A SHELAH GROUP IN ZFC

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Abstract. In a paper from 1980, Shelah constructed an uncountable group all of whose proper subgroups are countable. Assuming the continuum hypothesis, he constructed an uncountable group G that moreover admits an integer n satisfying that for every uncountable $X \subseteq G$, every element of G may be written as a group word of length n in the elements of X. The former is called a *Jónsson group* and the latter is called a *Shelah group*.

In this paper, we construct a Shelah group on the grounds of ZFC alone, that is, without assuming the continuum hypothesis. More generally, we identify a combinatorial condition (coming from the theories of negative square-bracket partition relations and strongly unbounded subadditive maps) sufficient for the construction of a Shelah group of size κ, and prove that the condition holds true for all successors of regular cardinals (such as $\kappa = \aleph_1, \aleph_2, \aleph_3, \ldots$). This also yields the first consistent example of a Shelah group of size a limit cardinal.

1. Introduction

For a prime number p, the Prüfer p-group

$$\{x \in \mathbb{C} \mid \exists n \in \mathbb{N}\, (x^{p^n} = 1)\}$$

is an example of an infinite subgroup of $(\mathbb{C}, \cdot)$ all of whose proper subgroups are finite. In [Ols80], Ol'šanskiĭ constructed finitely generated non-cyclic infinite groups in which every nontrivial proper subgroup is a finite cyclic group (the *Tarski monsters*). In [She80], answering a question of Kurosh, Shelah constructed an uncountable group in which every nontrivial proper subgroup is countable. All of those are examples of so-called *Jónsson groups*, i.e., an infinite group G having no proper subgroups of full size. An even more striking concept is that of a *boundedly-Jónsson group*, that is, a group G admitting a positive integer n such that for every $X \subseteq G$ of full size, it is the case that $X^n = G$, i.e., every element of G may be written as a group word of length exactly n in the elements of X. In [She80], Shelah constructed a boundedly-Jónsson group of size $\aleph_1$ with the aid of Continuum Hypothesis (CH). More generally, Shelah proved that $2^\lambda = \lambda^+$ yields a boundedly-Jónsson group of size λ^+. By now, the concept of boundedly-Jónsson groups is named after him:

Definition 1.1. A group G is *n-Shelah* if $X^n = G$ for every $X \subseteq G$ of full size.

A group is *Shelah* if it is n-Shelah for some positive integer n.

Along the years, variations of this concept were studied quite intensively, and from various angles. A group G is said to be *Cayley bounded* with respect to a subset

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$S \subseteq G$ if there exists a positive integer n_S such that $G = \bigcup_{i=1}^{n_S}(S \cup S^{-1})^i$, i.e., every element of G may be written as a group word of length at most n_S in the elements of S and inverses of elements of S. Extending the work of Macpherson and Neumann [MN90], Bergman proved [Ber06] that the permutation group $\mathrm{Sym}(\Omega)$ of an infinite set Ω is Cayley bounded with respect to all of its generating sets. Soon after, the notion *Bergman property* was coined as the assertion of being Cayley bounded with respect to all generating sets. Since then it has received a lot of attention, see [DG05, DH05, Tol06a, Tol06b, RR07, DHU08, DT09, MMR09, BTV12, TZ12]. More recent examples include the work of Dowerk [Dow20] on von Neumann algebras with unitary groups possessing the property of n*-strong uncountable cofinality* (i.e., having a common Cayley bound n for all generating sets, and the group is not the union of an infinite countable strictly increasing sequence of subgroups), and Shelah's work on locally finite groups [She20]. It is worth mentioning that the notion of strong uncountable cofinality has also geometric reformulations, e.g, by Cornulier [dC06], Pestov (see [Ros09, Theorem 1.2]) and Rosendal [Ros09, Proposition 3.3].

Shelah's 1980 construction from CH was of a 6640-Shelah group. It left open two independent questions:[1]

(1) Can CH be used to construct an n-Shelah group for a small number of n?
(2) Is CH necessary for the construction of an n-Shelah group?

Recently, in [Ban22], Banakh addressed the first question, using CH to construct a 36-Shelah group. Even more recently, Corson, Ol'šanskiĭ and Varghese [COV24] addressed the second question, constructing the first ZFC example of a Jónsson group of size $\aleph_1$ to have the Bergman property. Unfortunately, the new example stops short from being Shelah, as every generating set S of this group has its own n_S. In this paper, an affirmative answer to the second question is finally given, where a Shelah group of size $\aleph_1$ is constructed within ZFC.

Theorem A. *For every infinite regular cardinal λ, there exists a* 10120*-Shelah group of size λ^+. In particular, there exist Shelah groups of size $\aleph_1, \aleph_2, \aleph_3, \ldots$.*

The proof of Theorem A reflects advances both in small cancellation theory and in the study of infinite Ramsey theory. Towards it, we prove a far-reaching extension of Hesse's amalgamation lemma, and we obtain two maps, one coming from the theory of negative square-bracket partition relations, the other coming from the theory of strongly subadditive functions, and the two maps have the property that they may be triggered simultaneously, making them 'active' over each other.

The connection to infinite Ramsey theory should not come as a surprise. First, note that an n-Shelah group of size $\aleph_0$ does not exist, since such a group would have induced a coloring $c : [\mathbb{N}]^n \to k$ for a large enough integer k admitting no infinite homogeneous set,[2] in particular contradicting Ramsey's theorem $\aleph_0 \to (\aleph_0)^n_k$.[3]

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[1] See https://mathoverflow.net/questions/313516/ for a MathOverflow discussion initiated by Taras Banakh in October 2018. However, the second question was brought to the second author's attention in an email exchange with Ol'ga Sipacheva back in May 2006.

[2] An upper bound is $k = (n^n + 1)^{n^n}$, as shown in the proof of Corollary 5.24 below.

[3] For a (finite or infinite) cardinal λ, the Hungarian arrow notation $\lambda \to (\lambda)^n_k$ stands for the assertion that for every set X of size λ, whenever the family $[X]^n$ of all n-sized subsets of X is partitioned into k-many cells $[X]^n = \biguplus_{i=1}^k P_i$, then there exists a subset $Y \subseteq X$ of full size all of whose n-sized subsets belong to the same cell, i.e., $[Y]^n \subseteq P_i$ for one of the i's. Equivalently, for

A deeper connection to (additive) Ramsey theory is in the fact that the existence of a Jónsson group of size κ is equivalent to a very strong failure of the higher analog of Hindman's finite sums theorem [Hin74]. Indeed, by [FR17, Corollary 2.8], if there exists a Jónsson group of size κ, then for *every* Abelian group G of size κ, there exists a map $c : G \to G$ such that for every $X \subseteq G$ of full size, $c \restriction \mathrm{FS}(X)$ is surjective, i.e.,

$$\{c(x_1 + \cdots + x_n) \mid n \in \mathbb{N},\ \{x_1, \ldots, x_n\} \in [X]^n\} = G.$$

Conversely, if G is an Abelian group of size κ admitting a map $c : G \to G$ as above, then the structure $(G, +, c)$ is easily an example of a so-called *Jónsson algebra* [Jón72] of size κ, which by Corson's work [Cor22] implies the existence of a Jónsson group of size κ.

The fact that the elimination of CH goes through advances in the theory of partition calculus of uncountable cardinals should not come as a surprise, either. To give just one example, we mention that three decades after Juhász and Hajnal [HJ74] constructed an L-space with the aid of CH, Moore [Moo06] gave a ZFC construction of an L-space by establishing a new unbalanced partition relation for the first uncountable cardinal.

Having discussed Shelah groups of size $\aleph_0$ and of size a successor cardinal, the next question is whether it is possible to construct a Shelah group of size an uncountable *limit* cardinal. To compare, a natural ingredient available for transfinite constructions of length a successor cardinal $\kappa = \lambda^+$ is the existence of λ-filtrations of all ordinals less than κ. We overcome this obstruction at the level of a limit cardinal κ by employing subadditive strongly unbounded maps $d : [\kappa]^2 \to \lambda$ having arbitrarily large gaps between λ and κ. This way, we obtain the first consistent example of a Shelah group of size a limit cardinal. More generally:

Theorem B. *For every regular uncountable cardinal κ satisfying the combinatorial principle $\square(\kappa)$, there exists a Shelah group of size κ.*

By a seminal work of Jensen [Jen72], in Gödel's model of set theory known as *the constructible universe* [Göd40], the combinatorial principle $\square(\kappa)$ holds for every regular uncountable cardinal κ that is not weakly compact. As the reader may anticipate, a cardinal κ is *weakly compact* if it is a regular uncountable cardinal satisfying the higher analog of Ramsey's theorem $\kappa \to (\kappa)^2_2$. Altogether, we arrive at the following optimal result:

Theorem C. *In Gödel's constructible universe, for every regular uncountable cardinal κ, the following are equivalent:*

- *There exists a Shelah group of size κ;*
- *Ramsey's partition relation $\kappa \to (\kappa)^2_2$ fails.*

We conclude the introduction by discussing additional features that the groups constructed here possess. A group is said to be *topologizable* if it admits a non-discrete Hausdorff group topology; otherwise, it is *non-topologizable*. The first consistent instance of a non-topologizable group was the group constructed by Shelah in [She80] using CH. Shortly after, an uncountable ZFC example was given by Hesse [Hes79]. Then a countable such group was given by Ol'šanskiĭ [Ols12, Theorem 31.5] (an account of his construction may be found in [Adi06, §13.4]).

every coloring $c : [X]^n \to k$, there exists a subset $Y \subseteq X$ of full size that is c-homogeneous, i.e., $c \restriction [Y]^n$ is constant.

Ol'šanskiĭ's group is periodic; a torsion-free example was given by Klyachko and Trofimov in [KT05].

The Shelah group we construct in this paper is torsion-free and non-topologizable. The latter follows combining the property of Shelah-ness together with the fact that there will be a filtration of the group consisting of malnormal subgroups (see Definition 3.3). Moreover, our group contains a nonalgebraic unconditionally closed set, which can be shown by proving that small sets can be covered by a topologizable subgroup, similarly to the argument by Sipacheva [Sip06, Lemmas 1 and A.4].

1.1. **Can't you do better than $n = 10120$?** We believe a better n is achievable, but that is not the focus of this paper. In this paper, we establish a two-dimensional construction scheme for producing a group G of cardinality κ as a limit of a coherent system of subgroups $\langle G_{\gamma,i} \mid \gamma < \kappa,\ i < \theta\rangle$, where $G_{\gamma+1,i+1}$ is obtained as a particular amalgamation of the groups $G_{\gamma,i}$ and $G_{\gamma,i+1}$ over $G_{\gamma+1,i}$. The number $n = 10120$ comes from our amalgamation lemma, and so by plugging in alternative amalgamation lemmas to our construction scheme, we expect groups of various characteristics may be produced, including n-Shelah groups with $n < 10120$.

1.2. **Organization of this paper.** In Section 2, we fix our notations and conventions, and provide some necessary background from small cancellation theory.

In Section 3, we prove an amalgamation lemma that will serve as a building block in our two-dimensional recursive construction of a Shelah group.

In Section 4, we provide set-theoretic sufficient conditions for the existence of two types of maps to exist, and moreover be active over each other. The first type comes from the classical theory of negative square-bracket partition relations [EHR65, §18], and enables to eliminate the need for CH in the construction of a Shelah group of size $\aleph_1$. The second type comes from the theory of subadditive strongly unbounded functions [LR23], and enables to push the construction to higher cardinals including limit cardinals. At the level of successors of regulars, both of these colorings are obtained in ZFC using the method of *walks on ordinals* [Tod07] that did not exist at the time Shelah's paper [She80] was written.

In Section 5, we provide a transfinite construction of a Shelah group guided by the colorings given by Section 4, and using the amalgamation lemma of Section 3.

2. Preliminaries

2.1. **Notations and conventions.** Under ordinals we always mean von Neumann ordinals, and for a set X the symbol $|X|$ always refers to the smallest ordinal with the same cardinality. For a set X the symbol $\mathcal{P}(X)$ denotes the power set of X, while if θ is a cardinal we use the standard notation $[X]^\theta$ for $\{Y \in \mathcal{P}(X) \mid |Y| = \theta\}$, similarly for $[X]^{<\theta}$ and $[X]^{\leq\theta}$. We let $\mathcal{H}_\theta$ denote the collection of all sets of hereditary cardinality less than θ. A set D is a *club* in a cardinal κ iff $D \subseteq \kappa$ and for every $\epsilon < \kappa$, $\sup(D \cap \epsilon) \in D \cup \{0\}$ and $D \setminus \epsilon \neq \emptyset$. For a function f and a subset $A \subseteq \operatorname{dom}(f)$, we either write $f[A]$ or $f``A$ for $\{f(a) \mid a \in A\}$. By a sequence we mean a function on an ordinal, where for a sequence $\overline{s} = \langle s_\alpha \mid \alpha < \operatorname{dom}(\overline{s})\rangle$ the length of $\overline{s}$ (in symbols $\ell(\overline{s})$) denotes $\operatorname{dom}(\overline{s})$. We denote the empty sequence by $\langle\rangle$. For a set X and an ordinal α we use ${}^\alpha X = \{\overline{s} \mid \ell(\overline{s}) = \alpha,\ \operatorname{Im}(\overline{s}) \subseteq X\}$.

2.2. **Small cancellation theory.** The main algebraic tool we are going to use is small cancellation theory. In this regard the paper is self-contained, but for more details and proofs the interested reader can consult [LS77, §5. 11] and [She80, §1].

By convention, the free group with a set of generators A is denoted here by F_A, and the normal closure of a set S in a group G is denoted here by $\mathrm{ncl}(S, G)$.

Definition 2.1. Given groups H, K, L such that $K \cap L = H$ (as sets), in particular $H \leq K, L$, then one constructs the free amalgamation of K and L over H as

$$K *_H L = F_{K \cup L}/N,$$

where $N = \mathrm{ncl}(E_K \cup E_L, F_{K \cup L})$, and for $G \in \{K, L\}$,

$$E_G = \{g_1 g_2 g_3^{-1} \mid g_1, g_2, g_3 \in G,\ g_1 g_2 = g_3\}.$$

We invoke basic results about the structure of groups of the form $K *_H L$.

Definition 2.2. If $g = g_0^* g_1^* \cdots g_{n-1}^* \in K *_H L$, where $g_i^* \in K \cup L$, then we call the sequence of g_i^*'s the *canonical form* of the group element of g, if

- either $n = 1$, or
- $n > 1$, and for each $i < n$
  (1) $g_i^* \notin H$,
  (2) $i + 1 < n \rightarrow (g_i^* \in K \iff g_{i+1}^* \in L)$,

We note the following fact.

Fact 2.3 ([Sch71, Lemma 2.1])**.** *Suppose that $g_0^* g_1^* \cdots g_{n-1}^*$ and $g_0^{**} g_1^{**} \cdots g_{m-1}^{**}$ are canonical representations of the same element in $K *_H L$. Then $g_0^*, g_0^{**} \in K$, or $g_0^*, g_0^{**} \in L$, and $(g_0^*)^{-1} g_0^{**} \in H$.*

Using the above fact it is not difficult to verify that the canonical form is unique in the following sense.

Fact 2.4. *Suppose that $g_0^* g_1^* \cdots g_{n-1}^*$ and $g_0^{**} g_1^{**} \cdots g_{m-1}^{**}$ are canonical representations of the same element in $K *_H L$. Then $n = m$, and there exist $h_0, h_1, h_2, \ldots, h_n \in H$ with $h_0 = h_n = \mathbb{1}$ such that*

$$(\forall i < n)[g_i^{**} = h_i^{-1} g_i^* h_{i+1}].$$

Definition 2.5. Fix $g \in K *_H L$ distinct from $\mathbb{1}$, and the canonical representation $g = g_0^* g_1^* \cdots g_{n-1}^*$. We say that $g_0^* g_1^* \cdots g_{n-1}^*$ is *weakly cyclically reduced* if

- $n = 1$, or
- n is even, or
- $g_{n-1}^* g_0^* \notin H$, equivalently, g has no conjugate that has a canonical representation shorter than $n - 1$.

Recalling Fact 2.4, it is not difficult to see that the property of being weakly cyclically reduced is a property of the group element $g \in K *_H L$, so it does not depend on the particular choice of the canonical representation $g = g_0^* g_1^* \cdots g_{n-1}^*$.

Observation 2.6.

(1) If $g_0^* g_1^* \cdots g_{n-1}^*$ is a canonical representation of an element $g \neq \mathbb{1}$, $n \geq 2$, then g has a conjugate g' that has a canonical representation of length $m = 1$, or $m = 2k$ for some $k \geq 1$. Moreover, each conjugate g'' of g has length at least m.

