THE VANISHING LEVELS OF A TREE

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Abstract. We initiate the study of the spectrum $\mathrm{Vspec}(\kappa)$ of sets that can be realized as the vanishing levels $V(\mathbf{T})$ of a normal κ-tree $\mathbf{T}$. The latter is an invariant in the sense that if $\mathbf{T}$ and $\mathbf{T}'$ are club-isomorphic, then $V(\mathbf{T})\triangle V(\mathbf{T}')$ is nonstationary. Additional features of this invariant imply that $\mathrm{Vspec}(\kappa)$ is closed under finite unions and intersections.

The set $V(\mathbf{T})$ must be stationary for an homogeneous normal κ-Aronszajn tree $\mathbf{T}$, and if there exists a special κ-Aronszajn tree, then there exists one $\mathbf{T}$ that is homogeneous and satisfies $V(\mathbf{T}) = \kappa$ (modulo clubs). It is consistent (from large cardinals) that there is an $\aleph_2$-Souslin tree, and yet $V(\mathbf{T})$ is co-stationary for every $\aleph_2$-tree $\mathbf{T}$. Both $V(\mathbf{T}) = \emptyset$ and $V(\mathbf{T}) = \kappa$ (modulo clubs) are shown to be feasible using κ-Souslin trees, even at some large cardinal close to a weakly compact. It is also possible to have a family of 2^κ many κ-Souslin trees for which the corresponding family of vanishing levels forms an antichain modulo clubs.

1. Introduction

Throughout this paper, κ denotes a regular uncountable cardinal. Recall that a poset $\mathbf{T} = (T, <_T)$ is a *κ-tree* iff all of the following hold:

(1) For every $x \in T$, the set $x_\downarrow := \{y \in T \mid y <_T x\}$ is well-ordered by $<_T$. Hereafter, write $\mathrm{ht}(x) := \mathrm{otp}(x_\downarrow, <_T)$;
(2) For every ordinal $\alpha < \kappa$, the set $T_\alpha := \{x \in T \mid \mathrm{ht}(x) = \alpha\}$ is nonempty and has size less than κ, and the set T_κ is empty.

A subset $B \subseteq T$ is an *α-branch* iff $(B, <_T)$ is linearly ordered and $\{\mathrm{ht}(x) \mid x \in B\} = \alpha$; it is said to be *vanishing* iff it has no upper bound in $\mathbf{T}$.

Definition (Vanishing levels)**.** For a κ-tree $\mathbf{T} = (T, <_T)$, let $V(\mathbf{T})$ denote the set of all $\alpha \in \mathrm{acc}(\kappa)$ such that for any $x \in T$ with $\mathrm{ht}(x) < \alpha$ there exists a vanishing α-branch containing x.[1]

The above is an invariant of trees in the sense that if two κ-trees $\mathbf{T}, \mathbf{T}'$ are isomorphic on a club, then $V(\mathbf{T})$ is equal to $V(\mathbf{T}')$ modulo a club. It also satisfies that $V(\mathbf{T} \otimes \mathbf{T}') = V(\mathbf{T}) \cup V(\mathbf{T}')$ and $V(\mathbf{T} + \mathbf{T}') = V(\mathbf{T}) \cap V(\mathbf{T}')$ for any two normal κ-trees $\mathbf{T}, \mathbf{T}'$.

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[1]The definition of $\mathrm{acc}(\kappa)$ may be found in Subsection 1.2 below.

The importance of this invariant became apparent in [RS23], where it was shown that if $\mathbf{T}$ is a κ-Souslin tree, i.e., a κ-tree with no κ-branches and no κ-sized antichains, then the combinatorial principle $\clubsuit_{\mathrm{AD}}(S)$ holds for some subset $S \subseteq \kappa$ that is equal to $V(\mathbf{T})$ modulo a club.[2] In particular, if $V(\mathbf{T})$ is stationary, then a nontrivial instance of $\clubsuit_{\mathrm{AD}}$ holds true, and this has important applications in set-theoretic topology.

Surprisingly enough, the first main result of this paper shows that $V(\mathbf{T})$ need not be stationary. This is demonstrated in Gödel's constructible universe, L, where we obtain the following characterization:

Theorem A. *In L, for every (regular uncountable cardinal) κ that is not weakly compact, the following are equivalent:*

- *there exists a κ-Souslin tree $\mathbf{T}$ such that $V(\mathbf{T}) = \emptyset$;*
- *there exists a normal and splitting κ-tree $\mathbf{T}$ such that $V(\mathbf{T}) = \emptyset$;*
- *κ is not the successor of a cardinal of countable cofinality.*

On the other extreme, it is possible to have a κ-Souslin tree $\mathbf{T}$ with $V(\mathbf{T})$ as large as possible. Again, we obtain a complete characterization:

Theorem B. *In L, for every (regular uncountable cardinal) κ that is not weakly compact, the following are equivalent:*

- *there exists a κ-Souslin tree $\mathbf{T}$ such that $V(\mathbf{T}) = \mathrm{acc}(\kappa)$;*
- *there exists a κ-tree $\mathbf{T}$ such that $V(\mathbf{T}) = \mathrm{acc}(\kappa)$;*
- *κ is not subtle.*

An interesting feature of the proof of Theorem B is that it goes through a pump-up theorem generating κ-Souslin trees from other input trees with weaker properties. For a κ-tree $\mathbf{T}$, let $V^-(\mathbf{T})$ denote the set of all $\alpha \in \mathrm{acc}(\kappa)$ such that there exists a vanishing α-branch. If $\mathbf{T}$ is homogeneous, then $V^-(\mathbf{T})$ coincides with $V(\mathbf{T})$, but in contrast with Theorem A, for every normal κ-Aronszajn tree $\mathbf{T}$, the set $V^-(\mathbf{T})$ is necessarily stationary.[3]

Our first pump-up theorem asserts that the existence of a special κ-Aronszajn tree $\mathbf{T}$ is equivalent to the existence of one with $V(\mathbf{T}) = \mathrm{acc}(\kappa)$. Our second pump-up theorem asserts that for every κ-tree $\mathbf{K}$ there exists a κ-tree $\mathbf{T}$ such that $V^-(\mathbf{K}) \setminus V(\mathbf{T})$ is nonstationary. Our third pump-up theorem asserts that assuming an instance of the proxy principle $\mathrm{P}(\ldots)$ from [BR17a],[4] the corresponding tree $\mathbf{T}$ may moreover be made to be κ-Souslin:

Theorem C. *Suppose that $\mathrm{P}(\kappa, 2, \sqsubseteq^*, 1)$ holds. Then:*

(1) For every κ-tree $\mathbf{K}$, there exists a κ-Sousin tree $\mathbf{T}$ such that $V^-(\mathbf{K}) \setminus V(\mathbf{T})$ is nonstationary. In particular:
(2) There exists a κ-Sousin tree $\mathbf{T}$ such that $V(\mathbf{T})$ is stationary.

[2]The definition of $\clubsuit_{\mathrm{AD}}$ may be found in the paper's Appendix.
[3]Note that any κ-Souslin must be normal on a tail end.
[4]See Definitions 3.3 and 3.4 below.

The preceding addresses the problem of ensuring $V(\mathbf{T})$ to cover some stationary set S. The next theorem addresses the dual problem. Along the way, it provides a cheap way to obtain a family of 2^κ-many κ-Souslin trees that are not pairwise club-isomorphic.

Theorem D. *If $\diamondsuit(S)$ holds for some nonreflecting stationary subset S of a strongly inaccessible cardinal κ, then there is an almost disjoint family $\mathcal{S}$ of 2^κ many stationary subsets of S such that, for each $S' \in \mathcal{S}$, there is a κ-Souslin tree $\mathbf{T}$ with $V(\mathbf{T}) = S'$.*

Let us now come back to the motivating problem of getting instances of $\clubsuit_{\mathrm{AD}}$. By [RS23, Theorem 2.30], if κ is weakly compact, then $\clubsuit_{\mathrm{AD}}(S)$ fails for every S with $\mathrm{Reg}(\kappa) \subseteq S \subseteq \kappa$. This raises the question as to whether $\clubsuit_{\mathrm{AD}}(S)$ may hold over a large subset S of a cardinal κ that is close to being weakly compact. We answer this question in the affirmative:

Theorem E. *Assuming the consistency of a weakly compact cardinal, it is consistent that for some strongly inaccessible cardinal κ satisfying $\chi(\kappa) = \omega$,*[5] *there is a κ-Souslin tree $\mathbf{T}$ such that $V(\mathbf{T}) = \mathrm{acc}(\kappa)$.*

In the appendix to this paper, we improve a result from [RS23] concerning the connection between Ostaszewski's principle $\clubsuit$ and the principle $\clubsuit_{\mathrm{AD}}$. As a byproduct, we obtain the following unexpected result:

Theorem F. *If $\clubsuit(S)$ holds over a nonreflecting stationary $S \subseteq \kappa$, then there exists a Dowker space of size κ.*

1.1. **Organization of this paper.** In Section 2, we develop the basic theory of vanishing levels of trees. It is proved that if κ is not a strong limit, then $V^-(\mathbf{T})$ is stationary for every normal and splitting κ-tree $\mathbf{T}$. It is proved that for every κ-tree $\mathbf{K}$, there exists a κ-tree $\mathbf{T}$ such that $V^-(\mathbf{K}) \setminus V(\mathbf{T})$ is nonstationary, and that the existence of a special κ-Aronszajn tree $\mathbf{T}$ is equivalent to the existence of an homogeneous one with $V(\mathbf{T}) = \mathrm{acc}(\kappa)$.

In Section 3, we prove Theorem C and some variations of it. As a corollary, we get Theorem B and infer that if $\square_\lambda + \diamondsuit(\lambda^+)$ holds for an infinite cardinal λ, or if $\square(\lambda^+) + \mathsf{GCH}$ holds for a regular uncountable λ, then there exists a λ^+-Souslin tree $\mathbf{T}$ with $V(\mathbf{T}) = \mathrm{acc}(\lambda^+)$.

In Section 4, we address the problem of realizing a given nonreflecting stationary subset of κ as $V(\mathbf{T})$ for some κ-Souslin tree $\mathbf{T}$. The proof of Theorem D will be found there.

In Section 5, we address the problem of constructing an homogeneous κ-Souslin tree $\mathbf{T}$ such that $V(\mathbf{T}) = \{\alpha < \kappa \mid \mathrm{cf}(\alpha) \in x\}$ for a prescribed nonempty finite set $x \subseteq \mathrm{Reg}(\kappa)$. In particular, this is shown to be feasible in L whenever κ is $<\max(x)$-inaccessible. The proof of Theorem A will be found there.

[5] $\chi(\kappa)$ can be understood as measuring how far κ is from being weakly compact; see Definition 6.6 below.

In Section 6, we deal with Souslin trees admitting an ascent path. It is proved that for every uncountable cardinal λ, $\square_\lambda + \mathsf{GCH}$ entails that for every $\mu \in \mathrm{Reg}(\mathrm{cf}(\lambda))$ there exists a λ^+-Souslin tree $\mathbf{T}$ with a μ-ascent path such that $V(\mathbf{T}) = \mathrm{acc}(\lambda^+)$. The proof of Theorem E will be found there.

Section A is a short appendix where we improve [RS23, Lemma 2.10], from which we obtain the proof of Theorem F.

1.2. **Notation and conventions.** H_κ denotes the collection of all sets of hereditary cardinality less than κ. $\mathrm{Reg}(\kappa)$ denotes the set of all infinite regular cardinals $< \kappa$. For $\chi \in \mathrm{Reg}(\kappa)$, E^κ_χ denotes the set $\{\alpha < \kappa \mid \mathrm{cf}(\alpha) = \chi\}$, and $E^\kappa_{\geq\chi}$, $E^\kappa_{<\chi}$, $E^\kappa_{\neq\chi}$, are defined analogously.

For a set of ordinals C, we write $\mathrm{ssup}(C) := \sup\{\alpha + 1 \mid \alpha \in C\}$, $\mathrm{acc}^+(C) := \{\alpha < \mathrm{ssup}(C) \mid \sup(C \cap \alpha) = \alpha > 0\}$, $\mathrm{acc}(C) := C \cap \mathrm{acc}^+(C)$, and $\mathrm{nacc}(C) := C \setminus \mathrm{acc}(C)$. For a set S, we write $[S]^\chi$ for $\{A \subseteq S \mid |A| = \chi\}$, and $[S]^{<\chi}$ is defined analogously. For a set of ordinals S, we identify $[S]^2$ with $\{(\alpha, \beta) \mid \alpha, \beta \in S, \alpha < \beta\}$, and we let $\mathrm{Tr}(S) := \{\beta < \mathrm{ssup}(S) \mid \mathrm{cf}(\beta) > \omega \ \&\ S \cap \beta \text{ is stationary in } \beta\}$.

We define four binary relations over sets of ordinals, as follows:

- $D \sqsubseteq C$ iff there exists some ordinal β such that $D = C \cap \beta$;
- $D \sqsubseteq^* C$ iff $D \setminus \varepsilon \sqsubseteq C \setminus \varepsilon$ for some $\varepsilon < \sup(D)$;
- $D \,{}^S\!\sqsubseteq\, C$ iff $D \sqsubseteq C$ and $\sup(D) \notin S$;
- $D \,{}_\chi\!\sqsubseteq\, C$ iff $D \sqsubseteq C$ or $\mathrm{cf}(\sup(D)) < \chi$.

A *list* over a set of ordinals S is a sequence $\vec{A} = \langle A_\alpha \mid \alpha \in S\rangle$ such that, for each $\alpha \in S$, A_α is a subset of α. It is said to be *thin* if $|\{A_\alpha \cap \varepsilon \mid \alpha \in S\}| < \mathrm{ssup}(S)$ for every $\varepsilon < \mathrm{ssup}(S)$. It is said to be ξ-*bounded* if $\mathrm{otp}(A_\alpha) \leq \xi$ for all $\alpha \in S$. A *ladder system* over S is a list $\vec{A} = \langle A_\alpha \mid \alpha \in S\rangle$ such that $\sup(A_\alpha) = \sup(\alpha)$ for every $\alpha \in S$. It is said to be *almost disjoint* if $\sup(A_\alpha \cap A_{\alpha'}) < \alpha$ for all $\alpha \neq \alpha'$ in S. A C-*sequence* over S is a ladder system $\vec{C} = \langle C_\alpha \mid \alpha \in S\rangle$ such that each C_α is a closed subset of α. Finally, a (resp. thin/ξ-bounded/almost-disjoint) $\mathcal{C}$-*sequence* over S is a sequence $\vec{\mathcal{C}} = \langle \mathcal{C}_\alpha \mid \alpha \in S\rangle$ of nonempty sets such that every element of $\prod_{\alpha\in S} \mathcal{C}_\alpha$ is a (resp. thin/ξ-bounded/almost-disjoint) C-sequence.

2. The basic theory of vanishing levels

Definition 2.1. A tree $\mathbf{T} = (T, <_T)$ is said to be:

- *Hausdorff* iff for every limit ordinal α and all $x, y \in T_\alpha$, if $x_\downarrow = y_\downarrow$, then $x = y$;
- *normal* iff for every pair $\alpha < \beta$ of ordinals, if $T_\beta \neq \emptyset$, then for every $x \in T_\alpha$ there exists $y \in T_\beta$ with $x <_T y$;
- χ-*complete* iff any $<_T$-increasing sequence of elements of $\mathbf{T}$, and of length $< \chi$, has an upper bound in $\mathbf{T}$;
- ς-*splitting* iff every node of $\mathbf{T}$ admits at least ς-many immediate successors, that is, for every $x \in T$, $|\{y \in T \mid x <_T y, \mathrm{ht}(y) = \mathrm{ht}(x) + 1\}| \geq \varsigma$. By *splitting*, we mean 2-splitting;

- *κ-Aronszajn* iff $\mathbf{T}$ is a κ-tree with no κ-branches;
- *special κ-Aronszajn tree* iff it is a κ-Aronszajn and there exists a map $\rho : T \to T$ satisfying the following:
  - for every non-minimal $x \in T$, $\rho(x) <_T x$;
  - for every $y \in T$, $\rho^{-1}\{y\}$ is covered by less than κ many antichains.

Remark 2.2. All the κ-Souslin trees constructed in this paper will be Hausdorff, normal and splitting.

Definition 2.3. For a κ-tree $\mathbf{T} = (T, <_T)$:

(1) $V^-(\mathbf{T})$ denotes the set of all $\alpha \in \mathrm{acc}(\kappa)$ such that there exists a vanishing α-branch;
(2) $V(\mathbf{T})$ denotes the set of all $\alpha \in \mathrm{acc}(\kappa)$ such that for every $x \in T$ with $\mathrm{ht}(x) < \alpha$ there exists a vanishing α-branch containing x.
(3) $\mathrm{Vspec}(\kappa) := \{V(\mathbf{T}) \mid \mathbf{T} \text{ is a normal } \kappa\text{-tree}\}$;
(4) For $A \subseteq \kappa$, we write $T \restriction A := \{x \in T \mid \mathrm{ht}(x) \in A\}$.

Note that if $\mathbf{T}$ is a κ-tree such that $V(\mathbf{T})$ is cofinal in κ, then $\mathbf{T}$ is normal.

Lemma 2.4. *Suppose that $\mathbf{T}$ is a κ-tree such that $V^-(\mathbf{T})$ (resp. $V(\mathbf{T})$) covers a club in κ. Then there exists a subtree $\mathbf{T}'$ of $\mathbf{T}$ such that $V^-(\mathbf{T})$ (resp. $V(\mathbf{T})$) is equal to* $\mathrm{acc}(\kappa)$.

Proof. Let $D \subseteq \kappa$ be a club as in the hypothesis. Then $\mathbf{T}' := (T \restriction D, <_T)$ is a subtree as sought. □

Proposition 2.5. *For a κ-tree $\mathbf{T} = (T, <_T)$:*

(1) If $\mathbf{T}$ is a normal κ-Aronszajn tree, then $V^-(\mathbf{T})$ is stationary;
(2) If $\mathbf{T}$ is homogeneous,[6] *then $V^-(\mathbf{T}) = V(\mathbf{T})$.*

Proof. (1) Suppose not, and fix a club $D \subseteq \kappa$ disjoint from $V^-(\mathbf{T})$. We shall construct a $<_T$-increasing sequence $\langle t_\alpha \mid \alpha \in D\rangle$ in such a way that $t_\alpha \in T_\alpha$ for all $\alpha \in D$, contradicting the fact that $\mathbf{T}$ is κ-Aronszajn. We start by letting $t_{\min(D)}$ be an arbitrary element of $T_{\min(D)}$. Next, for every $\alpha \in D$ such that t_α has already been successfully defined, we set $\beta := \min(D \setminus (\alpha+1))$, and use the normality of $\mathbf{T}$ to pick t_β in T_β extending t_α. For every $\alpha \in \mathrm{acc}(D)$ such that $\langle t_\epsilon \mid \epsilon \in D \cap \alpha\rangle$ has already been defined, the latter clearly induces an α-branch, so the fact that $\alpha \notin V^-(\mathbf{T})$ implies that there exists some $t_\alpha \in T_\alpha$ such that $t_\epsilon <_T t_\alpha$ for all $\epsilon \in D \cap \alpha$. This completes the description of the recursion.

(2) Suppose that $\mathbf{T}$ is homogeneous. Let $\alpha \in V^-(\mathbf{T})$, and fix a vanishing α-branch b. Now, given a node x of $\mathbf{T}$ of height less than α, let y be the unique element of b to have the same height as x. Since $\mathbf{T}$ is homogeneous, there exists an automorphism π of $\mathbf{T}$ sending y to x, and it is clearly the case that $\pi[b]$ is a vanishing α-branch through x. □

[6] That is, for all $\alpha < \kappa$ and $s, t \in T_\alpha$, there is an automorphism of $\mathbf{T}$ sending s to t.

Proposition 2.6. *If $\square(\kappa)$ holds, then there exists a κ-Aronszajn tree $\mathbf{T}$ such that $V(\mathbf{T}) = E^\kappa_\omega$.*

Proof. By [Kön03, Theorem 3.9], $\square(\kappa)$ yields a sequence of functions $\langle f_\beta : \beta \to \beta \mid \beta \in \mathrm{acc}(\kappa)\rangle$ such that:

- for every $(\beta,\gamma) \in [\mathrm{acc}(\kappa)]^2$, $\{\alpha < \beta \mid f_\beta(\alpha) \neq f_\gamma(\alpha)\}$ is finite;
- there is no cofinal $B \subseteq \mathrm{acc}(\kappa)$ such that $\{f_\beta \mid \beta \in B\}$ is linearly ordered by $\subseteq$.

