1(\kappa,2,\theta,\chi)$, suppose that we are given $\epsilon < \theta$, $\sigma < \chi$ and a κ-sized pairwise disjoint subfamily $\mathcal{A} \subseteq [\kappa]^\sigma$; we need to find $\tau > \epsilon$ and $(a,b) \in [\mathcal{A}]^2$ such that $c[a \times b] = \{\tau\}$. Recall that χ is a regular cardinal smaller than κ and that $\chi_1(\vec{C}) = \sup(\mathrm{Reg}(\kappa))$, so that $\sigma < \chi \le \chi_1(\vec{C})$. In particular, we may fix a stationary subset $\Delta \subseteq \kappa$ and an ordinal $\eta < \kappa$ such that, for every $\delta \in \Delta$, there exists $b \in \mathcal{A}$ with $\min(b) > \delta$ such that $\lambda(\delta,\beta) = \eta$ for every $\beta \in b$. Set $\eta' := \max\{\eta,\epsilon\}$.

Consider the club $C := \{\gamma < \kappa \mid \sup\{\min(a) \mid a \in \mathcal{A} \cap \mathcal{P}(\gamma)\} = \gamma\}$. For all $\gamma \in C$ and $\varepsilon < \gamma$, fix $a^\gamma_\varepsilon \in \mathcal{A} \cap \mathcal{P}(\gamma)$ with $\min(a^\gamma_\varepsilon) > \varepsilon$; as $|a^\gamma_\varepsilon| < \mu$, $\tau^\gamma_\varepsilon := \sup\{d(\alpha,\gamma) \mid \alpha \in a^\gamma_\varepsilon\}$ is $< \mu$. As $\kappa \ge \aleph_2$, we may fix some stationary $\Gamma \subseteq C \cap E^\kappa_{\neq\mu}$ along with $\tau_0 < \mu$ such that, for every $\gamma \in \Gamma$, $\sup\{\varepsilon < \gamma \mid \tau^\gamma_\varepsilon \le \tau_0\} = \gamma$.

By Clause (4), for each $i < \mu$, $H_i := \{\alpha < \kappa \mid h(\alpha) = i\}$ is stationary, so, fix $\delta \in \Delta \cap \bigcap_{i<\mu} \mathrm{acc}^+(H_i \cap \mathrm{acc}^+(\Gamma \setminus \eta'))$. Pick $b \in \mathcal{A}$ with $\min(b) > \delta$ such that $\lambda(\delta,\beta) = \eta$ for every $\beta \in b$. As $|b| < \mu$, $\tau_1 := \sup\{d(\delta,\beta) \mid \beta \in b\}$ is $< \mu$. If $\epsilon < \mu$, then pick $\zeta \in H_{\tau_0+\tau_1+\epsilon+1} \cap \mathrm{acc}^+(\Gamma \setminus \eta')$; otherwise, pick $\zeta \in H_{\tau_0+\tau_1+1} \cap \mathrm{acc}^+(\Gamma \setminus \eta')$. Next, pick $\gamma \in \Gamma$ above $\max\{\lambda_2(\zeta,\delta),\eta'\}$. Finally, pick $\varepsilon < \gamma$ above $\max\{\lambda_2(\gamma,\zeta),\lambda_2(\zeta,\delta),\eta'\}$ such that $\tau^\gamma_\varepsilon \le \tau_0$, and then set $a := a^\gamma_\varepsilon$.

Claim 3.1.2. *Let $\alpha \in a$ and $\beta \in b$. Then:*

(i) $\max\{\lambda_2(\gamma,\zeta),\lambda_2(\zeta,\delta),\lambda(\delta,\beta),\epsilon\} < \varepsilon < \alpha < \gamma < \zeta < \delta < \beta$;
(ii) $c(\alpha,\beta) = c(\gamma,\delta) > \epsilon$.

Proof. (i) This is clear, recalling that $\eta' = \max\{\lambda(\delta,\beta),\epsilon\}$.