(2) If $g_0^* g_1^* \cdots g_{n-1}^*$ is a canonical representation of an element $g \neq \mathbb{1}$, n is even, and g' is a weakly cyclically reduced conjugate of g, then g' has a canonical representation in the following form:

$$g' = x_i' g_{i+1}^* g_{i+2}^* \cdots g_{n-1}^* g_0^* \cdots g_{i-1}^* x_i'',$$

where:

- *if* $g_i^* \in K$*, then* $x_i', x_i'' \in K$ *and* $K \models x_i'' x_i' = g_i^*$*,*
- *if* $g_i^* \in L$*, then* $x_i', x_i'' \in L$ *and* $L \models x_i'' x_i' = g_i^*$*.*

In particular, the length of any canonical representation of g' *is either* n *or* $n+1$*.*

Definition 2.7 ([LS77, p. 286])**.** Let $H \leq K, L$ be groups such that $L \cap K = H$, and fix $R \subseteq K *_H L$. We say that R is *symmetrized* if for every $g \in R$:

(1) $g^{-1} \in R$, and
(2) for each g' that is conjugate to g and weakly cyclically reduced, $g' \in R$.

Definition 2.8 ([LS77, p. 286])**.** Let $X \subseteq K *_H L$, and $\chi \in (0,1)$. We say that X satisfies $C'(\chi)$, if whenever

(1) $g_{n-1}^* g_{n-2}^* \cdots g_1^* g_0^*, g_0^{**} g_1^{**} \cdots g_{m-1}^{**} \in X$,
(2) $g_{n-1}^* g_{n-2}^* \cdots g_1^* g_0^* g_0^{**} g_1^{**} \cdots g_{m-1}^{**} \neq \mathbb{1}$,
(3) $\ell < n, m$, and
(4) $g_{\ell-1}^* g_{\ell-2}^* \cdots g_0^* g_0^{**} g_1^{**} \cdots g_{\ell-1}^{**} \in H$,

then $\ell < \min(n,m) \cdot \chi$ and moreover, $\min(n,m) > \frac{1}{\chi}$.

Definition 2.9 ([LS77, p. 286])**.** Let H, K, L be as in Definition 2.1, and let $g \in K *_H L$. We say that the word $w_0 w_1 \cdots w_{m-1}$ is a *part* of g if

(1) $w_0 w_1 \cdots w_{m-1} \in K *_H L$ is in canonical form,
(2) for some weakly cyclically reduced conjugate g' of g, the word $w_0 w_1 \cdots w_{m-1}$ is a subword of a canonical representation of g' (i.e., *for some* canonical representation $v_0 v_1 \cdots v_{n-1}$ of g' and some $k \leq n-m$, we have $v_k = w_0$, $v_{k+1} = w_1, \ldots, v_{k+m-2} = w_{m-2}$ and $v_{k+m-1} = w_{m-1}$).

We cite the following lemma, which is our key technical tool borrowed from small cancellation theory.

Fact 2.10 ([LS77, Theorem 11.2])**.** *Let* $H \leq K, L$ *be groups,* $K \cap L = H$*,* $k \geq 6$*, and assume that* $R \subseteq K *_H L$ *is symmetrized and satisfies* $C'(\frac{1}{k})$*.*

Then, letting $N = \mathrm{ncl}(R, K *_H L)$ *be the normal subgroup generated by* R*, for every weakly cyclically reduced* $w \in N$ *that is nontrivial (i.e.* $w \neq \mathbb{1}$*), there exist* $r \in R$ *and a part* p *of* r*, which is also a part of* w*, and* $\ell(p) > \frac{k-3}{k}\ell(r)$*.*

Corollary 2.11. *If* H, K, L, R *are as in Fact 2.10, then for the canonical projection map* $\pi : K *_H L \to (K *_H L)/N$*, it is the case that* $\pi \restriction K$ *and* $\pi \restriction L$ *are injective, and* $\pi``K \cap \pi``L = \pi``H$ *(where* K, L *are identified with the subgroups of* $K *_H L$*).* $\square$

3. Finding the right amalgam

The main result of this section is Lemma 3.4 below. It originates from the lemma by G. Hesse appearing in the Appendix of [She80]. The lemma will serve as a building block in the recursive construction of Section 5.

Definition 3.1. Let $\varrho(x,y)$ denote the word $xyx^2yx^3y \cdots x^{80}y$.

Note that $\ell(\varrho(x,y)) = 3320$.

Definition 3.2. For all $j < \omega$ and x, y, we shall define a word $\varrho_j(x,y)$ over the alphabet $\{x,y\}$. First, define a sequence $\langle n_j \mid j < \omega \rangle$ of integers via $n_j = 3320^j$. Then, let $\varrho_j(x,y) = \varrho(x^{n_j}, y^{n_j})$, so that $\varrho_0 = \varrho$.

Definition 3.3. Let $G \leq H$ be a pair of groups.

- Define an equivalence relation $\sim_G$ over H via
$$a \sim_G b \text{ iff } a \in Gb^{\pm 1}G,$$
where $Gb^{\pm 1}G$ denotes the set $GbG \cup Gb^{-1}G$.
- We say that G is a *malnormal* subgroup of H, and denote it by $G \leq_{\mathrm{m}} H$, if for all $g \in G \setminus \{\mathbb{1}\}$ and $h \in H \setminus G$, it is the case that $h^{-1}gh \notin G$.

Note that $\leq_{\mathrm{m}}$ is a transitive relation.

Lemma 3.4. *Let $H \leq K$, $H \leq_{\mathrm{m}} L$ be groups, $K \cap L = H$ and suppose that we are given a system of quadruples*
$$S = \{(h_\sigma, a_\sigma, b_\sigma, b'_\sigma) \mid \sigma \in \Sigma\} \subseteq H \times (K \setminus H) \times (L \setminus H) \times (L \setminus H)$$
that satisfies the following two:

(1) for every $\sigma \in \Sigma$, $b_\sigma \not\sim_H b'_\sigma$;
(2) for all $\sigma \neq \sigma^*$ in Σ, at least one of the following holds:
 $(\ominus)_a$ *$a_\sigma \not\sim_H a_{\sigma^*}$ (in K);*
 $(\ominus)_b$ *$b_\sigma \not\sim_H b_{\sigma^*}$;*
 $(\ominus)_c$ *$b_\sigma = b_{\sigma^*}$ and $a_\sigma \neq a_{\sigma^*}$;*
 $(\ominus)_d$ *there are subgroups $H_\sigma \leq H$ and $K_\sigma \leq K$ such that all of the following hold:*
  (i) $K_\sigma \cap H = H_\sigma$;
  (ii) $a_\sigma, a_{\sigma^*} \in K_\sigma \setminus H = K_\sigma \setminus H_\sigma$;
  (iii) $b_\sigma \not\sim_{H_\sigma} b_{\sigma^*}$ (although typically $b_\sigma \sim_H b_{\sigma^*}$);
  (iv) $b_\sigma \not\sim_H b'_{\sigma^*}$;
  (v) $(K_\sigma \setminus H) \cdot (H \setminus K_\sigma) \cdot (K_\sigma \setminus H) \subseteq (K \setminus H)$.

***Then**, letting R be the symmetric closure of $\{h_\sigma^{-1}\varrho(b_\sigma a_\sigma, b'_\sigma a_\sigma) \mid \sigma \in \Sigma\}$, $M = K *_H L$, $N = \mathrm{ncl}(R, M)$ be the generated normal subgroup and $M^* = M/N$, all of the following hold:*

(A) R satisfies the condition $C'(\frac{1}{10})$, consequently, the group M^* embeds both K and L with
$$M^* \models K \cap L = H$$
and $K \cup L$ generates M^*. Moreover, the set R^+ defined to be the symmetric closure of
$$\{h_\sigma^{-1}\varrho(b_\sigma a_\sigma, b'_\sigma a_\sigma),\ \varrho_j(b_\sigma a_\sigma, b'_\sigma a_\sigma) \mid \sigma \in \Sigma,\ j \in \omega \setminus \{0\}\}$$
also satisfies $C'(\frac{1}{10})$;
(B) $K \leq_{\mathrm{m}} M^*$, and if $H \leq_{\mathrm{m}} K$, then $L \leq_{\mathrm{m}} M^*$;
(C) for all $b, b^* \in L \setminus H$ and $z \in K \setminus H$, if $b \sim_H b^*$, then $M^* \models b^*z \not\sim_K bzbz$;
(D) if $b, b' \in L \setminus H$, $a \in K \setminus H$, then $M^* \models bab' \notin K$, $ba \notin K$ (and similarly the parallel statement with interchanging K and L);
(E) if $a, a' \in K \setminus H$, $a \not\sim_{H'} a'$ for subgroups $H' \leq H$ and $L' \leq L$ such that $L' \cap K = L' \cap H = H'$, then $a \not\sim_{L'} a'$ holds too (in M^*);
(F) similarly, if $b, b' \in L \setminus H$, $b \not\sim_{H'} b'$ for subgroups $H' \leq H$ and $K' \leq K$ such that $K' \cap L = K' \cap H = H'$, then $b \not\sim_{K'} b'$ holds (in M^*);
(G) if K and L are torsion-free, then so is M^*.

Proof. First we note that for all $a \in K \setminus H$, $b, b' \in L \setminus H$, the word $\varrho(ba, b'a)$ is an alternating word (over the union of $K \setminus H$ and $L \setminus H$) of length 6640.

(A) By Corollary 2.11 (and $R \subseteq R^+$), it is enough to argue that R^+ satisfies $C'(\frac{1}{10})$. To this end, fix two elements $g \neq g^*$ in R^+, as well as some canonical representations

$$\begin{aligned} g &= g_0 g_1 \cdots g_{n-1}, \\ g^* &= g_0^* g_1^* \cdots g_{m-1}^*. \end{aligned}$$

By Clause (2) of Observation 2.6, there are $i, i^* \in \omega$ such that $n \in \{6640n_i, 6640n_i + 1\}$, $m \in \{6640n_{i^*}, 6640n_{i^*}+1\}$.

Let $l \in \omega$, and assume that

$$(\star) \qquad \bigwedge_{k \leq l} (K *_H L \models g_{k-1}^{-1} g_{k-2}^{-1} \cdots g_0^{-1} g_0^* g_1^* \cdots g_{k-1}^* \in H),$$

so we have to show that $l < 664 \cdot \min(n_i, n_{i^*})$.

Assume on the contrary that $l \geq 664 \cdot n_i$. We can choose $\sigma \in \Sigma$, $\varepsilon \in \{1, -1\}$, such that letting $r = h_\sigma^{-1} \varrho_{n_i}(b_\sigma a_\sigma, b'_\sigma a_\sigma)$, g is a weakly cyclically reduced conjugate of $r^\varepsilon = (h_\sigma^{-1} \varrho_{n_i}(b_\sigma a_\sigma, b'_\sigma a_\sigma))^\varepsilon$ if $n_i = 0$, or of $r^\varepsilon = (\varrho_{n_i}(b_\sigma a_\sigma, b'_\sigma a_\sigma))^\varepsilon$, and similarly for g^*, r^* and σ^*, ε^*. If we fix the canonical representations

$$r = u_0 u_1 \cdots u_{6640n_i - 1},$$

where $u_j \in \{b_\sigma, b'_\sigma, a_\sigma, h_\sigma^{-1} b_\sigma\}$, and similarly

$$r^* = u_0^* u_1^* \cdots u_{6640n_{i^*}-1}^*,$$

then again recalling Observation 2.6(2), we can assume that there exist $j < 6640n_i$, $j^* < 6640n_{i^*}$, such that whenever $0 < k < 6640n_i - 1$, then $g_k = u_{j+\varepsilon k}^\varepsilon$ and if $0 < k < 6640n_{i^*} - 1$, then $g_k^* = (u_{j^*+\varepsilon^* k}^*)^{\varepsilon^*}$.

We first observe that $i = i^*$, since otherwise if, say, $i < i^*$ did hold, then for some $1 \leq k, k' \leq 81n_i$ with $u_{j+\varepsilon k} = b_\sigma$, $u_{j+\varepsilon k'} = b'_\sigma$, while $u_{j+\varepsilon k}^* = u_{j+\varepsilon k'}^* \in \{b_{\sigma^*}, b'_{\sigma^*}\}$, and so by $(\star)$ we get

$$b_\sigma = u_{j+\varepsilon k} \sim_H u_{j+\varepsilon k}^* = u_{j+\varepsilon k'}^* \sim_H u_{j+\varepsilon k'} = b'_\sigma,$$

contradicting $b_\sigma \not\sim_H b'_\sigma$. From now on n will denote the common value of $n_i = n_{i^*}$.

Now note that $b_\sigma \sim_H b_{\sigma^*}$: there is a k with $1 \leq k \leq 10n$ such that $u_{j+\varepsilon k} \in \{b_\sigma, h_\sigma^{-1} b_\sigma\}$, and $u_{j^*+\varepsilon^* k}^* \in \{b_{\sigma^*}, h_{\sigma^*}^{-1} b_{\sigma^*}\}$, so by $(\star)$ for some $h \in H$ we have $b_\sigma^{-\varepsilon} h b_{\sigma^*}^{\varepsilon^*} \in H$, implying that $b_\sigma \sim_H b_{\sigma^*}$. Similarly, for some $k^\bullet$, $1 \leq k^\bullet \leq 2n$, $u_{j+\varepsilon k^\bullet} = a_\sigma$, and $u_{j^*+\varepsilon^* k^\bullet}^* = a_{\sigma^*}$, and by the same line of reasoning $a_\sigma \sim_H a_{\sigma^*}$.

We clearly get that

($\boxminus$) either $(\ominus)_c$, or $(\ominus)_d$, or $\sigma = \sigma^*$ holds, and in each case $b_\sigma \not\sim_H b'_{\sigma^*}$.

Now, note that if $j \neq j^*$ or $\varepsilon \neq \varepsilon^*$, then there exists k with $1 \leq k < 500n$ such that $u_{j+\varepsilon k} \in \{b_\sigma, h_\sigma^{-1} b_\sigma\}$, and $u_{j^*+\varepsilon^* k}^* = b'_{\sigma^*} = b'_\sigma$, and for some $h \in H$ we have $b_\sigma^{-\varepsilon} h (b'_{\sigma^*})^{\varepsilon^*} \in H$ (or $(h_\sigma^{-1} b_\sigma)^{-\varepsilon} h (b'_{\sigma^*})^{\varepsilon^*}$), so $b_\sigma \sim_H b'_{\sigma^*}$, contradicting ($\boxminus$). Therefore hereafter we can assume that $j = j^*$ and $\varepsilon = \varepsilon^*$.

We now divide our analysis into a few cases and subcases:

- ► If either $(\ominus)_c$ or $\sigma = \sigma^*$, then necessarily $b_\sigma = b_{\sigma^*}$ and $b'_\sigma = b'_{\sigma^*}$. But now for some k with $1 \leq k \leq 10n$, $g_k = g_k^* = b_\sigma^\varepsilon$, so for

$$h = g_{k-1}^{-1} g_{k-2}^{-1} \cdots g_0^{-1} g_0^* g_1^* \cdots g_{k-1}^* \in H$$

we have

$$g_k^{-1} h g_k^* \in H,$$

but then $H \leq_{\mathrm{m}} L$ together with $b_\sigma \in L \setminus H$ imply that $h = \mathbb{1}$.

►► If $\sigma = \sigma^*$, then invoking Observation 2.6(2) again (and recalling that g and g^* are cyclically reduced conjugates of $h_\sigma^{-1} \varrho(b_\sigma a_\sigma, b'_\sigma a_\sigma)$), it is straightforward to check that $j = j^*$ and $\varepsilon = \varepsilon^*$ imply $g = g^*$, which is a contradiction.

►► If $\sigma \neq \sigma^*$ and $a_\sigma \neq a_{\sigma^*}$, then $g_k^{-1} h g_k^* = \mathbb{1}$ implies that

$$g_{k+1}^{-1}(g_k^{-1} h g_k^*) g_{k+1}^* = a_\sigma^{-\varepsilon} a_{\sigma^*}^{\varepsilon} \neq \mathbb{1},$$

and in the following step (conjugating by $b_\sigma = b_{\sigma^*}$ again) we get a contradiction.

► If the pair σ, σ^* satisfies condition $(\ominus)_d$, then we argue as follows. First we claim that the there exists a k with $1 \leq k < 10n + 2$ such that following three hold:

$(\boxtimes)_1$ $g_k = u_{j+\varepsilon k}^{\varepsilon} = a_\sigma^{\varepsilon}$,

$(\boxtimes)_2$ $g_k^* = (u_{j+\varepsilon k}^*)^{\varepsilon} = a_{\sigma^*}^{\varepsilon}$,

$(\boxtimes)_3$ $h = g_{k-1}^{-1} g_{k-2}^{-1} \cdots g_0^{-1} g_0^* g_1^* \cdots g_{k-1}^* \in H \setminus K' = H \setminus H'$.