Set $T := \{f \in {}^\alpha\alpha \mid \alpha \leq \beta < \kappa, f \text{ disagrees with } f_\beta \text{ on a finite set}\}$. Then $\mathbf{T} = (T, \subseteq)$ is a uniformly coherent κ-Aronszajn tree. By [RS23, Remark 2.20], then, $V(\mathbf{T}) = E^\kappa_\omega$. $\square$

Definition 2.7. For a κ-tree $\mathbf{T} = (T, <_T)$ and a subset $S \subseteq \kappa$, we say that $\mathbf{T}$ is *S-regressive* iff there exists a map $\rho : T \restriction S \to T$ satisfying the following:

- for every $x \in T \restriction S$, $\rho(x) <_T x$;
- for all $\alpha \in S$ and $x, y \in T_\alpha$, if $\rho(x) <_T y$ and $\rho(y) <_T x$, then $x = y$.

Remark 2.8. If ρ is as above, then every map $\varrho : T \restriction S \to T$ satisfying $\rho(x) \leq_T \varrho(x) <_T x$ for all $x \in T \restriction S$ is as well a witness to $\mathbf{T}$ being S-regressive.

The next lemma generalizes [RS23, Lemmas 2.19 and 2.21].

Lemma 2.9. *Suppose that:*

- *$\mathbf{T}$ is a normal, ς-splitting κ-tree, for some fixed cardinal $\varsigma < \kappa$;*
- *$S \subseteq E^\kappa_\chi$ is stationary for some fixed regular cardinal $\chi < \kappa$;*
- *Either of the following:*
  - *(1) $\varsigma^\chi \geq \kappa$;*
  - *(2) T is S-regressive and $\varsigma^{<\chi} < \varsigma^\chi$;*
  - *(3) T is S-regressive, $\chi = \varsigma$ and there exists a weak χ-Kurepa tree.*[7]

Then, for every $\alpha \in S$, either $\alpha \in V(\mathbf{T})$ or ($\mathrm{cf}(\alpha) > \omega$ and) $V^-(\mathbf{T}) \cap \alpha$ is stationary in α. In particular, $V^-(\mathbf{T}) \cap E^\kappa_{\leq\chi}$ is stationary.

Proof. Write $\mathbf{T} = (T, <_T)$. Towards a contradiction, suppose that $\alpha \in S$ is a counterexample. As $\alpha \notin V(\mathbf{T})$, we may fix $x \in T$ with $\mathrm{ht}(x) < \alpha$ such that every α-branch B with $x \in B$ has an upper bound in $\mathbf{T}$. Since either $\mathrm{cf}(\alpha) \leq \omega$ or $V^-(\mathbf{T}) \cap \alpha$ is nonstationary in α, we may fix a club C in α of order-type χ such that $\min(C) = \mathrm{ht}(x)$ and such that $\mathrm{acc}(C) \cap V^-(\mathbf{T}) = \emptyset$.

Let $\langle \alpha_i \mid i < \chi\rangle$ denote the increasing enumeration of C. We shall recursively construct an array of nodes $\langle t_s \mid s \in {}^{<\chi}\varsigma\rangle$ in such a way that $t_s \in T_{\alpha_{\mathrm{dom}(s)}}$. Set $t_\emptyset := x$. For every $i < \chi$ and every $s : i \to \varsigma$ such that t_s has already been defined, since T is normal and ς-splitting, we may find an injective sequence $\langle t_{s^\frown\langle j\rangle} \mid j < \varsigma\rangle$ of nodes of $T_{\alpha_{i+1}}$ all extending t_s. For every $i \in \mathrm{acc}(\chi)$ such that $\langle t_s \mid s \in {}^{<i}\varsigma\rangle$ has already been defined, for every $s : i \to \varsigma$, since $\{t_{s\restriction\iota} \mid \iota < i\}$ induces an α_i-branch, the fact that $\alpha_i \notin V^-(\mathbf{T})$

[7] That is, a tree of height and size χ admitting at least χ^+-many branches.

implies that we may find $t_s \in T_{\alpha_i}$ that is a limit of that α_i-branch. This completes the recursive construction of our array.

For every $s \in {}^{\chi}\varsigma$, $B_s := \{t \in T \mid \exists i < \chi\,(t <_T t_{s\restriction i})\}$ is an α-branch containing x, and hence there must be some $b_s \in T_\alpha$ extending all elements of B_s. Our construction also ensures that $B_s \neq B_{s'}$ whenever $s \neq s'$. We now consider a few options:

(1) Suppose that $\varsigma^\chi \geq \kappa$. Then $|T_\alpha| \geq |\{b_s \mid s \in {}^{\chi}\varsigma\}| = \varsigma^\chi \geq \kappa$. This is a contradiction.
(2) Suppose that $\mathbf{T}$ is S-regressive, as witnessed by $\rho : T \restriction S \to T$. For every $s \in {}^{\chi}\varsigma$, $\rho(b_s)$ belongs to B_s, but by Remark 2.8, we may assume that $\rho(b_s) = t_{s\restriction i}$ for some $i < \chi$.
   - ▶ If $\varsigma^{<\chi} < \varsigma^\chi$, then we may now find $s \neq s'$ in ${}^{\chi}\varsigma$ such that $\rho(b_s) = \rho(b_{s'})$. Then, $\rho(b_{s'}) <_T t_s$ and $\rho(b_s) <_T t_{s'}$, contradicting the fact that $b_s \neq b_{s'}$.
   - ▶ If $\chi = \varsigma$ and there exists a weak χ-Kurepa tree, then this may be witnessed by a tree of the form $(K, \subseteq)$ for some $K \subseteq {}^{<\chi}\varsigma$. Let $\langle s_\beta \mid \beta < \chi^+\rangle$ be an injective enumeration of branches through $(K, \subseteq)$. Since $|K| \leq \chi$, there must exist $\beta \neq \beta'$ such that $\rho(b_{s_\beta}) = \rho(b_{s_{\beta'}})$, which yields a contradiction as in the previous case. □

Corollary 2.10. *If κ is not a strong limit, then for every normal and splitting κ-tree $\mathbf{T}$, $V^-(\mathbf{T})$ is stationary.*

Proof. Suppose that κ is not a strong limit. It is not hard to see that there exists some infinite cardinal $\varsigma < \kappa$ for which there exists a regular cardinal $\chi < \kappa$ such that $\varsigma^\chi \geq \kappa$. Now, given a normal and splitting κ-tree $\mathbf{T} = (T, <_T)$, as shown in the proof of [RS23, Proposition 2.16], the club $D := \{\alpha < \kappa \mid \alpha = \varsigma^\alpha\}$ satisfies that $\mathbf{T}' = (T \restriction D, <_T)$ is normal and ς-splitting. By Lemma 2.9, $V^-(\mathbf{T}')$ is stationary. As D is a club in κ, this means that $V^-(\mathbf{T})$ is stationary, as well. □

Corollary 2.11. *If $\kappa = \lambda^+$ is a successor cardinal and $\lambda^{\aleph_0} \geq \kappa$, then for every normal and splitting κ-tree $\mathbf{T}$, $E^\kappa_\omega \setminus V(\mathbf{T})$ is nonstationary.*

Proof. Suppose that κ and λ are as above. Now, given a normal and splitting κ-tree $\mathbf{T} = (T, <_T)$, the club $D := \{\alpha < \kappa \mid \alpha = \lambda^\alpha\}$ satisfies that $\mathbf{T}' = (T \restriction D, <_T)$ is normal and λ-splitting. By Lemma 2.9, $V(\mathbf{T}') \supseteq E^\kappa_\omega$. As D is a club in κ, this means that $E^\kappa_\omega \setminus V(\mathbf{T})$ is nonstationary. □

Definition 2.12 ([BR21])**.** A *streamlined κ-tree* is a subset $T \subseteq {}^{<\kappa}H_\kappa$ such that the following two conditions are satisfied:

(1) T is downward-closed, i.e, for every $t \in T$, $\{t \restriction \alpha \mid \alpha < \kappa\} \subseteq T$;
(2) for every $\alpha < \kappa$, the set $T_\alpha := T \cap {}^{\alpha}\kappa$ is nonempty and has size $< \kappa$.

For every $\alpha \leq \kappa$, we denote $\mathcal{B}(T \restriction \alpha) := \{f \in {}^{\alpha}H_\kappa \mid \forall \beta < \alpha\,(f \restriction \beta \in T)\}$.

Note that every streamlined tree is Hausdorff.

Convention 2.13. We identify a streamlined tree T with the poset $\mathbf{T} = (T, \subseteq)$.

Definition 2.14. For two elements s, t of H_κ, we define $s * t$ to be the emptyset, unless $s, t \in {}^{<\kappa}H_\kappa$ with $\mathrm{dom}(s) \leq \mathrm{dom}(t)$, in which case $s * t : \mathrm{dom}(t) \to H_\kappa$ is defined by stipulating:

$$(s * t)(\beta) := \begin{cases} s(\beta), & \text{if } \beta \in \mathrm{dom}(s); \\ t(\beta), & \text{otherwise.} \end{cases}$$

Definition 2.15. A streamlined κ-tree T is *uniformly homogeneous* iff for all $\alpha < \beta < \kappa$, $s \in T_\alpha$ and $t \in T_\beta$, $s * t$ is in T.

The next proposition should be clear, but we include a proof sketch.

Proposition 2.16. *Suppose that T is a streamlined κ-tree that is uniformly homogeneous. Then T is indeed homogeneous.*

Proof. Let $\alpha < \kappa$ and $s, s' \in T_\alpha$. Define $\pi : T \to T$ via:

$$\pi(t) := \begin{cases} s' \restriction \mathrm{dom}(t), & \text{if } t \subseteq s; \\ s \restriction \mathrm{dom}(t), & \text{if } t \subseteq s'; \\ s' * t, & \text{if } t \supseteq s; \\ s * t, & \text{if } t \supseteq s'; \\ t, & \text{otherwise.} \end{cases}$$

Then π is a well-defined automorphism of T, sending s to s'. □

Lemma 2.17. *For a stationary $S \subseteq \kappa$, the following are equivalent:*

1. *There exist a club $D \subseteq \kappa$ and a thin ladder system $\langle A_\alpha \mid \alpha \in S \cap D\rangle$ such that, for every $(\alpha, \beta) \in [S \cap D]^2$, $\sup(A_\alpha \cap A_\beta) < \alpha$;*
2. *There exist a club $D \subseteq \kappa$ and a thin ladder system $\langle A_\alpha \mid \alpha \in S \cap D\rangle$ such that, for every $(\alpha, \beta) \in [S \cap D]^2$, $A_\alpha \neq A_\beta \cap \alpha$;*
3. *There exist a club $D \subseteq \kappa$ and a uniformly homogeneous streamlined κ-tree T such that $V(T) \supseteq S \cap D$;*
4. *There exist a club $D \subseteq \kappa$ and a κ-tree $\mathbf{T}$ such that $V^-(\mathbf{T}) \supseteq S \cap D$.*

Proof. (1) $\implies$ (2): This is immediate.

(2) $\implies$ (3): Suppose that D and $\langle A_\alpha \mid \alpha \in S \cap D\rangle$ are as in (2). Let $\langle x_i \mid i < \kappa\rangle$ be an injective enumeration of $\langle A_\alpha \cap \varepsilon \mid \varepsilon < \alpha, \alpha \in S \cap D\rangle$. For each $\alpha \in S \cap D$, let $k_\alpha : \alpha \to \kappa$ be the unique function to satisfy for all $\varepsilon < \alpha$:

$$A_\alpha \cap \varepsilon = x_{k_\alpha(\varepsilon)}.$$

Define first an auxiliary collection K by letting

$$K := \{k_\beta \restriction \alpha \mid \alpha < \beta, \beta \in S \cap D\}.$$

Note that $\{\mathrm{dom}(y) \mid y \in K\} = \kappa$ and that K is closed under taking initial segments. So K is a streamlined κ-tree because otherwise there must exist some $\varepsilon < \kappa$ such that $\{k_\beta \restriction \varepsilon \mid \beta \in S \cap D\}$ has size κ, contradicting the fact

that $\langle A_\beta \mid \beta \in S \cap D\rangle$ is thin. We shall use K to construct a uniformly homogeneous streamlined κ-tree T by defining its levels T_α by recursion on $\alpha < \kappa$.

Start by letting $T_0 := K_0$. Clearly, $T_0 = \{\emptyset\}$, so that $|T_0| < \kappa$. Next, for every nonzero $\alpha < \kappa$ such that $T \restriction \alpha$ has already been defined and have size less than κ, let

$$T_\alpha := \{x * y \mid x \in T \restriction \alpha,\, y \in K_\alpha\}$$

and note that $|T_\alpha| < \kappa$. Altogether, T is a streamlined κ-tree.

Claim 2.17.1. *T is uniformly homogeneous.*

Proof. We prove that $x * y \in T$ for all $x, y \in T$ with $\mathrm{dom}(x) < \mathrm{dom}(y)$. The proof is by induction on $\mathrm{dom}(y)$. So suppose that $\alpha < \kappa$ is such that for all $x, y \in T$ with $\mathrm{dom}(x) < \mathrm{dom}(y) < \alpha$, it is the case that $x * y \in T$, and let $x, y \in T$ with $\mathrm{dom}(x) < \mathrm{dom}(y) = \alpha$. Recalling the definition of T_α, pick $x' \in T \restriction \alpha$ and $y' \in K_\alpha$ such that $y = x' * y'$.

▶ If $\mathrm{dom}(x) < \mathrm{dom}(x')$, then $x * y = x * (x' * y') = (x * x') * y'$. As $\mathrm{dom}(x) < \mathrm{dom}(x') < \alpha$, the induction hypothesis implies that $x * x' \in T \restriction \alpha$, and then the definition of T_α implies that $(x * x') * y'$ is in T.

▶ If $\mathrm{dom}(x) \geq \mathrm{dom}(x')$, then $x * y = x * (x' * y') = x * y'$, and then the definition of T_α implies that $x * y'$ is in T. □

By the preceding claim together with Proposition 2.5, it now suffices to prove that $V^-(T) \supseteq S \cap D \cap \mathrm{acc}(\kappa)$. To this end, let $\alpha \in S \cap D \cap \mathrm{acc}(\kappa)$. Clearly, $b := \{k_\alpha \restriction \varepsilon \mid \varepsilon < \alpha\}$ is an α-branch in K and hence in T. If b is not vanishing in T, then we may find $x \in T \restriction \alpha$ and $y \in K_\alpha$ such that $x * y = k_\alpha$. Recalling the definition of K_α, we may pick $\beta \in S \cap D$ above α such that $y = k_\beta \restriction \alpha$. As $\alpha < \beta$, it is the case that $A_\alpha \neq A_\beta \cap \alpha$, so we may pick $\delta \in A_\alpha \Delta (A_\beta \cap \alpha)$. Then $\varepsilon := \max\{\delta, \mathrm{dom}(x)\} + 1$ is smaller than α and satisfies $k_\alpha(\varepsilon) \neq k_\beta(\varepsilon)$, contradicting the fact that $k_\alpha(\varepsilon) = (x * y)(\varepsilon) = y(\varepsilon) = k_\beta(\varepsilon)$.

(3) $\implies$ (4): This is immediate.

(4) $\implies$ (1) Every κ-tree is order-isomorphic to an ordinal-based tree (see, e.g., [RS23, Proposition 2.16]), so we may assume that we are given a tree $\mathbf{T}$ of the form $(\kappa, <_T)$ and a club $D \subseteq \kappa$ such that $V^-(\mathbf{T}) \supseteq S \cap D$. By possibly shrinking D, we may also assume that $D \subseteq \mathrm{acc}\{\beta < \kappa \mid T \restriction \beta = \beta\}$. It follows that for every $\alpha \in D$, every α-branch is a cofinal subset of α. For every $\alpha \in S \cap D$, let A_α be a vanishing α-branch. As $\mathbf{T}$ is a κ-tree, the ladder system $\langle A_\alpha \mid \alpha \in S \cap D\rangle$ is thin. In addition, for every $(\alpha, \beta) \in [S \cap D]^2$, if it were the case that $\sup(A_\beta \cap A_\alpha) = \alpha$, then $\min(A_\beta \setminus A_\alpha)$ is a node extending all elements of A_α, contradicting the fact that A_α is vanishing. So, $\sup(A_\beta \cap A_\alpha) < \alpha$. □

When S is a club, the preceding is related to the subtle tree property:

Definition 2.18 (Weiß, [Wei10])**.** κ has the *subtle tree property* (κ-STP for short) iff for every thin list $\langle A_\alpha \mid \alpha \in D\rangle$ over a club $D \subseteq \kappa$, there exists a pair $(\alpha, \beta) \in [D]^2$ such that $A_\alpha = A_\beta \cap \alpha$.

Corollary 2.19. *All of the following are equivalent:*

- κ-STP *fails;*
- *there is a* κ*-tree* $\mathbf{T}$ *with* $V^-(\mathbf{T}) = \mathrm{acc}(\kappa)$*;*
- *there is an homogeneous* κ*-tree* $\mathbf{T}$ *with* $V(\mathbf{T}) = \mathrm{acc}(\kappa)$*;*
- *there is a uniformly homogeneous streamlined* κ*-tree* T *such that* $V(T)$ *covers a club in* κ*.*

Proof. By Lemmas 2.17 and 2.4. □

Remark 2.20. By [Wei10, Theorem 3.2.5], PFA implies that $\aleph_2$-STP holds. By [HS20, Theorem 1.2], if λ is the singular limit of supercompact cardinals then λ^+-STP fails.[8]

Corollary 2.21. *Assuming the consistency of a subtle cardinal, it is consistent that the conjunction of the following holds true:*

- *there exists an* $\aleph_2$*-Souslin tree;*
- *for every normal and splitting* $\aleph_2$*-tree* $\mathbf{T}$*,* $E^{\aleph_2}_{\aleph_1} \setminus V(\mathbf{T})$ *is stationary.*

Proof. Fix a subtle cardinal κ that is not weakly compact in L, and work in the forcing extension by Mitchell's forcing of length κ. By [Wei10, Theorem 2.3.1], $\aleph_2$-STP holds, and hence $V(\mathbf{T})$ cannot contain a club for every $\aleph_2$-tree $\mathbf{T}$. In addition, this is a model in which $2^{\aleph_0} = \aleph_2$ and hence Corollary 2.11 implies that $E^{\aleph_2}_{\aleph_0} \setminus V(\mathbf{T})$ is nonstationary for every normal and splitting $\aleph_2$-tree $\mathbf{T}$. Therefore, $E^{\aleph_2}_{\aleph_1} \setminus V(\mathbf{T})$ is stationary for every normal and splitting $\aleph_2$-tree $\mathbf{T}$. In addition, this is a model in which $\mathfrak{b} = \aleph_1$, $2^{\aleph_1} = \aleph_2$, and (since κ is not weakly compact in L) $\square(\aleph_2)$ holds. So, by [Rin22, Theorem A], there exists an $\aleph_2$-Souslin tree. □

Corollary 2.22. *Suppose that* S *is a stationary subset of a strongly inaccessible* κ*. Then there exists a* κ*-tree* $\mathbf{T}$ *such that* $V(\mathbf{T}) \cap S$ *is stationary.*

Proof. By Lemma 2.17, it suffices to find a stationary $S^- \subseteq S$ that carries a thin almost disjoint C-sequence. We consider two cases:

▶ If $S \cap E^\kappa_\omega$ is stationary, then set $S^- := S \cap E^\kappa_\omega$, and let $\langle C_\alpha \mid \alpha \in S^- \rangle$ be some ω-bounded C-sequence over S^-.

