A COUNTEREXAMPLE RELATED TO A THEOREM OF KOMJÁTH AND WEISS

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This paper is dedicated to István Juhász on the occasion of his 80th birthday

Abstract. In a paper from 1987, Komjáth and Weiss proved that for every regular topological space X of character less than $\mathfrak{b}$, if $X \rightarrow (\mathrm{top}\ \omega+1)^1_\omega$, then $X \rightarrow (\mathrm{top}\ \alpha)^1_\omega$ for all $\alpha<\omega_1$. In addition, assuming $\diamondsuit$, they constructed a space X of size continuum, of character $\mathfrak{b}$, satisfying $X \rightarrow (\mathrm{top}\ \omega+1)^1_\omega$, but not $X \rightarrow (\mathrm{top}\ \omega^2+1)^1_\omega$. Here, a counterexample space with the same characteristics is obtained outright in ZFC.

1. Introduction

For two topological spaces X, Y and a cardinal θ, the arrow notation

$$X \rightarrow (\mathrm{top}\ Y)^1_\theta$$

asserts that for every coloring $c : X \rightarrow \theta$, there exists an homeomorphism ϕ from Y to X such that c is constant over $\mathrm{Im}(\phi)$.

In [KW87], Komjáth and Weiss studied the partition relation $X \rightarrow (\mathrm{top}\ \alpha)^1_\omega$, where α is a countable ordinal endowed with the usual order topology. The first result of their paper is a pump-up theorem for regular topological spaces of character less than $\mathfrak{b}$;[1] the theorem asserts that for any such space X, if $X \rightarrow (\mathrm{top}\ \omega+1)^1_\omega$, then moreover $X \rightarrow (\mathrm{top}\ \alpha)^1_\omega$ for all $\alpha<\omega_1$.[2]

To show that the bound $\mathfrak{b}$ cannot be improved, Theorem 4 of [KW87] gives an example, assuming $\diamondsuit$, of a regular topological space X of size and character $\aleph_1$ such that $X \rightarrow (\mathrm{top}\ \omega+1)^1_\omega$, but not $X \rightarrow (\mathrm{top}\ \omega^2+1)^1_\omega$. Question 2 of the same paper asks whether there is a ZFC example of a regular space X satisfying $X \rightarrow (\mathrm{top}\ \omega+1)^1_\omega$ and failing $X \rightarrow (\mathrm{top}\ \alpha)^1_\omega$ for some countable ordinal $\alpha>\omega^2$. The first main result of this note answers this question in the affirmative.

Theorem A. *There exists a zero-dimensional regular space X of size continuum, of character $\mathfrak{b}$, satisfying $X \rightarrow (\mathrm{top}\ \omega+1)^1_\omega$, but not $X \rightarrow (\mathrm{top}\ \omega^2+1)^1_1$.*

In [CFJ23], the Komjáth-Weiss counterexample was addressed from a different angle. There, a weakening of $\diamondsuit$ called $\clubsuit_F$ was introduced and shown to be sufficient for the construction of the same $\aleph_1$-sized example. Furthermore, it is established there that $\clubsuit_F$ is consistent with the failure of CH. Here, we provide an alternative way to get an $\aleph_1$-sized countexample space together with a large continuum:

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[1] Recall that $\mathfrak{b}$ denotes the least size of an unbounded submfaily of ${}^\omega\omega$, where a subfamily $\mathcal{F} \subseteq {}^\omega\omega$ is *bounded* if, for some function $g:\omega\rightarrow\omega$, $\{n<\omega \mid f(n)\leq g(n)\}$ is finite for all $f\in\mathcal{F}$.

[2] The published proof had a small gap that was later rectified in [CFJ23] based on a suggestion of Weiss.

Theorem B. *After forcing to add any number of Cohen reals, there exists a zero-dimensional regular space X of size $\aleph_1$, of character $\mathfrak{b}$, satisfying $X \to (\mathrm{top}\ \omega+1)^1_\omega$, but not $X \to (\mathrm{top}\ \omega^2+1)^1_1$.*

The preceding is a special case of a general theorem that identifies a class of notions of forcing that inevitably add consequences of higher analogs of $\clubsuit_F$. These notions of forcing include Cohen forcing, but also Prikry and Magidor forcing.

1.1. **Notation and conventions.** For a regular cardinal κ, we denote by H_κ the collection of all sets of hereditary cardinality less than κ. E^κ_χ denotes the set $\{\alpha<\kappa \mid \mathrm{cf}(\alpha)=\chi\}$, and $E^\kappa_{\geq\chi}$, $E^\kappa_{<\chi}$, $E^\kappa_{\neq\chi}$, etc. are defined analogously.