As before, for some $k^\bullet < 10n$ we have $u_{j+\varepsilon k^\bullet}^{\varepsilon} = a_\sigma^{\varepsilon}$, and $(u_{j+\varepsilon k^\bullet}^*)^{\varepsilon} = a_{\sigma^*}^{\varepsilon}$, $u_{j+\varepsilon(k^\bullet+1)}^{\varepsilon} = b_\sigma^{\varepsilon}$, and $u_{j+\varepsilon(k^\bullet+1)}^* = b_{\sigma^*}^{\varepsilon}$. Suppose that

$$h = g_{k^\bullet-1}^{-1} g_{k^\bullet-2}^{-1} \cdots g_0^{-1} g_0^* g_1^* \cdots g_{k^\bullet-1}^* \in H'.$$

Then $h' = a_\sigma^{-\varepsilon} h a_{\sigma^*}^{\varepsilon} \in K'H'K' = K'$, and by our indirect assumptions $a_\sigma^{-\varepsilon} h a_\sigma^{\varepsilon} \in H$, so h' lies in the intersection $K' \cap H = H'$. Now

$$u_{j+\varepsilon(k^\bullet+1)}^{-\varepsilon} h' u_{j+\varepsilon(k^\bullet+1)}^{\varepsilon} = b_\sigma^{-\varepsilon} h' b_{\sigma^*}^{\varepsilon} \in b_\sigma^{-\varepsilon} H' b_{\sigma^*}^{\varepsilon},$$

so by $(\ominus)_d$ (iii) this product is not in H', thus we can assume that some $k < 10n + 2$ satisfies $(\boxtimes)_1$–$(\boxtimes)_3$.

But then using $a_\sigma, a_{\sigma^*} \in K' \setminus H'$,

$$\begin{aligned} g_k^{-1} g_k^{-1} \cdots g_0^{-1} g_0^* g_1^* \cdots g_k^* &= a_\sigma^{-\varepsilon} h a_{\sigma^*}^{\varepsilon} \\ &\in (K' \setminus H) \cdot (H \setminus H') \cdot (K' \setminus H) \subseteq K \setminus H. \end{aligned}$$

This is a contradiction.

(B) Fix $g, g' \in K \setminus \{\mathbb{1}\} \subseteq M^*$, and $z \in M^* \setminus K$, with a canonical form $z = z_0 z_1 \cdots z_{m-1}$ satisfying it does not contain any subsequence $z_{\sigma_0} z_{\sigma_0+1} \cdots z_{\sigma_0+j-1}$ that is a subsequence of a canonical form of an element $r \in R$, where $j > \frac{6640}{2} + 1$ (we can assume this, since otherwise we could insert the entire sequence of the inverse of this fixed canonical form of r). Now suppose that $zgz^{-1}g' = \mathbb{1}$ holds in M^*, i.e.,

$$M \models zgz^{-1}g' \in N.$$

W.l.o.g. $z_0, z_{m-1} \in L$ (thus m is odd), since otherwise we can replace g with $z_{m-1} g z_{m-1}^{-1} \in K \setminus \{\mathbb{1}\}$, and g' with $z_0^{-1} g' z_0 \in K \setminus \{\mathbb{1}\}$. Now as $g, g' \in K$, if g, g' are not in H, then the product $z_0 z_1 \cdots z_{m-1} g z_{m-1}^{-1} \cdots z_0^{-1} g'$ is in a weakly cyclically reduced form, otherwise if $g \in H$, then $z_{m-1} g z_{m-1}^{-1} \in L \setminus H$ (here we use $H \leq_{\mathrm{m}} L$ and that $z_{m-1} \in L \setminus H$), and similarly, $g' \in H$ implies $z_0^{-1} g' z_0 \in L \setminus H$. So by these reductions we obtain a product in a cyclically reduced form of length $2m + 2$ or $2m$, or $2m - 2$. A cyclic conjugate of this word contains a long ($> 7/10$) subword of some canonical form of an $r \in R$. By our assumptions on z (not containing more than half of a canonical representation of r) this has to involve either g or g', in fact either the word $z_j z_{j+1} \cdots z_{m-1} g z_{m-1}^{-1} z_{m-2}^{-1} \cdots z_j^{-1}$ (or $z_j z_{j+1} \cdots z_{m-2}(z_{m-1} g z_{m-1}^{-1}) z_{m-2}^{-1} \cdots z_j^{-1}$ if $g \in H$), or $z_{j_*}^{-1} z_{j_*-1}^{-1} \cdots z_0^{-1} g' z_0 z_1 \cdots z_{j_*}$ contains a long ($> 2/10$ fraction) subword of a canonical form of some $r \in R$ (in the latter case if $g' \in H$, then of course we mean the word $z_{j_*}^{-1} z_{j_*-1}^{-1} \cdots (z_0^{-1} g' z_0) z_1 \cdots z_{j_*}$). But this is impossible since in any $r = r_0 r_1 \cdots r_{n-1} \in R$ ($n \in \{6640, 6641\}$) at any fixed $t \in [\frac{6640}{10}, \frac{6640 \cdot 9}{10}]$ there exists $k < 250$ such that (for some $\sigma \in \Sigma$) $r_{t-k} \in H b_\sigma^{\pm 1} H$, $r_{t+k} \in H (b'_\sigma)^{\pm 1} H$, and so $r_{t-k} \not\sim_H r_{t+k}$, while z_k, z_k^{-1} are clearly $\sim_H$-related.

(C) Suppose otherwise, e.g. for some $k, k' \in K$ either

$$y = (b^* z) k (z^{-1} b^{-1} z^{-1} b^{-1}) k' = \mathbb{1} \text{ in } M^*,$$

or

$$y = (b^* z) k (bzbz) k' = \mathbb{1}.$$

Observe that after performing the cancellations in the free amalgam M and writing $y = y_0 y_1 \cdots y_{m-1}$ as a reduced (alternating) word, in both cases (regardless of whether $k, k' \in H$) there is at most one j for which $y_j \in L \backslash H$ and $y_j \not\sim_H b$. Now possibly replacing $y_0 y_1 \cdots y_{m-1}$ with a weakly cyclically reduced conjugate of it (if the reduced form of $y_0 y_1 \cdots y_{m-1}$ is not weakly cyclically reduced) this clause remains true (and the resulting word similarly belongs to N in M). It is not difficult to see, that there exists at least one j' such that $y_{j'} \sim_H b$. Again, $y_0 y_1 \cdots y_{m-1}$ (or a cyclical permutation of it) contains a long subword of a canonical form of some $r \in R$, but any such subword (if longer than 400) contains at least two-two occurrences of b_σ and b'_σ (for some $\sigma \in \Sigma$), and b cannot be $\sim_H$-equivalent with both b_σ and b'_σ (since $b_\sigma \not\sim_H b'_\sigma$).

(D) This is the same as above. Assuming that $M^* \models bab' \in K$, then for some $a' \in K$, $M^* \models bab'a' = \mathbb{1}$, so

$$M \models bab'a' \in N.$$

Now if $a' \in K \setminus H$, then the word $bab'a'$ is weakly cyclically reduced, so any weakly cyclically reduced conjugate to it is of length either 4 or 5, and clearly cannot contain a long subword of any $r \in R$.

If $a' \in H$, then depending on whether $b'' = b'a'b \in H$, we have that either $b^{-1}(bab'a')b = ah \in K \setminus H$ is weakly cyclically reduced (so $M \models bab'a' = b(ah)b^{-1} \notin bNb^{-1} = N$), or $b^{-1}bab'a'b = ab'a'b = ab''$ (where $b'' \notin H$), which is weakly cyclically reduced, and similarly cannot lie in N.

(E) Let $a, a' \in K \setminus H$ be such that $a \not\sim_{H'} a'$, and fix $l, l' \in L'$. Suppose that $M^* \models ala'l' = \mathbb{1}$, that is,

$$M \models w = ala'l' \in N.$$

We can write w as a reduced word. If $l \in H$, then $l \in H'$, and since $a \not\sim_{H'} a'$ we have $ala' \in K \setminus H'$, so either $w = (ala')l'$ is a product of an element of $K \setminus H'$ and $L \setminus H$ (if $l' \notin H$) which has to lie in $K(L \setminus H)$ (which is disjoint to N), or $(ala')l' \in (K \setminus H') \cdot H' = K \setminus H'$, and we are done.

So w.l.o.g. $l \notin H$. (Similarly, $M^* \models a'l'al = \mathbb{1}$ implies that w.l.o.g. $l' \notin H$). So any weakly cyclically reduced conjugate of $w \in M$ has length at most 5, and contains at least 2 entries from $K \setminus H$. But $w \in N$ implies that some weakly cyclically reduced conjugate contains a long subword of some $r \in R$, which is clearly impossible.

(*F*) The proof of (*E*) works here too.

(*G*) Let $g \in M^*$, $n \in \omega$, $n > 1$ be such that $g \neq \mathbb{1}$, $M^* \models g^n = \mathbb{1}$. Recalling Observation 2.6, we can write g as an alternating product of elements of $K \setminus H$ and $L \setminus H$

$$g = g_0 g_1 \cdots g_{2m-1}.$$

W.l.o.g. there exists no conjugate ygy^{-1} of g, and g' with $g'(ygy^{-1})^{-1} \in N$ such that g' has a shorter canonical representation than $2m$, since we can replace g with g' and get a torsion element. Therefore there is no $r \in R$, $\sigma_0 < 2m$ with the sequence $g_{\sigma_0} g_{\sigma_0+1} \cdots g_{2m-1} g_0 g_1 \cdots g_{\sigma_0 - 1}$ containing a subsequence of a canonical representation of r of length $j > \frac{6640}{2} + 1$.

Now, since

$$M \models (g_0 g_1 \cdots g_{2m-1})^n \in N,$$

there exists a cyclic conjugate of $(g_0 g_1 \cdots g_{2m-1})^n$ and a subsequence $s_0 s_1 \cdots s_j$ of it that is also a subsequence of a canonical form of some $s \in R$ with $j \geq \frac{7}{10} \cdot 6640$. Our assumptions above on $g_0 g_1 \cdots g_{2m-1}$ easily imply

$$2m \leq \frac{6640}{2} + 1,$$

thus,

$$2m + \frac{2}{10} \cdot 6640 - 1 \leq j,$$

in particular, $2m + 330 \leq j$. This way we get that $s_\ell \sim_H s_{\ell+2m}$ for each $\ell \leq 330$, but as s is a cyclically reduced conjugate of $h_\sigma^{-1} \varrho(b_\sigma a_\sigma, b'_\sigma a_\sigma)$ or of its inverse (for some $\sigma \in \Sigma$), we get that for some $\ell \in [1, 330]$ $s_\ell \in H(b_\sigma)^{\pm 1} H$, $s_{\ell+2m} \in H(b'_\sigma)^{\pm 1} H$, thus $s_\ell \not\sim_H s_{\ell+2m}$. This is a contradiction. $\square_{\text{Lemma 3.4}}$

4. A set-theoretic interlude

In this section, $\chi, \theta, \mu, \lambda$ and κ all denote nonzero cardinals. Recall that $[\kappa]^2$ stands for the collection of all unordered pairs $\{\alpha, \beta\}$ of ordinals in κ, but here we identify it with the collection of all ordered pairs (α, β) with $\alpha < \beta$.

Definition 4.1. A map $d : [\kappa]^2 \to \theta$ is *subadditive* if the following inequalities hold for all $\alpha < \beta < \gamma < \kappa$:

(1) $d(\alpha, \gamma) \leq \max\{d(\alpha, \beta), d(\beta, \gamma)\}$;
(2) $d(\alpha, \beta) \leq \max\{d(\alpha, \gamma), d(\beta, \gamma)\}$.

Notation 4.2. Whenever the map $d : [\kappa]^2 \to \theta$ is clear from the context, we define for all $\gamma < \kappa$ and $i \leq \theta$, the following sets:

- $D^\gamma_{<i} = \{\beta < \gamma \mid d(\beta, \gamma) < i\}$, and
- $D^\gamma_{\leq i} = \{\beta < \gamma \mid d(\beta, \gamma) \leq i\}$.

Lemma 4.3. *If $d : [\kappa]^2 \to \theta$ is subadditive, then for all $\gamma < \kappa$, $i \leq \theta$, and $\beta \in D^\gamma_{<i}$, it is the case that $D^\gamma_{<i} \cap \beta = D^\beta_{<i}$.*

Proof. Suppose that $d : [\kappa]^2 \to \theta$ is subadditive, and let γ, i and β be as above.

▶ By Definition 4.1(1), for every $\alpha \in D^{\beta}_{<i}$, $d(\alpha, \gamma) \leq \max\{d(\alpha, \beta), d(\beta, \gamma)\}$, so, since $\alpha \in D^{\beta}_{<i}$ and $\beta \in D^{\gamma}_{<i}$, we infer that $d(\alpha, \gamma) < i$ and $\alpha \in D^{\gamma}_{<i} \cap \beta$.

▶ By Definition 4.1(2), for every $\alpha \in D^{\gamma}_{<i} \cap \beta$, $d(\alpha, \beta) \leq \max\{d(\alpha, \gamma), d(\beta, \gamma)\}$, so, since $\alpha, \beta \in D^{\gamma}_{<i}$, we infer that $d(\alpha, \beta) < i$ and $\alpha \in D^{\beta}_{<i}$. □

Theorem 4.4. *Suppose that λ is an infinite regular cardinal. Then there exist two maps $c : [\lambda^+]^2 \to \lambda^+$ and $d : [\lambda^+]^2 \to \lambda$ such that:*

- *d is subadditive;*
- *for every $A \in [\lambda^+]^{\lambda^+}$, there exists a club $D \subseteq \lambda^+$ such that for every $\delta \in D$, for every $\beta \in \lambda^+ \setminus \delta$, for every $\xi < \delta$, for every $i < \lambda$, there are cofinally many $\alpha < \delta$ such that $\alpha \in A$, $c(\alpha, \beta) = \xi$ and $d(\alpha, \beta) > i$.*

Proof. Let d be the function $\rho : [\lambda^+]^2 \to \lambda$ defined in [Tod07, §9.1]. By [Tod07, Lemma 9.1.1], d is subadditive. By [Tod07, Lemma 9.1.2], d is also *locally small*, i.e., $|D^{\gamma}_{\leq i}| < \lambda$ for all $\gamma < \lambda^+$ and $i < \lambda$.

Next, by [RT13], we may fix a coloring $c : [\lambda^+]^2 \to \lambda^+$ witnessing $\lambda^+ \nrightarrow [\lambda^+; \lambda^+]^2_{\lambda^+}$. By [IR24, Lemma 3.16], this means that for every $A \in [\lambda^+]^{\lambda^+}$, there exists an $\epsilon < \lambda^+$ such that, for all $\beta \in \lambda^+ \setminus \epsilon$ and $\xi < \epsilon$, there exists $\alpha \in A \cap \epsilon$ such that $c(\alpha, \beta) = \xi$.

We now verify that c and d are as sought.

Claim 4.4.1. *Let $A \in [\lambda^+]^{\lambda^+}$. Then there exists a club $D \subseteq \lambda^+$ such that for every $\delta \in D$, for every $\beta \in \lambda^+ \setminus \delta$, for every $\xi < \delta$, for every $i < \lambda$, there are cofinally many $\alpha < \delta$ such that $\alpha \in A$, $c(\alpha, \beta) = \xi$ and $d(\alpha, \beta) > i$.*

Proof. Let $\langle M_\gamma \mid \gamma < \lambda^+ \rangle$ be a sequence of elementary submodels of $\mathcal{H}_{\lambda^{++}}$, each of size λ, such that $\{A, e\} \in M_0$, such that $M_\gamma \in M_{\gamma+1}$ for every $\gamma < \lambda^+$, and such that $M_\delta = \bigcup_{\gamma<\delta} M_\gamma$ for every limit nonzero $\delta < \lambda^+$. It follows that $C = \{\gamma < \lambda^+ \mid M_\gamma \cap \lambda^+ = \gamma\}$ is a club in λ^+.

We claim that the following club is as sought:

$$D = \{\delta < \lambda^+ \mid \operatorname{otp}(C \cap \delta) = \lambda^\delta\}.$$

To this end, let $\delta \in D$, $\beta \in \lambda^+ \setminus \delta$, $\xi < \delta$, $i < \lambda$, and $\eta < \delta$. We shall find an $\alpha \in A \cap \delta$ above η such that $c(\alpha, \beta) = \xi$ and $d(\alpha, \beta) > i$.

