A NEW SMALL DOWKER SPACE

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Abstract. It is proved that a strong instance of the guessing principle $\clubsuit_{\mathrm{AD}}$ on the first uncountable cardinal follows from either the principle ⫯, or the principle $\diamondsuit(\mathfrak{b})$, or the existence of a Luzin set. In particular, any of the above hypotheses entails the existence of a Dowker space of size $\aleph_1$.

1. Introduction

A *Dowker space* is a normal topological space $\mathbb{X}$ whose product with the unit interval $\mathbb{X} \times [0,1]$ is not normal. Whether such a space exists was asked by Dowker back in 1951 [Dow51]. As of now, there are just three constructions of Dowker spaces in ZFC: Rudin's space of size $(\aleph_\omega)^{\aleph_0}$ [Rud72], Balogh's space of size continuum [Bal96], and the Kojman-Shelah space of size $\aleph_{\omega+1}$ [KS98]. Rudin's *Conjecture 4* from [Rud90], asserting that there exists a Dowker space of size $\aleph_1$ remains open.

In [RS23], the guessing principle $\clubsuit_{\mathrm{AD}}$ was introduced, and it was shown that for every regular uncountable cardinal κ, each of the following two conditions implies the existence of a Dowker space of size κ:

(i) $\clubsuit_{\mathrm{AD}}(\mathcal{S}, 1, 2)$ holds for a partition $\mathcal{S}$ of some nonreflecting stationary subset of κ into infinitely many stationary sets;
(ii) $\clubsuit_{\mathrm{AD}}(\{E^\kappa_\lambda\}, \lambda, 1)$ holds, where κ is the successor of a regular cardinal λ.

It was also shown that in each of the scenarios of [Rud74, dC77, Wei81, Goo95] in which there exists a Dowker space of size κ, either (i) or (ii) indeed hold.

The definition of the guessing principle under discussion reads as follows:

Definition 1.1. Let $\mathcal{S}$ be a collection of stationary subsets of a regular uncountable cardinal κ, and μ, θ be nonzero cardinals below κ. The principle $\clubsuit_{\mathrm{AD}}(\mathcal{S}, \mu, \theta)$ asserts the existence of a sequence $\langle \mathcal{A}_\alpha \mid \alpha \in \bigcup \mathcal{S} \rangle$ such that:

(1) For every $\alpha \in \mathrm{acc}(\kappa) \cap \bigcup \mathcal{S}$, $\mathcal{A}_\alpha$ is a pairwise disjoint family of μ many cofinal subsets of α;
(2) For every $\mathcal{B} \subseteq [\kappa]^\kappa$ of size θ, for every $S \in \mathcal{S}$, there are stationarily many $\alpha \in S$ such that $\sup(A \cap B) = \alpha$ for all $A \in \mathcal{A}_\alpha$ and $B \in \mathcal{B}$;
(3) For all $A \neq A'$ from $\bigcup_{S \in \mathcal{S}} \bigcup_{\alpha \in S} \mathcal{A}_\alpha$, $\sup(A \cap A') < \sup(A)$.

Remark 1.2. The variation $\clubsuit_{\mathrm{AD}}(\mathcal{S}, \mu, {<}\theta)$ is defined in the obvious way.

In [JKR76], Juhász, Kunen and Rudin constructed a Dowker space of size $\aleph_1$ assuming the continuum hypothesis. This was then improved by the third author [Tod89, p. 53] who got such a space from the existence of a Luzin set (cf. [Sze94]).

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The first main result of this paper shows that also in the above scenarios (when the continuum hypothesis holds or merely a Luzin set exists), Clauses (i) and (ii) hold. This answers Question 2.35 of [RS23] in the negative.

Theorem A. *Let $\lambda = \lambda^{<\lambda}$ be an infinite regular cardinal.*

Then (1) $\implies$ (2) $\implies$ (3):

(1) There exists a λ^+-Luzin subset of ${}^{\lambda}\lambda$;
(2) There exists a tight strongly unbounded coloring $c : \lambda \times \lambda^+ \to \lambda$;
(3) For every partition $\mathcal{S}$ of $E^{\lambda^+}_{\lambda}$ into stationary sets, $\clubsuit_{\mathrm{AD}}(\mathcal{S}, \lambda, {<}\lambda)$ holds.

Note that as in the case $\lambda := \aleph_0$, for every infinite cardinal $\lambda = \lambda^{<\lambda}$, the existence of a λ^+-Luzin subset of ${}^{\lambda}\lambda$ follows from $2^{\lambda} = \lambda^+$. Our second main result derives $\clubsuit_{\mathrm{AD}}$ from another consequence of $2^{\lambda} = \lambda^+$, namely, from the *stick* principle:

Theorem B. *Let λ be an infinite regular cardinal.*

Then $\stick(\lambda^+)$ implies that for every partition $\mathcal{S}$ of $E^{\lambda^+}_{\lambda}$ into stationary sets, $\clubsuit_{\mathrm{AD}}(\mathcal{S}, \lambda, \lambda)$ holds.

What's interesting about Theorem B is that it uncovers a scenario for the existence of a Dowker space that was not known before. In particular, it yields the following contribution to the small Dowker space problem:

Corollary C. *$\stick$ entails the existence of a Dowker spaces of size $\aleph_1$.*

Let us expand on our topological application. In Clause (i) above, the Dowker space is obtained as a ladder-system space, and in Clause (ii), it is a de Caux type space of hereditary density λ and Lindelöf degree κ (i.e., an S-space). At the end of this paper, we shall tweak the construction of Clause (i) to obtain many pairwise nonhomeomorphic Dowker spaces. Altogether, it will follow that $\stick$ entails the existence of $2^{\aleph_1}$ many pairwise nonhomeomorphic Dowker spaces of size $\aleph_1$.

1.1. **Organization of this paper.** In Section 2, we define *unbounded*, *strongly unbounded* and *tight* colorings, and provide sufficient conditions for their existence. The proof of the implication (1) $\implies$ (2) of Theorem A will be found there.

In Section 3, we get an instance of $\clubsuit_{\mathrm{AD}}(\dots)$ from a tight strongly unbounded coloring. In particular, the proof of the implication (2) $\implies$ (3) of Theorem A will be found there.

In Section 4, we get instances of $\clubsuit_{\mathrm{AD}}(\dots)$ from the principles $\stick(\lambda^+)$ and $\diamondsuit(\mathfrak{b})$. In particular, the proof of Theorem B will be found there.

In the Appendix, we slightly extend Clause (i) above, showing that for every regular uncountable cardinal κ, if $\clubsuit_{\mathrm{AD}}(\mathcal{S}, 1, 2)$ holds for an infinite partition $\mathcal{S}$ of some nonreflecting stationary subset of κ into μ many stationary sets, then there are 2^{μ} many pairwise nonhomeomorphic Dowker spaces of size κ.

2. Unbounded colorings and generalized Luzin sets

In this section, κ denotes a regular uncountable cardinal, and ν and λ denote infinite cardinals $< \kappa$.

Definition 2.1. Suppose that $c : \nu \times \kappa \to \lambda$ is a coloring.

- For each $\beta < \kappa$, derive the fiber map $c_\beta : \nu \to \lambda$ via $c_\beta(\eta) := c(\eta, \beta)$;
- c is *unbounded* iff for every cofinal $B \subseteq \kappa$, there is an $\eta < \nu$ such that

$$\sup\{c_\beta(\eta) \mid \beta \in B\} = \lambda;$$

- c is *strongly unbounded* iff for every cofinal $B \subseteq \kappa$, there are an $\eta < \nu$ and a map $t : \eta \to \lambda$ such that
$$\sup\{c_\beta(\eta) \mid \beta \in B \ \&\ t \subseteq c_\beta\} = \lambda;$$
- For every $\mathbb{T} \subseteq {}^{<\nu}\lambda$, let $[\mathbb{T}]_c := \{\beta < \kappa \mid \forall \eta < \nu\, (c_\beta \restriction \eta \in \mathbb{T})\}$;
- Set $\mathcal{T}_c := \{\mathbb{T} \subseteq {}^{<\nu}\lambda \mid \sup([\mathbb{T}]_c) = \kappa\}$;
- c is *tight* iff $\mathrm{cf}(\mathcal{T}_c, \supseteq) \leq \kappa$.

Remark 2.2. It is clear that ${}^{<\nu}\lambda \in \mathcal{T}_c$, so that $1 \leq |\mathcal{T}_c| \leq 2^{(\lambda^{<\nu})}$.

2.1. **From unbounded to strongly unbounded.**

Lemma 2.3. *Suppose that $\lambda = \aleph_0$ or λ is strongly inaccessible.*
Then any unbounded coloring $c : \lambda \times \kappa \to \lambda$ is strongly unbounded.