(ii) From $\lambda(\delta,\beta) < \alpha < \delta < \beta$ and Fact 2.2, we infer that $\mathrm{tr}(\alpha,\beta) = \mathrm{tr}(\delta,\beta)^\frown \mathrm{tr}(\alpha,\delta)$, so that $d(\alpha,\beta) = \max\{d(\delta,\beta), d(\alpha,\delta)\}$. By Clause (i) and Claim 3.1.1,

$$d(\alpha,\delta) = \max\{d(\alpha,\zeta), d(\zeta,\delta)\} \ge h(\zeta) > \tau_1 \ge d(\delta,\beta).$$

Consequently, $d(\alpha,\beta) = d(\alpha,\delta)$ and $c(\alpha,\beta) = c(\alpha,\delta)$. By Clause (i) and Claim 3.1.1, $\mathrm{Im}(\mathrm{tr}_h(\alpha,\delta)) = \mathrm{Im}(\mathrm{tr}_h(\alpha,\gamma) \cup \mathrm{Im}(\mathrm{tr}_h(\gamma,\delta))$. As $d(\alpha,\gamma) \le \tau_0 < h(\zeta) \le d(\gamma,\delta)$, it follows that $d(\alpha,\delta) = d(\gamma,\delta)$ and $c(\alpha,\delta) = d(\gamma,\delta)$. Altogether, $c(\alpha,\beta) = c(\gamma,\delta)$.

Now, if $\theta < \kappa$, then $\epsilon < \theta = \mu$ and $c = d$, so that $c(\alpha,\beta) = d(\gamma,\delta) \ge h(\zeta) > \epsilon$. Otherwise, $c(\alpha,\beta) \ge \min(\mathrm{Im}(\mathrm{tr}(\alpha,\beta))) > \alpha > \epsilon$. $\square$

Set $\tau := c(\gamma,\delta)$. Then $\tau > \epsilon$ and $c[a \times b] = \{\tau\}$, as sought. $\square$

Remark 3.2. The preceding proof makes it clear that the auxiliary coloring d witnesses $\mathrm{U}_1(\kappa,2,\mu,\chi)$. By Fact 2.4, the coloring d is moreover *closed* in the sense that, for all $\beta<\kappa$ and $i<\theta$, the set $\{\alpha<\beta\mid c(\alpha,\beta)\le i\}$ is closed below β. So, by [LHR18, Lemma 4.2], d witnesses $\mathrm{U}(\kappa,\kappa,\mu,\chi)$, as well.

4. Connecting U_1 with Pr_1

Throughout this section, $\chi<\kappa$ is a pair of infinite regular cardinals, and θ is a regular cardinal $\le\kappa$. Let $\mathbb{A}^\kappa_\chi$ denote the collection of all pairwise disjoint subfamilies $\mathcal{A}\subseteq\mathcal{P}(\kappa)$ such that $|\mathcal{A}|=\kappa$ and $\sup\{|a|\mid a\in\mathcal{A}\}<\chi$. Given a coloring $c:[\kappa]^2\to\theta$, for every $\mathcal{A}\subseteq\mathcal{P}(\kappa)$, let $T_c(\mathcal{A})$ be the set of all $\tau<\theta$ such that, for some $(a,b)\in[\mathcal{A}]^2$, $c[a\times b]=\{\tau\}$. The next definition appears (with a slightly different notation) in Stage B in the proof of [She21, Theorem 1.1]:

Definition 4.1. For every coloring $c:[\kappa]^2\to\theta$, let

$$F_{c,\chi}:=\{T\subseteq\theta\mid\exists\mathcal{A}\in\mathbb{A}^\kappa_\chi\,[T_c(\mathcal{A})\subseteq T]\}.$$

Proposition 4.2. *Suppose that a coloring $c:[\kappa]^2\to\theta$ witnesses $\mathrm{U}_1(\kappa,2,\theta,\chi)$, and λ is some cardinal. Then:*

1. *$F_{c,\chi}$ is a χ-complete uniform filter on θ;*
2. *If every χ-complete uniform filter on θ is not weakly λ-saturated, then $\mathrm{Pr}_1(\kappa,\kappa,\lambda,\chi)$ holds.*

Proof. (1) It is clear that $F_{c,\chi}$ is upward-closed. To see that it is χ-complete, suppose that we are given a sequence $\langle X_i\mid i<\delta\rangle$ of elements of $F_{c,\chi}$, for some $\delta<\chi$. For each $i<\delta$, fix $\mathcal{A}_i\in\mathbb{A}^\kappa_\chi$ such that $T_c(\mathcal{A}_i)\subseteq X_i$. Pick $\mathcal{A}\in\mathbb{A}^\kappa_\chi$ such that, for every $a\in\mathcal{A}$, there is a sequence $\langle a_i\mid i<\delta\rangle\in\prod_{i<\delta}\mathcal{A}_i$ such that $a=\bigcup_{i<\delta}a_i$. Then, $T_c(\mathcal{A})\subseteq\bigcap_{i<\delta}T_c(\mathcal{A}_i)\subseteq\bigcap_{i<\delta}X_i$ and hence the latter is in $F_{c,\chi}$. Finally, since c witnesses $\mathrm{U}_1(\kappa,2,\theta,\chi)$, for every $\mathcal{A}\in\mathbb{A}^\kappa_\chi$ and every $\epsilon<\theta$, $T_c(\mathcal{A})\setminus\epsilon$ is nonempty. So $F_{c,\chi}$ consists of cofinal subset of θ. Since θ is regular, $F_{c,\chi}$ is uniform.