For every $\gamma \in C \setminus \xi$, the set $A_\gamma = A \setminus \gamma$ is in $[\lambda^+]^{\lambda^+} \cap M_{\gamma+1}$, and hence there exists $\epsilon \in \lambda^+ \cap M_{\gamma+1}$ such that, for all $\beta' \in \lambda^+ \setminus \epsilon$ and $\xi' < \epsilon$, there exists $\alpha' \in A_\gamma \cap \epsilon$ such that $c(\alpha', \beta') = \xi'$. In particular, we may pick $\alpha_\gamma \in A \cap M_{\gamma+1} \setminus \gamma$ such that $c(\alpha_\gamma, \beta) = \xi$. It follows that $\gamma \mapsto \alpha_\gamma$ is a strictly increasing function from $C \cap \delta$ to $A \cap \delta$. As $\delta \in D$, we infer that $A' = \{\alpha \in A \cap \delta \mid \eta < \alpha \ \&\ c(\alpha, \beta) = \xi\}$ has size λ. As d is locally small, we may now pick $\alpha \in A' \setminus D^{\beta}_{\leq i}$. Then $\alpha \in A \cap \delta$ above η, $d(\alpha, \beta) > i$ and $c(\alpha, \beta) = \xi$, as sought. □

This completes the proof. □

Remark 4.5. The preceding result does not generalize to the case when λ is a singular cardinal. Indeed, it follows from [LR23, Lemma 3.38] that if λ is the singular limit of strongly compact cardinals, then for every infinite cardinal $\theta \leq \lambda$, for every subadditive map $d : [\lambda^+]^2 \to \theta$, there must exist an $A \in [\lambda^+]^{\lambda^+}$ such that $\sup\{d(\alpha, \beta) \mid \alpha < \beta \text{ in } A\} < \theta$.

Definition 4.6 ([She88])**.** $\mathrm{Pr}_1(\kappa,\kappa,\theta,\chi)$ asserts the existence of a coloring $c : [\kappa]^2 \to \theta$ such that for every $\sigma < \chi$, every pairwise disjoint subfamily $\mathcal{A} \subseteq [\kappa]^\sigma$ of size κ, and every $\tau < \theta$, there are $a, b \in \mathcal{A}$ with $\sup(a) < \min(b)$ such that $c[a \times b] = \{\tau\}$.

Definition 4.7 ([LR18])**.** $\mathrm{U}(\kappa,\mu,\theta,\chi)$ asserts the existence of a coloring $d : [\kappa]^2 \to \theta$ such that for every $\sigma < \chi$, every pairwise disjoint subfamily $\mathcal{A} \subseteq [\kappa]^\sigma$ of size κ, and every $\tau < \theta$, there exists $\mathcal{B} \in [\mathcal{A}]^\mu$ such that, for all $a, b \in \mathcal{B}$ with $\sup(a) < \min(b)$, it is the case that $\min(d[a \times b]) \geq \tau$.

Theorem 4.8. *Suppose that:*

- *$\theta < \kappa$ are infinite regular cardinals;*
- *$c : [\kappa]^2 \to \kappa$ is a coloring witnessing $\mathrm{Pr}_1(\kappa,\kappa,\kappa,4)$;*
- *$d : [\kappa]^2 \to \theta$ is a subadditive coloring witnessing $\mathrm{U}(\kappa,2,\theta,2)$.*

Then, for every $A \in [\kappa]^\kappa$, there exists a club $D \subseteq \kappa$ such that for every $\delta \in D$, for every $\beta \in \kappa \setminus \delta$, for every $\xi < \delta$, for every $i < \theta$, there are cofinally many $\alpha < \delta$ such that $\alpha \in A$, $c(\alpha,\beta) = \xi$ and $d(\alpha,\beta) > i$.

Proof. We start by verifying a special case.

Claim 4.8.1. *Let $A \in [\kappa]^\kappa$, $\xi < \kappa$ and $i < \theta$. There exists $\gamma < \kappa$ such that for every $\beta \in \kappa \setminus \gamma$, there exists $\alpha \in A \cap \gamma$ such that $c(\alpha,\beta) = \xi$ and $d(\alpha,\beta) > i$.*

Proof. For every $\epsilon < \kappa$, $A \setminus \epsilon$ is in $[\kappa]^\kappa$, and as $d : [\kappa]^2 \to \theta$ witnesses $\mathrm{U}(\kappa,2,\theta,2)$, it is the case that $d``[A \setminus \epsilon]^2$ is cofinal in θ. It thus follows that we may fix a κ-sized pairwise disjoint subfamily $\mathcal{A}$ of $[A]^2$ such that $d(a) > i$ for all $a \in \mathcal{A}$. Note that for all $\beta < \kappa$ and $a \in \mathcal{A} \cap \mathcal{P}(\beta)$, there must exist some $\alpha \in a$ such that $d(\alpha,\beta) > i$, because, by subadditivity,

$$i < d(a) \leq \max\{d(\min(a),\beta), d(\max(a),\beta)\}.$$

Therefore, it now suffices to prove that there exists some $\gamma < \kappa$ such that for every $\beta \in \kappa \setminus \gamma$, there exists $a \in \mathcal{A} \cap \mathcal{P}(\gamma)$ such that $c[a \times \{\beta\}] = \{\xi\}$. Towards a contradiction, suppose that this is not the case. For every $\gamma < \kappa$, fix $\beta_\gamma \in \kappa \setminus \gamma$ such that there exists no $a \in \mathcal{A} \cap \mathcal{P}(\gamma)$ with $c[a \times \{\beta_\gamma\}] = \{\xi\}$. For each $\gamma < \kappa$, set $a_\gamma = \{\beta_\gamma\} \cup a$ for some $a \in \mathcal{A}$ such that $\min(a) > \beta_\gamma$. Fix a club $C \subseteq \kappa$ such that for every $\gamma \in C$, for every $\bar{\gamma} < \gamma$, $\max(a_{\bar{\gamma}}) < \gamma$. It follows that $\mathcal{A}' = \{a_\gamma \mid \gamma \in C\}$ is a collection of κ-many pairwise disjoint elements of $[\kappa]^3$. So, since c witnesses $\mathrm{Pr}_1(\kappa,\kappa,\kappa,4)$, we may find $a, b \in \mathcal{A}'$ with $\max(a) < \min(b)$ such that $c[a \times b] = \{\xi\}$. Pick $\bar{\gamma}, \gamma$ in C such that $a = a_{\bar{\gamma}}$ and $b = a_\gamma$. From $\max(a_{\bar{\gamma}}) < \min(a_\gamma)$, it follows that $\bar{\gamma} < \gamma$, and

$$\max(a_{\bar{\gamma}}) < \gamma \leq \beta_\gamma = \min(a_\gamma).$$

In particular, $a' = a_{\bar{\gamma}} \setminus \{\beta_\gamma\}$ is an element of $\mathcal{A} \cap \mathcal{P}(\gamma)$ and $c[a' \times \{\beta_\gamma\}] = \{\xi\}$. This is a contradiction. □

Now, given $A \in [\kappa]^\kappa$, let $\langle M_\gamma \mid \gamma < \kappa\rangle$ be a sequence of elementary submodels of $\mathcal{H}_{\kappa^+}$, each of size less than κ, such that $\{A, c, d\} \cup \theta \subseteq M_0$, such that $M_\gamma \in M_{\gamma+1}$ for every $\gamma < \kappa$, and such that $M_\delta = \bigcup_{\gamma<\delta} M_\gamma$ for every limit nonzero $\delta < \kappa$. We claim that the following club is as sought:

$$D = \{\delta < \kappa \mid M_\delta \cap \kappa = \delta\}.$$

To see it, let $\beta \in \kappa \setminus \delta$, $\xi < \delta$, $i < \theta$, and $\epsilon < \delta$; we must find $\alpha \in A$ with $\epsilon \leq \alpha < \delta$ such that $c(\alpha,\beta) = \xi$ and $d(\alpha,\beta) > i$. The set $A' = A \setminus \epsilon$ is in $[\kappa]^\kappa \cap M_\delta$, and so are ξ and i. It thus follows from Claim 4.8.1 that there exists $\gamma \in \kappa \cap M_\delta$ such that for every $\beta' \in \kappa \setminus \gamma$, there exists $\alpha \in A' \cap \gamma$ such that $c(\alpha, \beta') = \xi$ and $d(\alpha, \beta') > i$. As $\gamma < \delta \leq \beta$, it follows that there exists $\alpha \in A' \cap \gamma$ such that $c(\alpha, \beta) = \xi$ and $d(\alpha, \beta) > i$. Evidently, $\epsilon \leq \alpha < \delta$. $\square$

In reading the next definition, recall that for a set X of ordinals, $\mathrm{acc}(X)$ stands for the set of all nonzero $\xi \in X$ such that $\sup(X \cap \xi) = \xi$.

Definition 4.9 ([BR19])**.** For infinite regular cardinals $\theta < \kappa$, the principle $\square(\kappa, \sqsubseteq_\theta)$ asserts the existence of a sequence $\vec{C} = \langle C_\alpha \mid \alpha < \kappa \rangle$ satisfying the following:

- for every $\alpha < \kappa$, C_α is a closed subset of α with $\sup(C_\alpha) = \sup(\alpha)$;
- for all $\alpha < \kappa$ and $\bar{\alpha} \in \mathrm{acc}(C_\alpha)$, if $\mathrm{otp}(C_\alpha) \geq \theta$, then $C_{\bar{\alpha}} = C_\alpha \cap \bar{\alpha}$;
- for every club D in κ, there exists some $\alpha \in \mathrm{acc}(D)$ such that $D \cap \alpha \neq C_\alpha$.

Note that $\square(\kappa, \sqsubseteq_\vartheta)$ implies $\square(\kappa, \sqsubseteq_\theta)$ whenever $\vartheta < \theta$. The strongest instance $\square(\kappa, \sqsubseteq_\omega)$ is commonly denoted by $\square(\kappa)$.

Corollary 4.10. *Suppose that $\theta < \kappa$ are infinite regular cardinals.*

If either $\square(\kappa, \sqsubseteq_\theta)$ holds or if there exists a uniformly coherent κ-Souslin tree, then there exist two maps $c : [\kappa]^2 \to \kappa$ and $d : [\kappa]^2 \to \theta$ such that:

- *d is subadditive;*
- *for every $A \in [\kappa]^\kappa$, there exists a club $D \subseteq \kappa$ such that for every $\delta \in D$, for every $\beta \in \kappa \setminus \delta$, for every $\xi < \delta$, for every $i < \theta$, there are cofinally many $\alpha < \delta$ such that $\alpha \in A$, $c(\alpha, \beta) = \xi$ and $d(\alpha, \beta) > i$.*

Proof. By Theorem 4.4, we may assume that $\theta^+ < \kappa$. Also, by Theorem 4.8, it suffices to find a map $c : [\kappa]^2 \to \kappa$ witnessing $\mathrm{Pr}_1(\kappa, \kappa, \kappa, 4)$, and a subadditive map $d : [\kappa]^2 \to \theta$ witnessing $\mathrm{U}(\kappa, 2, \theta, 2)$.

By [Rin14, Theorem B], $\square(\kappa)$ implies $\mathrm{Pr}_1(\kappa, \kappa, \kappa, \theta)$. Inspecting the proof of [Rin14, Theorem 3.3] makes it clear that the same conclusion already follows from $\square(\kappa, \sqsubseteq_\theta)$. In addition, by [LR23, Theorem A], $\square(\kappa, \sqsubseteq_\theta)$ yields a subadditive witness to $\mathrm{U}(\kappa, 2, \theta, 2)$.

Next, by [LR23, Corollary 3.29], the existence of a uniformly coherent κ-Souslin tree yields a subadditive witness to $\mathrm{U}(\kappa, 2, \theta, 2)$. It is also well known that the existence of a uniformly coherent κ-Souslin tree induces a witness to $\mathrm{Pr}_1(\kappa, \kappa, \kappa, \omega)$. $\square$

Remark 4.11. Coming back to the limitation highlighted in Remark 4.5, we point out that the conclusion of Corollary 4.10 is nevertheless compatible with a *bounded* amount of large cardinals. The point is that $\square(\kappa, \sqsubseteq_\theta)$ may be added by means of a θ-directed-closed and κ-strategically-closed forcing, so by Laver's indestructibility theorem, $\square(\kappa, \sqsubseteq_\theta)$ is compatible with θ being supercompact. Parallelly, the existence of a uniformly coherent κ-Souslin tree is compatible with κ possessing a generically-large cardinal property that refutes $\square(\kappa, \sqsubseteq_\theta)$ for all $\theta < \kappa$ (see [LR21, Theorem 3.3]).

5. A construction of a Shelah group

This section is devoted to proving the core result of this paper. The assumptions of the upcoming theorem are motivated by the results of the previous section.

Theorem 5.1. *Suppose:*

- *$\theta < \kappa$ is a pair of infinite regular cardinals;*
- *$c : [\kappa]^2 \to \kappa$ is a coloring;*
- *$d : [\kappa]^2 \to \theta$ is a subadditive coloring;*
- *for every $A \in [\kappa]^\kappa$, there exists a club $B \subseteq \kappa$ such that for every $\beta \in B$ there exists $\gamma \in A$ above β such that for all $\xi < \beta$ and $i < \theta$, there are cofinally many $\alpha < \beta$ such that $\alpha \in A$, $c(\alpha, \gamma) = \xi$ and $d(\alpha, \gamma) > i$.*

Then there exists a torsion-free Shelah group G of size κ.

Before embarking on the proof, we make a few promises and unfold some of their consequences.

5.1. **Promises and their consequences.** We start by listing our promises:

$(p)_1$ We shall recursively construct distinguished group elements $\langle x_\alpha \mid \alpha < \kappa\rangle$ generating the whole group G. For every subset $A \subseteq \kappa$, G_A will denote the group generated by $\{x_\alpha \mid \alpha \in A\}$, so that $G_\emptyset = \{\mathbb{1}\}$ and $G_\kappa = G$;

$(p)_2$ For every $\gamma \leq \kappa$, the underlying set of G_γ will be an initial segment of κ;

$(p)_3$ For all $\gamma < \kappa$ and $i < \theta$, $G_{D^\gamma_{<i} \cup \{\gamma\}}$ is torsion-free;[4]

$(p)_4$ For all $\gamma < \kappa$ and $i < \theta$, $G_{D^\gamma_{<i} \cup \{\gamma\}} \cap G_{D^\gamma_{\leq i}} = G_{D^\gamma_{<i}}$;

$(p)_5$ For all $\gamma < \kappa$ and $i < \theta$, $G_{D^\gamma_{<i}} \leq_{\mathrm{m}} G_{D^\gamma_{<i} \cup \{\gamma\}}$;

$(p)_6$ For all $\gamma \in [1, \kappa)$ and $i \in [1, \theta)$, $G_{D^\gamma_{\leq i} \cup \{\gamma\}}$ is the group M^* given by Lemma 3.4 when invoked with the groups
- $H = G_{D^\gamma_{<i}}$,
- $K = G_{D^\gamma_{\leq i}}$,
- $L = G_{D^\gamma_{<i} \cup \{\gamma\}}$,

and an appropriate (possibly empty) system S.

At the outset, we also agree on the following pieces of notation.

Notation 5.2. For every subset $A \subseteq \kappa$, we shall denote by $\equiv_A$ the relation $\sim_{G_A}$ of Definition 3.3. That is, $g \equiv_A h$ iff there are $y_0, y_1 \in G_A$ and $\varepsilon \in \{1, -1\}$ such that $g = y_0 \cdot h^\varepsilon \cdot y_1$.

Notation 5.3. For all $\gamma < \kappa$ and $g \in G_\gamma$, let

$$i^\gamma_g = \min\{i < \theta \mid g \in G_{D^\gamma_{\leq i}}\}.$$

We shall also record the first appearance of an element $g \in G_\kappa \setminus \{\mathbb{1}\}$ by letting

$$\alpha_g = \min\{\alpha < \kappa \mid g \in G_{\alpha+1}\}.$$

Since $g \in G_{\alpha_g \cup \{\alpha_g\}}$ and $\alpha_g = \bigcup_{i<\theta} D^{\alpha_g}_{\leq i}$, it also makes sense to define

$$i_g = \min\{i < \theta \mid g \in G_{D^{\alpha_g}_{\leq i} \cup \{\alpha_g\}}\}.$$

As for $g = \mathbb{1}$, since $G_0 = G_\emptyset = \{\mathbb{1}\}$, we agree to let $\alpha_{\mathbb{1}} = -1$ and $i_{\mathbb{1}} = 0$.

Remark 5.4. By possibly replacing $d : [\kappa]^2 \to \theta$ with the map $(\alpha, \beta) \mapsto 1 + d(\alpha, \beta)$, we may assume that $0 \notin \mathrm{Im}(d)$. This tacit assumption will ensure that for every $g \in G$, if $i_g = 0$, then either $g = \mathbb{1}$ or g is an element of the cyclic group $\langle x_{\alpha_g}\rangle$.

Notation 5.3 induces a well-ordering $\prec$ of G, as follows.

Definition 5.5. For $g \neq h$ in G, we shall let $g \prec h$ if one of the following holds:

[4] Recall Notation 4.2.

- $\alpha_g < \alpha_h$;
- $\alpha_g = \alpha_h$ and $i_g < i_h$;
- $\alpha_g = \alpha_h$ and $i_g = i_h$ and $g \in h$.[5]

Note that $\min(G, \prec) = \mathbb{1}$.