Proof. Suppose that $c : \lambda \times \kappa \to \lambda$ is a given unbounded coloring. Let B be some cofinal subset of κ. By hypothesis, there exists an $\eta < \lambda$ such that $\sup\{c_\beta(\eta) \mid \beta \in B\} = \lambda$. For the least such η, it follows that there exists some ordinal $\mu < \lambda$ such that
$$\{c_\beta(i) \mid \beta \in B, i < \eta\} \subseteq \mu.$$
As $|{}^{\eta}\mu| < \mathrm{cf}(\lambda) = \lambda$, there must exist some $t \in {}^{\eta}\mu$ such that
$$\sup\{c_\beta(\eta) \mid \beta \in B \ \&\ t \subseteq c_\beta\} = \lambda,$$
as sought. □

Definition 2.4 (Shelah, [She83, §2])**.** For a regular uncountable cardinal λ:
1. $D\ell_\lambda$ asserts the existence of a sequence $\langle \mathcal{P}_\eta \mid \eta < \lambda\rangle$ such that:
   - for every $\eta < \lambda$, $\mathcal{P}_\eta \subseteq \mathcal{P}(\eta)$ and $|\mathcal{P}_\eta| < \lambda$;
   - for every $A \subseteq \lambda$, for stationarily many $\eta < \lambda$, $A \cap \eta \in \mathcal{P}_\eta$.
2. $D\ell^*_\lambda$ asserts the existence of a sequence $\langle \mathcal{P}_\eta \mid \eta < \lambda\rangle$ such that:
   - for every $\eta < \lambda$, $\mathcal{P}_\eta \subseteq \mathcal{P}(\eta)$ and $|\mathcal{P}_\eta| < \lambda$;
   - for every $A \subseteq \lambda$, for club many $\eta < \lambda$, $A \cap \eta \in \mathcal{P}_\eta$.

Fact 2.5 (Shelah, [She00, Claim 3.2] and [She10, Claim 2.5])**.** *For a regular uncountable cardinal λ:*
1. *If λ is strongly inaccessible, then $D\ell^*_\lambda$ holds;*
2. *$\diamondsuit_\lambda$ implies $D\ell_\lambda$, and $\diamondsuit^*_\lambda$ implies $D\ell^*_\lambda$;*
3. *If $\lambda \geq \beth_\omega$ then $D\ell_\lambda$ iff $\lambda^{<\lambda} = \lambda$;*
4. *If λ is a successor of an uncountable cardinal, then $D\ell_\lambda$ iff $\lambda^{<\lambda} = \lambda$.*

Lemma 2.6. *Suppose that λ is a regular uncountable cardinal and $D\ell^*_\lambda$ holds. Then there exists a strongly unbounded coloring $c : \lambda \times \kappa \to \lambda$, for $\kappa := \mathfrak{b}_\lambda$.*

Proof. We commence with verifying the following variation of weak diamond.

Claim 2.6.1. *For every function $F : {}^{<\lambda}\lambda \to \lambda$, there exists a function $g : \lambda \to \lambda$ with the property that for every function $f : \lambda \to \lambda$, the following set covers a club:*
$$\{\eta < \lambda \mid F(f \restriction \eta) \leq g(\eta)\}.$$

Proof. Using $D\ell^*_\lambda$, we may fix a sequence $\langle \mathcal{F}_\eta \mid \eta < \lambda\rangle$ such that:
- for every $\eta < \lambda$, $\mathcal{F}_\eta \subseteq {}^{\eta}\eta$ and $|\mathcal{F}_\eta| < \lambda$;
- for every function $f : \lambda \to \lambda$, for club many $\eta < \lambda$, $f \restriction \eta \in \mathcal{F}_\eta$.

Now, given any function $F : {}^{<\lambda}\lambda \to \lambda$, define an oracle function $g : \lambda \to \lambda$ via

$$g(\eta) := \sup\{F(t) \mid t \in \mathcal{F}_\eta\}.$$

Next, given any function $f : \lambda \to \lambda$, the set $C := \{\eta < \lambda \mid f \restriction \eta \in \mathcal{F}_\eta\}$ covers a club, and it is clear that, for every $\eta \in C$, $g(\eta) \geq F(f \restriction \eta)$. □

For functions $f, g \in {}^\lambda\lambda$, let $f <_{cl} g$ iff $\{\alpha < \lambda \mid f(\alpha) < g(\alpha)\}$ covers a club. By [CS95, Theorem 6], $\mathfrak{b}_\lambda$ coincides with the least size of a family of functions from λ to λ that is not bounded with respect to $<_{cl}$. It follows that we may construct a $<_{cl}$-increasing sequence of functions $\langle f_\beta \mid \beta < \kappa\rangle$ for which $\{f_\beta \mid \beta < \kappa\}$ is not bounded with respect to $<_{cl}$. Define $c : \lambda \times \kappa \to \lambda$ via $c(\eta, \beta) := f_\beta(\eta)$. Then, for every function $g : \lambda \to \lambda$, for a tail of $\beta < \kappa$, $S_\beta(g) := \{\eta < \lambda \mid g(\eta) \leq c_\beta(\eta)\}$ is stationary. We claim that c is strongly unbounded. Towards a contradiction, suppose that this is not the case, as witnessed by a cofinal set B. Then, we may define a function $F : {}^{<\lambda}\lambda \to \lambda$ by letting for all $\eta < \lambda$ and $t : \eta \to \lambda$,

$$F(t) := \sup\{c_\beta(\eta) \mid \beta \in B, t \subseteq c_\beta\} + 1.$$

Now, pick a corresponding oracle $g : \lambda \to \lambda$ such that for every function $f : \lambda \to \lambda$, the following set covers a club:

$$C_f := \{\eta < \lambda \mid F(f \restriction \eta) \leq g(\eta)\}.$$

Pick $\beta \in B$ such that $S_\beta(g)$ is stationary. Then, find $\eta \in S_\beta(g) \cap C_{c_\beta}$. Altogether, $c_\beta(\eta) < F(c_\beta \restriction \eta) \leq g(\eta) \leq c_\beta(\eta)$. This is a contradiction. □

Lemma 2.7. *Suppose that λ is a regular uncountable cardinal and $D\ell_\lambda$ holds.*
If $\kappa = \mathfrak{b}_\lambda = \mathfrak{d}_\lambda$, then there exists a strongly unbounded coloring $c : \lambda \times \kappa \to \lambda$.

Proof. As $\kappa = \mathfrak{b}_\lambda = \mathfrak{d}_\lambda$, it is possible to construct a coloring $c : \lambda \times \kappa \to \lambda$ with the property that for every function $g : \lambda \to \lambda$, for a tail of $\beta < \kappa$,

$$C_\beta(g) := \{\eta < \lambda \mid g(\eta) \leq c_\beta(\eta)\}$$

is co-bounded in κ.[1] We claim that c is strongly unbounded. Towards a contradiction, suppose that this is not the case, as witnessed by a cofinal set B. Then, we may define a function $F : {}^{<\lambda}\lambda \to \lambda$ by letting for all $\eta < \lambda$ and $t : \eta \to \lambda$,

$$F(t) := \sup\{c_\beta(\eta) \mid \beta \in B, t \subseteq c_\beta\} + 1.$$

As $D\ell_\lambda$ holds, we may pick a corresponding oracle $g : \lambda \to \lambda$ such that for every function $f : \lambda \to \lambda$, the following set is stationary:

$$S_f := \{\eta < \lambda \mid F(f \restriction \eta) \leq g(\eta)\}.$$

Pick $\beta \in B$ such that $C_\beta(g)$ covers a club. Then, find $\eta \in C_\beta(g) \cap S_{c_\beta}$. Altogether, $c_\beta(\eta) < F(c_\beta \restriction \eta) \leq g(\eta) \leq c_\beta(\eta)$. This is a contradiction. □

2.2. **Tightness.**

Lemma 2.8. *Suppose that $\mathfrak{d} = \mathfrak{c} = \kappa$. Then there exists a tight strongly unbounded coloring $c : \omega \times \kappa \to \omega$.*

Proof. It is easy to construct an unbounded coloring $c : \omega \times \mathfrak{d} \to \omega$ (see [IR22, §6]). By Lemma 2.3, c is moreover strongly unbounded. As $\mathcal{T}_c \subseteq \mathcal{P}({}^{<\omega}\omega)$, it follows that $|T_c| \leq \mathfrak{c}$. So, if $\mathfrak{d} = \mathfrak{c} = \kappa$, then c is tight. □

[1] See [IR22, §6]: for λ regular, $\mathfrak{b}_\lambda = \mathfrak{d}_\lambda = \kappa$ implies that $\mathsf{unbounded}([\lambda]^\lambda, J^{\mathrm{bd}}[\kappa], \lambda)$ holds.

Corollary 2.9. *Suppose that $\lambda = \lambda^{<\lambda}$ is an infinite cardinal satisfying any of the following:*

- $\lambda = \aleph_0$, *or*
- $\lambda = \aleph_1$ *and* $\diamondsuit_\lambda$ *holds, or*
- $\lambda > \aleph_1$ *is a successor cardinal, or*
- $\lambda \geq \beth_\omega$, *or*
- λ *is strongly inaccessible.*

If $\kappa = \mathfrak{b}_\lambda = 2^\lambda$, then there exists a tight strongly unbounded coloring $c : \lambda \times \kappa \to \lambda$.