▶ Otherwise, let $S^- := S \setminus (E^\kappa_\omega \cup \mathrm{Tr}(S))$. Then S^- is stationary, and for every $\alpha \in S^-$, we may pick a club C_α in α that is disjoint from S. Evidently, $\sup(C_{\alpha'} \cap C_\alpha) < \alpha'$ for every $(\alpha, \alpha') \in [S^-]^2$. □

Lemma 2.23. *If* $\theta \in \mathrm{Reg}(\kappa)$ *is such that* $\lambda^{<\theta} < \kappa$ *for all* $\lambda < \kappa$*, then there exists an almost disjoint thin* C*-sequence over* E^κ_θ*.*

Proof. Just take a θ-bounded C-sequence over E^κ_θ. □

Building on the work of Todorčević [Tod07] and Krueger [Kru13], we obtain the following pump-up theorem for special κ-Aronszajn trees.

[8]The statement of the theorem in [HS20] is limited to countable cofinality, but the proof works unconditionally.

Theorem 2.24. *The following are equivalent:*

(i) There exists a special κ-Aronszajn tree;

(ii) There exists a streamlined κ-Aronszajn tree K, a club $D \subseteq \mathrm{acc}(\kappa)$ and a function $f : K \restriction D \to \kappa$ such that all of the following hold:
- *$V^-(K) \supseteq D$;*
- *$f(x) < \mathrm{dom}(x)$ for all $x \in K \restriction D$;*
- *$f(x) \neq f(y)$ for every pair $x \subsetneq y$ of nodes from $K \restriction D$;*
- *for all $x, y \in K$ and $\varepsilon \in \mathrm{dom}(x) \cap \mathrm{dom}(y)$, if $x(\varepsilon) = y(\varepsilon)$, then $x \restriction \varepsilon = y \restriction \varepsilon$.*

(iii) There exists a streamlined uniformly homogeneous special κ-Aronszajn tree T for which $V(T)$ covers a club in κ;

(iv) There exists an homogeneous special κ-Aronszajn tree $\mathbf{T}$ with $V(\mathbf{T}) = \mathrm{acc}(\kappa)$.

Proof. $(i) \implies (ii)$ Assuming that there exists a special κ-Aronszajn tree, by [Kru13, Lemma 1.2 and Theorem 2.5], we may fix a C-sequence $\vec{C} = \langle C_\beta \mid \beta < \kappa \rangle$ and a club $C \subseteq \mathrm{acc}(\kappa)$ satisfying the following:

(1) for every $\beta \in C$, $\min(C_\beta) > \mathrm{otp}(C_\beta)$;
(2) for every $\beta \in \mathrm{acc}(\kappa) \setminus C$, $\min(C_\beta) > \sup(C \cap \beta)$;
(3) for every $\epsilon < \kappa$, $|\{C_\beta \cap \epsilon \mid \beta < \kappa\}| < \kappa$.

Consider the following additional requirement:

(4) $\min(C_\beta) = \mathrm{otp}(C_\beta) + 1$ for every $\beta \in C$.

Claim 2.24.1. *We may moreover assume that Clause (4) holds.*

Proof. For every $\beta \in C$, let $C^\bullet_\beta := C_\beta \cup \{\mathrm{otp}(C_\beta)+1\}$, and for every $\beta \in \kappa \setminus C$, let $C^\bullet_\beta := C_\beta$. We just need to verify that $|\{C^\bullet_\beta \cap \epsilon \mid \beta < \kappa\}| < \kappa$ for every $\epsilon < \kappa$. Towards a contradiction, suppose that ϵ is a counterexample. From (3), it follows that we may fix $B \in [C]^\kappa$ on which the map $\beta \mapsto C^\bullet_\beta \cap \epsilon$ is injective. We may moreover assume that $\beta \mapsto C_\beta \cap \epsilon$ is constant over B. By possibly removing one element of B, we may assume that $C^\bullet_\beta \cap \epsilon$ is nonempty for all $\beta \in B$. So, we may moreover assume the existence of $\tau < \epsilon$ such that $\min(C^\bullet_\beta) = \tau$ for every $\beta \in B$. But then $C^\bullet_\beta \cap \epsilon = (C_\beta \cap \epsilon) \cup \{\tau\}$ for every $\beta \in B$. This is a contradiction. □

Now, let ρ_0 be the characteristic function from [Tod07, §6] obtained by walking along $\vec{C}$ satisfying (1)–(4), and consider the following streamlined κ-tree

$$T(\rho_0) := \{\rho_{0\beta} \restriction \alpha \mid \alpha \leq \beta < \kappa\}.$$

Using (1)–(3), the proof of [Kru13, Theorem 4.4] provides a club $D \subseteq C$ and a function $g : T(\rho_0) \restriction D \to \kappa$ satisfying the following two:

- $g(t) < \mathrm{dom}(t)$ for all $t \in T(\rho_0) \restriction D$;
- for every pair $s \subsetneq t$ of nodes from $T(\rho_0) \restriction D$, $g(s) \neq g(t)$.

Next, consider the following subfamily of $T(\rho_0)$:

$$T := \{\rho_{0\beta} \restriction \alpha \mid \alpha < \beta < \kappa\}.$$

Clearly, T is downward-closed and $\{\mathrm{dom}(y) \mid y \in T\} = \kappa$, so that T is a streamlined κ-Aronszajn subtree of $T(\rho_0)$.

Claim 2.24.2. *$T \cap \{\rho_{0\alpha} \mid \alpha \in C\} = \emptyset$. In particular, $V^-(T) \supseteq C \supseteq D$.*

Proof. The "in particular" part will follow from the fact that $\{\rho_{0\alpha}\restriction\epsilon \mid \epsilon < \alpha\}$ is an α-branch of T for every $\alpha < \kappa$. Thus, let $\alpha \in C$ and we shall prove that $\rho_{0\alpha} \notin T$. Suppose not, and pick some $\beta > \alpha$ such that $\rho_{0\alpha} = \rho_{0\beta} \restriction \alpha$. Recall that for every $\gamma < \kappa$,

$$C_\gamma = \{\xi < \gamma \mid \rho_{0\gamma}(\xi) \text{ is a sequence of length } 1\}.$$

In particular, $\min(C_\alpha) = \min(C_\beta)$. As $\sup(C \cap \beta) \geq \alpha > \min(C_\alpha)$, it follows from Clause (2) that $\beta \in C$. So, by Clause (4), $\mathrm{otp}(C_\alpha) = \mathrm{otp}(C_\beta)$. It follows that may fix some $\delta \in C_\alpha \setminus C_\beta$. But then $\rho_{0\alpha}(\delta)$ is a sequence of length 1, whereas $\rho_{0\beta}(\delta)$ is a longer sequence. This is a contradiction. $\square$

For every $t \in T \restriction \mathrm{acc}(\kappa)$, define a function $k_t : \mathrm{dom}(t) \to T$ via

$$k_t(\varepsilon) := t \restriction \varepsilon.$$

Let K be the following downward-closed subfamily of ${}^{<\kappa}H_\kappa$:

$$K := \{k_t \restriction \alpha \mid \alpha \leq \mathrm{dom}(t), t \in T \restriction \mathrm{acc}(\kappa)\}.$$

Evidently, for all $x, y \in K$ and $\varepsilon \in \mathrm{dom}(x) \cap \mathrm{dom}(y)$, if $x(\varepsilon) = y(\varepsilon)$, then $x \restriction \varepsilon = y \restriction \varepsilon$. In addition, $t \mapsto k_t$ constitutes an isomorphism between $(T\restriction\mathrm{acc}(\kappa), \subseteq)$ and $(K\restriction\mathrm{acc}(\kappa), \subseteq)$, and hence K is a streamlined κ-Aronszajn tree with $V^-(K) \supseteq D$. The fact that the above map is an isomorphism also implies that a function $f : K \restriction D \to \kappa$ defined via $f(k_t) := g(t)$ satisfies that $f(x) < \mathrm{dom}(x)$ for all $x \in K \restriction D$, and that $f(x) \neq f(y)$ for every pair $x \subsetneq y$ of nodes from $K \restriction D$.

(ii) $\implies$ *(iii)*: Suppose that K and $f : K \restriction D \to \kappa$ are as in Clause (ii). By possibly shrinking D, we may assume that for all $\beta \in D$ and $\alpha < \beta$, it is the case that $\omega \cdot \alpha < \beta$.

The operation of Definition 2.14 is associative, so we may define a family T to be the collection of all elements of the form $x_0 * \cdots * x_n$ where[9]

(a) $n < \omega$,
(b) $x_i \in K$ for all $i \leq n$, and
(c) $\mathrm{dom}(x_i) < \mathrm{dom}(x_{i+1})$ for all $i < n$.

It is clear that $t \restriction \alpha \in T$ for all $t \in T$ and $\alpha < \kappa$. Thus, recalling the proof of Claim 2.17.1, to establish that T is a uniformly homogeneous streamlined κ-tree, it suffices to prove the following claim.

Claim 2.24.3. *$T_0 = \{\emptyset\}$ and $T_\alpha = \{x * y \mid x \in T \restriction \alpha, y \in K_\alpha\}$ for every nonzero $\alpha < \kappa$.*

[9]To clarify, in the special case that $n = 0$, $x_0 * \cdots * x_n$ stands for x_0.

Proof. Suppose that α is a nonzero ordinal such that $T_\epsilon = \{x * y \mid x \in T \restriction \alpha, y \in K_\epsilon\}$ for every $\epsilon < \alpha$. Let $t \in T_\alpha$. Pick a sequence $(x_0, \dots, x_n)$ satisfying (a)–(c) for which $t = x_0 * \cdots * x_n$.

▶ If $n = 0$, then $t = \emptyset * x_0$ with $\emptyset \in T \restriction \alpha$ and $x_0 \in K_\alpha$.

▶ If $n = m + 1$ for some $m < \omega$, then $t = x * y$ with $x := x_0 * \cdots * x_m$ in $T \restriction \alpha$ and $y := x_{m+1}$ in K_α. □

For each node $t \in T$, we define $n(t)$ and $x(t)$ by first letting $n(t)$ denote the least n for which there exists a sequence $(x_0, \dots, x_n)$ satisfying (a)–(c) for which $t = x_0 * \cdots * x_n$, and then letting $x(t)$ be such an x_n. Note that $\mathrm{dom}(x(t)) = \mathrm{dom}(t)$, and that $K = \{t \in T \mid n(t) = 0\}$.

Define a function $g : T \restriction D \to \kappa$ via

$$g(t) := (\omega \cdot f(x(t))) + n(t).$$

Claim 2.24.4. *(1) $g(t) < \mathrm{dom}(t)$ for all $t \in T \restriction D$;*

(2) Let $s \subsetneq t$ be a pair of nodes from $T \restriction D$. Then $g(s) \neq g(t)$.

Proof. (1) Since $\omega \cdot \alpha < \beta$ for all $\beta \in D$ and $\alpha < \beta$.

(2) Suppose not. Let $\tau < \kappa$ and $n < \omega$ be such that $f(x(s)) = \tau = f(x(t))$ and $n(s) = n = n(t)$. By the choice of f it follows that $x(s) \nsubseteq x(t)$, so since $s \subsetneq t$, it must be the case that $n = m + 1$ for some $m < \omega$. Fix a sequence $(x_0, \dots, x_m, x_{m+1})$ of nodes from K such that $s = x_0 * \cdots * x_m * x_{m+1}$ and $x_{m+1} = x(s)$. Likewise, fix a sequence $(y_0, \dots, y_m, y_{m+1})$ of nodes from K such that $t = y_0 * \cdots * y_m * y_{m+1}$ and $y_{m+1} = x(t)$.

▶ As $x_{m+1} \nsubseteq y_{m+1}$, we may fix $\delta \in \mathrm{dom}(x_{m+1})$ such that $x_{m+1}(\delta) \neq y_{m+1}(\delta)$.

▶ As $s \subseteq t = y_0 * \cdots * y_m * y_{m+1}$ and $n(s) > m$, it must be the case that $\mathrm{dom}(y_m) < \mathrm{dom}(s)$.

Altogether, $\varepsilon := \max\{\delta + 1, \mathrm{dom}(x_m), \mathrm{dom}(y_m)\}$ is an ordinal less than $\mathrm{dom}(s)$, satisfying $x_{m+1}(\varepsilon) = s(\varepsilon) = t(\varepsilon) = y_{m+1}(\varepsilon)$, but then $x_{m+1} \restriction \varepsilon = y_{m+1} \restriction \varepsilon$, contradicting the fact that $\delta < \varepsilon$. □

It is easy to see that the two features of g together imply that T admits no κ-branch. The beginning of the proof of [Kru13, Theorem 4.4] shows furthermore that T must be a special κ-Aronszajn tree.

Claim 2.24.5. *$V(T) \supseteq D$.*

Proof. Let $\alpha \in D$. As $D \subseteq V^-(K)$, we may fix a function $t : \alpha \to H_\kappa$ such that $\{t \restriction \epsilon \mid \epsilon < \alpha\} \subseteq K$, but $t \notin K$. As $K \subseteq T$, it thus suffices to prove that $t \notin T$. Towards a contradiction, suppose that $t \in T$. In particular, $n(t) > 0$. Fix $m < \omega$ and a sequence $(x_0, \dots, x_m, x_{m+1})$ of nodes from K such that $t = x_0 * \cdots * x_m * x_{m+1}$. As $x_{m+1} \neq t$, we may fix some $\delta < \alpha$ such that $t(\delta) \neq x_{m+1}(\delta)$. Pick $\varepsilon < \alpha$ above $\max\{\delta, \mathrm{dom}(x_m)\}$. Then $t(\varepsilon) = x_{m+1}(\varepsilon)$. But $t \restriction (\varepsilon + 1)$ and $x_{m+1} \restriction (\varepsilon + 1)$ are two nodes in K that agree on ε and hence $t \restriction (\varepsilon + 1) = x_{m+1} \restriction (\varepsilon + 1)$, contradicting the fact that $\delta < \varepsilon$. □

The implication $(iii) \implies (iv)$ follows from the proof of Lemma 2.4 and the implication $(iv) \implies (i)$ is trivial. □

Definition 2.25 (Products)**.** For a sequence of κ-trees $\langle \mathbf{T}^i \mid i<\tau\rangle$ with $\mathbf{T}^i=(T^i,<_{T^i})$ for each $i<\tau$, the product $\bigotimes_{i<\tau}\mathbf{T}^i$ is defined to be the tree $\mathbf{T}=(T,<_T)$, where:

- $T=\bigcup\{\prod_{i<\tau}T^i_\alpha \mid \alpha<\kappa\}$;
- $\vec{s}<_T\vec{t}$ iff $\vec{s}(i)<_{T^i}\vec{t}(i)$ for every $i<\tau$.

Proposition 2.26. *For a sequence $\langle \mathbf{T}^i \mid i<\tau\rangle$ of normal κ-trees, if $\lambda^\tau<\kappa$ for all $\lambda<\kappa$, then:*

(1) $\bigotimes_{i<\tau}\mathbf{T}^i$ is a normal κ-tree;
(2) $V(\bigotimes_{i<\tau}\mathbf{T}^i)=\bigcup\{V(\mathbf{T}^i)\mid i<\tau\}$;
(3) $V^-(\bigotimes_{i<\tau}\mathbf{T}^i)=\bigcup\{V^-(\mathbf{T}^i)\mid i<\tau\}$.

Proof. Left to the reader. □

Definition 2.27 (Sums)**.** The *disjoint sum* $\sum\mathcal{P}$ of a family of posets $\mathcal{P}$ is the poset $(A,<_A)$ defined as follows:

- $A:=\{((P,<_P),x)\mid (P,<_P)\in\mathcal{P},x\in P\}$;
- $((P,<_P),x)<_A((Q,<_Q),y)$ iff $(P,<_P)=(Q,<_Q)$ and $x<_P y$.

In the special case of doubleton we write $\mathbf{T}+\mathbf{S}$ instead of $\sum\{\mathbf{T},\mathbf{S}\}$.

Proposition 2.28. *Suppose that $\mathcal{T}$ is a family of less than κ many κ-trees. Then:*

(1) $\sum\mathcal{T}$ is a κ-tree;
(2) $V(\sum\mathcal{T})=\bigcap\{V(\mathbf{T})\mid \mathbf{T}\in\mathcal{T}\}$;
(3) $V^-(\sum\mathcal{T})=\bigcup\{V^-(\mathbf{T})\mid \mathbf{T}\in\mathcal{T}\}$.

Proof. Left to the reader. □

It follows from Propositions 2.26 and 2.28 that $\mathrm{Vspec}(\kappa)$ is closed under finite unions and intersections.

Corollary 2.29. *Suppose $\chi\in\mathrm{Reg}(\kappa)$ is such that $\lambda^{<\chi}<\kappa$ for all $\lambda<\kappa$. Then there exists a κ-tree $\mathbf{T}$ with $V^-(\mathbf{T})\supseteq\mathrm{acc}(\kappa)\cap E^\kappa_{\leq\chi}$.*

Proof. Denote $\Theta:=\mathrm{Reg}(\chi+1)$. By Lemmas 2.23 and 2.17, for every $\theta\in\Theta$, we may pick a κ-tree $\mathbf{T}^\theta$ such that $V^-(\mathbf{T}^\theta)$ covers E^κ_θ modulo a club. In fact, the proof of (2) $\implies$ (3) of Lemma 2.17 shows that we may secure $V^-(\mathbf{T}^\theta)\supseteq E^\kappa_\theta$. Let $\mathbf{T}:=\sum\{\mathbf{T}^\theta\mid\theta\in\Theta\}$ be the disjoint sum of these trees. By Proposition 2.28, $V^-(\mathbf{T})=\bigcup_{\theta\in\Theta}V^-(\mathbf{T}^\theta)\supseteq\bigcup_{\theta\in\Theta}E^\kappa_\theta=\mathrm{acc}(\kappa)\cap E^\kappa_{\leq\chi}$. □

Remark 2.30. In Section 5, we provide sufficient conditions for getting an homogeneous κ-Souslin tree $\mathbf{T}$ with $V(\mathbf{T})=\bigcup_{\chi\in x}E^\kappa_\chi$ for a prescribed finite and nonempty $x\subseteq\mathrm{Reg}(\kappa)$.

Question 2.31. Is it consistent that for some regular uncountable cardinal κ, there are κ-Souslin trees, but $V(\mathbf{T})$ is nonstationary for every κ-Souslin tree $\mathbf{T}$?

By Proposition 2.5, Corollary 2.10 and [BR17b, Lemma 2.4], in such a model there cannot be an homogeneous κ-Souslin tree. A model with an $\aleph_1$-Souslin tree but no homogeneous one was constructed by Abraham and Shelah in [AS93].

3. Consulting another tree

The main result of this section is Theorem 3.7 below. A sample corollary of it reads as follows.

Corollary 3.1. *Suppose that $\kappa = \lambda^+$ for an infinite cardinal λ.*

(1) If $\square_\lambda + \diamondsuit(\kappa)$ holds, then there exists a κ-Souslin tree $\mathbf{T}$ with $V(\mathbf{T}) = \mathrm{acc}(\kappa)$;
(2) If $\square(\kappa)$ holds and $\aleph_0 < \lambda^{<\lambda} < \lambda^+ = 2^\lambda$, then there exists a κ-Souslin tree $\mathbf{T}$ with $V(\mathbf{T}) = \mathrm{acc}(\kappa)$;
(3) If $\mathrm{P}_\lambda(\kappa, \kappa, \sqsubseteq, 1)$ holds, then there exists a κ-Souslin tree $\mathbf{T}$ such that $V(\mathbf{T}) \supseteq E^\kappa_{>\omega}$.

Proof. (1) $\diamondsuit(\aleph_1)$ implies the existence of a normal and splitting $\aleph_1$-Souslin tree $\mathbf{T}$, and by Corollary 2.11, $V(\mathbf{T}) = \mathrm{acc}(\aleph_1)$. For $\lambda \geq \aleph_1$, by [BR17a, Corollary 3.9], $\square_\lambda + \mathrm{CH}_\lambda$ is equivalent to $\mathrm{P}_\lambda(\kappa, 2, \sqsubseteq, 1)$. In addition, by a theorem of Jensen, $\square_\lambda$ gives rise to a special λ^+-Aronszajn tree. Thus, we infer from Proposition 2.24 the existence of a κ-tree $\mathbf{K}$ for which $V^-(\mathbf{K}) = \mathrm{acc}(\kappa)$. It thus follows from Theorem 3.7(1) below that there exists a κ-Souslin tree $\mathbf{T}$ for which $V(\mathbf{T})$ is a club in κ. Finally, appeal to Lemma 2.4.