Lemma 5.6. *For all $\gamma < \kappa$ and $i \leq \theta$, $G_{D^\gamma_{<i} \cup \{\gamma\}} \cap G_\gamma = G_{D^\gamma_{<i}}$.*

Proof. Let $\gamma < \kappa$ and $i \leq \theta$. As $G_\gamma = \bigcup_{j<\theta} G_{D^\gamma_{<j}}$, it suffices to prove that for every $j \in (i, \theta]$,

$$G_{D^\gamma_{<i} \cup \{\gamma\}} \cap G_{D^\gamma_{<j}} = G_{D^\gamma_{<i}}. \tag{I}$$

The case $j = i + 1$ is immediate from $(p)_4$, and the case in which j is a limit ordinal follows from the fact that $G_{D^\gamma_{<j}} = \bigcup_{l<j} G_{D^\gamma_{<l}}$ for j limit. So, suppose that $j \in [i + 1, \theta)$ is such that (I) holds. By $(p)_4$, $G_{D^\gamma_{<j} \cup \{\gamma\}} \cap G_{D^\gamma_{\leq j}} = G_{D^\gamma_{<j}}$ holds, as well. Since $G_{D^\gamma_{<i} \cup \{\gamma\}} \subseteq G_{D^\gamma_{<j} \cup \{\gamma\}}$, altogether,

$$\begin{aligned} G_{D^\gamma_{<i} \cup \{\gamma\}} \cap G_{D^\gamma_{\leq j}} &= G_{D^\gamma_{<i} \cup \{\gamma\}} \cap G_{D^\gamma_{<j} \cup \{\gamma\}} \cap G_{D^\gamma_{\leq j}} \\ &= G_{D^\gamma_{<i} \cup \{\gamma\}} \cap G_{D^\gamma_{<j}} \\ &= G_{D^\gamma_{<i}}, \end{aligned}$$

as sought. □

By the preceding, and since $D^\gamma_{<0} = \emptyset$, the group $\langle x_\gamma \rangle$ generated by x_γ will have a trivial intersection with G_γ. Another consequence of the preceding is as follows.

Corollary 5.7. *For every $\gamma < \kappa$, $G_\gamma \leq_{\mathrm{m}} G_{\gamma+1}$.*

Proof. Let $g \in G_\gamma \setminus \{\mathbb{1}\}$ and $h \in G_{\gamma+1} \setminus G_\gamma$ for a given $\gamma < \kappa$. Find a large enough $i < \theta$ such that $g \in G_{D^\gamma_{<i}}$ and $h \in G_{D^\gamma_{<i} \cup \{\gamma\}} \setminus G_{D^\gamma_{<i}}$. Then, by $(p)_5$,

$$h^{-1}gh \in G_{D^\gamma_{<i} \cup \{\gamma\}} \setminus G_{D^\gamma_{<i}}.$$

Finally, Lemma 5.6 yields that $h^{-1}gh \notin G_\gamma$. □

The next consequence of our promises is the upcoming Lemma 5.9. In order to state it, we agree to say that a set $A \subseteq \kappa$ is *absorbent* if for every $\gamma \in A$, there exists some $i \leq \theta$ such that $A \cap \gamma = D^\gamma_{<i}$. To exemplify:

Proposition 5.8. *For all $\gamma < \kappa$ and $i \leq \theta$, $D^\gamma_{<i}$ is absorbent.*

Proof. By Lemma 4.3. □

Lemma 5.9. *Suppose that A, A' are absorbent subsets of κ.*

1. *For every $g \in G_A \setminus \{\mathbb{1}\}$, $D^{\alpha_g}_{\leq i_g} \cup \{\alpha_g\} \subseteq A$;*
2. *For all $\gamma < \kappa$, $i < \theta$, and $g \in G_{D^\gamma_{<i} \cup \{\gamma\}}$, we have $i_g < i$;*
3. *For all $\gamma < \kappa$ and $g \in G_\gamma \setminus \{\mathbb{1}\}$, we have $i^\gamma_g = \max\{d(\alpha_g, \gamma), i_g\}$;*
4. *$G_A \cap G_{A'} = G_{A \cap A'}$.*

[5]Recall $(p)_2$.

Proof. (1) Let $g \in G_A \setminus \{\mathbb{1}\}$. Denote by $\gamma \in A$ the minimal ordinal such that $g \in G_{A\cap(\gamma+1)}$. In particular, $g \notin G_{A\cap\gamma}$ and $\alpha_g \leq \gamma$. As A is absorbent, we may now fix $i \leq \theta$ such that $A \cap \gamma = D^\gamma_{<i}$. Consequently, $g \in G_{D^\gamma_{<i}\cup\{\gamma\}}$. If $\alpha_g < \gamma$, then $g \in G_{D^\gamma_{<i}\cup\{\gamma\}} \cap G_\gamma$, which, by Lemma 5.6 is equal to $G_{D^\gamma_{<i}} = G_{A\cap\gamma}$, contradicting the fact that $g \notin G_{A\cap\gamma}$. So, $\alpha_g = \gamma$, and hence $g \in G_{D^{\alpha_g}_{<i}\cup\{\alpha_g\}}$. As $G_{D^{\alpha_g}_{<i}\cup\{\alpha_g\}} = \bigcup_{j<i} G_{D^\gamma_{\leq j}\cup\{\gamma\}}$, the definition of i_g implies that $i_g < i$. Altogether, $D^{\alpha_g}_{\leq i_g} \cup \{\alpha_g\} \subseteq D^\gamma_{<i} \cup \{\gamma\} \subseteq A$.

(2) Let $\gamma < \kappa$, $i < \theta$, and $g \in G_{D^\gamma_{<i}\cup\{\gamma\}}$. By Clause (1), $D^{\alpha_g}_{\leq i_g}\cup\{\alpha_g\} \subseteq D^\gamma_{<i}\cup\{\gamma\}$. If $\alpha_g = \gamma$, then the inclusion implies that $i_g < i$. Otherwise, $\alpha_g \in D^\gamma_{<i}$, and then Lemma 4.3 implies that

$$D^{\alpha_g}_{\leq i_g} = (D^{\alpha_g}_{\leq i_g} \cup \{\alpha_g\}) \cap \alpha_g \subseteq D^\gamma_{<i} \cap \alpha_g = D^{\alpha_g}_{<i},$$

so, again $i_g < i$.

(3) Let $\gamma < \kappa$ and $g \in G_\gamma \setminus \{\mathbb{1}\}$. Clearly, $\alpha_g < \gamma$. Also, recalling Notation 5.3, $g \in G_{D^\gamma_{\leq i^\gamma_g}}$. So, Clause (1) together with Proposition 5.8 imply that $D^{\alpha_g}_{\leq i_g} \cup \{\alpha_g\} \subseteq D^\gamma_{\leq i^\gamma_g}$. In particular, $d(\alpha_g,\gamma) \leq i^\gamma_g$, and, by Lemma 4.3, also

$$D^{\alpha_g}_{\leq i_g} = (D^{\alpha_g}_{\leq i_g} \cup \{\alpha_g\}) \cap \alpha_g \subseteq D^\gamma_{\leq i^\gamma_g} \cap \alpha_g = D^{\alpha_g}_{\leq i^\gamma_g},$$

and hence $i_g \leq i^\gamma_g$. This shows that $i = \max\{d(\alpha_g,\gamma), i_g\}$ is $\leq i^\gamma_g$. On the other hand, since $\alpha_g \in D^\gamma_{\leq d(\alpha_g,\gamma)} \subseteq D^\gamma_{\leq i}$, Lemma 4.3 implies that

$$D^{\alpha_g}_{\leq i_g} \subseteq D^{\alpha_g}_{\leq i} = D^\gamma_{\leq i} \cap \alpha_g,$$

and hence $g \in G_{D^{\alpha_g}_{\leq i}\cup\{\alpha_g\}} \subseteq G_{D^\gamma_{\leq i}}$. Consequently, $i^\gamma_g \leq i$.

(4) By Clause (1), for every $g \in G_A \cap G_{A'}$, either $g = \mathbb{1}$ (and then $g \in G_\emptyset \subseteq G_{A\cap A'}$), or $D^{\alpha_g}_{\leq i_g} \cup \{\alpha_g\} \subseteq A \cap A'$, and then $g \in G_{D^{\alpha_g}_{\leq i_g}\cup\{\alpha_g\}} \subseteq G_{A\cap A'}$ by the definition of i_g and α_g. The other inclusion is trivial. $\square$

Corollary 5.10. *For all $\beta \leq \gamma < \kappa$, for all $j < i < \theta$, for all $g,h \in G_{D^\gamma_{<j}\cup\{\gamma\}} \setminus G_\gamma$, if $g \equiv_{D^\gamma_{<i}\cap\beta} h$, then $g \equiv_{D^\gamma_{<j}\cap\beta} h$.*

Proof. Let $\beta \leq \gamma < \kappa$ such that $\gamma \geq \theta$ and let $j < i < \theta$. Suppose that $g,h \in G_{D^\gamma_{<j}\cup\{\gamma\}} \setminus G_\gamma$ are such that $g \not\equiv_{D^\gamma_{<j}\cap\beta} h$, and we shall prove by induction on $l \in [j,i]$ that

$$g \not\equiv_{D^\gamma_{<l}\cap\beta} h. \tag{II}$$

Recalling Clause (4) of Lemma 5.9, $G_{D^\gamma_{<j}\cup\{\gamma\}} \cap G_\gamma = G_{D^\gamma_{<j}}$, so we infer that $g,h \in G_{D^\gamma_{<j}\cup\{\gamma\}} \setminus G_{D^\gamma_{<j}}$.

The case $l = j$ is trivial, and the case in which l is a limit ordinal follows from continuity. So, suppose that $l \in [j,i)$ is such that (II) holds, and we shall prove that $g \not\equiv_{D^\gamma_{\leq l}\cap\beta} h$. W.l.o.g. $l \geq 1$, otherwise $G_{D^\gamma_{<0}} = G_{D^\gamma_{\leq 0}} = \{\mathbb{1}\}$, so both relations $\equiv_{D^\gamma_{<0}}$ and $\equiv_{D^\gamma_{\leq 0}}$ are the identity on $G_{D^\gamma_{\leq 0}\cup\{\gamma\}}$.

By $(p)_6$, the group $G_{D^\gamma_{\leq l}\cup\{\gamma\}}$ was given by Lemma 3.4, when invoked with $H = G_{D^\gamma_{<l}}$, $K = G_{D^\gamma_{\leq l}}$ and $L = G_{D^\gamma_{<l}\cup\{\gamma\}}$. Consider $K' = G_{D^\gamma_{\leq l}\cap\beta}$, which is a subgroup of K, and then let $H' = K' \cap L$. By Lemma 5.9(4),

$$H' = G_{D^\gamma_{\leq l}\cap\beta} \cap G_{D^\gamma_{<l}\cup\{\gamma\}} = G_{D^\gamma_{<l}\cap\beta},$$

meaning that (II) asserts that $g \not\sim_{H'} h$.

As $g, h \in G_{D^{\gamma}_{<j} \cup \{\gamma\}} \subseteq L$, and $H' = K' \cap L = K' \cap H$ (since $K \cap L = H$), Clause (F) of Lemma 3.4 implies that $g \not\sim_{K'} h$. That is, $g \not\equiv_{D^{\gamma}_{\leq l} \cap \beta} h$, as sought. $\square$

Notation 5.11. As a last step of preparation, we fix a surjection $\vec{\pi} = (\pi_0, \pi_1, \pi_2, \pi_3, \pi_4)$ from κ to $\kappa \times \kappa \times \kappa \times \kappa \times \{1, -1\}$, i.e., for all $\eta_0, \eta_1, \eta_2, \eta_3 \in \kappa$ and $\varepsilon \in \{1, -1\}$ there exists a $\xi < \kappa$ such that

$$\vec{\pi}(\xi) = (\pi_0(\xi), \pi_1(\xi), \pi_2(\xi), \pi_3(\xi), \pi_4(\xi)) = (\eta_0, \eta_1, \eta_2, \eta_3, \varepsilon).$$

5.2. **The recursive construction.** We are now ready to start the recursive process. We start by letting x_0 generate an infinite cyclic group (i.e., $\mathbb{Z}$), and we assume this group G_1 has underlying set ω. Hereafter, we shall not worry about $(p)_2$, since it is clear it may be secured. Next, suppose that $\gamma \in [1, \kappa)$ is such that G_γ has already been defined and satisfies all of our promises. Note that $(p)_3$ implies that for every $\beta < \gamma$ the group $G_{\beta+1} = \bigcup_{i<\theta} G_{D^{\beta}_{<i} \cup \{\beta\}}$ is torsion-free, and so is $G_\gamma = \bigcup_{\beta<\gamma} G_{\beta+1}$. To construct $G_{\gamma+1}$, we first let $x_\gamma = \min(\kappa \setminus G_\gamma)$, and now we need to describe the group relationship between x_γ and the elements of G_γ. We will define $\langle G_{D^{\gamma}_{<i} \cup \{\gamma\}} \mid i < \theta \rangle$ by recursion on $i < \theta$, in such a way that all of our promises are kept.

Here we go. As $D^{\gamma}_{<0} = D^{\gamma}_{\leq 0} = \emptyset$ (recall Remark 5.4), we mean $G_{D^{\gamma}_{\leq 0}} = \{\mathbb{1}\}$, and we let $G_{D^{\gamma}_{\leq 0} \cup \{\gamma\}} = G_{\{\gamma\}}$ be the infinite group $\mathbb{Z}$ generated by x_γ. Note that $\{\mathbb{1}\} = G_{D^{\gamma}_{\leq 0}} \leq_{\mathrm{m}} G_{D^{\gamma}_{\leq 0} \cup \{\gamma\}}$ vacuously holds. Moving on, suppose that $i < \theta$ is such that $G_{D^{\gamma}_{<i} \cup \{\gamma\}}$ has already been defined. For all $j \leq i$ and $\beta \leq \gamma$, let $E^{\gamma}_{<j,\beta}$ be the restriction of the equivalence relation $\equiv_{(D^{\gamma}_{<j} \cap \beta)}$ to $G_{D^{\gamma}_{<j} \cup \{\gamma\}}$. Next, use Definition 5.5 to define a transversal $T^{\gamma}_{<j,\beta}$ for those equivalence classes of $E^{\gamma}_{<j,\beta}$ that lie in $G_{D^{\gamma}_{<j} \cup \{\gamma\}} \setminus G_\gamma = G_{D^{\gamma}_{<j} \cup \{\gamma\}} \setminus G_{D^{\gamma}_{<j}}$, as follows

$$T^{\gamma}_{<j,\beta} = \{\min([g]_{E^{\gamma}_{<j,\beta}}, \prec) \mid g \in G_{D^{\gamma}_{<j} \cup \{\gamma\}} \setminus G_\gamma\}.$$

Lemma 5.12. *For all $j \leq i$ and $\alpha \leq \beta \leq \gamma$:*

(1) $E^{\gamma}_{<i,\beta} \restriction (G_{D^{\gamma}_{<j} \cup \{\gamma\}} \setminus G_\gamma) = E^{\gamma}_{<j,\beta} \restriction (G_{D^{\gamma}_{<j} \cup \{\gamma\}} \setminus G_\gamma)$;
(2) $T^{\gamma}_{<j,\beta} \subseteq T^{\gamma}_{<i,\beta}$;
(3) $T^{\gamma}_{<i,\alpha} \supseteq T^{\gamma}_{<i,\beta}$.

Proof. (1) By Corollary 5.10.

(2) Let $t \in T^{\gamma}_{<j,\beta}$. As $G_{(D^{\gamma}_{<j} \cap \beta)} h^{\pm 1} G_{(D^{\gamma}_{<j} \cap \beta)} \subseteq G_\gamma$ for every $h \in G_\gamma$, it is the case that $[g]_{E^{\gamma}_{<j,\beta}}$ is disjoint from G_γ for every $g \in G_{D^{\gamma}_{<j} \cup \{\gamma\}} \setminus G_\gamma$. In particular, $t \in G_{\gamma+1} \setminus G_\gamma$, so that $i_t = i^{\gamma+1}_t < j$. Therefore, by Clause (1),

$$[t]_{E^{\gamma}_{<i,\beta}} \cap G_{D^{\gamma}_{<j} \cup \{\gamma\}} = [t]_{E^{\gamma}_{<j,\beta}}.$$

For every $g \in [t]_{E^{\gamma}_{<i,\beta}} \setminus G_{D^{\gamma}_{<j} \cup \{\gamma\}}$ we have $i_g = i^{\gamma+1}_g \geq j > i_t$, and then Definition 5.5 implies that $t \prec g$. Altogether,

$$\min([t]_{E^{\gamma}_{<i,\beta}}, \prec) = \min([t]_{E^{\gamma}_{<j,\beta}}, \prec) = t.$$

(3) This is clear from the definition of $T^{\gamma}_{<i,\alpha}$, $T^{\gamma}_{<i,\beta}$, as the equivalence relation $E^{\gamma}_{<j,\alpha}$ is a refinement of $E^{\gamma}_{<j,\beta}$. $\square$

Our next goal is to the define the system $S = \{(h_\sigma, a_\sigma, b_\sigma, b'_\sigma) \mid \sigma \in \Sigma\}$ that will yield the definition of $G_{D^\gamma_{\leq i} \cup \{\gamma\}}$, as per $(p)_6$. We start with a rough approximation Σ^{++} of Σ, we then refine it to $\Sigma^+ \subseteq \Sigma^{++}$, and finally we find the appropriate $\Sigma \subseteq \Sigma^+$.