Proof. As in the proof of the previous lemma, the fact that $\kappa = 2^{\lambda^{<\lambda}}$ implies that any strongly unbounded coloring $c : \lambda \times \kappa \to \lambda$ is tight. Assuming $\kappa = \mathfrak{b}_\lambda$, it is also easy to obtain an unbounded coloring $c : \lambda \times \kappa \to \lambda$. So the heart of the matter is to get a strongly unbounded one. Lemma 2.3 takes care of the first and last bullet. The remaining bullets follow from Lemma 2.7 together with Fact 2.5. □

Definition 2.10. A *κ-Luzin subset of ${}^\lambda\lambda$* is a subset $L \subseteq {}^\lambda\lambda$ of size κ having the property that for every $B \in [L]^\kappa$, there exists $t \in {}^{<\lambda}\lambda$ such that, for every $t' \in {}^{<\lambda}\lambda$ extending t, there exists an element of B extending t'.

It is well-known that MA implies the existence of a $\mathfrak{c}$-Luzin subset of ${}^\omega\omega$. More generally, $\mathrm{cov}(\mathcal{M}) = \mathrm{cof}(\mathcal{M}) = \kappa$ entails the existence of a κ-Luzin subset of ${}^\omega\omega$. Also, the following fact is standard:

Fact 2.11 (Luzin)**.** *For every infinite cardinal $\lambda = \lambda^{<\lambda}$, if $2^\lambda = \lambda^+$, then there exists a λ^+-Luzin subset of ${}^\lambda\lambda$.*

Lemma 2.12. *Suppose that there exists a κ-Luzin subset of ${}^\lambda\lambda$, with κ regular.*

If $\lambda^{<\lambda} < \kappa$, then there exists a tight strongly unbounded coloring $c : \lambda \times \kappa \to \lambda$.

Proof. Fix an injective enumeration $\vec{g} = \langle g_\beta \mid \beta < \kappa\rangle$ of a κ-Luzin subset of ${}^\lambda\lambda$. Let $c : \lambda \times \kappa \to \lambda$ denote the unique coloring such that $c_\beta = g_\beta$ for all $\beta < \kappa$.

Claim 2.12.1. *c is strongly unbounded.*

Proof. Let $B \in [\kappa]^\kappa$; we need to find $\eta < \lambda$ and a map $t : \eta \to \lambda$ such that $\sup\{c_\beta(\eta) \mid \beta \in B \ \&\ t \subseteq c_\beta\} = \lambda$. As $\mathrm{Im}(\vec{g})$ is a κ-Luzin subset of ${}^\lambda\lambda$, fix some $t \in {}^{<\lambda}\lambda$ such that, for every $t' \in {}^{<\lambda}\lambda$ extending t, there exists $l \in \{g_\beta \mid \beta \in B\}$ extending t'. Set $\eta := \mathrm{dom}(t)$. Then for every $\gamma < \lambda$, we can find $\beta \in B$ such that g_β extends $t^\frown\langle\gamma\rangle$. Altogether, $\{c_\beta(\eta) \mid \beta \in B \ \&\ t \subseteq c_\beta\} = \lambda$. □

For every $s \in {}^{<\lambda}\lambda$, denote $\mathbb{T}_s := \{t \in {}^{<\lambda}\lambda \mid s \subseteq t \text{ or } t \subseteq s\}$.

Claim 2.12.2. *For every $\mathbb{T} \in \mathcal{T}_c$, there exists $s \in {}^{<\lambda}\lambda$ such that $\mathbb{T} \supseteq \mathbb{T}_s \in \mathcal{T}_c$.*

Proof. Let $\mathbb{T} \in \mathcal{T}_c$, so that $[\mathbb{T}]_c = \{\beta < \kappa \mid \forall\eta < \lambda\,(c_\beta \restriction \eta \in \mathbb{T})\}$ is cofinal in κ. For every $t \in {}^{<\lambda}\lambda$, write $A_t := \{\beta \in [\mathbb{T}]_c \mid t \subseteq c_\beta\}$. As $\lambda^{<\lambda} < \mathrm{cf}(\kappa) = \kappa$, the set $N := \bigcup\{A_t \mid t \in {}^{<\lambda}\lambda, |A_t| < \kappa\}$ has size $< \kappa$. In particular, $B := [\mathbb{T}]_c \setminus N$ has size κ. As $\mathrm{Im}(\vec{g})$ is a κ-Luzin subset of ${}^\lambda\lambda$, fix some $s \in {}^{<\lambda}\lambda$ such that, for every $s' \in {}^{<\lambda}\lambda$ extending s, there exists $\beta \in B$ such that $s' \subseteq g_\beta$. As $B \subseteq [\mathbb{T}]_c$, it follows that $\mathbb{T}_s \subseteq \mathbb{T}$.

Finally, to show that $[\mathbb{T}_s]_c = \{\beta < \kappa \mid \forall\eta < \lambda\,(c_\beta \restriction \eta \in \mathbb{T}_s)\}$ is in $\mathcal{T}_c$, we need to prove that $\sup([\mathbb{T}_s]_c) = \kappa$. Recalling that s extends g_β for some $\beta \in B \subseteq (\kappa \setminus N)$, we infer that $|A_s| = \kappa$. As $[\mathbb{T}_s]_c$ clearly covers A_s, we infer that $\sup([\mathbb{T}_s]_c) = \kappa$. □

In particular, $\mathrm{cf}(\mathcal{T}_c, \supseteq) \leq \lambda^{<\lambda}$. So, we are done. □

3. Theorem A

Definition 3.1 ([LHR23, §3.3])**.** Let $\lambda < \kappa$ be a pair of infinite cardinals, $e : [\kappa]^2 \rightarrow \lambda$ be a coloring, and S be a subset of κ.

(1) e is *S-coherent* iff for all $\beta \leq \gamma < \delta < \kappa$ with $\beta \in S$,

$$\sup\{\xi < \beta \mid e(\xi, \gamma) \neq e(\xi, \delta)\} < \beta;$$

(2) $\partial(e) := \{\alpha \in \operatorname{acc}(\kappa) \mid \forall \gamma \in \kappa \setminus \alpha\, \forall \nu < \lambda\, [\sup\{\xi < \alpha \mid e(\xi, \gamma) \leq \nu\} < \alpha]\}$.

Fact 3.2 ([LHR23, Lemma 3.31])**.** *Let $\lambda < \kappa$ be a pair of infinite regular cardinals. For a stationary subset $S \subseteq E^\kappa_\lambda$, the following are equivalent:*

- *S is nonreflecting;*
- *There exists an S-coherent coloring $e : [\kappa]^2 \rightarrow \lambda$ such that $\partial(e) \supseteq S$.*

Theorem 3.3. *Suppose:*

(1) $\theta \leq \lambda < \kappa$ are infinite cardinals, with λ, κ regular,
(2) $c : \lambda \times \kappa \rightarrow \lambda$ is a strongly unbounded coloring,
(3) c is tight. Furthermore, $(\operatorname{cf}(\mathcal{T}_c, \supseteq))^{<\theta} \leq \kappa$, and
(4) $\mathcal{S}$ is a partition of some nonreflecting stationary subset of E^κ_λ into stationary sets.

Then there exists a club $C \subseteq \kappa$ such that $\clubsuit_{\mathrm{AD}}(\{S \cap C \mid S \in \mathcal{S}\}, \lambda, {<}\theta)$ holds. If either $\kappa = \lambda^+$ or $\lambda^{<\lambda} = \lambda$, then C can moreover be taken to be whole of κ.

Proof. The proof is an elaboration of a construction from [Tod89, §2]. Let $\mathcal{T}$ be a dense subfamily of $\mathcal{T}_c$ of minimal size. Let $\operatorname{Seq}_{<\theta}(\mathcal{T})$ denote the collection of all nonempty sequences of elements of $\mathcal{T}$ of length $< \theta$. By Clause (3) above, $|\operatorname{Seq}_{<\theta}(\mathcal{T})| \leq \kappa$. Let $\mathcal{S}$ be a given partition of some nonreflecting stationary subset $\mathbb{S}$ of E^κ_λ into stationary sets. Then, for every $S \in \mathcal{S}$, let $\langle S_\sigma \mid \sigma \in \operatorname{Seq}_{<\theta}(\mathcal{T})\rangle$ be a partition of S into stationary sets.

Claim 3.3.1. *The set $\Sigma := \{c_\beta \restriction \Lambda \mid \beta < \kappa, \Lambda < \lambda\}$ has size $< \kappa$.*

Proof. Suppose not. Since κ is a regular cardinal greater than λ, it follows that there exist $B \in [\kappa]^\kappa$ and $\Lambda < \lambda$ on which the map $\beta \mapsto c_\beta \restriction \Lambda$ is injective. By possibly shrinking B further, we may also assume the existence of some $\epsilon < \lambda$ such that $c_\beta[\Lambda] \subseteq \epsilon$ for all $\beta \in B$. But then c cannot be strongly unbounded. Indeed, for every $t : \eta \rightarrow \lambda$, if $\eta < \Lambda$, then $\{c_\beta(\eta) \mid \beta \in B, t \subseteq c_\beta\} \subseteq \epsilon$, and if $\eta \in [\Lambda, \lambda)$, then $|\{c_\beta(\eta) \mid \beta \in B, t \subseteq c_\beta\}| \leq 1$. □

Set $T^* := \{t \in \Sigma \mid |\{\beta < \kappa \mid t \subseteq c_\beta\}| = \kappa\}$. By Claim 3.3.1, $|T^*| < \kappa$, so we may fix a surjection $f : \kappa \rightarrow T^*$ with the property that for every $\epsilon < \kappa$, $\{f(\xi + 1) \mid \epsilon < \xi < \epsilon + |T^*|\} = T^*$. By Claim 3.3.1, we may also fix a large enough ordinal $\varrho < \kappa$ such that $\bigcup\{\{\beta < \kappa \mid t \subseteq c_\beta\} \mid t \in \Sigma \setminus T^*\} \subseteq \varrho$.