(2) By [Rin17, Corollary 4.4], the hypothesis implies that $\mathrm{P}^-(\kappa, 2, \sqsubseteq, 1)$ holds. In addition, by a theorem of Specker, $\lambda = \lambda^{<\lambda}$ implies the existence of a special λ^+-Aronszajn tree. Now, continue as in the proof of Clause (1).

(3) Similar to the proof of Clause (1), using Theorem 3.7(2), instead. $\square$

Remark 3.2. Sufficient conditions for $\mathrm{P}_\lambda(\kappa, \kappa, \sqsubseteq, 1)$ to hold are given by Corollaries 3.15 and 3.24 of [BR19c].

Before turning to the proofs of the main results of this section, we provide a few preliminaries.

Definition 3.3 (Proxy principle, [BR17a, BR21])**.** Suppose that $\mu, \theta \leq \kappa$ are cardinals, $\xi \leq \kappa$ is an ordinal, $\mathcal{R}$ is a binary relation over $[\kappa]^{<\kappa}$ and $\mathcal{S}$ is a collection of stationary subsets of κ. The principle $\mathrm{P}^-_\xi(\kappa, \mu, \mathcal{R}, \theta, \mathcal{S})$ asserts the existence of a ξ-bounded $\mathcal{C}$-sequence $\langle \mathcal{C}_\alpha \mid \alpha < \kappa \rangle$ such that:

- for every $\alpha < \kappa$, $|\mathcal{C}_\alpha| < \mu$;
- for all $\alpha < \kappa$, $C \in \mathcal{C}_\alpha$, and $\bar{\alpha} \in \mathrm{acc}(C)$, there exists some $D \in \mathcal{C}_{\bar{\alpha}}$ such that $D \mathrel{\mathcal{R}} C$;
- for every sequence $\langle B_i \mid i < \theta \rangle$ of cofinal subsets of κ, and every $S \in \mathcal{S}$, there are stationarily many $\alpha \in S$ such that for all $C \in \mathcal{C}_\alpha$ and $i < \min\{\alpha, \theta\}$, $\sup(\mathrm{nacc}(C) \cap B_i) = \alpha$.

Convention 3.4. We write $\mathrm{P}_\xi(\kappa, \mu, \mathcal{R}, \theta, \mathcal{S})$ to assert that $\mathrm{P}^-_\xi(\kappa, \mu, \mathcal{R}, \theta, \mathcal{S})$ and $\diamondsuit(\kappa)$ both hold.

Convention 3.5. If we omit ξ, then we mean $\xi := \kappa$. If we omit $\mathcal{S}$, then we mean $\mathcal{S} := \{\kappa\}$. In the case $\mu = 2$, we identify $\langle \mathcal{C}_\alpha \mid \alpha < \kappa \rangle$ with the unique element $\langle C_\alpha \mid \alpha < \kappa \rangle$ of $\prod_{\alpha<\kappa} \mathcal{C}_\alpha$.

Fact 3.6 ([BR17a, Lemma 2.2])**.** *The following are equivalent:*

(1) $\diamondsuit(\kappa)$*, i.e., there is a sequence* $\langle f_\beta \mid \beta < \kappa \rangle$ *such that for every function* $f : \kappa \to \kappa$*, the set* $\{\beta < \kappa \mid f \restriction \beta = f_\beta\}$ *is stationary in* κ*.*

(2) $\diamondsuit^-(H_\kappa)$*, i.e., there is a sequence* $\langle \Omega_\beta \mid \beta < \kappa \rangle$ *such that for all* $p \in H_{\kappa^+}$ *and* $\Omega \subseteq H_\kappa$*, there exists an elementary submodel* $\mathcal{M} \prec H_{\kappa^+}$ *such that:*
- $p \in \mathcal{M}$*;*
- $\mathcal{M} \cap \kappa \in \kappa$*;*
- $\mathcal{M} \cap \Omega = \Omega_{\mathcal{M}\cap\kappa}$*.*

(3) $\diamondsuit(H_\kappa)$*, i.e., there are a partition* $\langle R_i \mid i < \kappa \rangle$ *of* κ *and a sequence* $\langle \Omega_\beta \mid \beta < \kappa \rangle$ *such that for all* $p \in H_{\kappa^+}$*,* $\Omega \subseteq H_\kappa$*, and* $i < \kappa$*, there exists an elementary submodel* $\mathcal{M} \prec H_{\kappa^+}$ *such that:*
- $p \in \mathcal{M}$*;*
- $\mathcal{M} \cap \kappa \in R_i$*;*
- $\mathcal{M} \cap \Omega = \Omega_{\mathcal{M}\cap\kappa}$*.*

Theorem 3.7. *Suppose that* K *is some streamlined* κ*-tree.*

(1) If $\mathrm{P}(\kappa, 2, \sqsubseteq^*, 1)$ *holds, then there exists a normal and splitting streamlined* κ*-Souslin tree* T *such that* $V(T) \supseteq V^-(K)$*;*

(2) If $\mathrm{P}(\kappa, \kappa, \sqsubseteq, 1)$ *holds, then there exists a normal and splitting streamlined* κ*-Souslin tree* T *such that* $V(T) \supseteq V^-(K) \cap E^\kappa_{>\omega}$*.*

Proof. Fix a well-ordering $\lhd$ of H_κ, and a sequence $\langle \Omega_\beta \mid \beta < \kappa \rangle$ witnessing $\diamondsuit^-(H_\kappa)$. If $\mathrm{P}^-(\kappa, \kappa, \sqsubseteq, 1)$ holds, then let $\vec{\mathcal{C}} = \langle \mathcal{C}_\alpha \mid \alpha < \kappa \rangle$ be any $\mathrm{P}^-(\kappa, \kappa, \sqsubseteq, 1)$-sequence. If $\mathrm{P}^-(\kappa, 2, \sqsubseteq^*, 1)$ holds, then, by [BR21, Theorem 4.39], we may let $\vec{\mathcal{C}} = \langle \mathcal{C}_\alpha \mid \alpha < \kappa \rangle$ be a $\mathrm{P}^-(\kappa, \kappa, \sqsubseteq, 1)$-sequence with the added feature that for every $\alpha \in \mathrm{acc}(\kappa)$ for all $C, D \in \mathcal{C}_\alpha$, $\sup(C \triangle D) < \alpha$.

Following the proof of [BR19b, Proposition 2.2], we shall recursively construct a sequence $\langle T_\alpha \mid \alpha < \kappa \rangle$ such that $T := \bigcup_{\alpha<\kappa} T_\alpha$ will constitute the tree of interest whose α^{th}-level is T_α.

We start by letting $T_0 := \{\emptyset\}$, and once T_α has already been defined, we let

$$T_{\alpha+1} := \{t^\frown\langle 0\rangle, t^\frown\langle 1\rangle, t^\frown\langle \eta\rangle \mid t \in T_\alpha, \eta \in K_\alpha\}.$$

Next, suppose that $\alpha \in \mathrm{acc}(\kappa)$ is such that $T \restriction \alpha$ has already been defined. For all $C \in \mathcal{C}_\alpha$ and $x \in T \restriction C$, we shall identify a set of potential nodes $\{\mathbf{b}^{C,\eta}_x \mid \eta \in \mathcal{B}(K \restriction \alpha)\}$ and then let

$$(\star) \qquad T_\alpha := \{\mathbf{b}^{C,\eta}_x \mid C \in \mathcal{C}_\alpha, \eta \in K_\alpha, x \in T \restriction C\}.$$

To this end, fix $C \in \mathcal{C}_\alpha$, $x \in T \restriction C$ and $\eta \in \mathcal{B}(K \restriction \alpha)$. The node $\mathbf{b}^{C,\eta}_x$ will be obtained as the limit $\bigcup \mathrm{Im}(b^{C,\eta}_x)$ of a sequence $b^{C,\eta}_x \in \prod_{\beta\in C\setminus\mathrm{dom}(x)} T_\beta$, as follows:

- Let $b_x^{C,\eta}(\mathrm{dom}(x)) := x$.
- For every $\beta \in \mathrm{nacc}(C)$ above $\mathrm{dom}(x)$ such that $b_x^{C,\eta}(\beta^-)$ has already been defined for $\beta^- := \sup(C \cap \beta)$, let
$$Q_x^{C,\eta}(\beta) := \{t \in T_\beta \mid \exists s \in \Omega_\beta[(s \cup (b_x^{C,\eta}(\beta^-)^\frown\langle \eta \restriction \beta^- \rangle)) \subseteq t]\}.$$
Now, consider the two possibilities:
  - If $Q_x^{C,\eta}(\beta) \neq \emptyset$, then let $b_x^{C,\eta}(\beta)$ be its $\lhd$-least element;
  - Otherwise, let $b_x^{C,\eta}(\beta)$ be the $\lhd$-least element of T_β that extends $b_x^{C,\eta}(\beta^-)^\frown\langle \eta \restriction \beta^- \rangle$. Such an element must exist, as the level T_β was constructed so as to preserve normality.
- For every $\beta \in \mathrm{acc}(C \setminus \mathrm{dom}(x))$ such that $b_x^{C,\eta} \restriction \beta$ has already been defined, let $b_x^{C,\eta}(\beta) := \bigcup \mathrm{Im}(b_x^{C,\eta} \restriction \beta)$.

For the last case, we need to argue that $b_x^{C,\eta}(\beta)$ is indeed an element of T_β. As $\vec{\mathcal{C}}$ is $\sqsubseteq$-coherent, the set $\bar{C} := C \cap \beta$ is in $\mathcal{C}_\beta$. Also, K is a tree and hence $\bar{\eta} := \eta \restriction \beta$ is in K_β. So, since $\mathbf{b}_x^{\bar{C},\eta\restriction\beta} \in T_\beta$, to show that $b_x^{C,\eta}(\beta) \in T_\beta$, it suffices to prove the following.

Claim 3.7.1. $b_x^{C,\eta}(\beta) = \mathbf{b}_x^{\bar{C},\bar{\eta}}$.

Proof. Clearly, $\mathrm{dom}(b_x^{C,\eta}(\beta)) = C \cap \beta \setminus \mathrm{dom}(x) = \bar{C} \setminus \mathrm{dom}(x) = \mathrm{dom}(b_x^{\bar{C},\bar{\eta}})$. So, we are left with showing that $b_x^{C,\eta}(\delta) = b_x^{\bar{C},\bar{\eta}}(\delta)$ for all $\delta \in \bar{C} \setminus \mathrm{dom}(x)$. The proof is by induction on $\delta \in \bar{C} \setminus \mathrm{dom}(x)$:

- For $\delta = \mathrm{dom}(x)$, we have that $b_x^{\eta,C}(\delta) = x = b_x^{\bar{C},\bar{\eta}}(\delta)$.
- Given $\delta \in \mathrm{nacc}(\bar{C})$ above $\mathrm{dom}(x)$ such that $b_x^{C,\eta}(\delta^-) = b_x^{\bar{C},\bar{\eta}}(\delta^-)$ for $\delta^- := \sup(\bar{C} \cap \delta)$, we argue as follows. Since
$$b_x^{C,\eta}(\delta^-)^\frown\langle \eta \restriction \delta^- \rangle = b_x^{\bar{C},\bar{\eta}}(\delta^-)^\frown\langle \bar{\eta} \restriction \delta^- \rangle,$$
the definitions of $b_x^{C,\eta}(\delta)$ and $b_x^{\bar{C},\bar{\eta}}(\delta)$ coincide.
- If $\delta \in \mathrm{acc}(\bar{C} \setminus \mathrm{dom}(x))$, then we take the limit of two identical sequences, and the unique limit is identical. $\square$

This completes the definition of $b_x^{C,\eta}$. For all $\eta \in \mathcal{B}(K \restriction \alpha)$, let $\mathbf{b}_x^{C,\eta} := \bigcup \mathrm{Im}(b_x^{C,\eta})$, and then we define T_α as promised in $(\star)$.

Clearly, $T := \bigcup_{\alpha<\kappa} T_\alpha$ is a normal and splitting κ-tree. The verification of Souslin-ness is standard (see [BR19b, Claims 2.2.2 and 2.2.3]).

Claim 3.7.2. *Suppose that $\alpha \in V^-(K)$ is such that $\sup(C \cap D) = \alpha$ for all $C, D \in \mathcal{C}_\alpha$. Then $\alpha \in V(T)$.*

Proof. As $\alpha \in V^-(K)$, we may fix $\eta \in \mathcal{B}(K \restriction \alpha) \setminus K_\alpha$. Let $x \in T \restriction \alpha$, and we shall find a vanishing α-branch through x in T. First fix $C \in \mathcal{C}_\alpha$. Using normality and by possibly extending x, we may assume that $x \in T \restriction C$. We have already established that $\{\mathbf{b}_x^{C,\eta} \restriction \epsilon \mid \epsilon < \alpha\}$ is an α-branch through x. Towards a contradiction, suppose that it is not vanishing, so that $\bigcup \mathrm{Im}(b_x^{C,\eta})$ is in T_α. It follows from $(\star)$ that we may pick $D \in \mathcal{C}_\alpha$,

$y \in T \restriction D$ and $\xi \in K_\alpha$ such that $\bigcup \mathrm{Im}(b_x^{C,\eta}) = \mathbf{b}_y^{D,\xi}$. Fix $\beta \in C \cap D$ large enough such that $\beta > \max\{\mathrm{dom}(x), \mathrm{dom}(y)\}$ and $\eta \restriction \beta \neq \xi \restriction \beta$. In particular, $\beta \in \mathrm{dom}(b_x^{C,\eta}) \cap \mathrm{dom}(b_y^{D,\xi})$. Consider $\beta^C := \min(C \setminus \beta + 1)$, the successor of β in C and $\beta^D := \min(D \setminus \beta + 1)$, the successor of β in D. Then the definition of the successor stage of $b_x^{C,\eta}$ ensures that $b_x^{C,\eta}(\beta^C)$ extends $b_x^{C,\eta}(\beta)^\frown\langle \eta \restriction \beta \rangle$, so that $b_x^{C,\eta}(\beta^C)(\beta) = \eta \restriction \beta$. Likewise, $b_y^{D,\xi}(\beta^D)(\beta) = \xi \restriction \beta$. From $\mathbf{b}_x^{C,\eta} = \mathbf{b}_y^{D,\xi}$, we infer that $b_x^{C,\eta}(\beta^C)(\beta) = \mathbf{b}_x^{C,\eta}(\beta) = \mathbf{b}_y^{D,\xi}(\beta) = b_y^{D,\xi}(\beta^D)(\beta)$, contradicting the fact that $\eta \restriction \beta \neq \xi \restriction \beta$. □

This completes the proof. □

We now arrive at Theorem C:

Corollary 3.8. *Suppose that* $\mathrm{P}(\kappa, 2, \sqsubseteq^*, 1)$ *holds. Then:*

(1) For every $\chi \in \mathrm{Reg}(\kappa)$ *such that* $\lambda^{<\chi} < \kappa$ *for all* $\lambda < \kappa$*, and every* κ*-tree* $\mathbf{K}$*, there exists a* κ*-Sousin tree* $\mathbf{T}$ *such that* $(E^\kappa_{\leq\chi} \cup V^-(\mathbf{K})) \setminus V(\mathbf{T})$ *is nonstationary;*

(2) There exists a κ*-Sousin tree* $\mathbf{T}$ *such that* $V(\mathbf{T})$ *is stationary.*

Proof. (1) Suppose χ and $\mathbf{K}$ are as above. By Corollary 2.29, we may fix a κ-tree $\mathbf{H}$ with $V^-(\mathbf{H}) \supseteq \mathrm{acc}(\kappa) \cap E^\kappa_{\leq\chi}$. By Proposition 2.28, $\mathbf{K} + \mathbf{H}$ is a κ-tree with $V^-(\mathbf{K} + \mathbf{H}) = V^-(\mathbf{K}) \cup V^-(\mathbf{H})$. By [BR21, Lemma 2.5], we may fix a streamlined κ-tree that K that is club-isomorphic to $\mathbf{K} + \mathbf{H}$. Now, appeal to Theorem 3.7(1) with K.

(2) Appeal to Clause (1) with $\chi = \omega$. □

Definition 3.9 (Jensen-Kunen, [JK69])**.** A cardinal κ is *subtle* iff for every list $\langle A_\alpha \mid \alpha \in D \rangle$ over a club $D \subseteq \kappa$, there is a pair $(\alpha, \beta) \in [D]^2$ such that $A_\alpha = A_\beta \cap \alpha$.

We now arrive at Theorem B:

Corollary 3.10. *We have* (1) $\implies$ (2) $\implies$ (3) $\implies$ (4)*:*

(1) there exists a κ*-Souslin tree* $\mathbf{T}$ *such that* $V(\mathbf{T}) = \mathrm{acc}(\kappa)$*;*

(2) there exists a κ*-tree* $\mathbf{T}$ *such that* $V(\mathbf{T}) = \mathrm{acc}(\kappa)$*;*

(3) there exists a κ*-tree* $\mathbf{T}$ *such that* $V^-(\mathbf{T})$ *contains a club in* κ*;*

(4) κ *is not subtle.*

In addition, in L*, for* κ *not weakly compact,* (4) $\implies$ (1)*.*

Proof. (1) $\implies$ (2) $\implies$ (3): This is immediate.

(3) $\implies$ (4): By Lemma 2.17.

Next, work in L and suppose that κ is a regular uncountable cardinal that is not subtle and not weakly compact. If κ is a successor cardinal, then by Corollary 3.1(1), Clause (1) holds, so assume that κ is inaccessible. By GCH, κ is moreover strongly inaccessible, and then Lemma 2.17 yields that Clause (3) holds. Since we work in L and κ is not weakly compact, by [BR17a, Theorem 3.12], $\mathrm{P}(\kappa, 2, \sqsubseteq, 1)$ holds. So by Corollary 3.8(1),

Clause (3) yields a κ-Souslin tree $\mathbf{T}$ such that $V(\mathbf{T})$ covers a club in κ. Now, appeal to Lemma 2.4. $\square$

Corollary 3.11. *In* L*, if* κ *is not weakly compact, then for every stationary* $S \subseteq \kappa$*, there exists a* κ*-Souslin tree* $\mathbf{T}$ *for which* $V(\mathbf{T}) \cap S$ *is stationary.*

Proof. By Corollary 3.1(1), we may assume that κ is (strongly) inaccessible. By Corollary 2.22, we may fix a κ-tree $\mathbf{K}$ such that $V^-(\mathbf{K}) \cap S$ is stationary. By [BR17a, Theorem 3.12], $\mathrm{P}(\kappa, 2, \sqsubseteq, 1)$ holds. Finally, appeal to Corollary 3.8(1). $\square$

4. Realizing a nonreflecting stationary set

In this section, we provide conditions concerning a set $S \subseteq \kappa$ sufficient to ensure the existence of a κ-Souslin tree $\mathbf{T}$ with $V(\mathbf{T}) \supseteq S$ and possibly $V(\mathbf{T}) = S$. As a corollary, we obtain Theorem D:

Corollary 4.1. *If* $\diamondsuit(S)$ *holds for some nonreflecting stationary subset* S *of a strongly inaccessible cardinal* κ*, then there is an almost disjoint family* $\mathcal{S}$ *of* 2^κ *many stationary subsets of* S *such that, for every* $S' \in \mathcal{S}$*, there is a* κ*-Souslin tree* $\mathbf{T}$ *with* $V^-(\mathbf{T}) = V(\mathbf{T}) = S'$*.*

Proof. By Corollary 4.9 below, it suffices to prove that there exists a family $\mathcal{S}$ of 2^κ many stationary subsets of S such that:

- for every $S' \in \mathcal{S}$, $\diamondsuit(S')$ holds.
- $|S' \cap S''| < \kappa$ for all $S' \neq S''$ from $\mathcal{S}$.

Now, as $\diamondsuit(S)$ holds, we may easily fix a sequence $\langle (A_\beta, B_\beta) \mid \beta \in S\rangle$ such that, for all $A, B \in \mathcal{P}(\kappa)$, the following set is stationary

$$G_A(B) := \{\beta \in S \mid A \cap \beta = A_\beta \ \&\ B \cap \beta = B_\beta\}.$$

Set $\mathcal{S} := \{S_A \mid A \in \mathcal{P}(\kappa)\}$, where $S_A := \{\beta \in S \mid A \cap \beta = A_\beta\}$. Then $\mathcal{S}$ is an almost disjoint family of 2^κ many stationary subsets of S, and for every $S' \in \mathcal{S}$, $\diamondsuit(S')$ holds, as witnessed by $\langle B_\beta \mid \beta \in S'\rangle$. $\square$

Definition 4.2 ([BR17a])**.** A streamlined tree $T \subseteq {}^{<\kappa}H_\kappa$ is *prolific* iff for all $\alpha < \kappa$ and $t \in T_\alpha$, $\{t^\frown\langle i\rangle \mid i < \max\{\omega, \alpha\}\} \subseteq T$.