Definition 5.13. Let:

- $\Sigma^{++} = \{(a,t) \mid a \in G_{D^\gamma_{\leq i}} \setminus G_{D^\gamma_{<i}},\ \alpha_a \in D^\gamma_{\leq i} \setminus D^\gamma_{<i},\ t \in T^\gamma_{<i,\alpha_a}\}$;
- $\Sigma^+ = \{(a,t) \in \Sigma^{++} \mid \forall l < 4\, [\pi_l(c(\alpha_a, \gamma)) \in G_\gamma]\}$.

Definition 5.14. For each $\sigma = (a,t) \in \Sigma^+$, we attach the following objects:

- ▷ $a_\sigma = a$;
- ▷ $t_\sigma = t$;
- ▷ $h_\sigma = \pi_0(c(\alpha_a, \gamma))$;
- ▷ $y_{\sigma,0} = \pi_1(c(\alpha_a, \gamma))$;
- ▷ $y_{\sigma,1} = \pi_2(c(\alpha_a, \gamma))$;
- ▷ $z_\sigma = \pi_3(c(\alpha_a, \gamma))$;
- ▷ $\varepsilon_\sigma = \pi_4(c(\alpha_a, \gamma))$;
- ▷ $b_\sigma = y_{\sigma,0} \cdot t^{\varepsilon_\sigma} \cdot y_{\sigma,1} \cdot z_\sigma$;
- ▷ $b'_\sigma = b_\sigma \cdot b_\sigma$;
- ▷ $K_\sigma = G_{D^\gamma_{\leq i}} \cap G_{(\alpha_a+1)}$.

We then let Σ be the set of all $\sigma \in \Sigma^+$ for which all of the following hold:

(1) $\max\{\alpha_{y_{\sigma,0}}, \alpha_{y_{\sigma,1}}, \alpha_{z_\sigma}\} < \alpha_{a_\sigma}$;
(2) $\max\{i_t, i^\gamma_{y_{\sigma,0}}, i^\gamma_{y_{\sigma,1}}\} < i^\gamma_{z_\sigma} < i$ (in particular, $i^\gamma_{z_\sigma} \geq 1$);
(3) $h_\sigma \in G_{D^\gamma_{<i}}$.

Remark 5.15. Clause (1) implies that $y_{\sigma,0}, y_{\sigma,1}, z_\sigma \in G_{\alpha_{a_\sigma}}$, and Clause (2) implies that, for some $j < i$, $y_{\sigma,0}, y_{\sigma,1} \in G_{D^\gamma_{<j}}$, $t \in G_{D^\gamma_{<j} \cup \{\gamma\}}$, and $z_\sigma \notin G_{D^\gamma_{<j}}$.

Definition 5.16. Denote $H = G_{D^\gamma_{<i}}$, $K = G_{D^\gamma_{\leq i}}$, $L = G_{D^\gamma_{<i} \cup \{\gamma\}}$, and

$$S = \{(h_\sigma, a_\sigma, b_\sigma, b'_\sigma) \mid \sigma \in \Sigma\}.$$

Lemma 5.17. *$H \leq K$, $H \leq_{\mathrm{m}} L$, $K \cap L = H$ and $S \subseteq H \times (K \setminus H) \times (L \setminus H) \times (L \setminus H)$.*

Proof. It is clear that $H = G_{D^\gamma_{<i}} \leq G_{D^\gamma_{\leq i}} = K$. By $(p)_5$, $H \leq_{\mathrm{m}} L$, and by $(p)_4$, $K \cap L = H$.

Next, let $\sigma \in \Sigma$. By Definition 5.14(3), $h_\sigma \in H$. Since $\sigma \in \Sigma^{++}$, $a_\sigma \in K \setminus H$. Recall that $t \in T^\gamma_{<i,\alpha} \subseteq G_{D^\gamma_{<i} \cup \{\gamma\}} \setminus G_\gamma = L \setminus G_\gamma$. By Lemma 5.9(4), $H = L \cap G_\gamma$, and hence $t \in L \setminus H$. By Definition 5.14(2), $y_{\sigma,0}, y_{\sigma,1}, z_\sigma$ are in $H \leq L$, so, altogether, b_σ and b'_σ are in L, as well. Since $t \notin H$, we get that $b_\sigma \notin H$. Finally, to see that $b'_\sigma \notin H$, it suffices to verify that $b'_\sigma \in G_{D^\gamma_{\leq j} \cup \{\gamma\}} \setminus G_{D^\gamma_{\leq j}}$ for some $j < i$, since $H = G_{D^\gamma_{<i}} \leq G_\gamma$, and $G_{D^\gamma_{\leq j} \cup \{\gamma\}} \cap G_\gamma = G_{D^\gamma_{\leq j}}$ by Lemma 5.9(4).

By the definition of Σ and since $y_{\sigma,0}, y_{\sigma,1} \in G_{D^\gamma_{<i^\gamma_{z_\sigma}}}$, we have that $y_{\sigma,0} \cdot t^{\varepsilon_\sigma} \cdot y_{\sigma,1} \in G_{D^\gamma_{<i^\gamma_{z_\sigma}} \cup \{\gamma\}} \setminus G_{D^\gamma_{<i^\gamma_{z_\sigma}}}$ and $z_\sigma \in G_{D^\gamma_{\leq i^\gamma_{z_\sigma}}} \setminus G_{D^\gamma_{<i^\gamma_{z_\sigma}}}$. By $(p)_6$, $G_{D^\gamma_{\leq i^\gamma_{z_\sigma}} \cup \{\gamma\}}$ has been obtained by invoking Lemma 3.4 (note that $i^\gamma_{z_\sigma} \geq 1$ necessarily) with $\bar{K} = G_{D^\gamma_{\leq i^\gamma_{z_\sigma}}}$, $\bar{L} = G_{D^\gamma_{<i^\gamma_{z_\sigma}} \cup \{\gamma\}}$, and $\bar{H} = G_{D^\gamma_{<i^\gamma_{z_\sigma}}}$, and then Clause (D) of that lemma implies that

$$(y_{\sigma,0} \cdot t^{\varepsilon_\sigma} \cdot y_{\sigma,1}) \cdot z_\sigma \cdot (y_{\sigma,0} \cdot t^{\varepsilon_\sigma} \cdot y_{\sigma,1}) \notin \bar{K} = G_{D^\gamma_{\leq i^\gamma_{z_\sigma}}},$$

and then the fact that $z_\sigma \in G_{D^\gamma_{\le i^\gamma_{z_\sigma}}}$ implies that

$$b'_\sigma = (y_{\sigma,0} \cdot t^{\varepsilon_\sigma} \cdot y_{\sigma,1}) \cdot z_\sigma \cdot (y_{\sigma,0} \cdot t^{\varepsilon_\sigma} \cdot y_{\sigma,1}) \cdot z_\sigma \notin \bar{K} = G_{D^\gamma_{\le i^\gamma_{z_\sigma}}},$$

as sought. □

Lemma 5.18. *For every $\sigma \in \Sigma$, $b_\sigma \not\sim_H b'_\sigma$.*

Proof. Let $\sigma = (a, t)$ in Σ. Set

$$j = \max\{i_t, i^\gamma_{y_{\sigma,0}}, i^\gamma_{y_{\sigma,1}}\}.$$

From $\sigma \in \Sigma$, we infer that $j < i^\gamma_{z_\sigma} < i$, and

$$y_{\sigma,0}, y_{\sigma,1} \in G_{D^\gamma_{\le j}} \subseteq G_{D^\gamma_{\le j} \cup \{\gamma\}}.$$

Recall that $t \in T^\gamma_{<i,\alpha_a} \subseteq G_{D^\gamma_{<i} \cup \{\gamma\}} \setminus G_\gamma$, therefore

$$y_{\sigma,0} \cdot t^{\varepsilon_\sigma} \cdot y_{\sigma,1} \in G_{D^\gamma_{\le j} \cup \{\gamma\}} \le G_{D^\gamma_{<i^\gamma_{z_\sigma}} \cup \{\gamma\}}.$$

By Lemma 5.9(4), $G_{D^\gamma_{<i^\gamma_{z_\sigma}}} = G_{D^\gamma_{<i^\gamma_{z_\sigma}} \cup \{\gamma\}} \cap G_\gamma$, so since $t \notin G_\gamma$,

$$y_{\sigma,0} \cdot t^{\varepsilon_\sigma} \cdot y_{\sigma,1} \in G_{D^\gamma_{<i^\gamma_{z_\sigma}} \cup \{\gamma\}} \setminus G_{D^\gamma_{<i^\gamma_{z_\sigma}}}.$$

By $(p)_6$, $G_{D^\gamma_{\le i^\gamma_{z_\sigma}} \cup \{\gamma\}}$ has been obtained by invoking Lemma 3.4 (note that $i^\gamma_{z_\sigma} \ge 1$ necessarily) with $\bar{K} = G_{D^\gamma_{\le i^\gamma_{z_\sigma}}}$, $\bar{L} = G_{D^\gamma_{<i^\gamma_{z_\sigma}} \cup \{\gamma\}}$, and $\bar{H} = G_{D^\gamma_{<i^\gamma_{z_\sigma}}}$, and then Clause (C) of that lemma together with the facts that $y_{\sigma,0} \cdot t^{\varepsilon_\sigma} \cdot y_{\sigma,1} \in \bar{L} \setminus \bar{H}$ and $z_\sigma \in \bar{K} \setminus \bar{H}$ imply that for $b = b^* = y_{\sigma,0} \cdot t^{\varepsilon_\sigma} \cdot y_{\sigma,1}$ and $z = z_\sigma$, it is the case that $b^* z \not\sim_{\bar{K}} bzbz$. That is,

$$b_\sigma \not\sim_{G_{D^\gamma_{\le i^\gamma_{z_\sigma}}}} b'_\sigma,$$

which is the same as $\neg(b_\sigma\ E^\gamma_{<i^\gamma_{z_\sigma}+1,\gamma}\ b'_\sigma)$. By Lemma 5.12(1),

$$E^\gamma_{<i,\gamma} \restriction (G_{D^\gamma_{<i^\gamma_{z_\sigma}+1} \cup \{\gamma\}} \setminus G_\gamma) = E^\gamma_{<i^\gamma_{z_\sigma}+1,\gamma} \restriction (G_{D^\gamma_{<i^\gamma_{z_\sigma}+1} \cup \{\gamma\}} \setminus G_\gamma),$$

and hence $b_\sigma \not\sim_{G_{D^\gamma_{<i} \cap \gamma}} b'_\sigma$. which concludes our proof (since $G_{D^\gamma_{<i} \cap \gamma} = H$). □

Lemma 5.19. *For all $\sigma \neq \sigma^*$ in Σ, at least one of the following holds:*

- $(\ominus)_a$ *$a_\sigma \not\sim_H a_{\sigma^*}$;*
- $(\ominus)_b$ *$b_\sigma \not\sim_H b_{\sigma^*}$;*
- $(\ominus)_c$ *$b_\sigma = b_{\sigma^*}$ and $a_\sigma \neq a_{\sigma^*}$;*
- $(\ominus)_d$ *all of the following hold:*
  - (i) *$\alpha_{a_\sigma} = \alpha_{a_{\sigma^*}}$ (so $K_\sigma = K_{\sigma^*}$);*
  - (ii) *$a_\sigma, a_{\sigma^*} \in K_\sigma \setminus H$;*
  - (iii) *$b_\sigma \not\sim_{H_\sigma} b_{\sigma^*}$, where $H_\sigma = K_\sigma \cap H$;*
  - (iv) *$b_\sigma \not\sim_H b'_{\sigma^*}$;*
  - (v) *$K \models (K_\sigma \setminus H) \cdot (H \setminus K_\sigma) \cdot (K_\sigma \setminus H) \subseteq (K \setminus H)$.*

Proof. We start with two general claims.

Claim 5.19.1. *Suppose $a, a^* \in G_{D^\gamma_{\le i}}$ are such that $\alpha_a < \alpha_{a^*} < \gamma$ and $\alpha_a, \alpha_{a^*} \in D^\gamma_{\le i} \setminus D^\gamma_{<i}$. Then $a \not\sim_H a^*$.*

Proof. Since $\alpha_a, \alpha_{a^*} \in D^\gamma_{\leq i} \setminus D^\gamma_{<i}$, Lemma 5.9(1) implies that a and a^* are not in $G_{D^\gamma_{<i}}$. We shall prove by induction on $\beta \in [\alpha_{a^*}, \gamma]$ that:

$$a \notin G_{D^\gamma_{<i} \cap \beta} (a^*)^{\pm 1} G_{D^\gamma_{<i} \cap \beta}. \tag{III}$$

The base case $\beta = \alpha_{a^*}$ follows from the following constellation:

- $a \in G_{D^\gamma_{\leq i}} \cap G_{\alpha_a+1} = G_{D^\gamma_{\leq i} \cap (\alpha_a+1)} \subseteq G_{\alpha_a+1} \subseteq G_{\alpha_{a^*}}$,
- $a^* \in G_{\alpha_{a^*}+1} \setminus G_{\alpha_{a^*}}$, and
- $G_{D^\gamma_{<i} \cap \alpha_{a^*}} \subseteq G_{\alpha_{a^*}}$.

The case that β is a limit follows from continuity, so suppose that $\beta \in [\alpha_{a^*}, \gamma]$ satisfies (III), and we shall show that

$$a \notin G_{D^\gamma_{<i} \cap (\beta+1)} (a^*)^{\pm 1} G_{D^\gamma_{<i} \cap (\beta+1)}.$$

To avoid trivialities, we may assume that $\beta \in D^\gamma_{<i}$, so that, by Lemma 4.3, $D^\gamma_{<i} \cap \beta = D^\beta_{<i}$ and $D^\gamma_{\leq i} \cap \beta = D^\beta_{\leq i}$. Therefore, $\alpha_a, \alpha_{a^*} \in D^\beta_{\leq i} \setminus D^\beta_{<i}$. So Notation 5.3 together with Lemma 5.9(4) imply that $a \in G_{D^\gamma_{\leq i} \cap (\alpha_a+1)} \subseteq G_{D^\beta_{\leq i}}$ and $a^* \in G_{D^\gamma_{\leq i} \cap (\alpha_{a^*}+1)} \subseteq G_{D^\beta_{\leq i}}$. As $\alpha_a, \alpha_{a^*} \notin D^\beta_{<i}$, Lemma 5.9(1) implies that $a, a^* \notin G_{D^\beta_{<i}}$. Altogether,

$$i^\beta_a = i = i^\beta_{a^*}.$$

Now $\beta > \alpha_{a^*} \geq 0$ and $i \geq 1$, since $\beta \in D^\gamma_{<i}$, so $(p)_6$ tells us that $G_{D^\beta_{\leq i} \cup \{\beta\}}$ was constructed by invoking Lemma 3.4 with $\bar{H} = G_{D^\beta_{<i}}$, $\bar{K} = G_{D^\beta_{\leq i}}$ and $\bar{L} = G_{D^\beta_{<i} \cup \{\beta\}}$. By Lemma 5.9,

$$\bar{L} \cap \bar{K} = G_{D^\beta_{<i} \cup \{\beta\}} \cap G_{D^\beta_{\leq i}} = G_{D^\beta_{<i}} = \bar{H},$$

so, taking (III) into account, Clause (E) of Lemma 3.4 implies that

$$a \notin \bar{L}(a^*)^{\pm 1} \bar{L}.$$

However, $\bar{L} = G_{D^\beta_{<i} \cup \{\beta\}} = G_{D^\gamma_{<i} \cap (\beta+1)}$, so we are done. $\square_{\text{Claim 5.19.1}}$

Claim 5.19.2. *Let $\alpha \in D^\gamma_{\leq i} \setminus D^\gamma_{<i}$ and $a, a^* \in G_{D^\gamma_{\leq i} \cap (\alpha+1)} \setminus G_\alpha$.*

(1) For every $g \in G_{D^\gamma_{<i} \cap (\beta+1)} \setminus G_{D^\gamma_{<i} \cap \beta}$ with $\beta \in D^\gamma_{<i} \setminus \alpha$, we have $a \cdot g \cdot a^* \notin G_{D^\gamma_{<i} \cap (\beta+1)}$;
(2) For every $g \in G_{D^\gamma_{<i}} \setminus G_\alpha$, we have $a \cdot g \cdot a^* \notin G_{D^\gamma_{<i}}$.