Next, we turn to recursively define an injective matrix $\langle \beta_{\alpha,j} \mid \alpha < \kappa, j < \zeta(\alpha)\rangle$ of ordinals in κ, as follows. Suppose that $\alpha < \kappa$ and that $\langle \beta_{\bar{\alpha},\bar{j}} \mid \bar{\alpha} < \alpha, \bar{j} < \zeta(\bar{\alpha})\rangle$ has already been defined.

- ▶ If $\{(S, \sigma) \in \mathcal{S} \times \operatorname{Seq}_{<\theta}(\mathcal{T}) \mid \alpha \in S_\sigma\}$ is nonempty, then it is a singleton, so let (S, σ) denote its unique element. Write $\sigma = \langle \mathbb{T}_j \mid j < \zeta\rangle$. Set $\zeta(\alpha) := \zeta$, and then, by recursion on $j < \zeta(\alpha)$, set

$$(\star) \qquad \beta_{\alpha,j} := \min([\mathbb{T}_j]_c \setminus (\varrho \cup \{\beta_{\bar{\alpha},\bar{j}}, \beta_{\alpha,j'} \mid \bar{\alpha} < \alpha, \bar{j} \leq \zeta(\bar{\alpha}), j' < j\})).$$

▶ Otherwise, set $\zeta(\alpha) := 1$ and

$$(\star\star) \qquad \beta_{\alpha,0} := \min(\{\beta < \kappa \mid f(\alpha) \subseteq c_\beta\} \setminus (\varrho \cup \{\beta_{\bar\alpha,\bar j} \mid \bar\alpha < \alpha, \bar j < \zeta(\bar\alpha)\})).$$

Having constructed the above injective matrix of ordinals in κ, we derive a corresponding injective matrix $\vec d = \langle d_{\alpha,j} \mid \alpha < \kappa, j < \zeta(\alpha)\rangle$ by setting $d_{\alpha,j} := c_{\beta_{\alpha,j}}$.

For all $x \neq y$ in ${}^\lambda\lambda$, denote $\Delta(x,y) := \min\{\eta < \lambda \mid x(\eta) \neq y(\eta)\}$. As λ is regular, for every $x \in {}^\lambda\lambda$, we may attach a strictly increasing function $\hat x : \lambda \to \lambda$ satisfying $\hat x(\eta) \geq x(\eta)$ for all $\eta < \lambda$.

Next, as $\mathbb{S}$ is a nonreflecting stationary subset of E^κ_λ, by Fact 3.2, we may fix a coloring $e : [\kappa]^2 \to \lambda$ that is $\mathbb{S}$-coherent and such that $\mathbb{S} \subseteq \partial(e)$.

Fix a surjection $\varsigma : \kappa \to \lambda$ such that $\partial(e) \cap \varsigma^{-1}\{i\}$ is stationary for every $i < \lambda$. Next, for every $\alpha < \kappa$ and $j < \zeta(\alpha)$, define a map $h_{\alpha,j} : \alpha \to \lambda$ via:

$$h_{\alpha,j}(\xi) := \varsigma(\min\{\gamma \in (\xi,\alpha] \mid \gamma = \alpha \text{ or } e(\gamma,\alpha) \leq \Delta(d_{\xi,0}, d_{\alpha,j})\}).$$

For all $\alpha < \kappa$ and $i < \lambda$, let $A^i_{\alpha,*} := \bigcup_{j<\zeta(\alpha)} A^i_{\alpha,j}$, where for every $j < \zeta(\alpha)$:

$$A^i_{\alpha,j} := \{\xi < \alpha \mid h_{\alpha,j}(\xi) = i \;\&\; e(\xi,\alpha) \leq \hat d_{\alpha,j}(\Delta(d_{\xi,0}, d_{\alpha,j}))\}.$$

Claim 3.3.2. *Suppose that $\alpha \in \partial(e)$, $\beta < \kappa$, $j < \zeta(\alpha), j' < \zeta(\beta)$, and $i, i' < \lambda$. If $(\alpha, j) \neq (\beta, j')$. Then $\sup(A^i_{\alpha,j} \cap A^{i'}_{\beta,j'}) < \alpha$.*

Proof. Suppose that $(\alpha, j) \neq (\beta, j')$, and then let $\eta := \Delta(d_{\alpha,j}, d_{\beta,j'})$. Towards a contradiction, suppose that $A^i_{\alpha,j} \cap A^{i'}_{\beta,j'}$ is cofinal in α. Set $\nu := \hat d_{\alpha,j}(\eta)$. As $\alpha \in \partial(e)$, the following set is cofinal in α:

$$Y := \{\xi \in A^i_{\alpha,j} \cap A^{i'}_{\beta,j'} \mid e(\xi,\alpha) > \nu\}.$$

For every $\xi \in Y$, $\hat d_{\alpha,j}(\eta) = \nu < e(\xi,\alpha) \leq \hat d_{\alpha,j}(\Delta(d_{\xi,0}, d_{\alpha,j}))$, so since $\hat d_{\alpha,j}$ is strictly increasing, $\Delta(d_{\xi,0}, d_{\alpha,j}) > \eta = \Delta(d_{\alpha,j}, d_{\beta,j'})$, and hence $\Delta(d_{\xi,0}, d_{\beta,j'}) = \eta$. Set $\tau := \hat d_{\beta,j'}(\eta)$. As $Y \subseteq \alpha \cap A^{i'}_{\beta,j'}$, altogether $Y \subseteq \{\xi < \alpha \mid e(\xi,\beta) \leq \tau\}$. As $\alpha \in \partial(e)$, Y is bounded in α. This is a contradiction. □

Claim 3.3.3. *Let $\alpha \in \mathbb{S}$ and $i \neq i'$ in λ. Then $\sup(A^i_{\alpha,*} \cap A^{i'}_{\alpha,*}) < \alpha$.*

Proof. Suppose not. As $\zeta(\alpha) < \theta \leq \lambda = \mathrm{cf}(\alpha)$, there must exist $j, j' < \zeta(\alpha)$ such that $\sup(A^i_{\alpha,j} \cap A^{i'}_{\alpha,j'}) = \alpha$. As $\alpha \in \mathbb{S} \subseteq \partial(e)$, Claim 3.3.2 implies that $j = j'$. But it is evident that $A^i_{\alpha,j}$ and $A^{i'}_{\alpha,j}$ are disjoint. □

For all $\alpha \in \mathbb{S}$ and $i < \lambda$, let

$$A^i_\alpha := A^i_{\alpha,*} \setminus \bigcup_{i'<i} A^{i'}_{\alpha,*}.$$

Clearly, $\langle A^i_\alpha \mid i < \lambda\rangle$ consists of pairwise disjoint subsets of α.

Claim 3.3.4. *Let $(\alpha,\beta) \in [\mathbb{S}]^2$ and $i, i' < \lambda$. Then $\sup(A^i_\alpha \cap A^{i'}_\beta) < \alpha$.*

Proof. Suppose not. In particular, $\sup(A^i_{\alpha,*} \cap A^{i'}_{\beta,*}) = \alpha$. However, $\zeta(\alpha), \zeta(\beta) < \theta \leq \lambda = \mathrm{cf}(\alpha)$, so there must exist $j < \zeta(\alpha)$ and $j' < \zeta(\beta)$ such that $\sup(A^i_{\alpha,j} \cap A^{i'}_{\beta,j'}) = \alpha$, contradicting Claim 3.3.2. □

Now, we turn to inspect the guessing features of the matrix $\langle A^i_\alpha \mid \alpha \in \mathbb{S}, i < \lambda\rangle$.

Claim 3.3.5. *Let $S \in \mathcal{S}$, and let $\langle X_j \mid j < \zeta\rangle$ be any sequence of cofinal subsets of κ with $0 < \zeta < \theta$. Then $\{\alpha \in S \mid \forall i < \lambda \forall j < \zeta \ \sup(A^i_\alpha \cap X_j) = \alpha\}$ is stationary.*

Proof. For all $j<\zeta$ and $t\in\Sigma$, denote $X_j^t:=\{\xi\in X_j\mid t\subseteq d_{\xi,0}\}$. Set

$$T_j:=\{t\in\Sigma\mid |X_j^t|=\kappa\}.$$

By Claim 3.3.1, $N_j:=\bigcup\{X_j^t\mid t\in\Sigma\setminus T_j\}$ is the small union of sets of size $<\kappa$, so that $|X_j\setminus N_j|=\kappa$. For all $\xi\in X_j\setminus N_j$ and $\eta<\lambda$, $c_{\beta_{\xi,0}}\restriction\eta=d_{\xi,0}\restriction\eta\in T_j$, so that $[T_j]_c$ covers $\{\beta_{\xi,0}\mid\xi\in X_j\setminus N_j\}$, and hence $T_j\in\mathcal{T}_c$. Recalling that $\mathcal{T}$ is dense in $\mathcal{T}_c$, we may now pick $\mathbb{T}_j\in\mathcal{T}$ with $\mathbb{T}_j\subseteq T_j$. In particular, $\sigma:=\langle\mathbb{T}_j\mid j<\zeta\rangle$ is in $\mathrm{Seq}_{<\theta}(\mathcal{T})$, and S_σ is stationary.