For a set of ordinals a, we write $\mathrm{ssup}(a) := \sup\{\alpha+1 \mid \alpha\in a\}$, $\mathrm{acc}^+(a) := \{\alpha < \mathrm{ssup}(a) \mid \sup(a\cap\alpha)=\alpha>0\}$, $\mathrm{acc}(a) := a\cap\mathrm{acc}^+(a)$, and $\mathrm{nacc}(a) := a\setminus\mathrm{acc}(a)$.

2. Topological spaces based on trees

Following [BR21], we say that T is a *streamlined tree* iff there exists some cardinal κ such that $T\subseteq {}^{<\kappa}H_\kappa$ and, for all $t\in T$ and $\alpha<\mathrm{dom}(t)$, $t\restriction\alpha\in T$. For a subset $E\subseteq\kappa$, we let $T\restriction E := \{t\in T \mid \mathrm{dom}(t)\in E\}$. For a subset $T'\subseteq T$, a *ladder system* over T' is a sequence $\vec{A}=\langle A_t \mid t\in T'\rangle$ such that, for every $t\in T'$, A_t is a cofinal subset of $t_\downarrow := \{s\in T \mid s\subsetneq t\}$ with $\mathrm{otp}(A_t)=\mathrm{cf}(\mathrm{dom}(t))$. For every ladder system $\vec{A}=\langle A_t \mid t\in T'\rangle$, we attach a symmetric relation $E_{\vec{A}}\subseteq[T]^2$, as follows:

$$E_{\vec{A}} = \{\{s,t\} \mid t\in T', s\in A_t\}.$$

Theorem 2.1. *Suppose that:*

- *$T\subseteq {}^{<\kappa}H_\kappa$ is a streamlined tree;*
- *$\vec{A}$ is a ladder system over $T'=T\restriction E^\kappa_\omega$;*
- *The graph $(T,E_{\vec{A}})$ is uncountably chromatic.*

Then there exists a zero-dimensional topology τ on T such that $X:=(T,\tau)$ is a regular space of character $\mathfrak{b}$ satisfying $X\to(\mathrm{top}\,(\omega+1))^1_\omega$ and $X\nrightarrow(\mathrm{top}\,(\omega^2+1))^1_1$.

Proof. Let $\vec{A}=\langle A_t \mid t\in T'\rangle$ denote the above ladder system. We now build another ladder system $\langle B_t \mid t\in T'\rangle$ with the property that $A_t\cap T'\subseteq B_t\subseteq T\restriction(E^\kappa_1\cup E^\kappa_\omega)$ for all $t\in T'$. To this end, for each $t\in T'$, we consider three options:

- ▶ If $A_t\cap T'$ is infinite, then let $B_t := A_t\cap T'$.
- ▶ If $A_t\cap T'$ is finite, but $t\in T\restriction(E^\kappa_\omega\cap\mathrm{acc}^+(E^\kappa_\omega))$, then let B_t be some cofinal subset of $t_\downarrow\cap E^\kappa_\omega$ of order-type ω, with $A_t\cap T'\subseteq B_t$.
- ▶ Otherwise, let B_t be some cofinal subset of $t_\downarrow$ of order-type ω all of whose nodes s with $\mathrm{cf}(\mathrm{dom}(s))\neq 1$ are the ones from $A_t\cap T'$.

Next, for every $t\in T$ and every $i<\omega$, we consider two cases depending on whether $B_t(i)$ — the i^{th}-element of B_t — belongs to T':

- ▶ If $B_t(i)\in T'$, then let $\langle a_{t,i}(j) \mid j<\omega\rangle$ be a strictly increasing sequence of nodes converging to $B_t(i)$. We also require that $a_{t,i+1}(0)$ be bigger than $B_t(i)$ for all $i<\omega$.
- ▶ Otherwise, let $\langle a_{t,i}(j) \mid j<\omega\rangle$ be the constant sequence whose sole element is $B_t(i)\restriction(\max(\mathrm{dom}(B_t(i))))$.

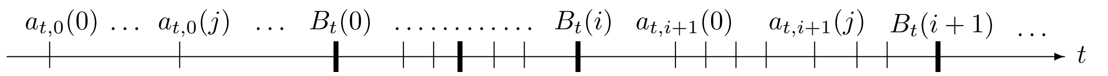

Figure 1: Illustration of the ladders assigned to a node $t \in T \restriction E^\kappa_\omega \cap \mathrm{acc}^+(E^\kappa_\omega)$.