(2) Suppose that no χ-complete uniform filter on θ is weakly λ-saturated. In particular, by Clause (1), we may pick a map $\psi:\theta\to\lambda$ such that that the preimage of any singleton is $F_{c,\theta}$-positive. Then $\psi\circ c$ witnesses $\mathrm{Pr}_1(\kappa,\kappa,\lambda,\chi)$. □

Definition 4.3 (folklore)**.** For a regular uncountable cardinal λ, $\mathrm{Refl}(\lambda,\lambda,\mathrm{Reg}(\lambda))$ asserts that for every sequence $\langle S_i\mid i<\lambda\rangle$ of stationary subsets of λ, there exists an inaccessible cardinal $\beta<\lambda$ such that $S_i\cap\beta$ is stationary in β for every $i<\beta$.

Corollary 4.4. *Suppose that λ is a regular uncountable cardinal. If $\mathrm{Refl}(\lambda,\lambda,\mathrm{Reg}(\lambda))$ fails, then the following are equivalent:*

1. *$\mathrm{Pr}_1(\lambda^+,\lambda^+,\lambda^+,\lambda)$;*
2. *$\mathrm{Pr}_1(\lambda^+,\lambda^+,\lambda,\lambda)$;*
3. *$\mathrm{U}_1(\lambda^+,2,\lambda,\lambda)$.*

Proof. The implication (1) $\implies$ (2) $\implies$ (3) is trivial, and the implication (2) $\implies$ (1) is well-known (this follows from [KRS22, Theorem A]). By the preceding proposition, to see the implication (3) $\implies$ (2), it suffices to prove that under our hypothesis on λ, no λ-complete uniform filter on λ is weakly λ-saturated. By [IR22, Theorem A], the latter follows from $\chi(\lambda)>1$. By [LHR21, Lemma 2.12], if $\chi(\lambda)\le 1$, then $\mathrm{Refl}(\lambda,\lambda,\mathrm{Reg}(\lambda))$ holds. □

Corollary 4.5. *If λ is a regular uncountable cardinal that is not greatly Mahlo, then $\mathrm{U}_1(\lambda^+, 2, \lambda, \lambda)$ is equivalent to $\mathrm{Pr}_1(\lambda^+, \lambda^+, \lambda^+, \lambda)$.*

Proof. For every regular uncountable cardinal λ, the reflection principle $\mathrm{Refl}(\lambda, \lambda, \mathrm{Reg}(\lambda))$ implies that λ is greatly Mahlo (see [LHR21, Proposition 2.11]). $\square$

Lemma 4.6. *Suppose that λ is a regular uncountable cardinal and $\square(\lambda^+, {<}\lambda)$ holds. Then every λ-complete uniform filter on λ^+ is not weakly λ-saturated.*

Proof. Fix a $\square(\lambda^+, {<}\lambda)$-sequence $\vec{\mathcal{C}} = \langle \mathcal{C}_\alpha \mid \alpha < \lambda^+\rangle$. For each $\alpha < \lambda^+$, fix an injective enumeration $\langle C_{\alpha,i} \mid i < |\mathcal{C}_\alpha|\rangle$ of $\mathcal{C}_\alpha$.

Towards a contradiction, suppose that F is a λ-complete uniform filter on λ^+ that is weakly λ-saturated. Since F is λ-complete, F is moreover λ-saturated. Hence, $\mathcal{P}(\lambda^+)/F$ is a λ-cc notion of forcing.

Let G be $\mathcal{P}(\lambda^+)/F$-generic over V. Then G is a uniform V-ultrafilter over λ^+ extending F. By [For10, Propositions 2.9 and 2.14], $\mathrm{Ult}(V, G)$ is well-founded and $j : V \to M \simeq \mathrm{Ult}(V, G)$ satisfies $\mathrm{crit}(j) = \lambda$.