Towards a contradiction, suppose that $\{\alpha\in S\mid\forall i<\lambda\forall j<\zeta\ \sup(A_\alpha^i\cap X_j)=\alpha\}$ is nonstationary. As S_σ is a stationary subset of S, we may fix $i<\lambda$ and $j<\zeta$ for which the following set is stationary:

$$S^0:=\{\alpha\in S_\sigma\mid\sup(A_\alpha^i\cap X_j)<\alpha\}.$$

For every $\alpha\in S^0$, since $i<\lambda=\mathrm{cf}(\alpha)$ and since $A_\alpha^i=A_{\alpha,*}^i\setminus\bigcup_{i'<i}A_{\alpha,*}^{i'}$, Claim 3.3.3 implies that $\sup(A_{\alpha,*}^i\cap X_j)<\alpha$. In particular, $\sup(A_{\alpha,j}^i\cap X_j)<\alpha$. So by Fodor's lemma, we may fix an $\epsilon<\kappa$ such that the following set is stationary:

$$S^1:=\{\alpha\in S_\sigma\mid\sup(A_{\alpha,j}^i\cap X_j)=\epsilon<\alpha\}.$$

By the choice of the map ς, the set Γ of all $\gamma\in\partial(e)\cap\varsigma^{-1}\{i\}$ for which there exists an elementary submodel $M_\gamma\prec H_{\kappa^+}$ containing $\{\vec{d},X_j,\Sigma\}$ and satisfying $\gamma=M_\gamma\cap\kappa$ is stationary. Fix $\delta\in\mathbb{S}\cap\mathrm{acc}^+(\Gamma\setminus\epsilon)$. As $\delta\in\mathbb{S}$ and e is $\mathbb{S}$-coherent, we may fix $S^2\in[S^1\setminus\delta]^\kappa$ along with some $\varepsilon<\delta$ such that, for every $\alpha\in S^2$,

$$\{\xi<\delta\mid e(\xi,\alpha)\neq e(\xi,\delta)\}\subseteq\varepsilon.$$

Pick $\gamma\in\Gamma\cap\delta$ above $\max\{\epsilon,\varepsilon\}$, and then fix a model M_γ witnessing that $\gamma\in\Gamma$.

Put $\nu:=e(\gamma,\delta)$. By Claim 3.3.1, we may find a cofinal subset of $S^3\subseteq S^2$ on which the map $\alpha\mapsto d_{\alpha,j}\restriction\nu$ is constant.

Next, as c is strongly unbounded and $\{\beta_{\alpha,j}\mid\alpha\in S^3\}$ is cofinal in κ, we may find an ordinal $\eta<\lambda$ and a map $t:\eta\to\lambda$ such that

$$\sup\{c_{\beta_{\alpha,j}}(\eta)\mid\alpha\in S^3,t\subseteq c_{\beta_{\alpha,j}}\}=\lambda.$$

Equivalently, for every $\tau<\lambda$, for some $\alpha\in S^3$, $d_{\alpha,j}\restriction\eta=t$ and $d_{\alpha,j}(\eta)>\tau$. Clearly, $\eta\geq\nu$.

Pick for a moment $\alpha^*\in S^3$ such that $t\subseteq d_{\alpha^*,j}$. Since $\alpha^*\in S^3\subseteq S_\sigma$, Equation ($\star$) and the definition of σ implies that $\beta_{\alpha^*,j}$ is in $[\mathbb{T}_j]_c$. Recalling Definition 2.1, from $c_{\beta_{\alpha^*,j}}\restriction\eta=d_{\alpha^*,j}\restriction\eta=t$, we infer that $t\in\mathbb{T}_j$. As $\mathbb{T}_j\subseteq T_j$, this means that $|X_j^t|=\kappa$. It thus follows from $\{\vec{d},X_j,\Sigma\}\in M_\gamma$ that $\sup(X_j^t\cap\gamma)=\gamma$. Now, as $\gamma\in\partial(e)$, $G:=\{\bar\gamma<\gamma\mid e(\bar\gamma,\delta)\leq\eta\}$ is bounded below γ. Altogether, we may find $\xi\in X_j^t\cap\gamma$ above $\max\{\epsilon,\varepsilon,\sup(G)\}$.

Set $\tau:=\max\{e(\xi,\delta),d_{\xi,0}(\eta)\}$, and then pick $\alpha\in S^3$ such that $d_{\alpha,j}\restriction\eta=t$ and $d_{\alpha,j}(\eta)>\tau$. As $d_{\alpha,j}\restriction\eta=t=d_{\xi,0}\restriction\eta$ and $d_{\alpha,j}(\eta)>d_{\xi,0}(\eta)$, we infer that $\Delta(d_{\xi,0},d_{\alpha,j})=\eta$. As $\varepsilon<\xi<\delta$, altogether,

$$\hat{d}_{\alpha,j}(\Delta(d_{\xi,0},d_{\alpha,j}))=\hat{d}_{\alpha,j}(\eta)\geq d_{\alpha,j}(\eta)>\tau\geq e(\xi,\delta)=e(\xi,\alpha).$$

Next, from $\Delta(d_{\xi,0},d_{\alpha,j})=\eta$ and the fact that $\xi>\varepsilon$, we also infer that

$$h_{\alpha,j}(\xi)=\varsigma(\min\{\bar\gamma\in(\xi,\alpha]\mid\bar\gamma=\alpha\text{ or }e(\bar\gamma,\delta)\leq\eta\}).$$

Since $e(\gamma,\delta)=\nu\le\eta$ and $\varsigma(\gamma)=i$, it follows that if $h_{\alpha,j}(\xi)\neq i$, then there exists $\bar\gamma\in(\xi,\gamma)$ such that $e(\bar\gamma,\delta)\le\eta$, contradicting the fact that $\xi>\sup(G)$. So, it is the case that $h_{\alpha,j}(\xi)=i$. Consequently, $\xi\in A^i_{\alpha,j}$.

Altogether, we established that ξ is an element of $A^i_{\alpha,j}\cap X_j$ above ϵ, contradicting the fact that $\alpha\in S^3\subseteq S^1$. □

The next claim implies that there exists a club $C\subseteq\kappa$ such that, for every $\alpha\in C$, for every $i<\lambda$, $\sup(A^i_\alpha)=\alpha$.

Claim 3.3.6. *Let $S\subseteq\mathbb{S}$ be stationary. Then $\{\alpha\in S\mid\forall i<\lambda\ \sup(A^i_\alpha\cap S)=\alpha\}$ is stationary.*

Proof. Suppose not, and fix $i<\lambda$ for which the following set is stationary:

$$S^0:=\{\alpha\in S\mid\sup(A^i_\alpha\cap S)<\alpha\}.$$

It follows that there exists an $\epsilon<\kappa$ such that

$$S^1:=\{\alpha\in S\mid\sup(A^i_{\alpha,0}\cap S)=\epsilon<\alpha\}.$$

Similarly to the proof of the previous claim, find ordinals $\varepsilon<\gamma<\delta$ and a set $S^2\in[S^1\setminus\delta]^\kappa$ such that:

- $\gamma,\delta\in\partial(e)$;
- $\delta=M_\delta\cap\kappa$ for some elementary submodel $M_\delta\prec H_{\kappa^+}$ containing $\{\vec d,S,\Sigma\}$;
- $\gamma=M_\gamma\cap\kappa$ for some elementary submodel $M_\gamma\prec H_{\kappa^+}$ containing $\{\vec d,S,\Sigma\}$;
- for every $\alpha\in S^2$, $\{\xi<\delta\mid e(\xi,\alpha)\neq e(\xi,\delta)\}\subseteq\varepsilon$.

Put $\nu:=e(\gamma,\delta)$. Then find a cofinal subset of $S^3\subseteq S^2$ on which the map $\alpha\mapsto d_{\alpha,0}\restriction\nu$ is constant. As $\{\beta_{\alpha,0}\mid\alpha\in S^3\}$ is cofinal in κ, the choice of the coloring c provides an ordinal $\eta<\lambda$ and a map $t:\eta\to\lambda$ such that, for every $\tau<\lambda$, for some $\alpha\in S^3$, $d_{\alpha,0}\restriction\eta=t$ and $d_{\alpha,0}(\eta)>\tau$. The same analysis is true for any final segment of S^3 and hence, by Clause (1) and the pigeonhole principle, we may fix some $t:\eta\to\lambda$ such that, for every $\tau<\lambda$, for cofinally many $\alpha\in S^3$, $d_{\alpha,0}\restriction\eta=t$ and $d_{\alpha,0}(\eta)>\tau$. Clearly, $\eta\ge\nu$.