Proof. (1) Fix g and β as above. Now in the same line of reasoning as in Claim 5.19.1, $D^\gamma_{<i} \cap \beta = D^\beta_{<i}$, $D^\gamma_{\leq i} \cap (\beta+1) = D^\beta_{\leq i} \cup \{\beta\}$. Therefore, $G_{D^\gamma_{<i} \cap (\beta+1)}$ is equal to $G_{D^\beta_{<i} \cup \{\beta\}}$, and the latter (recalling $\beta \geq 1$, and $i \geq 1$ which is true since $\beta \in D^\gamma_{<i}$) was obtained by invoking Lemma 3.4 with $\bar{H} = G_{D^\beta_{<i}}$, $\bar{K} = G_{D^\beta_{\leq i}}$, $\bar{L} = G_{D^\beta_{<i} \cup \{\beta\}}$ (where $g \in \bar{L} \setminus \bar{H}$). So just apply (the parallel of) Clause (D) of the said Lemma.

(2) Suppose not, and let g be a counterexample. Let β be minimal such that $g \in G_{\beta+1}$. By Lemma 5.9(4), $G_{D^\gamma_{<i}} \cap G_{\beta+1} = G_{D^\gamma_{<i} \cap (\beta+1)}$, and hence $g \in G_{D^\gamma_{<i} \cap (\beta+1)} \setminus G_{D^\gamma_{<i} \cap \beta}$. Then, By Lemma 5.9(1), $\beta \in D^\gamma_{<i}$, so that $\beta \in D^\gamma_{<i} \setminus \alpha$. As $a \cdot g \cdot a^* \in G_{\alpha+1} \cdot G_{\beta+1} \cdot G_{\alpha+1} = G_{\beta+1}$, and recalling that $a \cdot g \cdot a^* \in G_{D^\gamma_{<i}}$, we infer that $a \cdot g \cdot a^* \in G_{D^\gamma_{<i}} \cap G_{\beta+1} = G_{D^\gamma_{<i} \cap (\beta+1)}$, contradicting Clause (1). $\square_{\text{Claim 5.19.2}}$

Suppose now that $\sigma = (a,t)$ and $\sigma^* = (a^*,t^*)$ are two distinct elements of Σ. We assume that alternatives $(\ominus)_a$–$(\ominus)_c$ fail, and we shall verify alternative $(\ominus)_d$. Note that our assumptions have the following immediate consequences.

Claim 5.19.3. *$b_\sigma \sim_H b_{\sigma^*}$, $t \neq t^*$, and $\alpha_a = \alpha_{a^*}$.*

Proof. The first part follows from the failure of alternative $(\ominus)_b$, and the last part follows from failure of alternative $(\ominus)_a$ together with Claim 5.19.1.

In addition, if t were to equal t^*, Definition 5.14 (using $\alpha_a = \alpha_{a^*}$) would have implied that alternative $(\ominus)_c$ holds. So $t \neq t^*$. $\square_{\text{Claim 5.19.3}}$

It thus follows from Definition 5.14 that

$$(h_\sigma, y_{\sigma,0}, y_{\sigma_1}, z_\sigma, K_\sigma) = (h_{\sigma^*}, y_{\sigma^*,0}, y_{\sigma_1^*}, z_{\sigma^*}, K_{\sigma^*}).$$

Consequently,

$$\max\{i_t, i_{t^*}, i^\gamma_{y_{\sigma,0}}, i^\gamma_{y_{\sigma^*,0}}, i^\gamma_{y_{\sigma,1}}, i^\gamma_{y_{\sigma^*,1}}\} < i^\gamma_{z_\sigma},$$

and hence the next two elements are in $G_{D^\gamma_{<i^\gamma_{z_\sigma}} \cup \{\gamma\}}$:

- $b = y_{\sigma,0} \cdot t^{\varepsilon_\sigma} \cdot y_{\sigma,1}$,
- $b^* = y_{\sigma^*,0} \cdot (t^*)^{\varepsilon_{\sigma^*}} \cdot y_{\sigma^*,1}$,

moreover,

$$b, b^* \notin G_{D^\gamma_{<i^\gamma_{z_\sigma}}}, \tag{IV}$$

since $y_{\sigma,0}, y_{\sigma,1}, y_{\sigma^*,0}, y_{\sigma^*,1} \in G_{D^\gamma_{<i^\gamma_{z_\sigma}}}$. Note that

$$K_\sigma = G_{D^\gamma_{\leq i}} \cap G_{(\alpha_a+1)} = G_{D^\gamma_{\leq i} \cap (\alpha_a+1)},$$

and

$$H_\sigma = K_\sigma \cap H = G_{D^\gamma_{\leq i} \cap (\alpha_a+1)} \cap G_{D^\gamma_{<i}} = G_{D^\gamma_{<i} \cap \alpha_a}$$

by Lemma 5.9(4).

As $\sigma \in \Sigma$, it is also the case that $z_\sigma \in G_{D^\gamma_{\leq i^\gamma_{z_\sigma}}} \leq G_{D^\gamma_{<i}} = H$ and

$$y_{\sigma,0} \cdot t^{\varepsilon_\sigma} \cdot y_{\sigma,1} = b_\sigma \cdot z_\sigma^{-1},$$

so that $b \sim_H b_\sigma$. Likewise, $b^* \sim_H b_{\sigma^*}$. Recalling that $b_\sigma \sim_H b_{\sigma^*}$, altogether

$$b \sim_H b^*.$$

Now, $(p)_6$ tells us that $G_{D^\gamma_{\leq i^\gamma_{z_\sigma}} \cup \{\gamma\}}$ was constructed by invoking Lemma 3.4 with $\bar{H} = G_{D^\gamma_{<i^\gamma_{z_\sigma}}}$, $\bar{K} = G_{D^\gamma_{\leq i^\gamma_{z_\sigma}}}$ and $\bar{L} = G_{D^\gamma_{<i^\gamma_{z_\sigma}} \cup \{\gamma\}}$ (again, $i^\gamma_{z_\sigma} \geq 1$, by Clause (2) of Definition 5.14). Trivially, $z_\sigma \notin \bar{H}$. In addition, $b, b^* \notin \bar{H}$ by (IV). Thus Clause (C) of that lemma implies that $b^* \cdot z_{\sigma^*} \not\sim_{\bar{K}} b \cdot z_\sigma \cdot b \cdot z_\sigma$, and hence $b'_\sigma \not\sim_{\bar{K}} b_{\sigma^*}$. Finally, $b'_\sigma \not\sim_H b_{\sigma^*}$ (i.e., $b'_\sigma \not\sim_{G_{D^\gamma_{<i}}} b_{\sigma^*}$) by Lemma 5.12(1).

On the other hand, by the definition of $T^\gamma_{<i,\alpha_{a_\sigma}}$, we get that $t \not\sim_{K_\sigma \cap H} t^*$. As $K_\sigma \cap H = G_{D^\gamma_{\leq i} \cap (\alpha_{a_\sigma}+1)} \cap G_{D^\gamma_{<i}} = G_{D^\gamma_{<i} \cap (\alpha_{a_\sigma}+1)} = G_{D^\gamma_{<i} \cap \alpha_{a_\sigma}}$, we also get that $b_\sigma \not\sim_{K_\sigma \cap H} b_{\sigma^*}$, since $z_\sigma = z_{\sigma^*}$, $y_{\sigma,0}$, $y_{\sigma,1} \in G_{D^\gamma_{<i} \cap \alpha_{a_\sigma}}$ (by recalling the definition of Σ). At this stage, it remains to check Clause (v), but this follows from Claim 5.19.2(2). $\square_{\text{Lemma 5.19}}$

By Lemmas 5.17, 5.18 and 5.19, the tuple (H, K, L, S) satisfies all of the assumptions of Lemma 3.4. Adhering to $(p)_6$, we then let $G_{D^\gamma_{\leq i}\cup\{\gamma\}}$ be the outcome M^* of Lemma 3.4 when invoked with this tuple. By Clause (A) of that lemma, $K, L \leq M^*$,

$$M^* \models K \cap L = H,$$

and M^* is generated by $K \cup L = G_{D^\gamma_{\leq i}} \cup G_{D^\gamma_{<i}\cup\{\gamma\}}$. This means that M^* is generated by the set of generators $\{x_\beta \mid \beta \in D^\gamma_{\leq i} \cup \{\gamma\}\}$, and hence $(p)_4$ is preserved. Also, Clause (B) implies that $K \leq_{\mathrm{m}} M^*$, hence $(p)_5$ is preserved, as well.

Our promise $(p)_3$ implies that $L = G_{D^\gamma_{<i}\cup\{\gamma\}}$ and

$$\bigcup_{\beta<\gamma} G_{\beta+1} = \bigcup_{\beta<\gamma}\bigcup_{j<\theta} G_{D^\beta_{<j}\cup\{\beta\}}$$

are both torsion-free. In particular, K, being a subgroup of $G_\gamma = \bigcup_{\beta<\gamma} G_{\beta+1}$ is torsion-free, as well. It now follows from Clause (G) of Lemma 3.4 that we have maintained $(p)_3$.

This completes the description of the recursive construction of our group G.

5.3. **Verification.** We now turn to show that G is an n-Shelah group for $n = 10120$.

Lemma 5.20. *Let $Z \in [G]^\kappa$. Then $Z^{10120} = G$.*

Proof. By possibly thinning out (using the pigeonhole principle), we may assume the existence of some $j < \theta$ such that $i_z = j$ for all $z \in Z$. Set $A = \{\alpha_z \mid z \in Z \setminus \{\mathbb{1}\}\}$, so that $A \in [\kappa]^\kappa$. For each $\alpha \in A$, pick $z_\alpha \in Z$ such that $\alpha_{z_\alpha} = \alpha$.

Recalling the hypothesis of Theorem 5.1, we now let B be a club in κ such that for every $\beta \in B$, there exists a $\gamma \in A$ above β such that

(V) $$\forall \xi < \beta \forall i < \theta\, [\sup\{\alpha \in A \cap \beta \mid c(\alpha,\gamma) = \xi \;\&\; d(\alpha,\gamma) > i\} = \alpha].$$

Recalling $(p)_2$ and the surjection $\vec{\pi}$ of Notation 5.11, the following is yet another club in κ:

$$C = \{\beta < \kappa \mid G_\beta = \beta \;\&\; \vec{\pi}[\beta] = \beta \times \beta \times \beta \times \beta \times \{-1, 1\}\}.$$

Now, let h be an arbitrary element of G, and we shall show that h is in Z^{10120}. Pick a large enough $\beta \in B \cap C$ such that $h \in G_\beta$, and then pick $\gamma \in A$ above β satisfying (V). As $i_{z_\gamma} = j$, we consider the unique $t \in T^\gamma_{<j+1,\beta}$ such that $t\; E^\gamma_{<j+1,\beta}\; z_\gamma$. By the choice of t, we may pick

(VI) $$y_0, y_1 \in G_{D^\gamma_{<j+1}\cap\beta}$$

and $\varepsilon \in \{-1, 1\}$ such that

$$z_\gamma = y_0 \cdot t^\varepsilon \cdot y_1.$$

It follows that $\max\{i^\gamma_{y_0}, i^\gamma_{y_1}\} \leq j$, and as $t \in T^\gamma_{<j+1,\beta} \subseteq G_{D^\gamma_{<j+1}\cup\{\gamma\}}$, Lemma 5.9(2) implies that $i_t \leq j$, as well.

As γ was chosen to satisfy (V), we may fix $\bar{\alpha} \in A \cap \beta$ with $d(\bar{\alpha}, \gamma) > j$. Set $z = z_{\bar{\alpha}}$ and note that, by Lemma 5.9(3),

(VII) $$\max\{i_t, i^\gamma_{y_0}, i^\gamma_{y_1}\} \leq j < d(\bar{\alpha}, \gamma) \leq i^\gamma_z.$$

As $\alpha_z = \bar{\alpha} < \beta$, we may find a large enough $\zeta < \beta$ such that $y_0, y_1, z \in G_{\zeta+1}$. Altogether, $y_0, y_1, z \in G_{D^\gamma_{\leq i^\gamma_z}\cap(\zeta+1)}$.

As $\beta \in C$ and $z \in G_\beta$, it follows from (VI) that we may find a $\xi < \beta$ such that

$$(\pi_0(\xi), \pi_1(\xi), \pi_2(\xi), \pi_3(\xi), \pi_4(\xi)) = (h, y_0, y_1, z, \varepsilon).$$

Utilizing (V) once more, we now pick $\alpha \in A \cap \beta$ above $\max\{\alpha_h, \zeta\}$ such that $c(\alpha, \gamma) = \xi$ and $d(\alpha, \gamma) > \max\{i^\gamma_h, i^\gamma_z\}$. Consider $i = d(\alpha, \gamma)$, and note that by (VII),

$$\max\{i_t, i^\gamma_{y_0}, i^\gamma_{y_1}, i_{z_\alpha}, i^\gamma_h, i^\gamma_z\} < d(\alpha, \gamma) = i,$$

so that

$$y_0, y_1, h, z \in G_{D^\gamma_{<i}} \cap G_\alpha. \tag{VIII}$$

Next, consider the group elements $a = z_\alpha$, $b = z_\gamma \cdot z$, and $b' = b \cdot b$, and the pair $\sigma_* = (a, t)$.

Claim 5.20.1. *σ_* is in Σ^{++} of Definition 5.13.*

Proof. From $d(\alpha, \gamma) = i$, we get that $D^\gamma_{<i} \cap (\alpha + 1) = D^\gamma_{<i} \cap \alpha$. By Lemma 4.3, $D^\alpha_{\le i_{z_\alpha}} \subseteq D^\alpha_{\le d(\alpha,\gamma)} = D^\gamma_{\le i} \cap \alpha$, and hence $z_\alpha \in G_{D^\gamma_{\le i}}$. So, if z_α were to be in $G_{D^\gamma_{<i}}$, then since $\alpha_{z_\alpha} = \alpha$, Lemma 5.9(4) would imply that

$$z_\alpha \in G_{D^\gamma_{<i}} \cap G_{\alpha+1} = G_{D^\gamma_{<i} \cap (\alpha+1)} = G_{D^\gamma_{<i} \cap \alpha} \subseteq G_\alpha,$$

contradicting the fact that $\alpha_{z_\alpha} = \alpha$. Altogether, $z_\alpha \in G_{D^\gamma_{\le i}} \setminus G_{D^\gamma_{<i}}$.

Next, since $t \in T^\gamma_{<j+1,\beta}$ and $\alpha < \beta$, Lemma 5.12(3) implies that $t \in T^\gamma_{<j+1,\alpha}$. In addition, since $i = d(\alpha, \gamma) > i^\gamma_z > j$, Lemma 5.12(2) implies that $t \in T^\gamma_{<i,\alpha}$. Also, $i = d(\alpha, \gamma)$ amounts to saying that $\alpha_{z_\alpha} = \alpha \in D^\gamma_{\le i} \setminus D^\gamma_{<i}$, so we have established that $\sigma_* \in \Sigma^{++}$. □

Looking at Definition 5.14, we arrive at the following table of evaluations:

• a_{σ_*}	$=$	a	$=$	z_α	$=$	a
• t_{σ_*}	$=$	t	$=$	t	$=$	t
• h_{σ_*}	$=$	$\pi_0(c(\alpha_a, \gamma))$	$=$	$\pi_0(\xi)$	$=$	h
• $y_{\sigma_*,0}$	$=$	$\pi_1(c(\alpha_a, \gamma))$	$=$	$\pi_1(\xi)$	$=$	y_0
• $y_{\sigma_*,1}$	$=$	$\pi_2(c(\alpha_a, \gamma))$	$=$	$\pi_2(\xi)$	$=$	y_1
• z_{σ_*}	$=$	$\pi_3(c(\alpha_a, \gamma))$	$=$	$\pi_3(\xi)$	$=$	z
• ε_{σ_*}	$=$	$\pi_4(c(\alpha_a, \gamma))$	$=$	$\pi_4(\xi)$	$=$	ε
• b_{σ_*}	$=$	$y_{\sigma_*,0} \cdot t^{\varepsilon_{\sigma_*}} \cdot y_{\sigma_*,1} \cdot z_{\sigma_*}$	$=$	$z_\gamma \cdot z$	$=$	b
• b'_{σ_*}	$=$	$b_{\sigma_*} \cdot b_{\sigma_*}$	$=$	$z_\gamma \cdot z \cdot z_\gamma \cdot z$	$=$	b'
• K_{σ_*}	$=$	$G_{D^\gamma_{\le i}} \cap G_{(\alpha_a+1)}$	$=$	$G_{D^\gamma_{\le i}} \cap G_{(\alpha+1)}$	$=$	$G_{D^\gamma_{\le i} \cap (\alpha+1)}$

TABLE 1. Evaluations

It thus follows from (VIII) that $\pi_l(c(\alpha_a, \gamma)) \in G_\gamma$ for every $l < 4$, so that σ_* is moreover in Σ^+, as per Definition 5.13. Looking at Conditions (1)–(3) of Definition 5.14, we see that σ_* is a member of Σ, as well: conditions (1) and (3) follow from (VIII), and condition (2) follows from (VII) and the fact that $i^\gamma_z < i$.