A prolific tree is clearly splitting.

Theorem 4.3. *Suppose that* $\mathrm{P}(\kappa, \kappa, {}^S\sqsubseteq, 1)$ *holds for a given* $S \subseteq \mathrm{acc}(\kappa)$*. Then there exists a normal, prolific, streamlined* κ*-Souslin tree* T *such that* $V(T) \supseteq S$*.*

Proof. Fix a well-ordering $\lhd$ of H_κ, a sequence $\langle \Omega_\beta \mid \beta < \kappa\rangle$ witnessing $\diamondsuit^-(H_\kappa)$, and a sequence $\vec{\mathcal{C}} = \langle \mathcal{C}_\alpha \mid \alpha < \kappa\rangle$ witnessing $\mathrm{P}^-(\kappa, \kappa, {}^S\sqsubseteq, 2)$. By ${}^S\sqsubseteq$-coherence, we may assume that for every $\alpha \in S$, $\mathcal{C}_\alpha$ is a singleton.

Following the proof of [BR19b, Proposition 2.2], we shall recursively construct a sequence $\langle T_\alpha \mid \alpha < \kappa\rangle$ such that $T := \bigcup_{\alpha<\kappa} T_\alpha$ will constitute a normal prolific full streamlined κ-Souslin tree whose α^{th}-level is T_α.

Let $T_0 := \{\emptyset\}$, and for all $\alpha < \kappa$ let

$$T_{\alpha+1} := \{t^\frown\langle i\rangle \mid t \in T_\alpha, i < \max\{\omega, \alpha\}\}.$$

Next, suppose that $\alpha \in \mathrm{acc}(\kappa)$ is such that $T \restriction \alpha$ has already been defined. Constructing the level T_α involves deciding which branches through $T \restriction \alpha$ will have its limit placed into our tree. For all $C \in \mathcal{C}_\alpha$ and $x \in T \restriction C$, we first define two α-branches $\mathbf{b}^C_x$ and $\mathbf{d}^C_x$ such that $\{\mathbf{b}^C_x \mid x \in T \restriction C\} \cap \{\mathbf{d}^C_x \mid x \in T \restriction C\} = \emptyset$, and then we shall let:

$$(\star) \qquad T_\alpha := \begin{cases} \{\mathbf{b}^C_x \mid C \in \mathcal{C}_\alpha, x \in T \restriction C\}, & \text{if } \alpha \in S; \\ \{\mathbf{b}^C_x, \mathbf{d}^C_x \mid C \in \mathcal{C}_\alpha, x \in T \restriction C\}, & \text{otherwise.} \end{cases}$$

For every $\alpha \in S$, since $|\mathcal{C}| = 1$, this ensures that $\alpha \in V(T)$.

Let $C \in \mathcal{C}$ and $x \in T \restriction C$. We start by defining $\mathbf{b}^C_x$. It will be the limit $\bigcup \mathrm{Im}(b^C_x)$ of a sequence $b^C_x \in \prod_{\beta \in C \setminus \mathrm{dom}(x)} T_\beta$ obtained by recursion, as follows. Set $b^C_x(\mathrm{dom}(x)) := x$. At successor step, for every $\beta \in C \setminus (\mathrm{dom}(x)+1)$ such that $b^C_x(\beta^-)$ has already been defined with $\beta^- := \sup(C \cap \beta)$, we consult the following set:

$$Q^{C,\beta}_{x,0} := \{t \in T_\beta \mid \exists s \in \Omega_\beta[(s \cup (b^C_x(\beta^-)^\frown\langle 0\rangle)) \subseteq t]\}.$$

Now, consider the two possibilities:

- If $Q^{C,\beta}_{x,0} \neq \emptyset$, then let $b^C_x(\beta)$ be its $\lhd$-least element;
- Otherwise, let $b^C_x(\beta)$ be the $\lhd$-least element of T_β that extends $b^C_x(\beta^-)^\frown\langle 0\rangle$. Such an element must exist, as the tree constructed so far is prolific and normal.

Finally, for every $\beta \in \mathrm{acc}(C \setminus \mathrm{dom}(x))$ such that $b^C_x \restriction \beta$ has already been defined, we let $b^C_x(\beta) = \bigcup \mathrm{Im}(b^C_x \restriction \beta)$. By $(\star)$, ${}^S\sqsubseteq$-coherence and the exact same proof of [BR19b, Claim 2.2.1], $b^C_x(\beta)$ is indeed in T_β.

Next, we define $\mathbf{d}^C_x$ as the limit of a sequence $d^C_x \in \prod_{\beta \in C \setminus \mathrm{dom}(x)} T_\beta$ obtained by recursion, as follows. Set $d^C_x(\mathrm{dom}(x)) := x$. At successor step, for every $\beta \in C \setminus (\mathrm{dom}(x)+1)$ such that $d^C_x(\beta^-)$ has already been defined with $\beta^- := \sup(C \cap \beta)$, we consult the following set:

$$Q^{C,\beta}_{x,1} := \{t \in T_\beta \mid \exists s \in \Omega_\beta[(s \cup (d^C_x(\beta^-)^\frown\langle 1\rangle)) \subseteq t]\}.$$

Now, consider the two possibilities:

- If $Q^{C,\beta}_{x,1} \neq \emptyset$, then let $d^C_x(\beta)$ be its $\lhd$-least element;
- Otherwise, let $d^C_x(\beta)$ be the $\lhd$-least element of $T_\beta \setminus \{b^C_x(\beta)\}$ that extends $d^C_x(\beta^-)^\frown\langle 1\rangle$. Such an element must exist, as the tree constructed so far is prolific and normal.

Finally, for every $\beta \in \mathrm{acc}(C \setminus \mathrm{dom}(x))$ such that $d^C_x \restriction \beta$ has already been defined, we let $d^C_x(\beta) = \bigcup \mathrm{Im}(d^C_x \restriction \beta)$. By $(\star)$, ${}^S\sqsubseteq$-coherence and the exact same proof of [BR19b, Claim 2.2.1], $d^C_x(\beta)$ is indeed in T_β.

Claim 4.3.1. *For every $C \in \mathcal{C}_\alpha$, $\{\mathbf{b}^C_x \mid x \in T \restriction C_\alpha\} \cap \{\mathbf{d}^C_x \mid x \in T \restriction C_\alpha\} = \emptyset$.*

Proof. Let $C \in \mathcal{C}_\alpha$ and $x, y \in T \restriction C$. Fix a large enough $\beta \in \mathrm{nacc}(C)$ for which $\beta^- := \sup(C \cap \beta)$ is bigger than $\max\{\mathrm{dom}(x), \mathrm{dom}(y)\}$. By the definitions of b^C_x and d^C_y,

- $b^C_x(\beta)(\beta^-) = 0$, and
- $d^C_y(\beta)(\beta^-) = 1$.

In particular, $\mathbf{b}^C_x \neq \mathbf{d}^C_y$. □

This finishes the construction of T_α. Finally, by [BR19b, Claims 2.2.2 and 2.2.3], $T := \bigcup_{\alpha<\kappa} T_\alpha$ is a κ-Souslin tree. □

Theorem 4.4. *Suppose that χ is a cardinal such that $\lambda^\chi < \kappa$ for all $\lambda < \kappa$, and that $\mathrm{P}(\kappa, \kappa, {}^S\sqsubseteq, 1, \{S \cup E^\kappa_{>\chi}\})$ holds for a given $S \subseteq \mathrm{acc}(\kappa) \cap E^\kappa_{\leq\chi}$. Then there exists a normal, prolific, streamlined κ-Souslin tree T such that $V^-(T) \cap E^\kappa_{\leq\chi} = V(T) \cap E^\kappa_{\leq\chi} = S$.*

Proof. The proof is almost identical to that of Theorem 4.3, where the only change is in that now, the definition of T_α for a limit α splits into three:

$$T_\alpha := \begin{cases} \{\mathbf{b}^C_x \mid C \in \mathcal{C}_\alpha, x \in T \restriction C\}, & \text{if } \alpha \in S; \\ \{\mathbf{b}^C_x, \mathbf{d}^C_x \mid C \in \mathcal{C}_\alpha, x \in T \restriction C\}, & \text{if } \alpha \in E^\kappa_{>\chi}; \\ \mathcal{B}(T \restriction \alpha), & \text{otherwise.} \end{cases}$$

The details are left to the reader. □

Remark 4.5. Sufficient conditions for the existence of $S \subseteq \kappa$ for which $\mathrm{P}(\kappa, \kappa, {}^S\sqsubseteq, 1, \{S\})$ holds are given by [BR21, Corollary 4.22] and [BR21, Theorem 4.28]. In particular, for every (nonreflecting) stationary $E \subseteq \kappa$, if $\square(E)$ and $\diamondsuit(E)$ both hold, then there exists a stationary $S \subseteq E$ such that $\mathrm{P}(\kappa, \kappa, {}^S\sqsubseteq, 1, \{S\})$ holds.

Corollary 4.6. *Suppose that $2^{2^{\aleph_0}} = \aleph_2$, and that S is a nonreflecting stationary subset of $E^{\aleph_2}_{\aleph_0}$. Then there exists a normal prolific streamlined $\aleph_2$-Souslin tree T such that $V(T) = S \cup E^{\aleph_2}_{\aleph_1}$.*

Proof. By [BR19c, Lemma 3.2], the hypotheses implies that $\mathrm{P}(\aleph_2, \aleph_2, {}^S\sqsubseteq, 1, \{S\})$ holds. Appealing to Theorem 4.4 with $(\kappa, \chi) := (\aleph_2, \aleph_0)$ provides us with a normal, prolific, streamlined $\aleph_2$-Souslin tree T such that $V^-(T) \cap E^{\aleph_2}_{\aleph_0} = V(T) \cap E^{\aleph_2}_{\aleph_0} = S$. As $V^-(T) \cap E^{\aleph_2}_{\aleph_0}$ is a nonreflecting stationary set, Lemma 2.9(1) (using $(\varsigma, \chi, \kappa) := (2, \aleph_1, \aleph_2)$) implies that $V(T) \cap E^{\aleph_2}_{\aleph_1} = E^{\aleph_2}_{\aleph_1}$. □

Corollary 4.7. *Suppose* CH *and $\boxdiamond_{\aleph_1}$ both hold. For every stationary $S \subseteq E^{\aleph_2}_{\aleph_0}$, there exists an $\aleph_2$-Souslin tree $\mathbf{T}$ such that $V(\mathbf{T})$ is a stationary subset of S.*

Proof. $\boxdiamond_{\aleph_1}$ implies $\square_{\aleph_1}$ which implies that for every stationary $S \subseteq E^{\aleph_2}_{\aleph_0}$ there exists a stationary $R \subseteq S$ that is nonreflecting. It thus follows from Corollary 4.6 that for every stationary $S \subseteq E^{\aleph_2}_{\aleph_0}$ there exist a stationary

$R \subseteq S$ and an $\aleph_2$-Souslin tree $\mathbf{T}$ such that $V(\mathbf{T}) = R \cup E^{\aleph_2}_{\aleph_1}$. In addition, $\boxed{\diamondsuit}_{\aleph_1}$ yields a uniformly coherent $\aleph_2$-Souslin tree $\mathbf{S}$ (see [Vel86, Theorem 7] or [BR17a, Proposition 2.5 and Theorem 3.6]). By [RS23, Remark 2.20], then, $V(\mathbf{S}) = E^{\aleph_2}_{\aleph_0}$. Clearly, $\mathbf{T} + \mathbf{S}$ is an $\aleph_2$-Souslin tree, and, by Proposition 2.28(2), $V(\mathbf{T} + \mathbf{S}) = R$. □

Theorem 4.8. *Suppose that κ is a strongly inaccessible cardinal, and that $\mathrm{P}(\kappa, \kappa, {}^S\sqsubseteq, 1, \{S\})$ holds for a given $S \subseteq \mathrm{acc}(\kappa)$. Then there exists a normal, prolific, streamlined κ-Souslin tree T such that $V^-(T) = V(T) = S$.*

Proof. The proof is almost identical to that of Theorem 4.3, where the only change is that now, the definition of T_α for a limit α does not explicitly mention the $\mathbf{d}^C_x$'s. Instead, it is:

$$T_\alpha := \begin{cases} \{\mathbf{b}^C_x \mid C \in \mathcal{C}_\alpha, x \in T \restriction C\}, & \text{if } \alpha \in S; \\ \mathcal{B}(T \restriction \alpha), & \text{otherwise.} \end{cases}$$

The details are left to the reader. □

Corollary 4.9. *Suppose that κ is a strongly inaccessible cardinal, and S is a nonreflecting stationary subset of $\mathrm{acc}(\kappa)$ on which $\diamondsuit$ holds. Then there exists a normal prolific streamlined κ-Souslin tree T such that $V^-(T) = V(T) = S$.*

Proof. By Theorem 4.8 together with [BR21, Theorem 4.26]. □

5. Realizing all points of some fixed cofinality

The main result of this section is Theorem 5.9 below. A sample corollary of it reads as follows.

Corollary 5.1. *In L, for every regular uncountable cardinal κ that is not weakly compact, for every finite nonempty $x \subseteq \mathrm{Reg}(\kappa)$ with $\max(x) \leq \mathrm{cf}(\sup(\mathrm{Reg}(\kappa)))$, there exists a uniformly homogeneous κ-Souslin tree $\mathbf{T}$ such that $V^-(\mathbf{T}) = \bigcup_{\chi \in x} E^\kappa_\chi$.*

Proof. Work in L. Let κ be regular uncountable cardinal that is not weakly compact, and let $\langle \chi_i \mid i \leq n \rangle$ be a strictly increasing finite sequence of regular cardinals with $\chi_n \leq \mathrm{cf}(\sup(\mathrm{Reg}(\kappa)))$.

By [BR17a, Theorem 3.6] and [BR19a, Corollary 4.12], $\mathrm{P}(\kappa, 2, \sqsubseteq, \kappa, \{E^\kappa_{\geq \chi_n}\})$ holds. By GCH, $\lambda^{<\chi_n} < \kappa$ for all $\lambda < \kappa$. So, by Theorem 5.9 below, using $S := {}^{<\kappa}1$, we may pick a streamlined, normal, 2-splitting, uniformly homogeneous, χ_0-complete, χ_0-coherent, $E^\kappa_{\geq \chi_0}$-regressive κ-Souslin tree T^0. Furthermore, T^0 is $\mathrm{P}^-(\kappa, 2, \sqsubseteq, \kappa, \{E^\kappa_{\geq \chi_n}\})$-respecting.

Claim 5.1.1. $V^-(T^0) = E^\kappa_{\chi_0}$.

Proof. Since T^0 is χ_0-complete, $V^-(T^0) \cap E^\kappa_{<\chi_0} = \emptyset$, so that $\mathrm{Tr}(\kappa \setminus V^-(T^0))$ covers $E^\kappa_{\geq \chi_0}$. By GCH, $2^{<\chi_0} < 2^{\chi_0}$. Together with the fact that T is $E^\kappa_{\chi_0}$-regressive, it follows from Lemma 2.9(2) that $E^\kappa_{\chi_0} \subseteq V^-(T^0)$. Finally, since

T^0 is χ_0-coherent and uniformly homogeneous, we get from Lemma 5.3 below that $V^-(T^0) \cap E^\kappa_{>\chi_0} = \emptyset$. $\square$

If $n = 0$, then our proof is complete. Otherwise, one can continue by recursion, where the successive step is as follows: Suppose that $i < n$ is such that $\bigotimes_{j\leq i} T^j$ is a streamlined uniformly homogeneous normal κ-Souslin tree that is $\mathrm{P}^-(\kappa, 2, \sqsubseteq, \kappa, \{E^\kappa_{\geq\chi_n}\})$-respecting, and that $V(\bigotimes_{j\leq i} T^j) = \bigcup_{j\leq i} E^\kappa_{\chi_j}$. By Theorem 5.9 below, using $S := \bigotimes_{j\leq i} T^j$, we may pick a streamlined, normal, 2-splitting, uniformly homogeneous, χ_{i+1}-complete, χ_{i+1}-coherent, $E^\kappa_{\geq\chi_{i+1}}$-regressive κ-Souslin tree T^{i+1}. Furthermore, $S \otimes T^{i+1}$ is a normal $\mathrm{P}^-(\kappa, 2, \sqsubseteq, \kappa, \{E^\kappa_{\geq\chi_n}\})$-respecting κ-Souslin tree. By an analysis similar to that of Claim 5.1.1, $V^-(T^{i+1}) = E^\kappa_{\chi_{i+1}}$. Therefore, $\bigotimes_{j\leq i+1} T^j$ is a uniformly homogeneous normal κ-Souslin tree that is $\mathrm{P}^-(\kappa, 2, \sqsubseteq, \kappa, \{E^\kappa_{\geq\chi_n}\})$-respecting. In addition, by Proposition 2.26(2), $V(\bigotimes_{j\leq i+1} T^j) = \bigcup_{j\leq i+1} E^\kappa_{\chi_j}$. $\square$

We start by giving a definition.

Definition 5.2. A streamlined κ-tree T is *χ-coherent* iff for all $s, t \in T$, $\{\xi \in \mathrm{dom}(s) \cap \mathrm{dom}(t) \mid s(\xi) \neq t(\xi)\}$ has size $< \chi$.

Lemma 5.3. *Suppose that $\chi < \kappa$ is a cardinal, and that T is a streamlined, χ-coherent uniformly homogeneous κ-tree. Then $V^-(T) \subseteq E^\kappa_{\leq\chi}$.*

Proof. Let $\alpha \in E^\kappa_{>\chi}$. Suppose that $B \subseteq T$ is an α-branch, and we shall show it is not vanishing.

For every $\beta < \alpha$, let t_β denote the unique element of $T_\beta \cap B$. Fix a node $t \in T_\alpha$. For every $\beta \in E^\alpha_\chi$, by χ-coherence, the following ordinal is smaller than β:

$$\epsilon_\beta := \sup\{\xi < \beta \mid t_\beta(\xi) \neq t(\xi)\}.$$

As $\mathrm{cf}(\alpha) > \chi$, E^α_χ is a stationary subset of α, so we may fix a large enough $\epsilon < \alpha$ for which $R := \{\beta \in E^\alpha_\chi \mid \epsilon_\beta < \epsilon\}$ is stationary. As T is uniformly homogeneous, $t_\epsilon * t$ is in T_α. For every $\beta \in R$, $t_\beta = (t_\epsilon * t) \restriction \beta$. But since R is cofinal in α, it is the case that $t_\epsilon * t$ constitutes a limit for B. Therefore, B is not vanishing. $\square$

In the context of streamlined κ-trees, there is a neater way of presenting the operation of product (compare with Definition 2.25):

Definition 5.4 ([BR21, §6.7])**.** For every function $x : \alpha \to {}^\tau H_\kappa$ and every $i < \tau$, we let $(x)_i : \alpha \to H_\kappa$ be $\langle x(\beta)(i) \mid \beta < \alpha\rangle$. Using this notation, for every sequence $\langle T^i \mid i < \tau\rangle$ of streamlined κ-trees, one may identify $\bigotimes_{i<\tau} T^i$ with the streamlined tree $T := \{x \in {}^{<\kappa}({}^\tau H_\kappa) \mid \forall i < \tau\, [(x)_i \in T^i]\}$.

Remark 5.5. The product of two uniformly homogeneous κ-trees is uniformly homogeneous.

Before we can state the main result of this section, we need one more definition.