Claim 2.1.1. *There exist a family $\mathcal{F} \subseteq {}^\omega\omega$ of size $\mathfrak{b}$ such that:*

- *for every $A \in [\omega]^\omega$, for every function $g : A \to \omega$, there exists $f \in \mathcal{F}$ for which $\{n \in A \mid g(n) \leq f(n)\}$ is infinite;*
- *$\mathcal{F}$ is closed under pointwise maximum, i.e., for all $f, g \in \mathcal{F}$, the function $n \mapsto \max\{f(n), g(n)\}$ is in $\mathcal{F}$, as well.*

Proof. This is well-known, but we include an argument anyway. By [Rin22, Proposition 2.4], $m_f(\omega, \omega, \omega, \omega) = \mathfrak{b}$, hence, we may fix a family $\mathcal{H}$ of functions from ω to $[\omega]^{<\omega}$ such that, for every $A \in [\omega]^\omega$, and every function $g : A \to \omega$, there exists $h \in \mathcal{H}$ for which $\{n \in A \mid g(n) \in h(n)\}$ is infinite. Now, let $\mathcal{F}$ denote the smallest subfamily of ${}^\omega\omega$ that covers $\{\sup \circ h \mid h \in \mathcal{H}\}$ and that is closed under pointwise maximum.[3] Clearly, $\mathcal{F}$ is as sought. □

Let $\mathcal{F}$ be given by the claim. For all $s, t \in T$, denote $(s, t] := \{x \in T \mid s \subseteq x \subsetneq t\}$. We shall now define a topology τ over T by defining a system $\langle \mathcal{N}_t \mid t \in T \rangle$ of local bases. For every $t \in T \setminus T'$, set $\mathcal{N}_t := \{\{t\}\}$. For every $t \in T'$, set $\mathcal{N}_t := \{N_t(f, j) \mid f \in \mathcal{F}, j < \omega\}$, where

$$N_t(f, j) = \{t\} \cup \biguplus \{(a_{t,i}(f(i)), B_t(i)] \mid j \leq i < \omega\}.$$

Since $\mathcal{F}$ is closed under pointwise maximum, $\mathcal{N}_t$ is indeed closed under intersections. In addition, for every element s of a neighborhood $N_t(f, j)$, there exists $N \in \mathcal{N}_s$ with $N \subseteq N_t(f, j)$. Specifically:

- If $s \in T \setminus T'$, then $N := \{s\}$ does the job;
- If $s \in T' \setminus \{t\}$, then there exists a unique $i \in \omega \setminus j$ such that $s \in (a_{t,i}(f(i)), B_t(i)]$, and so by picking a large enough k to satisfy $(a_{t,i}(f(i)) \subseteq B_s(k)$, we get that $N_s(g, k+1) \subseteq N_t(f, j)$ for any choice of $g \in \mathcal{F}$.

As $\bigcap \mathcal{N}_t = \{t\}$ for every $t \in T$, we altogether conclude that $X = (T, \tau)$ is a T_1 topological space. As $|\mathcal{N}_t| \leq |\mathcal{F} \times \omega| = \mathfrak{b}$ for every $t \in T$, we get that $\chi(X) \leq \mathfrak{b}$. Since X is T_1, to show that X is regular, it suffices to prove that the space X is zero-dimensional.

Claim 2.1.2. *Every $N \in \bigcup_{t \in T} \mathcal{N}_t$ is τ-closed.*

Proof. Let $t \in T'$, $f \in \mathcal{F}$, $j < \omega$, and we shall show that that $N_t(f, j)$ is τ-closed. To this end, let $s \in T \setminus N_t(f, j)$.

▶ If $s \notin T'$, then $\{s\}$ is a neighborhood of s disjoint from $N_t(f, j)$.

▶ If $s \in T'$ and $s \subseteq B_t(0)$, then $N_s(g, 0)$ is readily disjoint from $N_t(f, j)$ for any choice of $g \in \mathcal{F}$.

▶ If $s \in T'$ and $B_t(i) \subseteq s \subseteq B_t(i+1)$, then find a large enough $k < \omega$ such that $B_t(i) \subseteq B_s(k)$, and note that $N_s(g, k+1)$ is disjoint from $N_t(f, j)$ for any choice of $g \in \mathcal{F}$.

[3] We use sup instead of max, since $\sup(x)$ is meaningful for any set x, including $x = \emptyset$.