As $\{\vec d,S,\Sigma\}\in M_\delta$ and $S^3\cap M_\delta=\emptyset$, by elementarity, the set of $\xi\in S\cap M_\delta$ such that $d_{\xi,0}\restriction\eta=t$ is cofinal in γ. As $\gamma\in\partial(e)$, $G:=\{\bar\gamma<\gamma\mid e(\bar\gamma,\delta)\le\eta\}$ is bounded below γ. So, we may find $\xi\in S\cap\gamma$ above $\max\{\epsilon,\varepsilon,\sup(G)\}$ such that $d_{\xi,0}\restriction\eta=t$. Set $\tau:=\max\{e(\xi,\delta),d_{\xi,0}(\eta)\}$, and then pick $\alpha\in S^3$ such that $d_{\alpha,0}\restriction\eta=t$ and $d_{\alpha,0}(\eta)>\tau$. From this point on, a verification identical to that of Claim 3.3.5 shows that ξ is an element of $A^i_{\alpha,0}\cap S$ above ϵ, contradicting the fact that $\alpha\in S^3\subseteq S^1$. □

In summary, we have shown that there exists a club $C\subseteq\kappa$ such that:

(a) For every $\alpha\in C$, for every $i<\lambda$, $\sup(A^i_\alpha)=\alpha$;
(b) For every $S\in\mathcal{S}$, for every sequence $\langle X_j\mid j<\zeta\rangle$ of cofinal subsets of κ with $0<\zeta<\theta$, the set $\{\alpha\in S\mid\forall i<\lambda\forall j<\zeta\ \sup(A^i_\alpha\cap X_j)=\alpha\}$ is stationary;
(c) For all $(\alpha,\beta)\in[\mathbb{S}]^2$ and $i,i'<\lambda$, $\sup(A^i_\alpha\cap A^{i'}_\beta)<\alpha$.

Suppose now that either $\kappa=\lambda^+$ or $\lambda^{<\lambda}=\lambda$, and let us prove that $\clubsuit_{\mathrm{AD}}(\mathcal{S},\lambda,{<}\theta)$ holds. For this, it suffices to define for every $\alpha\in\mathbb{S}\setminus C$, a sequence $\langle a^i_\alpha\mid i<\lambda\rangle$ of pairwise disjoint cofinal subsets of α such that the amalgam of $\langle\langle a^i_\alpha\mid i<\lambda\rangle\mid\alpha\in\mathbb{S}\setminus C\rangle$ and $\langle\langle A^i_\alpha\mid i<\lambda\rangle\mid\alpha\in\mathbb{S}\cap C\rangle$ will form an almost-disjoint system. To this end, let $\alpha\in\mathbb{S}\setminus C$.

Claim 3.3.7. *For every $\eta<\lambda$,* $\sup\{\xi<\alpha\mid\Delta(d_{\xi,0},d_{\alpha,0})\geq\eta\}=\alpha$.

Proof. Let $\eta<\lambda$, $t:=d_{\alpha,0}\restriction\eta$ and $\epsilon<\alpha$; we need to find ξ with $\epsilon<\xi<\alpha$ such that $t\subseteq d_{\xi,0}$. Now, recall that by the construction of $\vec{d}$, $d_{\alpha,0}=c_\beta$ for some ordinal $\beta\in\kappa\setminus\varrho$. Consequently, $t=c_\beta\restriction\eta\in T^*$. So since either $\kappa=\lambda^+$ or $\lambda^{<\lambda}=\lambda$, Claim 3.3.1 implies that $|T^*|\leq\lambda=\mathrm{cf}(\alpha)$. Then, since the surjection f was chosen to satisfy $\{f(\xi+1)\mid\epsilon<\xi<\epsilon+|T^*|\}=T^*$, we may find some ξ with $\epsilon<\xi<\xi+1<\alpha$ such that $f(\xi+1)=t$. As $\bigcup\mathcal{S}\subseteq\mathbb{S}$, we get from Equation ($\star\star$) that $d_{\xi+1,0}=c_{\beta_{\xi+1,0}}\supseteq t$, and hence $\Delta(d_{\xi+1,0},d_{\alpha,0})\geq\eta$. $\square$

Using the preceding claim, fix a strictly increasing sequence $\langle\xi^\alpha_\eta\mid\eta<\lambda\rangle$ of ordinals, converging to α, such that, for every $\eta<\lambda$, $\Delta(d_{\xi^\alpha_\alpha,0},d_{\alpha,0})\geq\eta$. Then, let $\langle a^i_\alpha\mid i<\lambda\rangle$ be some partition of $\{\xi^\alpha_\eta\mid\eta<\lambda\}$ into λ many sets of size λ.

As each a^i_α has order-type λ, the verification of almost-disjointness of the merged systems boils down to verifying the following case.

Claim 3.3.8. *Let $\alpha\in\mathbb{S}\setminus C$ and $\beta\in\mathbb{S}\cap C$ above α. Let $i,i'<\lambda$. Then* $\sup(a^i_\alpha\cap A^{i'}_\beta)<\alpha$.

Proof. Suppose not. Fix $j'<\zeta(\beta)$ such that $a^i_\alpha\cap A^{i'}_{\beta,j'}$ is cofinal in α. Set $\eta:=\Delta(d_{\alpha,0},d_{\beta,j'})$. By the choice of a^i_α, $\{\xi\in a^i_\alpha\mid\Delta(d_{\xi,0},d_{\alpha,0})\leq\eta\}$ is bounded in α, and hence the following set is cofinal in α:
$$Y:=\{\xi\in a^i_\alpha\cap A^{i'}_{\beta,j'}\mid\Delta(d_{\xi,0},d_{\alpha,0})>\eta\}.$$
For every $\xi\in Y$, $\Delta(d_{\xi,0},d_{\alpha,0})>\eta=\Delta(d_{\alpha,0},d_{\beta,j'})$, and hence $\Delta(d_{\xi,0},d_{\beta,j'})=\eta$. Set $\tau:=\hat{d}_{\beta,j'}(\eta)$. As $Y\subseteq\alpha\cap A_{\beta,j'}$, altogether $Y\subseteq\{\xi<\alpha\mid e(\xi,\beta)\leq\tau\}$. As $\alpha\in\mathbb{S}\subseteq\partial(e)$, Y is bounded in α. This is a contradiction. $\square$

This completes the proof. $\square$

3.1. **Variations.** A second reading of the proof of Theorem 3.3 makes it clear that the conclusion remains valid even after relaxing Clause (3) in the hypothesis to $\mathrm{cov}(\mathrm{cf}(\mathcal{T}_c,\supseteq),\lambda,\theta,2)\leq\kappa$. In the other direction, by waiving Clause (3) completely, the above proof yields the following:

Theorem 3.4. *Suppose $\lambda<\kappa$ is a pair of infinite regular cardinals, and $\mathbb{S}$ is a nonreflecting stationary subset of E^κ_λ.*

If there exists a strongly unbounded coloring $c:\lambda\times\kappa\to\lambda$, then there exists a club $C\subseteq\kappa$ and a matrix $\langle A^i_\alpha\mid\alpha\in\mathbb{S}\cap C,i<\alpha\rangle$ such that:

1. *For every $\alpha\in\mathbb{S}\cap C$, $\langle A^i_\alpha\mid i<\alpha\rangle$ is a sequence of pairwise disjoint cofinal subsets of α;*
2. *For every stationary $S\subseteq\mathbb{S}$, there are stationarily many $\alpha\in S\cap C$ such that* $\sup(A^i_\alpha\cap S)=\alpha$ *for all $i<\alpha$;*
3. *For all $(\alpha,\alpha')\in[\mathbb{S}\cap C]^2$, $i<\alpha$ and $i'<\alpha'$,* $\sup(A^i_\alpha\cap A^{i'}_{\alpha'})<\alpha$.

In the special case that $\kappa=\lambda^+$ or $\lambda^{<\lambda}=\lambda$, one can take C to be whole of κ. $\square$

Let $\clubsuit_{\mathrm{AD}^*}(\mathcal{S},\mu,{<}\theta)$ denote the strengthening of $\clubsuit_{\mathrm{AD}}(\mathcal{S},\mu,{<}\theta)$ obtained by replacing Clause (3) of Definition 1.1 by:

(3^*) For all $A\neq A'$ from $\bigcup_{S\in\mathcal{S}}\bigcup_{\alpha\in S}\mathcal{A}_\alpha$, $|A\cap A'|<\mathrm{cf}(\sup(A))$.

In the special case that $\kappa=\lambda^+$, one can use in the proof of Theorem 3.3 a *locally small* coloring $e:[\kappa]^2\to\lambda$ (such as the map ρ_1 from [Tod07, §6.2]), and then get:

Theorem 3.5. *Suppose:*

1. *λ is an infinite regular cardinal,*
2. *there exists a tight strongly unbounded coloring $c : \lambda \times \lambda^+ \rightarrow \lambda$, and*
3. *$\mathcal{S}$ is a partition of $E^{\lambda^+}_{\lambda}$ into stationary sets.*

Then $\clubsuit_{\mathrm{AD}^*}(\mathcal{S}, \lambda, {<}\lambda)$ holds. □

4. Theorem B

In this section, we give two sufficient conditions for a strong form of $\clubsuit_{\mathrm{AD}}$ to hold. The strong form under discussion is a double strengthening of $\clubsuit_{\mathrm{AD}}(\mathcal{S}, \mu, {<}\kappa)$, and it reads as follows.