Definition 5.6 ([BR17b])**.** A streamlined κ-tree X is $\mathrm{P}^-_\xi(\kappa,\mu,\mathcal{R},\theta,\mathcal{S})$-*respecting* if there exists a subset $\S\subseteq\kappa$ and a sequence of mappings $\langle d^C:(X\restriction C)\to{}^\alpha H_\kappa\cup\{\emptyset\}\mid\alpha<\kappa,C\in\mathcal{C}_\alpha\rangle$ such that:

(1) for all $\alpha\in\S$ and $C\in\mathcal{C}_\alpha$, $X_\alpha\subseteq\mathrm{Im}(d^C)$;
(2) $\vec{\mathcal{C}}=\langle\mathcal{C}_\alpha\mid\alpha<\kappa\rangle$ witnesses $\mathrm{P}^-_\xi(\kappa,\mu,\mathcal{R},\theta,\{S\cap\S\mid S\in\mathcal{S}\})$;
(3) for all sets $D\sqsubseteq C$ from $\vec{\mathcal{C}}$ and $x\in X\restriction D$, $d^D(x)=d^C(x)\restriction\sup(D)$.

Remark 5.7. (1) If $\mathrm{P}^-_\xi(\kappa,\mu,\mathcal{R},\theta,\mathcal{S})$ holds, then the normal streamlined κ-tree $X:={}^{<\kappa}1$ is $\mathrm{P}^-_\xi(\kappa,\mu,\mathcal{R},\theta,\mathcal{S})$-respecting;
(2) If $\kappa=\lambda^+$ for an infinite regular cardinal λ, and $\mathrm{P}^-_\lambda(\kappa,\mu,{}_\lambda\sqsubseteq,\theta,\{E^\kappa_\lambda\})$ holds, then every κ-tree is $\mathrm{P}^-_\lambda(\kappa,\mu,{}_\lambda\sqsubseteq,\theta,\{E^\kappa_\lambda\})$-respecting.

Lemma 5.8. *Suppose that:*

- *X is a streamlined κ-tree that is $\mathrm{P}^-_\xi(\kappa,\mu,\mathcal{R},\kappa,\mathcal{S})$-respecting, as witnessed by some $\vec{\mathcal{C}}$ and $\S$;*
- *Y is a streamlined κ-tree that is $\mathrm{P}^-_\xi(\kappa,\mu,\mathcal{R},\kappa,\{S\cap\S\mid S\in\mathcal{S}\})$-respecting, as witnessed by the same $\vec{\mathcal{C}}$.*

Then the product $X\otimes Y$ is $\mathrm{P}^-_\xi(\kappa,\mu,\mathcal{R},\kappa,\mathcal{S})$-respecting.

Proof. In view of Definition 5.4, for every two functions x,y from an ordinal $\alpha<\kappa$ to H_κ, we denote by $\ulcorner(x,y)\urcorner$ the unique function $p:\alpha\to{}^2H_\kappa$ such that $(p)_0=x$ and $(p)_1=y$. Note that $X\otimes Y=\bigcup_{\alpha<\kappa}\{\ulcorner(x,y)\urcorner\mid(x,y)\in X_\alpha\times Y_\alpha\}$.

Write $\vec{\mathcal{C}}$ as $\langle\mathcal{C}_\alpha\mid\alpha<\kappa\rangle$. Fix a sequence of mappings $\langle d^C:(X\restriction C)\to{}^\alpha H_\kappa\cup\{\emptyset\}\mid\alpha<\kappa,C\in\mathcal{C}_\alpha\rangle$ such that:

(1) for all $\alpha\in\S$ and $C\in\mathcal{C}_\alpha$, $X_\alpha\subseteq\mathrm{Im}(d^C)$;
(2) $\vec{\mathcal{C}}=\langle\mathcal{C}_\alpha\mid\alpha<\kappa\rangle$ witnesses $\mathrm{P}^-_\xi(\kappa,\mu,\mathcal{R},\kappa,\{S\cap\S\mid S\in\mathcal{S}\})$;
(3) for all sets $D\sqsubseteq C$ from $\vec{\mathcal{C}}$ and $x\in X\restriction D$, $d^D(x)=d^C(x)\restriction\sup(D)$.

Fix a stationary $\S'\subseteq\S$ and a sequence of mappings $\langle e^C:(Y\restriction C)\to{}^\alpha H_\kappa\cup\{\emptyset\}\mid\alpha<\kappa,C\in\mathcal{C}_\alpha\rangle$ such that:

(4) for all $\alpha\in\S'$ and $C\in\mathcal{C}_\alpha$, $Y_\alpha\subseteq\mathrm{Im}(e^C)$;
(5) $\vec{\mathcal{C}}=\langle\mathcal{C}_\alpha\mid\alpha<\kappa\rangle$ witnesses $\mathrm{P}^-_\xi(\kappa,\mu,\mathcal{R},\kappa,\{S\cap\S'\mid S\in\mathcal{S}\})$;
(6) for all sets $D\sqsubseteq C$ from $\vec{\mathcal{C}}$ and $y\in Y\restriction D$, $e^D(y)=e^C(y)\restriction\sup(D)$.

Let $\vec{B}=\langle B_{x,y}\mid(x,y)\in X\times Y\rangle$ be a partition of κ into cofinal subsets of κ. Define a sequence of mappings $\langle b^C:(X\otimes Y)\restriction C\to{}^\alpha H_\kappa\cup\{\emptyset\}\mid\alpha<\kappa,C\in\mathcal{C}_\alpha\rangle$, as follows. Let $\alpha<\kappa$ and $C\in\mathcal{C}_\alpha$.

▶ For every $\beta\in C$, if there are $x\in X\restriction(C\cap\beta)$ and $y\in Y\restriction(C\cap\beta)$ such that $\beta\in B_{x,y}$, then since $\vec{B}$ is a sequence of pairwise disjoint sets, this pair (x,y) is unique, and we let $b^C(p):=\ulcorner(d^C(x),e^C(y))\urcorner$ for every $p\in(X\otimes Y)_\beta$.

▶ For every $\beta\in C$ for which there is no such pair (x,y), we let $b^C(p):=\emptyset$ for every $p\in(X\otimes Y)_\beta$.

Claim 5.8.1. *Suppose $D \sqsubseteq C$ are sets from $\vec{\mathcal{C}}$. For every $p \in (X \otimes Y) \restriction D$, $b^D(p) = b^C(p) \restriction \sup(D)$.*

Proof. Given $p \in (X \otimes Y) \restriction D$. Denote $\beta := \mathrm{dom}(p)$. Note that $D \cap \beta = C \cap \beta$. Now, there are two options:

▶ There are $x \in X \restriction (C \cap \beta)$ and $y \in Y \restriction (C \cap \beta)$ such that $\beta \in B_{x,y}$. Then $b^D(p) = \ulcorner (d^D(x), e^D(y)) \urcorner$ and $b^C(p) = \ulcorner (d^C(x), e^C(y)) \urcorner$. Since $D \sqsubseteq C$, we know that $d^D(x) = d^C(x) \restriction \sup(D)$ and $e^D(y) = e^C(y) \restriction \sup(D)$. Therefore, $b^D(p) = d^C(p) \restriction \sup(D)$.

▶ There are no such x and y. Then $b^D(p) = \emptyset = d^C(p)$. □

Consider the following set:

$$\S'' := \{\alpha \in \S' \mid \forall C \in \mathcal{C}_\alpha \forall x \in (X \restriction \alpha) \forall y \in (Y \restriction \alpha)\, [\sup(C_\alpha \cap B_{x,y}) = \alpha]\}.$$

Claim 5.8.2. $\vec{\mathcal{C}} = \langle \mathcal{C}_\alpha \mid \alpha < \kappa \rangle$ *witnesses* $\mathrm{P}^-_\xi(\kappa, \mu, \mathcal{R}, \kappa, \{S \cap \S'' \mid S \in \mathcal{S}\})$.

Proof. Let $\langle B_i \mid i < \kappa \rangle$ be a given sequence of cofinal subsets of κ. Let $\pi : \kappa \leftrightarrow \kappa \uplus (X \times Y)$ be a surjection. As X and Y are κ-tree, the set $D := \{\alpha < \kappa \mid \pi[\alpha] = \alpha \uplus ((X \restriction \alpha) \times (Y \restriction \alpha))\}$ is a club in κ. By Clause (5), then, for every $S \in \mathcal{S}$, there are stationarily many $\alpha \in S \cap \S' \cap D$ such that for all $C \in \mathcal{C}_\alpha$ and $i < \alpha = \min\{\alpha, \kappa\}$, $\sup(\mathrm{nacc}(C) \cap B_{\pi(i)}) = \alpha$. In particular, for every $S \in \mathcal{S}$, there are stationarily many $\alpha \in S \cap \S''$ such that for all $C \in \mathcal{C}_\alpha$ and $i < \alpha = \min\{\alpha, \kappa\}$, $\sup(\mathrm{nacc}(C) \cap B_i) = \alpha$. □

Claim 5.8.3. *Let $\alpha \in \S''$ and $C \in \mathcal{C}_\alpha$. Then $(X \otimes Y)_\alpha \subseteq \mathrm{Im}(b^C)$.*

Proof. Let $(s, t) \in X_\alpha \times Y_\alpha$. As $\S'' \subseteq \S' \subseteq S$, using Clauses (1) and (4), we may fix $x \in X \restriction C$ and $y \in Y \restriction C$ such that $d^C(x) = s$ and $e^c(y) = t$. As $\alpha \in \S''$, we may pick $\beta \in C_\alpha \cap B_{x,y}$ above $\max\{\mathrm{dom}(x), \mathrm{dom}(y)\}$. Let p be an arbitrary element of $(X \otimes Y) \restriction C$. Then $b^C(p) := \ulcorner (d^C(x), e^C(y)) \urcorner = \ulcorner (s, t) \urcorner$. □

This completes the proof. □

Theorem 5.9. *Suppose that:*

- *$\varsigma < \kappa$ is a cardinal;*
- *$\nu \leq \chi < \kappa$ are cardinals such that $\lambda^{<\chi} < \kappa$ for all $\lambda < \kappa$;*
- *S is a $\mathrm{P}^-(\kappa, 2, {}_\nu\sqsubseteq, \kappa, \{E^\kappa_{\geq\chi}\})$-respecting streamlined normal κ-tree with no κ-sized antichains;*
- *$\diamondsuit(\kappa)$ holds.*

Then there exists a streamlined, normal, ς-splitting, prolific, uniformly homogeneous, χ-complete, χ-coherent, $E^\kappa_{\geq\chi}$-regressive κ-Souslin tree T such that $S \otimes T$ is a normal $\mathrm{P}^-(\kappa, 2, {}_\nu\sqsubseteq, \kappa, \{E^\kappa_{\geq\chi}\})$-respecting κ-Souslin tree.

Proof. Fix a stationary $\S \subseteq \kappa$ and a sequence $\langle d^\alpha : S \restriction C_\alpha \to {}^\alpha H_\kappa \cup \{\emptyset\} \mid \alpha < \kappa \rangle$ such that:

(1) for all $\alpha \in \S$, $S_\alpha \subseteq \mathrm{Im}(d^\alpha)$;
(2) $\vec{C} := \langle C_\alpha \mid \alpha < \kappa \rangle$ witnesses $\mathrm{P}^-(\kappa, 2, {}_\nu\sqsubseteq, \kappa, \{\S\})$;

(3) for all $\alpha < \beta < \kappa$, if $C_\alpha \sqsubseteq C_\beta$, then $d^\alpha(x) = d^\beta(x) \restriction \alpha$ for every $x \in S \restriction C_\alpha$.

Without loss of generality, we may assume that $0 \in C_\alpha$ for all nonzero $\alpha < \kappa$.

The upcoming construction follows the proof of [BR17a, Proposition 2.5]. Let $\langle R_i \mid i < \kappa\rangle$ and $\langle \Omega_\beta \mid \beta < \kappa\rangle$ together witness $\diamondsuit(H_\kappa)$. Let $\pi : \kappa \to \kappa$ be such that $\alpha \in R_{\pi(\alpha)}$ for all $\alpha < \kappa$. From $\diamondsuit(\kappa)$, we have $|H_\kappa| = \kappa$, thus let $\lhd$ be some well-ordering of H_κ of order-type κ, and let $\phi : \kappa \leftrightarrow H_\kappa$ witness the isomorphism $(\kappa, \in) \cong (H_\kappa, \lhd)$. Put $\psi := \phi \circ \pi$.

We now recursively construct a sequence $\langle T_\alpha \mid \alpha < \kappa\rangle$ of levels whose union will ultimately be the desired tree T. Let $T_0 := \{\emptyset\}$, and for all $\alpha < \kappa$, let

$$T_{\alpha+1} := \{t^\frown\langle i\rangle \mid t \in T_\alpha, i < \max\{\varsigma, \omega, \alpha\}\}.$$

Next, suppose that $\alpha \in \mathrm{acc}(\kappa)$, and that $\langle T_\beta \mid \beta < \alpha\rangle$ has already been defined. We shall identify some $\mathbf{b}^\alpha \in \mathcal{B}(T \restriction \alpha)$, and then define the αth-level, as follows:

$$(\star) \qquad T_\alpha := \begin{cases} \mathcal{B}(T \restriction \alpha), & \text{if } \alpha \in E^\kappa_{<\chi}; \\ \{x * \mathbf{b}^\alpha \mid x \in T \restriction \alpha\}, & \text{if } \alpha \in E^\kappa_{\geq\chi}. \end{cases}$$

We shall obtain $\mathbf{b}^\alpha$ as a limit $\bigcup \mathrm{Im}(b^\alpha)$ of a sequence $b^\alpha \in \prod_{\beta\in C_\alpha} T_\beta$ that we define recursively, as follows. Let $b^\alpha(0) := \emptyset$. Next, suppose $\beta^- < \beta$ are two successive points of C_α, and that $b^\alpha(\beta^-)$ has already been defined. There are two possible options:

▶ If $\psi(\beta)$ happens to be a pair (y, x) lying in $(S \restriction \beta^-) \times (T \restriction \beta^-)$, and the following set happens to be nonempty:

$$Q^{\alpha,\beta} := \{t \in T_\beta \mid \exists(\bar{s}, \bar{t}) \in \Omega_\beta\, [\bar{s} \subseteq d^\alpha(y) \restriction \beta \;\&\; (\bar{t} \cup (x * b^\alpha(\beta^-))) \subseteq t]\},$$

then let t denote its $\lhd$-least element, and put $b^\alpha(\beta) := b^\alpha(\beta^-) * t$.

▶ Otherwise, let $b^\alpha(\beta)$ be the $\lhd$-least element of T_β that extends $b^\alpha(\beta^-)$.

As always, for all $\beta \in \mathrm{acc}(C_\alpha)$ such that $b^\alpha \restriction \beta$ has already been defined, we let $b^\alpha(\beta) := \bigcup \mathrm{Im}(b^\alpha \restriction \beta)$ and infer that it belongs to T_β. Indeed, either $\mathrm{cf}(\beta) < \chi$, and then $b^\alpha(\beta) \in \mathcal{B}(T \restriction \beta) = T_\beta$, or $\mathrm{cf}(\beta) \geq \chi \geq \nu$, and then $C_\beta = C_\alpha \cap \beta$ from which it follows that $b^\alpha(\beta) = \mathbf{b}^\beta \in T_\beta$. This completes the definition of b^α, hence also that of $\mathbf{b}^\alpha$. Finally, let T_α be defined as promised in $(\star)$.

It is clear that $T := \bigcup_{\alpha<\kappa} T_\alpha$ is a streamlined, normal, ς-splitting, prolific, uniformly homogeneous, χ-complete κ-tree.

Claim 5.9.1. *T is χ-coherent.*

Proof. Suppose not, and let α be the least ordinal to accommodate $s, t \in T_\alpha$ such that s differs from t on a set of size $\geq \chi$. Clearly, $\alpha \in E^\kappa_{\geq\chi}$. So $s = x * \mathbf{b}^\alpha$ and $t = y * \mathbf{b}^\alpha$ for nodes $x, y \in T \restriction \alpha$, and hence x and y differ on a set of size $\geq \chi$, contradicting the minimality of α. □

Claim 5.9.2. *T is $E^\kappa_{\geq\chi}$-regressive.*

Proof. To define $\rho : T \restriction E^\kappa_{\geq\chi} \to T$, let $\alpha \in E^\kappa_{\geq\chi}$. By the definition of T_α, for every $t \in T$, there exists some $x \in T \restriction \alpha$ such that $t = x * \mathbf{b}^\alpha$, so we let $\rho(t)$ be an element of $T \restriction \alpha$ such that $t = \rho(t) * \mathbf{b}^\alpha$. Now, if $s, t \in T_\alpha$ are such that $\rho(t) \subseteq s$ and $\rho(s) \subseteq t$, then $\rho(t) \subseteq \rho(s) * \mathbf{b}^\alpha$ and $\rho(s) \subseteq \rho(t) * \mathbf{b}^\alpha$. In particular, $\rho(s)$ is compatible with $\rho(t)$. Without loss of generality, $\rho(s) \subseteq \rho(t)$. Then $t = \rho(s) * \mathbf{b}^\alpha = s$. □

Claim 5.9.3. *T is $\mathrm{P}^-(\kappa, 2, {}_\nu\sqsubseteq, \kappa, \{\S\})$-respecting, as witnessed by $\vec{C}$.*

Proof. Define $\langle e^\alpha : T \restriction C_\alpha \to T_\alpha \mid \alpha < \kappa\rangle$ via:

$$e^\alpha(x) := x * \mathbf{b}^\alpha.$$

The second part of $(\star)$ implies that $S_\alpha = \mathrm{Im}(d^\alpha)$ for all $\alpha \in E^\kappa_{\geq\chi} \supseteq \S$. In addition, it is clear that for all $\alpha < \beta < \kappa$, if $C_\alpha \sqsubseteq C_\beta$, then $\mathbf{b}^\alpha = \mathbf{b}^\beta \restriction \alpha$, and hence $e^\alpha(x) = e^\beta(x) \restriction \alpha$ for every $x \in S \restriction C_\alpha$. □

It thus follows from Lemma 5.8 that $S \otimes T$ is $\mathrm{P}^-(\kappa, 2, {}_\nu\sqsubseteq, \kappa, \{E^\kappa_{\geq\chi}\})$-respecting. It is clear that $S \otimes T$ is normal, thus we are left with verifying that it is Souslin. To this end, let A be a maximal antichain in $S \otimes T$. As both S and T are normal, it follows that for every $z \in T$, the following (upward-closed) set is cofinal in S:

$$D_z := \{s \in S \mid \exists(\bar{s}, \bar{t}) \in A \exists t \in T \cap z^\uparrow [\mathrm{dom}(s) = \mathrm{dom}(t), \bar{s} \subseteq s, \bar{t} \subseteq t]\}.$$

As an application of $\diamondsuit(H_\kappa)$, using the parameter $p := \{\phi, S \otimes T, A, \langle D_z \mid z \in T\rangle\}$, we get that for every $i < \kappa$, the following set is cofinal (in fact, stationary) in κ:

$$B_i := \{\beta \in R_i \mid \exists \mathcal{M} \prec H_{\kappa^+} (p \in \mathcal{M}, \mathcal{M} \cap \kappa = \beta, \Omega_\beta = A \cap \mathcal{M})\}.$$

Note that $(S \restriction \beta) \otimes (T \restriction \beta) \subseteq \phi[\beta]$ for every $\beta \in \bigcup_{i<\kappa} B_i$. Now, as $\vec{C}$ witnesses $\mathrm{P}^-(\kappa, 2, {}_\nu\sqsubseteq, \kappa, \{\S\})$, we may fix some $\alpha \in \S$ such that, for all $i < \alpha$,

$$\sup(\mathrm{nacc}(C_\alpha) \cap B_i) = \alpha.$$

In particular, $(S \restriction \alpha) \otimes (T \restriction \alpha) \subseteq \phi[\alpha]$. As $\alpha \in \S$, we also know that $S_\alpha \subseteq \mathrm{Im}(d^\alpha)$ and that $\mathrm{cf}(\alpha) \geq \chi$.

Claim 5.9.4. *$A \subseteq (S \otimes T) \restriction \alpha$. In particular, $|A| < \kappa$.*

Proof. As A is an antichain, it suffices to prove that every element of $(S \otimes T)_\alpha$ extends some element of A. To this end, fix $(s', t') \in (S \otimes T)_\alpha$. Since $S_\alpha \subseteq \mathrm{Im}(d^\alpha)$, we may fix a $y \in S \restriction C_\alpha$ such that $d^\alpha(y) = s'$. Recalling $(\star)$, we may also fix some $x \in T \restriction C_\alpha$ such that $t' = x * \mathbf{b}^\alpha$.