▶ If $s \in T'$ and $s \notin t_{\downarrow}$, then $r := s \cap t$ is an element of T that constitutes the meet of s and t. Find a large enough k such that $r \subseteq B_s(k)$ and note that for any choice of $g \in \mathcal{F}$, $N_s(g, k+1)$ is disjoint from $t_{\downarrow}$, and hence from $N_t(f, j)$. □

Claim 2.1.3. $X \rightarrow (\mathrm{top}\,(\omega+1))^1_\omega$.

Proof. Let $c : T \rightarrow \omega$ be a given a coloring. It suffices to find a $t \in T'$ such that $\{s \in B_T \mid c(s) = c(t)\}$ is infinite. Towards a contradiction, suppose that $\{s \in B_T \mid c(s) = c(t)\}$ is finite for every $t \in T'$. It follows that we may define a function $d : T \rightarrow \omega \times 2 \times \omega$ by recursion on the levels of T, as follows:

$$d(t) := \begin{cases} \langle c(t), 1, \max\{0, n+1 \mid \exists s \in B_t\,[c(s) = c(t)\ \&\ d(s) = \langle c(s), 1, n\rangle]\}\rangle, & \text{if } t \in T' \\ \langle c(t), 0, 0\rangle, & \text{otherwise.} \end{cases}$$

Recalling that $(T, E_{\vec{A}})$ is uncountably chromatic, we may now find $\{s, t\} \in E_{\vec{A}}$ such that $d(s) = d(t)$. By possibly switching the roles of s and t, we may assume that $t \in T'$ and $s \in A_t$. As $t \in T'$, it follows that $d(t) = (c(t), 1, m)$ for some $m < \omega$. As $d(s) = d(t)$, it follows that $c(s) = c(t)$ and $s \in T'$, and hence $s \in B_t$. But then the definition of $d(t)$ implies that the third coordinate of $d(t)$ is bigger than the corresponding one of $d(s)$. This is a contradiction. □

Claim 2.1.4. $X \nrightarrow (\mathrm{top}\,(\omega^2+1))^1_1$.

Proof. Towards a contradiction, suppose that $\phi : \omega^2 + 1 \rightarrow X$ is an homeomorphism. For every $n < \omega$, since $\omega \cdot (n+1)$ is an accumulation point of the interval $A_n := (\omega \cdot n, \omega \cdot (n+1))$, the singleton $\{\phi(\omega \cdot (n+1))\}$ cannot be τ-open, so that the node $t_n := \phi(\omega \cdot (n+1))$ must be in T' and $\phi[A_n]$ must contain an infinite sequence converging to t_n. Likewise, $\{t_n \mid n < \omega\}$ must contain an infinite sequence converging to the node $t_\omega := \phi(\omega^2)$. It thus follows that there exists a strictly increasing and continuous map $\psi : \omega^2 + 1 \rightarrow \omega^2 + 1$ such that $\phi \circ \psi$ is a strictly increasing and continuous map from $\omega^2 + 1$ to T. For notational simplicity, we assume ψ is the identity, so that $\langle t_n \mid n < \omega\rangle$ is a strictly increasing sequence of nodes in T' converging to t_ω. In particular, $t_\omega \in T \restriction (E^\kappa_\omega \cap \mathrm{acc}^+(E^\kappa_\omega))$.

As $\mathrm{otp}(B_{t_\omega}) = \omega < \omega^2 = \mathrm{otp}(\phi[\omega^2])$, we may fix a map $d : \omega \rightarrow \phi[\omega^2] \setminus B_t$ such that $\langle d(n) \mid n < \omega\rangle$ is a strictly increasing increasing sequence of nodes converging to t_ω. Consequently, the following set is infinite:

$$A := \{i \in \omega \setminus \{0\} \mid (B_t(i-1), B_t(i)] \text{ has an element of } \mathrm{Im}(d)\}.$$

It follows that for every $i \in A$, we may let

$$m_i := \max\{m < \omega \mid B_t(i-1) \subsetneq d(m) \subseteq B_t(i)\}.$$

Define a function $g : A \rightarrow \omega$ defined via

$$g(i) := \min\{j < \omega \mid d(m_n) \subseteq a_{t,i}(j)\}.$$

Recalling that $\mathcal{F}$ was given by Claim 2.1.1, we now pick $f \in \mathcal{F}$ such that $I := \{n \in A \mid g(n) \leq f(n)\}$ is infinite. For every $i \in I$, it is the case that

$$B_t(i-1) \subsetneq d(m_i) \subseteq a_{t,i}(g(i)) \subseteq a_{t,i}(f(i)) \subsetneq B_t(i).$$

Therefore, for every node s in the set $D := \{d(m_i) \mid i \in I\}$, there exists an $i \in I$ such that $D \cap (B_t(i-1), B_t(i)] = \{s\}$. So D is an infinite discrete subset of the compact set $\phi[\omega^2+1]$. This is a contradiction. □

It now follows from [KW87, Theorem 1] that $\chi(X) \geq \mathfrak{b}$. Altogether, the space X is as sought. $\square$

We are now ready to prove Theorem A.