Definition 4.1. Let $\mathcal{S}$ be a collection of stationary subsets of a regular uncountable cardinal κ, and μ be a nonzero cardinal $< \kappa$. The principle $\clubsuit_{\mathrm{AD}^*}(\mathcal{S}, \mu, \kappa)$ asserts the existence of a sequence $\langle \mathcal{A}_\alpha \mid \alpha \in \bigcup \mathcal{S} \rangle$ such that:

1. For every $\alpha \in \mathrm{acc}(\kappa) \cap \bigcup \mathcal{S}$, $\mathcal{A}_\alpha$ is a pairwise disjoint family of μ many cofinal subsets of α;
2. For every sequence $\langle B_i \mid i < \kappa \rangle$ of cofinal subsets of κ, for every $S \in \mathcal{S}$, there are stationarily many $\alpha \in S$ such that, for all $A \in \mathcal{A}_\alpha$ and $i < \alpha$, $\sup(A \cap B_i) = \alpha$;
3. For all $A \neq A'$ from $\bigcup_{S \in \mathcal{S}} \bigcup_{\alpha \in S} \mathcal{A}_\alpha$, $|A \cap A'| < \mathrm{cf}(\sup(A))$.

An inspection of the proof [CGW20, §3] yields the following useful fact:

Fact 4.2 ([CGW20]). *For an infinite cardinal λ, the following are equivalent:*

1. *$\stick(\lambda^+)$ holds, i.e., there exists a sequence $\langle x_\beta \mid \beta < \lambda^+ \rangle$ of elements of $[\lambda^+]^\lambda$ such that, for every cofinal $X \subseteq \lambda^+$, there exists $\beta < \lambda^+$ such that $x_\beta \subseteq X$;*
2. *There exists a sequence $\langle x_\beta \mid \beta < \lambda^+ \rangle$ of elements of $[\lambda^+]^\lambda$ satisfying the following. For every sequence $\langle A_\alpha \mid \alpha < \lambda^+ \rangle$ of elements of $[\lambda^+]^{\leq \lambda}$ such that $|A_\alpha \cap A_\beta| < \lambda$ for all $\alpha < \beta < \lambda^+$, for every cofinal $X \subseteq \lambda^+$, there exists $\beta < \lambda^+$ such that $x_\beta \subseteq X$ and, for every $a \in [\lambda^+]^{<\mathrm{cf}(\lambda)}$, $|x_\beta \setminus \bigcup_{\alpha \in a} A_\alpha| = \lambda$.*

Theorem 4.3. *Suppose that $\stick(\lambda^+)$ holds for an infinite regular cardinal λ.*

For every partition $\mathcal{S}$ of $E^{\lambda^+}_{\lambda}$ into stationary sets, $\clubsuit_{\mathrm{AD}^*}(\mathcal{S}, \lambda, \lambda^+)$ holds.

Proof. Let $\vec{x} = \langle x_\beta \mid \beta < \lambda^+ \rangle$ be given by Fact 4.2(2). Fix a bijection $\pi : \lambda \leftrightarrow \lambda \times \lambda$ and then let π_0, π_1 be the unique maps from λ to λ to satisfy $\pi(j) = (\pi_0(j), \pi_1(j))$ for all $j < \lambda$. For all nonzero $\alpha < \lambda^+$, fix a surjection $e_\alpha : \lambda \rightarrow \alpha$.

Define a sequence $\langle A_\alpha \mid \alpha < \lambda^+ \rangle$ by recursion on $\alpha < \lambda^+$, as follows. Set $A_\alpha := \emptyset$. Next, given a nonzero $\alpha < \lambda^+$ such that $\langle A_{\bar\alpha} \mid \bar\alpha < \alpha \rangle$ has already been defined, put

$$J_\alpha := \{ j < \lambda \mid |x_{e_\alpha(\pi_0(j))} \cap \alpha \setminus \bigcup \{ A_{e_\alpha(j')} \mid j' \leq j \}| = \lambda \}.$$

Then, pick an injective sequence $\langle \xi_{\alpha,j} \mid j \in J_\alpha \rangle$ such that, for each $j \in J_\alpha$,

$$\xi_{\alpha,j} \in x_{e_\alpha(\pi_0(j))} \cap \alpha \setminus \bigcup \{ A_{e_\alpha(j')} \mid j' \leq j \}.$$

If $\{ \xi_{\alpha,j} \mid j \in J_\alpha \ \&\ \pi_1(j) = i \}$ happens to be cofinal in α for every $i < \lambda$, then we say that α is *good*, and let

- $A^i_\alpha := \{ \xi_{\alpha,j} \mid j \in J_\alpha \ \&\ \pi_1(j) = i \}$ for every $i < \lambda$, and
- $A_\alpha := \{ \xi_{\alpha,j} \mid j \in J_\alpha \}$.

Otherwise, we just let A_α be any cofinal subset of α of order-type $\mathrm{cf}(\alpha)$, and let $\langle A^i_\alpha \mid i < \mathrm{cf}(\alpha)\rangle$ be any partition of A_α into cofinal subsets of α.

Claim 4.3.1. *For all $\bar\alpha < \alpha < \lambda^+$, $|A_{\bar\alpha} \cap A_\alpha| < \lambda$.*

Proof. If α is not good, then $\mathrm{otp}(A_\alpha) = \mathrm{cf}(\alpha) \leq \lambda$, and the conclusion follows. Next, suppose that α is good. Find $j' < \lambda$ such that $e_\alpha(j') = \bar\alpha$. Then $A_{\bar\alpha} \cap A_\alpha \subseteq \{\xi_{\alpha,j} \mid j \in J_\alpha \cap j'\}$ □

Next, given a cofinal $X \subseteq \lambda^+$, for every $\epsilon < \lambda^+$, by Claim 4.3.1 and the choice of $\vec{x}$, we may let β_ϵ denote the least $\beta < \lambda^+$ to satisfy both $x_\beta \subseteq X \setminus \epsilon$ and $|x_\beta \setminus \bigcup_{\alpha\in a} A_\alpha| = \lambda$ for every $a \in [\lambda^+]^{<\lambda}$.

Fix a set $E \in [\lambda^+]^{\lambda^+}$ on which the map $\epsilon \mapsto \beta_\epsilon$ is strictly increasing. Consider the club
$$D := \{\delta \in \mathrm{acc}^+(E) \mid \forall \epsilon \in E \cap \delta\, (\beta_\epsilon \cup x_{\beta_\epsilon} \subseteq \delta)\}.$$

Claim 4.3.2. *Let $\delta \in D$. For every $i < \lambda$, $\sup(A^i_\delta \cap X) = \delta$.*

Proof. Let $i < \lambda$ and let $\epsilon < \delta$. We shall show that there exists $j \in J_\delta$ such that $\xi_{\delta,j}$ is an element of $A^i_\delta \cap X \setminus \epsilon$.

Here we go. By possibly increasing ϵ, we may assume that $\epsilon \in E \cap \delta$. Set $k := e_\delta^{-1}(\beta_\epsilon)$, and pick the unique $j < \lambda$ such that $\pi(j) = (k, i)$. Then
$$x_{e_\delta(\pi_0(j))} \cap \delta \setminus \bigcup\{A_{e_\delta(j')} \mid j' \leq j\} = x_{\beta_\epsilon} \setminus \bigcup_{\alpha\in a} A_\alpha$$
for the set $a := e_\delta[j + 1]$ which is an element of $[\lambda^+]^{<\lambda}$. Consequently, $j \in J_\delta$, and since $\pi_1(j) = i$, $\xi_{\delta,j}$ is an element of $x_{\beta_\epsilon} \subseteq X \setminus \epsilon$ that lies in A^i_δ. □

It follows that for every partition $\mathcal{S}$ of $E^{\lambda^+}_\lambda$ into stationary sets, $\langle\{A^i_\delta \mid i < \lambda\} \mid \delta \in E^{\lambda^+}_\lambda\rangle$ witnesses $\clubsuit_{\mathrm{AD}^*}(\mathcal{S}, \lambda, \lambda^+)$. □

Lemma 4.4. *Suppose that $\clubsuit_{\mathrm{AD}^*}(\{S\}, \mu, \lambda^+)$ holds for some stationary subset S of a successor cardinal λ^+. Then $\clubsuit_{\mathrm{AD}^*}(\mathcal{S}, \mu, \lambda^+)$ holds for some partition $\mathcal{S}$ of S into λ^+ many stationary sets.*

Proof. Let $\langle\{A^i_\alpha \mid i < \mu\} \mid \alpha \in S\rangle$ be an array witnessing that $\clubsuit_{\mathrm{AD}^*}(\{S\}, \mu, \lambda^+)$ holds. Let $\mathcal{I}$ denote the collection of all $T \subseteq S$ such that $\langle\{A^i_\alpha \mid i < \mu\} \mid \alpha \in T\rangle$ fails to witness that $\clubsuit_{\mathrm{AD}^*}(\{T\}, \mu, \lambda^+)$ holds. It is not hard to see that $\mathcal{I}$ is a λ^+-complete proper ideal on S. By Ulam's theorem, then, $\mathcal{I}$ is not weakly λ^+-saturated, meaning that we may fix a partition $\mathcal{S}$ of S into λ^+-many $\mathcal{I}^+$-sets. Then $\langle\{A^i_\alpha \mid i < \mu\} \mid \alpha \in S\rangle$ witnesses that $\clubsuit_{\mathrm{AD}^*}(\mathcal{S}, \mu, \lambda^+)$ holds. □

Definition 4.5 ([MHD04]). $\diamondsuit(\mathfrak{b})$ asserts that for every Borel map $F : {}^{<\omega_1}2 \to {}^\omega\omega$, there exists a function $g : \omega_1 \to {}^\omega\omega$ with the property that for every function $f : \omega_1 \to 2$, the set $\{\alpha < \omega_1 \mid F(f \restriction \alpha) \leq^* g(\alpha)\}$ is stationary.