As the pair (y, x) is an element of $(S \restriction \alpha) \times (T \restriction \alpha)$, we may find an $i < \alpha$ such that $\phi(i) = (y, x)$, and then find a $\beta \in \mathrm{nacc}(C_\alpha) \cap B_i$ such that $\beta^- := \sup(C_\alpha \cap \beta)$ is greater than $\max\{\mathrm{dom}(y), \mathrm{dom}(x)\}$. Note that $\psi(\beta) = \phi(\pi(\beta)) = \phi(i) = (y, x)$.

Subclaim 5.9.4.1. *$\Omega_\beta = A \cap ((S \otimes T) \restriction \beta)$, and $Q^{\alpha,\beta} \neq \emptyset$.*

Proof. As $\beta \in B_i$, we may fix $\mathcal{M} \prec H_{\kappa^+}$ such that all of the following hold:

- $\{\phi, S\otimes T, A, \langle D_x \mid x\in T\rangle\}\in\mathcal{M}$;
- $\mathcal{M}\cap\kappa=\beta$;
- $\Omega_\beta = A\cap\mathcal{M}$

By elementarity, $(T\otimes S)\cap\mathcal{M} = (S\otimes T)\restriction\beta$, and $\Omega_\beta = A\cap\mathcal{M} = A\cap((S\otimes T)\restriction\beta)$. Then $z := t'\restriction\beta^-$ is in $\mathcal{M}$, and hence, so is D_z. Pick in $\mathcal{M}$ a maximal antichain $\bar{D}$ in D_z. Since D_z is cofinal in S, $\bar{D}$ is a maximal antichain in S. Since S has no κ-sized antichains, we may find a large enough $\gamma\in\mathcal{M}\cap\kappa$ such that $\bar{D}\subseteq S\restriction\gamma$. It thus follows that $s'\restriction\gamma$ extends an element of $\bar{D}$, but since D_z is upward-closed, $s := s'\restriction\gamma$ is in D_z. It follows that we may fix $(\bar{s},\bar{t})\in A$ and $t\in T_\gamma\cap z^\uparrow$ such that $\bar{s}\subseteq s$ and $\bar{t}\subseteq t$. As $\Omega_\beta = A\cap((S\otimes T)\restriction\beta)$, $(d^\alpha(y)\restriction\beta)\restriction\gamma = s$ and $x * b^\alpha(\beta^-) = z\subseteq t$, we infer that $t\in Q^{\alpha,\beta}$. □

It follows that $b^\alpha(\beta) = b^\alpha(\beta^-) * t$ for some $t\in Q^{\alpha,\beta}$. This means that we may pick $(\bar{s},\bar{t})\in\Omega_\beta\subseteq A$ such that $\bar{s}\subseteq s'\restriction\beta$ and $\bar{t}\cup(x * b^\alpha(\beta^-))\subseteq t$. Therefore, $\bar{t}\subseteq x * b^\alpha(\beta)$. Altogether, $(\bar{s},\bar{t})\in A$, $\bar{s}\subseteq s'$ and $\bar{t}\subseteq t'$. □

This completes the proof. □

We now arrive at the proof of Theorem A:

Theorem 5.10. *We have* (1) $\implies$ (2) $\implies$ (3)*:*

(1) there exists a κ-Souslin tree $\mathbf{T}$ such that $V(\mathbf{T})=\emptyset$;
(2) there exists a normal and splitting κ-tree $\mathbf{T}$ such that $V(\mathbf{T})$ is nonstationary;
(3) κ is not the successor of a cardinal of countable cofinality.

In addition, in L*, for κ not weakly compact,* (3) $\implies$ (1)*.*

Proof. (1) $\implies$ (2): If $\mathbf{T} = (T, <_T)$ is a κ-Souslin tree, then a standard argument (see [BR17b, Lemma 2.4]) shows that for some club $D\subseteq\kappa$, $\mathbf{T}' = (T\restriction D, <_T)$ is normal and splitting. Clearly, if $V(\mathbf{T})=\emptyset$, then $V(\mathbf{T}')=\emptyset$, as well.

(2) $\implies$ (3): Suppose that $\mathbf{T}$ is a normal and splitting κ-tree. If κ is the successor of a cardinal of countable cofinality then by Corollary 2.11, $V(\mathbf{T})$ covers the stationary set E^κ_ω.

Hereafter, work in L, and suppose that κ is a regular uncountable cardinal that is not weakly compact and not the successor of a cardinal of countable cofinality. Then by Corollary 5.1 together with Proposition 2.5(2) there are κ-Souslin trees $\mathbf{T}^0,\mathbf{T}^1$ such that $V(\mathbf{T}^0)=E^\kappa_\omega$ and $V(\mathbf{T}^1)=E^\kappa_{\omega_1}$. The disjoint sum of the two $\mathbf{T} := \sum\{\mathbf{T}^0,\mathbf{T}^1\}$ is clearly κ-Souslin. In addition, by Proposition 2.28(2), $V(\mathbf{T}) = V(\mathbf{T}^0)\cap V(\mathbf{T}^1)=\emptyset$. □

Remark 5.11. The κ-Souslin tree $\mathbf{T}$ constructed in the preceding proof satisfies $V(\mathbf{T})=\emptyset$, yet it has a κ-Souslin subtree $\mathbf{T}'$ for which $V(\mathbf{T}')$ is stationary. A κ-tree $\mathbf{T}$ is said to be *full* iff for every $\alpha\in\mathrm{acc}(\kappa)$, there is no more than one vanishing α-branch in $\mathbf{T}$. It is clear that if $\mathbf{T}$ is a full κ-tree that is splitting (resp. Aronszajn), then $V(\mathbf{T})$ is empty (resp. nonstationary). In [RYY23],

we construct full κ-Souslin trees, thus giving an example of a κ-Souslin tree $\mathbf{T}$ such that $V(\mathbf{T}')$ is nonstationary for all of its κ-subtrees $\mathbf{T}'$.

We conclude this section by pointing out that by using [BR17a, Theorem 3.6] and a proof similar to that of Theorem 5.10, we get more information on the model studied in Corollary 4.7.

Corollary 5.12. *Suppose that* CH *and* $\diamondsuit_{\aleph_1}$ *both hold. Then there are* $\aleph_2$*-Souslin trees* $\mathbf{T}^0, \mathbf{T}^1, \mathbf{T}^2, \mathbf{T}^3$ *such that:*

- $V(\mathbf{T}^0) = \emptyset$*;*
- $V(\mathbf{T}^1) = E^{\aleph_2}_{\aleph_0}$*;*
- $V(\mathbf{T}^2) = E^{\aleph_2}_{\aleph_1}$*;*
- $V(\mathbf{T}^3) = \mathrm{acc}(\aleph_2)$. □

6. Souslin trees with an ascent path

The subject matter of this section is the following definition.

Definition 6.1 (Laver)**.** Suppose that $\mathbf{T} = (T, <_T)$ is a tree of some height κ. A *μ-ascent path* through $\mathbf{T}$ is a sequence $\vec{f} = \langle f_\alpha \mid \alpha < \kappa \rangle$ such that:

- for every $\alpha < \kappa$, $f_\alpha : \mu \to T_\alpha$ is a function;
- for all $\alpha < \beta < \kappa$, there is an $i < \mu$ such that $f_\alpha(j) <_T f_\beta(j)$ whenever $i \leq j < \mu$.

We will show that Souslin trees having a large set of vanishing levels are compatible with carrying an ascent path. For this, we shall make use of the following strengthening of $\mathrm{P}^-_\xi(\kappa, \mu^+, \sqsubseteq, \theta, \mathcal{S})$:

Definition 6.2 ([BR21, §4.6])**.** The principle $\mathrm{P}^-_\xi(\kappa, \mu^{\mathrm{ind}}, \sqsubseteq, \theta, \mathcal{S})$ asserts the existence of a ξ-bounded $\mathcal{C}$-sequence $\langle \mathcal{C}_\alpha \mid \alpha < \kappa \rangle$ together with a sequence $\langle i(\alpha) \mid \alpha < \kappa \rangle$ of ordinals in μ, such that:

- for every $\alpha < \kappa$, there exists a canonical enumeration $\langle C_{\alpha,i} \mid i(\alpha) \leq i < \mu \rangle$ of $\mathcal{C}_\alpha$ satisfying that the sequence $\langle \mathrm{acc}(C_{\alpha,i}) \mid i(\alpha) \leq i < \mu \rangle$ is $\subseteq$-increasing with $\bigcup_{i \in [i(\alpha), \mu)} \mathrm{acc}(C_{\alpha,i}) = \mathrm{acc}(\alpha)$;
- for all $\alpha < \kappa$, $i \in [i(\alpha), \mu)$ and $\bar{\alpha} \in \mathrm{acc}(C_{\alpha,i})$, it is the case that $i \geq i(\bar{\alpha})$ and $C_{\bar{\alpha},i} \sqsubseteq C_{\alpha,i}$;
- for every sequence $\langle B_\tau \mid \tau < \theta \rangle$ of cofinal subsets of κ, and every $S \in \mathcal{S}$, there are stationarily many $\alpha \in S$ such that for all $C \in \mathcal{C}_\alpha$ and $\tau < \min\{\alpha, \theta\}$, $\sup(\mathrm{nacc}(C) \cap B_\tau) = \alpha$.

Conventions 3.4 and 3.5 apply to the preceding, as well.

Lemma 6.3. *Suppose that:*

- $\mu < \kappa$ *is an infinite cardinal;*
- K *is a streamlined* κ*-tree;*
- $\mathrm{P}(\kappa, \mu^{\mathrm{ind}}, \sqsubseteq, 1)$ *holds.*

Then there exists a normal and splitting streamlined κ*-Souslin tree* T *with* $V(T) \supseteq V^-(K)$ *such that* T *admits a* μ*-ascent path.*

Proof. As a preparatory step, we shall need the following simple claim.

Claim 6.3.1. *We may assume that $\mathcal{B}(K) \neq \emptyset$.*

Proof. For every $\eta \in K$, define a function $\eta' : \mathrm{dom}(\eta) \to H_\kappa$ via $\eta'(\alpha) := (\eta(\alpha), 0)$. Then $K' := \{\eta' \mid \eta \in K\} \uplus {}^{<\kappa}1$ is a streamlined κ-tree with $V^-(K') = V^-(K)$ and, in addition, $\mathcal{B}(K') \neq \emptyset$. □

Let $\vec{\mathcal{C}} = \langle \mathcal{C}_\alpha \mid \alpha < \kappa \rangle$ and $\langle i(\alpha) \mid \alpha < \kappa \rangle$ witness together that $\mathrm{P}^-(\kappa, \mu^{\mathrm{ind}}, \sqsubseteq, 1)$ holds. In particular, $\vec{\mathcal{C}}$ is a $\mathrm{P}^-(\kappa, \kappa, \sqsubseteq, 1)$-sequence satisfying that, for all $\alpha \in \mathrm{acc}(\kappa)$ and $C, D \in \mathcal{C}_\alpha$, $\sup(C \cap D) = \alpha$. As always, we may also assume that $0 \in \bigcap_{0<\alpha<\kappa} \bigcap \mathcal{C}_\alpha$.

Using $\vec{\mathcal{C}}$ and K, construct the sequence of levels $\langle T_\alpha \mid \alpha < \kappa \rangle$ exactly as in the proof of Theorem 3.7, so that $T := \bigcup_{\alpha<\kappa} T_\alpha$ is a normal and splitting streamlined κ-Souslin tree. From Claim 3.7.2, we infer that $V(T) \supseteq V^-(K)$.

In addition, the construction of Theorem 3.7 ensures that for every $\alpha \in \mathrm{acc}(\kappa)$, it is the case that

$$T_\alpha = \{\mathbf{b}_x^{C,\eta} \mid C \in \mathcal{C}_\alpha, \eta \in K_\alpha, x \in T \restriction C\}.$$

Fix $\zeta \in \mathcal{B}(K)$. For every $\alpha \in \mathrm{acc}(\kappa)$, using the canonical enumeration $\langle C_{\alpha,i} \mid i(\alpha) \leq i < \mu \rangle$ of $\mathcal{C}_\alpha$, we define a function $f_\alpha : \mu \to T_\alpha$ via

$$f_\alpha(j) := \mathbf{b}_\emptyset^{C_{\alpha,\max\{j,i(\alpha)\}},\zeta\restriction\alpha}.$$

Claim 6.3.2. *Let $\beta < \alpha$ be a pair of ordinals in $\mathrm{acc}(\kappa)$. Then there exists an $i < \mu$ such that $f_\beta(j) \subseteq f_\alpha(j)$ whenever $i \leq j < \mu$.*

Proof. Note that by Claim 3.7.1, for all $C \in \mathcal{C}_\alpha$, $\eta \in K_\alpha$, and $x \in T \restriction (C \cap \beta)$, if $\beta \in \mathrm{acc}(C)$, then $\mathbf{b}_x^{C,\eta} \restriction \beta = \mathbf{b}_x^{C\cap\beta,\eta\restriction\beta}$.

Now, by Definition 6.2, we may fix a large enough $i \in [i(\alpha), \mu)$ such that $\beta \in \mathrm{acc}(C_{\alpha,j})$ whenever $i \leq j < \mu$. Let j be such an ordinal. Then $j \geq i(\beta)$ and $C_{\alpha,j} \cap \beta = C_{\beta,j}$, so that

$$f_\beta(j) = \mathbf{b}_\emptyset^{C_{\beta,j},\zeta\restriction\beta} = \mathbf{b}_\emptyset^{C_{\alpha,j},\zeta\restriction\alpha} \restriction \beta = f_\alpha(j) \restriction \beta,$$

as sought. □

It now easily follows that T admits a μ-ascent path. □

Corollary 6.4. *Suppose that:*

- *λ is an uncountable cardinal satisfying $\square_\lambda$ and $2^\lambda = \lambda^+$;*
- *$\mu < \lambda$ is an infinite regular cardinal satisfying $\lambda^\mu = \lambda$.*

Then there exists a streamlined λ^+-Souslin tree T with $V(T) = \mathrm{acc}(\lambda^+)$ such that T admits a μ-ascent path.

Proof. By [LHL18, Theorem 3.4], in particular, $\square^{\mathrm{ind}}(\lambda^+, \mu)$ holds. Then, by [BR21, Theorem 4.44], $\mathrm{P}^-(\lambda^+, \mu^{\mathrm{ind}}, \sqsubseteq, 1)$ holds. By Shelah's theorem, $2^\lambda = \lambda^+$ implies $\diamondsuit(\lambda^+)$, so that, altogether $\mathrm{P}(\lambda^+, \mu^{\mathrm{ind}}, \sqsubseteq, 1)$ holds. In addition, it is a classical theorem of Jensen that $\square_\lambda$ gives a special λ^+-Aronszajn tree, so by Lemma 2.24, $\mathrm{acc}(\lambda^+) \in \mathrm{Vspec}(\lambda^+)$. It now follows from Lemma 6.3 that

there exists a normal and splitting streamlined λ^+-Souslin tree T such that $V(T)$ covers a club in λ^+ and such that T admits a μ-ascent path. Finally, the proof of Lemma 2.4 completes this proof. $\square$

Remark 6.5. The conclusion of the preceding remains valid once relaxing $\square_\lambda$ to $\square_\lambda(\sqsubseteq_\mu)$. In particular, the conclusion of the preceding is compatible with μ being supercompact.

We now turn to combine the preceding construction with the study of large cardinals. The following cardinal characteristic $\chi(\kappa)$ provides a measure of how far κ is from being weakly compact.

Definition 6.6 (The C-sequence number of κ, [LHR21])**.** If κ is weakly compact, then let $\chi(\kappa) := 0$. Otherwise, let $\chi(\kappa)$ denote the least cardinal $\chi \leq \kappa$ such that, for every C-sequence $\langle C_\beta \mid \beta < \kappa\rangle$, there exist $\Delta \in [\kappa]^\kappa$ and $b : \kappa \to [\kappa]^\chi$ with $\Delta \cap \alpha \subseteq \bigcup_{\beta \in b(\alpha)} C_\beta$ for every $\alpha < \kappa$.

By [LHR21, Lemma 2.12(1)], if κ is an inaccessible cardinal satisfying $\chi(\kappa) < \kappa$, then κ is ω-Mahlo. The following is an expanded form of Theorem E.

Theorem 6.7. *Assuming the consistency of a weakly compact cardinal, it is consistent that for some strongly inaccessible cardinal κ satisfying $\chi(\kappa) = \omega$, the following two hold:*

- *Every κ-Aronszajn tree admits an ω-ascent path;*
- *There is a κ-Souslin tree $\mathbf{T}$ such that $V(\mathbf{T}) = \mathrm{acc}(\kappa)$.*

Proof. Suppose that κ is a non-subtle weakly compact cardinal. By possibly using a preparatory forcing, we may assume that the non-subtle weak compactness of κ is indestructible under forcing with $\mathrm{Add}(\kappa, 1)$. Following the proof of [LHR21, Theorem 3.4], let $\mathbb{P}$ be the standard forcing to add $\square^{\mathrm{ind}}(\kappa, \omega)$-sequence by closed initial segments, let G be $\mathbb{P}$-generic, and let $\vec{\mathcal{C}} = \langle C_{\alpha,i} \mid \alpha < \kappa,\ i(\alpha) \leq i < \omega\rangle$ denote the generically-added $\square^{\mathrm{ind}}(\kappa, \omega)$-sequence. Work in $V[G]$. By Clauses (1),(2) and (4) of [LHR21, Theorem 3.4], κ is strongly inaccessible, $\chi(\kappa) = \omega$, and every κ-Aronszajn tree admits an ω-ascent path.

For every $\alpha \in \mathrm{acc}(\kappa)$, let

$$B_\alpha := \{\beta \in C_{\alpha,i(\alpha)} \mid \forall l < \omega\, [\min(C_{\alpha,i(\alpha)} \setminus \beta + 1) + l \in C_{\alpha,i(\alpha)}]\}.$$

Claim 6.7.1. *For every cofinal $B \subseteq \kappa$, there exist $\alpha \in E^\kappa_\omega$ and $\epsilon < \alpha$ such that $(B_\alpha \setminus \epsilon) \subseteq B$, $i(\alpha) = 0$ and $\sup(\mathrm{nacc}(C_{\alpha,i}) \cap B_\alpha) = \alpha$ for all $i < \omega$.*

Proof. We follow the proof of [LH17, Lemma 3.9]. Work in V. For every $\alpha \in \mathrm{acc}(\kappa)$, let $\dot{B}_\alpha$ be the canonical $\mathbb{P}$-name for B_α. Next, let $\dot{B}$ be a $\mathbb{P}$-name for a cofinal subset of κ, and let p_0 be an arbitrary condition in $\mathbb{P}$. By possibly extending p_0, we may assume that $i(\gamma^{p_0})^{p_0} = 0$. We shall recursively define a decreasing sequence of conditions $\langle p_n \mid n < \omega\rangle$, and an increasing sequence of ordinals $\langle \beta_n \mid n < \omega\rangle$ such that for every $n < \omega$, all of the following hold:

(1) $p_{n+1} \leq p_n$;
(2) $i(\gamma^{p_{n+1}})^{p_{n+1}} = 0$;
(3) $p_{n+1} \Vdash$ "$\beta_n \in \dot{B}$ and $\dot{B}_{\gamma^{p_{n+1}}} \setminus (\gamma^{p_n}+1) = \{\beta_n\}$";
(4) For every $i \leq n$, $\beta_n \in \mathrm{nacc}(C^{p_{n+1}}_{\gamma^{p_{n+1}},i})$;
(5) For every $i < \omega$, $C^{p_{n+1}}_{\gamma^{p_{n+1}},i} \cap (\gamma^{p_n}+1) = C^{p_n}_{\gamma^{p_n},i} \cup \{\gamma^{p_n}\}$.