Now, work in $V[G]$. Denote $j(\vec{\mathcal{C}})$ by $\langle \mathcal{D}_\alpha \mid \alpha < j(\lambda^+)\rangle$. For every $\alpha < \lambda^+$, since $\mathrm{crit}(j) = \lambda > |\mathcal{C}_\alpha|$, it is the case that $\mathcal{D}_{j(\alpha)} = j(\mathcal{C}_\alpha) = j``\mathcal{C}_\alpha$. Since $G$ is uniform, $\gamma := \sup(j``\lambda^+)$ is $< j(\lambda^+)$, as witnessed by the identity map $\mathrm{id} : \lambda^+ \to \lambda^+$. As $V[G]$ is a λ-cc forcing extension of V, $\mathrm{cf}^V(\gamma) = \mathrm{cf}^{V[G]}(\gamma) = \lambda^+$, so that $\mathrm{cf}^M(\gamma) \geq \lambda^+$. Pick $D \in \mathcal{D}_\gamma$.

Claim 4.6.1. *$A := j^{-1}[\mathrm{acc}(D)]$ is a cofinal subset of λ^+.*

Proof. Given $\epsilon < \lambda^+$, we recursively define (in $V[G]$) an increasing sequence $\langle \alpha_n \mid n < \omega\rangle$ of ordinals below λ^+ such that:

(1) $\epsilon = \alpha$, and
(2) for all $n < \omega$, $(j(\alpha_n), j(\alpha_{n+1})] \cap D \neq \emptyset$.

Consider $\alpha^* := \sup_{n<\omega} \alpha_n$. Notice that $\mathrm{cf}^V(\alpha^*) < \lambda$, since if $\mathrm{cf}^V(\alpha^*) \geq \lambda$, then by the fact that $V[G]$ is a λ-cc forcing extension of V we have $\omega = \mathrm{cf}^{V[G]}(\alpha^*) \geq \lambda$ which is impossible. As a result, $\sup j``\alpha^* = j(\alpha^*) \in \mathrm{acc}(D)$, which implies that α^* is an element of A above ϵ. $\square$

For each $\alpha \in A$, $D \cap j(\alpha) \in \mathcal{D}_{j(\alpha)} = j``\mathcal{C}_\alpha$, so we may pick some $i_\alpha < \lambda$ such that $D \cap j(\alpha) = j(C_{\alpha,i_\alpha})$. Fix some $i < \lambda$ for which $A' := \{\alpha \in A \mid i_\alpha = i\}$ is cofinal in λ^+. For every $(\alpha, \beta) \in [A']^2$, $j(C_{\alpha,i}) = D \cap j(\alpha)$ and $j(C_{\beta,i}) = D \cap j(\beta)$, so, by elementarity, $C_{\alpha,i} = C_{\beta,i} \cap \alpha$. As A' is cofinal in λ^+, it follows that $C := \bigcup\{C_{\alpha,i} \mid \alpha \in A\}$ is a club in λ^+. Evidently, $C \cap \alpha \in \mathcal{C}_\alpha$ for every $\alpha \in \mathrm{acc}(C)$. However, $V[G]$ is a λ-cc forcing extension of V, contradicting Fact 2.9. $\square$

We are now ready to prove Theorem A:

Corollary 4.7. *Suppose that λ is a regular uncountable cardinal, and $\square(\lambda^+)$ holds. Then $\mathrm{Pr}_1(\lambda^+, \lambda^+, \lambda^+, \lambda)$ holds, as well.*

Proof. By Theorem 3.1, using $(\kappa, \theta, \chi) := (\lambda^+, \lambda^+, \lambda)$, $\mathrm{U}_1(\lambda^+, 2, \lambda^+, \lambda)$ holds. So, by Proposition 4.2 (using $\theta := \lambda^+$) and Lemma 4.6, $\mathrm{Pr}_1(\lambda^+, \lambda^+, \lambda, \lambda)$ holds. Then, again by [KRS22, Theorem A], $\mathrm{Pr}_1(\lambda^+, \lambda^+, \lambda^+, \lambda)$ holds, as well. $\square$

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Department of Mathematics, Bar-Ilan University, Ramat-Gan 5290002, Israel.
URL: http://www.assafrinot.com

Department of Mathematics, Bar-Ilan University, Ramat-Gan 5290002, Israel.
URL: https://jingjzzhang.github.io/