Corollary 4.6. *Suppose that $\diamondsuit(\mathfrak{b})$ holds. Then:*

(1) $\clubsuit_{\mathrm{AD}^*}(\mathcal{S}, 1, \omega_1)$ holds for some partition $\mathcal{S}$ of ω_1 into uncountably many stationary sets;

(2) There exist $2^{\aleph_1}$ many pairwise nonhomeomorphic Dowker spaces of size $\aleph_1$.

Proof. (1) By [MHD04, Theorem 5.5], $\diamondsuit(\mathfrak{b})$ implies that $\clubsuit_{\mathrm{AD}^*}(\{\omega_1\}, 1, \omega_1)$ holds. Now, the conclusion follows from Lemma 4.4.

(2) By Clause (1) and Theorem A.1 below. □

A. Appendix: Many Dowker spaces

In this section, κ denotes a regular uncountable cardinal. By [RS23, §3], if $\clubsuit_{\mathrm{AD}}(\mathcal{S},1,2)$ holds for a partition $\mathcal{S}$ of some nonreflecting stationary subset of κ into infinitely many stationary sets, then there exists a Dowker space of size κ. Here, we demonstrate the advantage of $\mathcal{S}$ being large.

Theorem A.1. *Suppose that $\clubsuit_{\mathrm{AD}}(\mathcal{S},1,2)$ holds, where $\mathcal{S}$ is a partition of a nonreflecting stationary subset of κ into infinitely many stationary sets. Denote $\mu := |\mathcal{S}|$. Then there are 2^μ many pairwise nonhomeomorphic Dowker spaces of size κ.*

Proof. Fix an injective enumeration $\langle S^\zeta_n \mid \zeta < \mu, n < \omega\rangle$ of the elements of $\mathcal{S}$. As $\clubsuit_{\mathrm{AD}}(\mathcal{S},1,2)$ holds, we may fix a sequence $\langle A_\alpha \mid \alpha \in \bigcup\mathcal{S}\rangle$ such that:

(i) For every $\alpha \in \bigcup\mathcal{S}$, A_α is a subset of α, and for every $\alpha' \in \alpha \cap \bigcup\mathcal{S}$, $\sup(A_{\alpha'} \cap A_\alpha) < \alpha'$;

(ii) For all $B_0, B_1 \in [\kappa]^\kappa$ and $(\zeta, n) \in \mu \times \omega$, the following set is stationary:
$$G(S^\zeta_n, B_0, B_1) := \{\alpha \in S^\zeta_n \mid \sup(A_\alpha \cap B_0) = \sup(A_\alpha \cap B_1) = \alpha\}.$$

For every nonempty $Z \subseteq \mu$, we shall want to define a topological space $\mathbb{X}^Z$. To this end, fix a nonempty $Z \subseteq \mu$. For every $n < \omega$, let $S^Z_{n+1} := \biguplus_{\zeta\in Z} S^\zeta_{n+1}$, and then let $S^Z_0 := \kappa \setminus \biguplus_{n<\omega} S^Z_{n+1}$. For every $\alpha < \kappa$, let $n^Z(\alpha)$ denote the unique $n < \omega$ such that $\alpha \in S^Z_n$. For each $n < \omega$, let $W^Z_n := \bigcup_{i\le n} S^Z_i$. Then, define a sequence $\vec{L^Z} = \langle L^Z_\alpha \mid \alpha < \kappa\rangle$ via:
$$L^Z_\alpha := \begin{cases} W^Z_{n^Z(\alpha)-1} \cap A_\alpha, & \text{if } n^Z(\alpha) > 0 \ \&\ \sup(W^Z_{n^Z(\alpha)-1} \cap A_\alpha) = \alpha; \\ \emptyset, & \text{otherwise.}\end{cases}$$

Denote $S^Z := \{\alpha \in \mathrm{acc}(\kappa) \mid \sup(L^Z_\alpha) = \alpha\}$. Finally, let $\mathbb{X}^Z = (\kappa, \tau^Z)$ be the ladder-system space determined by $\vec{L^Z}$, that is, a subset $U \subseteq \kappa$ is τ^Z-open iff, for every $\alpha \in U \cap S^Z$, $\sup(L^Z_\alpha \setminus U) < \alpha$.

Claim A.1.1. *Let Z and Z' be nonempty subsets of μ. Then:*

(1) For all $n < \omega$ and $\alpha \in S^Z_{n+1}$, $L^Z_\alpha \subseteq W^Z_n$;

(2) If $Z \setminus Z'$ is nonempty, then $S^Z \setminus S^{Z'}$ is stationary;

(3) For all $\alpha \neq \alpha'$ from S^Z, $\sup(L^Z_\alpha \cap L^Z_{\alpha'}) < \alpha$;

(4) For all $B_0, B_1 \in [\kappa]^\kappa$, there exists $m < \omega$ such that, for every $n \in \omega \setminus m$, the following set is stationary:
$$\{\alpha \in S^Z_n \mid \sup(L^Z_\alpha \cap B_0) = \sup(L^Z_\alpha \cap B_1) = \alpha\};$$

(5) S^Z is a nonreflecting stationary set.

Proof. (1) Clear.

(2) Suppose that $Z \setminus Z' \neq \emptyset$, and pick $\zeta \in Z \setminus Z'$. As $W^Z_0 = S^Z_0 \supseteq S^\zeta_0$, the former is cofinal. So, $S^Z \setminus S^{Z'}$ covers the stationary set $G(S^\zeta_1, W^Z_0, \kappa)$.

(3) For all $\alpha \neq \alpha'$ from S^Z, $\sup(L^Z_\alpha \cap L^Z_{\alpha'}) \le \sup(A_\alpha \cap A_{\alpha'}) < \alpha$.

(4) Pick $\zeta \in Z$. Given two cofinal subsets B_0, B_1 of κ, find $m_0, m_1 < \omega$ be such that $|B_0 \cap S^Z_{m_0}| = |B_1 \cap S^Z_{m_1}| = \kappa$. Set $m := \max\{m_0, m_1\} + 1$. Then, for every $n \in \omega \setminus m$,
$$G(S^\zeta_n, B_0 \cap S^Z_{m_0}, B_1 \cap S^Z_{m_1}) \subseteq \{\alpha \in S^Z_n \mid \sup(L^Z_\alpha \cap B_0) = \sup(L^Z_\alpha \cap B_1) = \alpha\}$$
and hence the latter is stationary.

(5) By Clause (4), S^Z is stationary. As $S^Z \subseteq \bigcup_{n<\omega} S^Z_{n+1} \subseteq \bigcup \mathcal{S}$, and since $\bigcup \mathcal{S}$ is a nonreflecting stationary set, so is S^Z. $\square$

By the preceding claim, and the results of [RS23, §3], for every nonempty $Z \subseteq \mu$, $\mathbb{X}^Z$ is a Dowker space. Thus we are left with proving the following:

Claim A.1.2. *Suppose that Z and Z' are two distinct nonempty subsets of μ. Then $\mathbb{X}^Z$ and $\mathbb{X}^{Z'}$ are not homeomorphic.*

Proof. Without loss of generality, we may pick $\zeta \in Z \setminus Z'$. Towards a contradiction, suppose that $f : \kappa \leftrightarrow \kappa$ forms an homeomorphism from $\mathbb{X}^Z$ to $\mathbb{X}^{Z'}$. As f is a bijection, there are club many $\alpha < \kappa$ such that $f^{-1}[\alpha] = \alpha$. By Claim A.1.1(2), then, we may pick some $\alpha \in S^Z \setminus S^{Z'}$ such that $f^{-1}[\alpha] = \alpha$. Set $\beta := f(\alpha)$.

► If $\beta \notin S^{Z'}$, then $U := \{\beta\}$ is a $\tau^{Z'}$-open neighborhood of β.

► If $\beta \in S^{Z'}$, then $\beta > \alpha + 1$ and the ordinal interval $U := [\alpha + 1, \beta + 1]$ is a $\tau^{Z'}$-open neighborhood of β.

In both cases, $U \subseteq \kappa \setminus \alpha$, so that $f^{-1}[U] \subseteq f^{-1}[\kappa \setminus \alpha] = \kappa \setminus \alpha$. As f is continuous and U is a $\tau^{Z'}$-open neighborhood of $f(\alpha)$, $f^{-1}[U]$ must be a τ^Z-open neighborhood of α, contradicting the fact that $f^{-1}[U]$ is disjoint from L^Z_α. $\square$

This completes the proof. $\square$

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