Suppose $n < \omega$ is such that $\langle p_m \mid m \leq n\rangle$ and $\langle \beta_m \mid m < n\rangle$ have already been successfully defined. Find a $p^*_n \leq p_n$ and a $\beta_n > \gamma^{p_n}$ such that $p^*_n \Vdash$ "$\beta_n \in \dot{B}$". Without loss of generality, $\gamma^{p^*_n} > \beta_n$. Now, let $\gamma := \gamma^{p^*_n} + \omega$, so that

$$\gamma^{p_n} < \beta_n < \gamma^{p^*_n} < \gamma^{p^*_n} + \omega = \gamma.$$

Let $m < \omega$ be the least such that $m \geq \max\{n, i(\gamma^{p^*_n})^{p^*_n}\}$ and $\gamma^{p_n} \in \mathrm{acc}(C^{p^*_n}_{\gamma^*,m})$. Then let p_{n+1} be the unique extension of p^*_n with $\gamma^{p_{n+1}} = \gamma$ and $i(\gamma)^{p_{n+1}} = 0$ to satisfy the following for all $i < \omega$:

$$C^{p_{n+1}}_{\gamma,i} := \begin{cases} C^{p_n}_{\gamma^{p_n},i} \cup \{\gamma^{p_n}, \beta_n\} \cup \{\gamma^{p^*_n} + l \mid l < \omega\}, & \text{if } i \leq m; \\ C^{p^*_n}_{\gamma^{p^*_n},i} \cup \{\gamma^{p^*_n} + l \mid l < \omega\}, & \text{otherwise.} \end{cases}$$

Thus, we have maintained requirements (1)–(5).

Once completing the above recursion, we obtain a decreasing sequence of conditions $\langle p_n \mid n < \omega\rangle$. Let $\alpha := \sup\{\gamma^{p_n} \mid n < \omega\}$, and let p be the unique lower bound of $\langle p_n \mid n < \omega\rangle$ to satisfy $\gamma^p = \alpha$, $i(\alpha)^p = 0$, and $C^p_{\alpha,i} = \bigcup_{n<\omega} C^{p_n}_{\gamma^{p_n},i}$ for every $i < \omega$. Then p is a legitimate condition satisfying $p \Vdash$ "$\dot{B}_\alpha \setminus (\gamma^{p_0}+1) = \{\beta_n \mid n < \omega\} \subseteq \dot{B}$". In addition, for all $i < \omega$, $\{\beta_n \mid i \leq n < \omega\} \subseteq \mathrm{nacc}(C^p_{\alpha,i})$. So we are done. □

We claim that $\vec{\mathcal{C}}$ is a $\mathrm{P}^-(\kappa, \omega^{\mathrm{ind}}, \sqsubseteq, 1)$-sequence. As we already know that $\vec{\mathcal{C}}$ is an $\square^{\mathrm{ind}}(\kappa,\omega)$-sequence, we just need to verify that it satisfies the last bullet of Definition 6.2 with $\theta := 1$ and $\mathcal{S} := \{\kappa\}$. But, by the same argument from the proof of [BR21, Corollary 3.4], this boils down to showing that for every cofinal $B \subseteq \kappa$, there exists at least one $\alpha \in \mathrm{acc}(\kappa)$ such that $\sup(\mathrm{nacc}(C_{\alpha,i}) \cap B) = \alpha$ for all $i \in [i(\alpha), \omega)$. This is covered by Claim 6.7.1.

Claim 6.7.2. *$\diamondsuit(E^\kappa_\omega)$ holds.*

Proof. This is a standard consequence of Claim 6.7.1 together with the fact that $\kappa^{<\kappa} = \kappa$, but we give the details. Let $\vec{X} = \langle X_\beta \mid \beta < \kappa\rangle$ be a repetitive enumeration of $[\kappa]^{<\kappa}$ such that each set appears cofinally often. Let us say that an ordinal $\alpha \in E^\kappa_\omega$ is *informative* if $\sup(B_\alpha) = \alpha$ and there are $\epsilon < \kappa$ and a subset $A_\alpha \subseteq \alpha$ such that $A_\alpha \cap \gamma = X_\beta \cap \gamma$ for every pair $\gamma < \beta$ of ordinals from $B_\alpha \setminus \epsilon$. Note that if α is informative, then the set A_α is uniquely determined. For a noninformative $\alpha \in E^\kappa_\omega$, we let $A_\alpha := \emptyset$.

To verify that $\langle A_\alpha \mid \alpha \in E^\kappa_\omega\rangle$ witnesses $\diamondsuit(E^\kappa_\omega)$, let A be a subset of κ and let C be a club in κ, and we shall find an $\alpha \in C \cap E^\kappa_\omega$ such that $A \cap \alpha = A_\alpha$.

By the choice of $\vec{X}$, we may fix a strictly increasing function $f : \kappa \to \kappa$ satisfying that $A \cap \xi = X_{f(\xi)}$ for every $\xi < \kappa$. Consider the club $D :=$

$\{\delta \in C \mid f[\delta] \subseteq \delta\}$. Let B be some cofinal subset of $\mathrm{Im}(f)$ sparse enough to satisfy that for every pair $\gamma < \beta$ of ordinals from B, there exists a $\delta \in D$ with $\gamma < \delta < \beta$. Using Claim 6.7.1, fix $\alpha \in E^\kappa_\omega$ and $\epsilon < \alpha$ such that $(B_\alpha \setminus \epsilon) \subseteq B$ and $\sup(B_\alpha) = \alpha$. Now, let $\gamma < \beta$ be a pair of ordinals in $B_\alpha \setminus \epsilon$. As $\gamma, \beta \in B$, we may pick a $\delta \in D$ with $\gamma < \delta < \beta$. As $\beta \in B \subseteq \mathrm{Im}(f)$, we may also pick a $\xi < \kappa$ such that $\beta = f(\xi)$. Since $f[\delta] \subseteq \delta \subseteq \beta$, it must be the case that $\xi \geq \delta > \gamma$. So $A \cap \gamma = (A \cap \xi) \cap \gamma = X_\beta \cap \gamma$. Thus, we showed that $A \cap \gamma = X_\beta \cap \gamma$ for every pair $\gamma < \beta$ of ordinals in $B_\alpha \setminus \epsilon$, and hence α is informative and $A_\alpha = A \cap \alpha$. In addition, for every pair $\gamma < \beta$ of ordinals in $B_\alpha \setminus \epsilon$, there exists $\delta \in D$ with $\gamma < \delta < \beta$, and hence $\alpha \in \mathrm{acc}^+(D) \subseteq C$. $\square$

Altogether, $\mathrm{P}(\kappa, \omega^{\mathrm{ind}}, \sqsubseteq, 1)$ holds. Since κ is a strongly inaccessible cardinal that is non-subtle, Corollary 2.19 implies that there exists a streamlined κ-tree K such that $V^-(K)$ covers a club in κ. So by appealing to Lemma 6.3 and then to Lemma 2.4, we infer that there exists a κ-Souslin tree $\mathbf{T}$ with $V(\mathbf{T}) = \mathrm{acc}(\kappa)$. $\square$

By [RS23, Theorem 2.30], $\chi(\kappa) = 0$ refutes $\clubsuit_{\mathrm{AD}}(\mathrm{Reg}(\kappa))$. An easy variant of that proof yields that $\chi(\kappa) = 0$ furthermore refutes $\clubsuit_{\mathrm{AD}}(\mathrm{Reg}(\kappa) \cap D)$ for every club $D \subseteq \kappa$. It follows from the preceding theorem together with the proof of [RS23, Theorem 2.23] that $\chi(\kappa) = \omega$ is compatible with $\clubsuit_{\mathrm{AD}}(D)$ holding for some club $D \subseteq \kappa$. Whether this can be improved to $\chi(\kappa) = 1$ remains an open problem.

A. A new sufficient condition for a Dowker space

Definition A.1 ([RS23])**.** Let $\mathcal{S}$ be a collection of stationary subsets of a regular uncountable cardinal κ, and μ, θ be nonzero cardinals below κ. The principle $\clubsuit_{\mathrm{AD}}(\mathcal{S}, \mu, \theta)$ asserts the existence of a sequence $\langle \mathcal{A}_\alpha \mid \alpha \in \bigcup \mathcal{S} \rangle$ such that:

(1) For every $\alpha \in \mathrm{acc}(\kappa) \cap \bigcup \mathcal{S}$, $\mathcal{A}_\alpha$ is a pairwise disjoint family of μ many cofinal subsets of α;
(2) For every $\mathcal{B} \subseteq [\kappa]^\kappa$ of size θ, for every $S \in \mathcal{S}$, there are stationarily many $\alpha \in S$ such that $\sup(A \cap B) = \alpha$ for all $A \in \mathcal{A}_\alpha$ and $B \in \mathcal{B}$;[10]
(3) For all $A \neq A'$ from $\bigcup_{S \in \mathcal{S}} \bigcup_{\alpha \in S} \mathcal{A}_\alpha$, $\sup(A \cap A') < \sup(A)$.

Remark A.2. The variation $\clubsuit_{\mathrm{AD}}(\mathcal{S}, \mu, {<}\theta)$ asserts the existence of a sequence simultaneously witnessing $\clubsuit_{\mathrm{AD}}(\mathcal{S}, \mu, \vartheta)$ for all $\vartheta < \theta$.

By [RS23, Lemma 2.10], for a pair $\chi < \kappa$ of infinite regular cardinals, for a stationary subset S of E^κ_χ, Ostaszewski's principle $\clubsuit(S)$ implies $\clubsuit_{\mathrm{AD}}(\mathcal{S}, \chi, {<}\omega)$ for some partition $\mathcal{S}$ of S into κ many stationary sets. The next theorem reduces the hypothesis "$S \subseteq E^\kappa_\chi$" down to "$S \cap \mathrm{Tr}(S) = \emptyset$".

Lemma A.3. *Suppose:*

[10] Note that the existence of stationarily many such $\alpha \in S$ is no stronger than the existence of just one $\alpha \in S$. See [BR21, Corollary 3.4] for the prototype argument.

- *$\mu,\theta<\kappa=\kappa^{<\theta}$ are infinite cardinals;*
- *$S\subseteq E^\kappa_{\geq\max\{\mu,\theta\}}$ is stationary and $\mathrm{Tr}(S)\cap S=\emptyset$;*
- *$\clubsuit(S)$ holds.*

Then $\clubsuit_{\mathrm{AD}}(\mathcal{S},\mu,<\theta)$ holds for some partition $\mathcal{S}$ of S into κ many stationary sets. More generally, for every $Z\subseteq\kappa$ such that $S\subseteq\mathrm{acc}^+(Z)$, there exists a matrix $\langle A_{\delta,i}\mid\delta\in S,i<\mu\rangle$ and a partition $\mathcal{S}$ of S into κ many pairwise disjoint stationary sets such that:

(1) For all $\delta\in S$, $\langle A_{\delta,i}\mid i<\mu\rangle$ is a sequence of pairwise disjoint subsets of $Z\cap\delta$, and $\sup(A_{\delta,i})=\delta$;
(2) For every $(\gamma,\delta)\in[S]^2$, for all $i,j<\mu$, $\sup(A_{\gamma,i}\cap A_{\delta,j})<\gamma$;
(3) For every $\vartheta<\theta$, every sequence $\langle B_\tau\mid\tau<\vartheta\rangle$ of cofinal subsets of Z and every $S'\in\mathcal{S}$, there exists $\delta\in S'$ such that $\sup(A_{\delta,i}\cap B_\tau)=\delta$ for all $i<\mu$ and $\tau<\vartheta$.

Proof. By [BR21, Theorem 3.7], since $\clubsuit(S)$ holds, we may find a partition $\langle S_{\vartheta,\iota}\mid\vartheta<\theta,\iota<\kappa\rangle$ of S into stationary sets such that $\clubsuit(S_{\vartheta,\iota})$ holds for all $\vartheta<\theta$ and $\iota<\kappa$. For all $\vartheta<\theta$ and $\iota<\kappa$, since $\clubsuit(S_{\vartheta,\iota})$ holds and $\kappa^\vartheta=\kappa$, by [RS23, Lemma 3.5], we may fix a matrix $\langle X^\tau_\delta\mid\delta\in S_{\vartheta,\iota},\tau<\vartheta\rangle$ such that, for every sequence $\langle X^\tau\mid\tau<\vartheta\rangle$ of cofinal subsets of κ, there are stationarily many $\delta\in S_{\vartheta,\iota}$, such that, for all $\tau<\vartheta$, $X^\tau_\delta\subseteq X^\tau\cap\delta$ and $\sup(X^\tau_\delta)=\delta$.

Now, let $Z\subseteq\kappa$ with $S\subseteq\mathrm{acc}^+(Z)$ be given. For all $\vartheta<\theta$, $\iota<\kappa$, $\delta\in S_{\vartheta,\iota}$ and $\tau<\vartheta$, we do the following:

- if $X^\tau_\delta\cap Z$ is a cofinal subset of δ, then let $Y^\tau_\delta:=X^\tau_\delta\cap Z$. Otherwise, let Y^τ_δ be an arbitrary cofinal subset of $Z\cap\delta$;
- since $\delta\in S\subseteq\kappa\setminus\mathrm{Tr}(S)$, we may fix a club $C_\delta\subseteq\delta$ disjoint from S, and then, by [BR21, Lemma 3.3], we may find a cofinal subset Z^τ_δ of Y^τ_δ such that in-between any two points of Z^τ_δ there exists a point of C_δ, so that $\mathrm{acc}^+(Z^\tau_\delta)\cap S=\emptyset$.

As $\mathrm{cf}(\delta)\geq\theta>\vartheta$ and by possibly thinning out, we may assume that $\langle Z^\tau_\delta\mid\tau<\vartheta\rangle$ consists of pairwise disjoint cofinal subsets of $Z\cap\delta$. As $\mathrm{cf}(\delta)\geq\mu$, for every $\tau<\vartheta$, we may fix a partition $\langle Z^{\tau,i}_\delta\mid i<\mu\rangle$ of Z^τ_δ into cofinal subsets of δ. For every $i<\mu$, let

$$A_{\delta,i}:=\bigcup_{\tau<\vartheta}Z^{\tau,i}_\delta.$$

For every $i<\mu$, since $\mathrm{acc}^+(Z^{\tau,i}_\delta)\cap S\subseteq\mathrm{acc}^+(Z^\tau_\delta)\cap S=\emptyset$, and since $\delta\in S\subseteq E^\kappa_{>\vartheta}$, we get that $\mathrm{acc}^+(A_{\delta,i})\cap S=\emptyset$. So $\langle A_{\delta,i}\mid i<\mu\rangle$ is a sequence of pairwise disjoint cofinal subsets of δ, and for every $\gamma\in S\cap\delta$ and every cofinal subset $A\subseteq\gamma$, $\sup(A\cap A_{\delta,i})<\gamma$. Thus, we have already taken care of Clauses (1) and (2).

Next, consider $\mathcal{S}:=\{\bigcup_{\vartheta<\theta}S_{\vartheta,\iota}\mid\iota<\kappa\}$ which is a partition of S into κ many stationary sets. Now, given $\vartheta<\theta$, a sequence $\langle B_\tau\mid\tau<\vartheta\rangle$ of cofinal subsets of Z, and some $S'\in\mathcal{S}$, we may find $\iota<\kappa$ such that $S'\supseteq S_{\vartheta,\iota}$, and find $\delta\in S_{\vartheta,\iota}$ such that, for all $\tau<\vartheta$, $X^\tau_\delta\subseteq B_\tau\cap\delta$ and $\sup(X^\tau_\delta)=\delta$.

In particular, for all $\tau < \vartheta$ and $i < \mu$, $Z^{\tau,i}_\delta \subseteq Z^\tau_\delta \subseteq Y^\tau_\delta = X^\tau_\delta \cap Z \subseteq B_\tau$. Therefore, for all $\tau < \vartheta$ and $i < \mu$, $\sup(A_{\delta,i} \cap B_\tau) = \delta$. □

Corollary A.4. *Suppose that $\clubsuit(S)$ holds for some nonreflecting stationary subset S of κ. Then $\clubsuit_{\mathrm{AD}}(\mathcal{S}, \omega, <\omega)$ holds for some partition $\mathcal{S}$ of S into κ many stationary sets.* □

The preceding yields the proof of Theorem F which in turn extends an old result of Good [Goo95] who got a Dowker space of size λ^+ from $\clubsuit(S)$ holding over a nonreflecting stationary $S \subseteq E^{\lambda^+}_\omega$.[11]

Corollary A.5. *If $\clubsuit(S)$ holds over a nonreflecting stationary $S \subseteq \kappa$, then there are 2^κ many pairwise nonhomeomorphic Dowker spaces of size κ.*

Proof. By [RST23, Theorem A.1], if $\clubsuit_{\mathrm{AD}}(\mathcal{S}, 1, 2)$ holds for a partition $\mathcal{S}$ of a nonreflecting stationary subset of κ into κ many stationary sets, then there are 2^κ many pairwise nonhomeomorphic Dowker spaces of size κ. □

Our last corollary deals with the problem of having $\clubsuit_{\mathrm{AD}}$ hold over a club subset of a successor cardinal.

Corollary A.6. *Suppose that $\kappa = \lambda^+$ for some infinite cardinal λ, and that $\clubsuit(E^\kappa_\theta)$ holds for every $\theta \in \mathrm{Reg}(\kappa)$. Then there exists a partition $\mathcal{S}$ of some club subset $D \subseteq \mathrm{acc}(\kappa)$ into κ many sets such that $\clubsuit_{\mathrm{AD}}(\mathcal{S}, \omega, 1)$ holds. Furthermore, there is a matrix $\langle A_{\delta,i} \mid \delta \in D, i < \mathrm{cf}(\delta)\rangle$ such that:*

(1) For every $\delta \in D$, $\langle A_{\delta,i} \mid i < \mathrm{cf}(\delta)\rangle$ is sequence of pairwise disjoint cofinal subsets of δ;
(2) For all $A \neq A'$ from $\{A_{\delta,i} \mid \delta \in D, i < \mathrm{cf}(\delta)\}$, $\sup(A \cap A') < \sup(A)$;
(3) For every cofinal $B \subseteq \kappa$, for every $S \in \mathcal{S}$, there are stationarily many $\delta \in S$ such that $\sup(A_{\delta,i} \cap B) = \delta$ for all $i < \mathrm{cf}(\delta)$.

Proof. Let $\langle Z_\mu \mid \mu \in \mathrm{Reg}(\kappa)\rangle$ be a partition of κ into cofinal sets. Let $D := \bigcap_{\mu \in \mathrm{Reg}(\kappa)} \mathrm{acc}^+(Z_\mu)$. For every $\mu \in \mathrm{Reg}(\kappa)$, by Lemma A.3, we may fix a matrix $\langle A_{\delta,i} \mid \delta \in E^\kappa_\mu, i < \mu\rangle$ and a partition $\langle S_{\mu,\iota} \mid \iota < \kappa\rangle$ of E^κ_μ into κ many pairwise disjoint stationary sets such that:

- For all $\delta \in E^\kappa_\mu$, $\langle A_{\delta,i} \mid i < \mu\rangle$ is a sequence of pairwise disjoint subsets of $Z_\mu \cap \delta$, and $\sup(A_{\delta,i}) = \delta$;
- For every $(\gamma, \delta) \in [E^\kappa_\mu]^2$, for all $i, j < \mu$, $\sup(A_{\gamma,i} \cap A_{\delta,j}) < \gamma$;
- For every cofinal $B \subseteq Z_\mu$, for every $\iota < \kappa$, there exists $\delta \in S_{\mu,\iota}$ such that $\sup(A_{\delta,i} \cap B) = \delta$ for all $i < \mu$.

Putting these matricies together, we get a matrix $\langle A_{\delta,i} \mid \delta \in D, i < \mathrm{cf}(\delta)\rangle$ satisfying Clause (1). In addition, since $Z_\mu \cap Z_{\mu'} = \emptyset$ for $\mu \neq \mu'$, Clause (2) is satisfied. Now, $\mathcal{S} := \{\bigcup_{\mu \in \mathrm{Reg}(\kappa)} S_{\mu,\iota} \mid \iota < \kappa\}$ is a partition of D into κ many stationary sets. By the pigeonhole principle, for every cofinal $B \subseteq \kappa$, there exists some $\mu \in \mathrm{Reg}(\kappa)$ such that $B \cap Z_\mu$ is cofinal in κ. So, for every

[11] Strictly speaking, the hypothesis in [Goo95] is $\clubsuit_{\lambda^+}(S, 2)$, but [BR21, Lemma 3.5] shows that this is no stronger than the vanilla $\clubsuit(S)$.

$S \in \mathcal{S}$, there exist $\iota < \kappa$ and $\delta \in S_{\mu,\iota} \subseteq S$ such that $\sup(A_{\delta,i} \cap B) = \delta$ for all $i < \mathrm{cf}(\delta)$. $\square$

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