Claim 5.20.2. *$h^{-1}\varrho(b \cdot a, b' \cdot a) = \mathbb{1}$ holds in $G_{D^\gamma_{\le i} \cup \{\gamma\}}$.*

Proof. Recall that the group $G_{D^{\gamma}_{\leq i}\cup\{\gamma\}}$ was obtained as the output group M^* of Lemma 3.4, when invoked with (H,K,L,S) of Definition 5.16. Specifically, $H = G_{D^{\gamma}_{<i}}$, $K = G_{D^{\gamma}_{\leq i}}$, $L = G_{D^{\gamma}_{<i}\cup\{\gamma\}}$ and $S = \{(h_\sigma, a_\sigma, b_\sigma, b'_\sigma) \mid \sigma \in \Sigma\}$ of Definition 5.14. But M^* is M/N, where M is the free amalgam $K *_H L$, and N is the least normal subgroup containing $\{h_\sigma^{-1}\varrho(b_\sigma \cdot a_\sigma, b'_\sigma \cdot a_\sigma) \mid \sigma \in \Sigma\}$, hence for each $\sigma \in \Sigma$ we have $h_\sigma^{-1}\varrho(b_\sigma \cdot a_\sigma, b'_\sigma \cdot a_\sigma) \in N$, and clearly

$$G_{D^{\gamma}_{\leq i}\cup\{\gamma\}} = M^* = M/N \models h_\sigma^{-1}\varrho(b_\sigma \cdot a_\sigma, b'_\sigma \cdot a_\sigma) = \mathbb{1}.$$

By Table 1, $b = b_{\sigma_*}$, $b' = b'_{\sigma_*}$, $a = a_{\sigma_*}$, and $h = h_{\sigma_*}$, hence $h^{-1}\varrho(b\cdot a, b'\cdot a) = \mathbb{1}$. □

Recall that for all $x, y \in G$, $\varrho(x,y)$ is a word of length 3320 over the alphabet $\{x,y\}$, so since $\varrho(ba, b'a) = h$, the fact that z_α, z_γ and z all come from the initial set Z implies that

$$\varrho(z_\gamma \cdot z \cdot z_\alpha\ ,\ z_\gamma \cdot z \cdot z_\gamma \cdot z \cdot z_\alpha) \in Z^{9720+400}.$$

Thus, we have verified that h is in Z^{10120}. □

Lemma 5.21. *(1) G admits no T_1 topology other than the discrete topology;*

(2) G is not well-behaved in the sense of [FR23, p. 624];[6]

(3) $G \setminus \{\mathbb{1}\}$ is a nonalgebraic unconditionally closed set (i.e., closed in each Hausdorff group topology).

Proof. (1) This is a standard consequence of the malnormality of the G_γ's ($\gamma < \kappa$). Suppose that τ is some T_1 topology on G. Fix $g \in G$ distinct from $\mathbb{1}$. Then $U = G \setminus \{g\}$ is τ-open, so there is a τ-open neighborhood V of $\mathbb{1}$ for which $V^n \subseteq U$, where n is the integer for which G is n-Shelah. Note that if $|V| = \kappa$, then $V^n = G$, which is a contradiction, so it must be the case that $|V| < \kappa$. But then $V \subseteq G_\gamma$ for some large enough $\gamma < \kappa$. Now $G_\gamma \leq_{\mathrm{m}} G_{\gamma+1}$ by Corollary 5.7, so for any choice of $h \in G_{\gamma+1} \setminus G_\gamma$, it is the case that $(h^{-1}Vh) \cap V = \{\mathbb{1}\}$ is a τ-open neighborhood of $\mathbb{1}$, and hence τ is discrete.

(2) Suppose not. This means that there exists a map $\varphi : G \to [G]^{<\omega}$ such that the following two hold:

(a) φ is countable-to-one;
(b) for all $x \neq y$ in G, $\varphi(x) \triangle \varphi(y) \subseteq \varphi(x \cdot y) \subseteq \varphi(x) \cup \varphi(y)$.

By Clause (a) and the Δ-system lemma, we may fix an $X \subseteq G$ of size κ, some $r \in [G]^{<\omega}$ and some $k < \omega$ such that $\langle \varphi(x) \setminus r \mid x \in X\rangle$ is a sequence of pairwise disjoint sets, each of size k. It then follows from Clause (b) that $|\varphi(x_1 \cdots x_l)| \geq k \cdot l$ for every injective finite sequence $\langle x_1, \ldots, x_l\rangle$ of elements of X. In particular, we may fix a $g \in G$ with $|\varphi(g)| > |r| + kn$, where n is the integer for which G is n-Shelah. Since $X^n = G$, we may fix a (possibly non-injective) sequence $\langle x_1, \ldots, x_n\rangle$ of elements of X such that $x_1 \cdots x_n = g$. However, Clause (b) implies that $|\varphi(x_1 \cdots x_n)| \leq |r| + kn$. This is a contradiction.

(3) We need to show that for no system $\{w_i \mid i \in I\}$ of words over $G \cup \{x\}$ (where x is an abstract variable outside G) do we have

$$G \setminus \{\mathbb{1}\} = \bigcap\nolimits_{i \in I} \{g \in G \mid f_{w_i}(g) = \mathbb{1}\},$$

[6] *Well-behaved* is a weakening of the assertion that a group admits a basis. For instance, any infinite commutative cancellative semigroup is well-behaved (follows from [Fuc70, Theorem 23.1]).

where the value of $f_{w_i}(g) \in G$ is given by substituting each occurrence of the symbol x in $w_i \in {}^{<\omega}(G \cup \{x, x^{-1}\})$ with g, and calculating the value in G. It is easy to see that it suffices to prove that for no such word w does the following equation holds true:

$$G \setminus \{\mathbb{1}\} = \{g \in G \mid f_w(g) = \mathbb{1}\}. \tag{IX}$$

Suppose that w satisfies (IX), and fix a finite subset $B \subseteq G$ with $w \in {}^{<\omega}(B \cup \{x, x^{-1}\})$. As $|B| < \theta$, we may find $\gamma \in [1, \kappa)$ and $i \in [1, \theta)$ such that

$$B \subseteq G_{D^{\gamma}_{\leq i} \cup \{\gamma\}},$$

so for each $g \in G_{D^{\gamma}_{\leq i} \cup \{\gamma\}}$ that is not the identity $f_w(g) = \mathbb{1}$.

We are going to prove (provided that Σ from Definition 5.14 in the construction of $G_{D^{\gamma}_{\leq i} \cup \{\gamma\}}$ is not empty) that the group $G_{D^{\gamma}_{\leq i} \cup \{\gamma\}}$ is topologizable (with a non-discrete T_1 topology), which will imply that $G_{D^{\gamma}_{\leq i} \cup \{\gamma\}} \setminus \{\mathbb{1}\}$ is closed (with respect to this nontrivial topology), contradicting that the topology was non-discrete.

To this end, it is enough to argue that there exists a sequence $\langle N^*_k \mid k \in \omega \rangle$ of normal subgroups of $G_{D^{\gamma}_{\leq i} \cup \{\gamma\}}$ such that for each k do $N^*_{k+1} \leq N^*_k$, $\bigcap_{k \in \omega} N^*_k = \{\mathbb{1}\}$ and $\{\mathbb{1}\} \lneq N^*_k$ hold. Now recall how $G_{D^{\gamma}_{\leq i} \cup \{\gamma\}}$ was constructed in Subsection 5.2 (appealing to Lemma 3.4 there):

$$G_{D^{\gamma}_{\leq i} \cup \{\gamma\}} = (G_{D^{\gamma}_{<i} \cup \{\gamma\}} *_{G_{D^{\gamma}_{<i}}} G_{D^{\gamma}_{\leq i}})/N,$$

where N was the normal closure of $\{h_\sigma^{-1} \varrho(b_\sigma a_\sigma, b'_\sigma a_\sigma) \mid \sigma \in \Sigma\}$ (Σ is from Definition 5.14). Let N_0 denote this N. Observe that it is enough to define a sequence $\langle N_k \mid k \in \omega \setminus \{0\} \rangle$ of normal subgroups in $G_{D^{\gamma}_{<i} \cup \{\gamma\}} *_{G_{D^{\gamma}_{<i}}} G_{D^{\gamma}_{\leq i}}$ that satisfies $N_{k+1} \leq N_k$ for $k \geq 1$, $\bigcap_{k \in \omega} N_k = N_0$ and $N_0 \lneq N_k$.

Recall that in Definition 3.2, we have the sequence $\langle n_\ell \mid \ell < \omega \rangle$ defined via $n_\ell = 3320^\ell$, and that we let $\varrho_\ell(x, y) = \varrho(x^{n_\ell}, y^{n_\ell})$ (in particular, $\varrho_0 = \varrho$), and

$$R_k = \{h_\sigma^{-1} \varrho_0(b_\sigma a_\sigma, b'_\sigma a_\sigma), \varrho_\ell(b_\sigma a_\sigma, b'_\sigma a_\sigma) \mid \ell \geq k,\ \sigma \in \Sigma\}.$$

Set N_k to be the normal closure of R_k. Now the following will complete the proof:

Claim 5.21.1. *(1) For all $\sigma \in \Sigma$ and $k > 0$,*

$$G_{D^{\gamma}_{<i} \cup \{\gamma\}} *_{G_{D^{\gamma}_{<i}}} G_{D^{\gamma}_{\leq i}} \models \varrho_k(b_\sigma a_\sigma, b'_\sigma a_\sigma) \in N_k \setminus N_0,$$

(2) R_1 satisfies $C'(\frac{1}{10})$, moreover, if a group element $g \in G_{D^{\gamma}_{<i} \cup \{\gamma\}} *_{G_{D^{\gamma}_{<i}}} G_{D^{\gamma}_{\leq i}}$ has a canonical representation of length $< \frac{7}{10} \cdot (n_k \cdot 6640) - 1$ for some $k \geq 1$, and $g \notin N_0$, then $g \notin N_k$.

Proof. Let us start with verifying the second clause. R_1 satisfies $C'(\frac{1}{10})$ just by the moreover part of (A) from Lemma 3.4. Suppose $k \in \omega$, $g \in G_{D^{\gamma}_{<i} \cup \{\gamma\}} *_{G_{D^{\gamma}_{<i}}} G_{D^{\gamma}_{\leq i}}$ is such that $g \notin N_0$, and g has a canonical representation of length

$$\ell < \frac{7}{10} \cdot (n_k \cdot 6640) - 1. \tag{X}$$

W.l.o.g. we can assume that whenever $g' \in G_{D^{\gamma}_{<i} \cup \{\gamma\}} *_{G_{D^{\gamma}_{<i}}} G_{D^{\gamma}_{\leq i}}$ satisfies

$$g' \in \{hgh^{-1} N_0 \mid h \in G_{D^{\gamma}_{<i} \cup \{\gamma\}} *_{G_{D^{\gamma}_{<i}}} G_{D^{\gamma}_{\leq i}}\},$$

then the length of g's canonical representation does not exceed that of g' (by possibly replacing g with a g' with a shorter representation, since $g \in N_k \setminus N_0 \Rightarrow g' \in N_k \setminus N_0$ by the normality of N_0, and N_k). Suppose on the contrary that $g \in N_k$. Now Fact 2.10 implies that for a weakly cyclically reduced conjugate g' of g, we have that g' has a canonical representation $w = w_0 w_1 \cdots w_{j-1}$, which, as a word contains a subword $s_0 s_1 \cdots s_{m-1}$ that is a subword of a representation $r_0 r_1 \cdots r_{n-1}$ of some r in the symmetric closure of R_k, and $m \geq \frac{7}{10}n$. (W.l.o.g. we can assume that $s_i = r_i$ for $i < m$, by possibly replacing r with a cyclical conjugate of it, as R_k is closed under such operations.)

Now clearly $m \leq j$, and $j \leq \ell + 1$ (by Definition 2.5), so it follows from $m \geq \frac{7}{10}n$ that $\ell + 1 \geq \frac{7}{10}n$. But $n \in \{6640 \cdot n_0, 6640 \cdot n_0 + 1, 6640 \cdot n_i, 6640 \cdot n_i + 1 \mid i \geq k\}$ since the lengths of the words in R_k form the set $\{6640 \cdot n_0, 6640 \cdot n_i \mid i \geq k\}$, r is a weakly cyclically reduced conjugate of some $r' \in R_k$, and this conjugation can only increase the length by at most one (by Observation 2.6 (2)). Therefore, by (X), $n = 6640 \cdot n_0$ holds necessarily, and

$$r_0 r_1 \cdots r_{n-1} \in N_0.$$

Finally, observe that substituting

$$(r_{n-1}^{-1} \cdot r_{n-2}^{-1} \cdots r_0^{-1}) \cdot (r_0 \cdot r_1 \cdots r_{m-1}) = r_{n-1}^{-1} \cdot r_{n-2}^{-1} \cdots r_m^{-1}$$

instead of $r_0 \cdot r_1 \cdots r_{m-1}$ in w yields an element in

$$\{hgh^{-1}N_0 \mid h \in G_{D^\gamma_{<i} \cup \{\gamma\}} *_{G_{D^\gamma_{<i}}} G_{D^\gamma_{\leq i}}\}$$

with a shorter representation (than that of g), a contradiction.

The first clause is immediate noting that the second clause implies $\varrho_k(b_\sigma a_\sigma, b'_\sigma a_\sigma) \notin N_{k+1}$. □

This completes the proof. □

Corollary 5.22. *For every infinite regular cardinal λ, there exists a torsion-free Shelah group of size λ^+.*

Proof. Invoke Theorem 5.1 with the pair $(\kappa, \theta) = (\lambda^+, \lambda)$, using Theorem 4.4. □

Corollary 5.23. *For every regular uncountable cardinal κ, if $\square(\kappa)$ holds, then there exists a torsion-free Shelah group of size κ.*

Proof. By Theorem 5.1 together with Corollary 4.10. □

Corollary 5.24. *In Gödel's constructible universe, for every regular uncountable cardinal κ, the following are equivalent:*

- *there exists a torsion-free Shelah group of size κ;*
- *there exists a Shelah group of size κ;*
- *κ is not weakly compact.*

Proof. By [Jen72, Theorem 6.1], in Gödel's constructible universe, every regular uncountable κ is either weakly compact, or $\square(\kappa)$ holds. By Corollary 5.23, it thus suffices to prove that weakly compact cardinals do not carry a Shelah group. To this end, suppose that there is an n-Shelah group of size κ.

Claim 5.24.1. *There is a system $\vec{f} = \langle f_j \mid j < n^n \rangle$ of functions from $[\kappa]^n$ to $n^n + 1$ such that $\bigcup_{j<n^n} f_j``[X]^n = n^n + 1$ for every $X \in [\kappa]^\kappa$.*

Proof. Fix an n-Shelah group G with underlying set κ. Let $\langle \psi_j \mid j < n^n \rangle$ list all possible maps from n to n. For every $j < n^n$, define $h_j : [\kappa]^n \to \kappa$ by letting for every n-tuple $(g_0, g_1, \ldots, g_{n-1})$ of elements of G, enumerated in $\in$-increasing order:

$$h_j(g_0, g_1, \ldots, g_{n-1}) = g_{\psi_j(0)} \cdot g_{\psi_j(1)} \cdots g_{\psi_j(n-1)}.$$

Evidently, for every infinite $X \subseteq \kappa$, $\bigcup_{j<n^n} h_j``[X]^n$ is nothing but the set of all group words of length $n$ in the elements of $X$. So, since $G$ is an $n$-Shelah group with underlying set $\kappa$, for every $X \subseteq \kappa$ of full size, $\bigcup_{j<n^n} h_j``[X]^n = \kappa$.

For every $j < n^n$, let $f_j : [\kappa]^n \to (n^n + 1)$ be the color-blind version of h_j obtained via

$$f_j(u) = \min(h_j(u), n^n).$$

Then, $\bigcup_{j<n^n} f_j``[X]^n = n^n + 1$ for every $X \in [\kappa]^\kappa$. □

Let $\vec{f}$ be given by the claim. Define $c : [\kappa]^n \to {}^{n^n}(n^n + 1)$ via:

$$c(u) = \langle f_j(u) \mid j < n^n \rangle.$$

Since κ is weakly compact, $\kappa \to (\kappa)^n_k$ holds for every cardinal $k < \kappa$, in particular, for $k = (n^n + 1)^{n^n}$. So, we may find a set $X \in [\kappa]^\kappa$ such that $c \restriction [X]^n$ is constant with value, say, $\langle m_j \mid j < n^n \rangle$. Pick an $m \in n^n + 1$ distinct from m_j for all $j < n^n$. Then $m \notin \bigcup_{j<n^n} f_j``[X]^n$, contradicting the choice of $\vec{f}$. □

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