Corollary 2.2. *There exists a zero-dimensional regular space X of size continuum, of character $\mathfrak{b}$, satisfying $X \to (\mathrm{top}\ \omega+1)^1_\omega$, but not $X \to (\mathrm{top}\ \omega^2+1)^1_1$.*

Proof. By Theorem 2.1, it suffices to find a streamlined tree $T \subseteq {}^{<\omega_1}\omega_1$ of size continuum, and a ladder system $\vec{A}$ over $T' := T \restriction \mathrm{acc}(\omega_1)$ such that the graph $(T, E_{\vec{A}})$ is uncountably chromatic. A tree with the same key features was constructed by D. Soukup in [Sou15, Theorem 3.5], though it was not streamlined. By abstract nonsense considerations (see [BR21, Lemma 2.5]), this should not make any difference. As the argument in [BR21] does not deal with the adjacent ladder system, we spell out the details in here.

Soukup's tree is the tree $T(S) := \{x \subseteq \omega_1 \mid \mathrm{acc}^+(x) \subseteq x \subseteq S\}$ for an arbitrary choice of a stationary and co-stationary subset S of ω_1, ordered by the end-extension relation, $\sqsubseteq$. It comes equipped with a sequence $\vec{C} = \langle C_x \mid x \in T(S)\rangle$ such that C_x is either a finite subset of $x_\downarrow$ or a cofinal subset of $x_\downarrow$ of order-type ω. In addition, the corresponding graph $(T(S), \{\{y,x\} \mid x \in T(S), y \in C_x\})$ is uncountably chromatic.

As S is stationary, $T(S)$ contains infinite sets. As S is co-stationary, every element of $T(S)$ is countable. Altogether $|T(S)| = 2^{\aleph_0}$. As every $x \in T(S)$ is a closed countable set of countable ordinals, its corresponding collapsing map $\pi_x : \mathrm{otp}(x) \to x$ is an element of $\bigcup_{\beta\in\mathrm{nacc}(\omega_1)} {}^\beta\omega_1$. In addition, for every pair $x \sqsubset y$ of nodes in $T(S)$, it is the case that $\pi_x \subset \pi_y$. Thus, altogether,

$$T := \{\pi_x \restriction \alpha \mid x \in T(S), \alpha < \omega_1\}$$

is a streamlined tree satisfying:

- $x \mapsto \pi_x$ forms an order-isomorphism from $(T(S), \sqsubseteq)$ to $(T \restriction \mathrm{nacc}(\omega_1), \subseteq)$;
- every element of $T \restriction \mathrm{acc}(\omega_1)$ admits a unique immediate successor.[4]

We shall now define the ladder system $\vec{A} = \langle A_t \mid t \in T'\rangle$, for $T' := T \restriction \mathrm{acc}(\omega_1)$, as follows. Given $t \in T \restriction \mathrm{acc}(\omega_1)$, let x_t denote the unique element of $T(S)$ such that π_{x_t} is the immediate successor of t. Now consider the following possibilities:

▶ If $|C_{x_t}| < \aleph_0$, then let A_t be an arbitrary cofinal subset of $t_\downarrow$ of order-type ω.

▶ Otherwise, C_{x_t} is a cofinal subset of $(x_t)_\downarrow$ of order-type ω, and hence

$$A_t := \{\pi_y \restriction \sup(\mathrm{otp}(y)) \mid y \in C_{x_t}\}$$

is a cofinal subset of $t_\downarrow$ of order-type ω.

Claim 2.2.1. *The graph $(T, E_{\vec{A}})$ is uncountably chromatic.*

Proof. Let $c : T \to \omega$ be given, and we shall find $s \subset t$ such that $c(s) = c(t)$.

As in the proof of Claim 2.1.3, by recursion on the levels of the tree we may construct a coloring $d : T(S) \to \omega$ satisfying the following for every $x \in T(S)$:

(1) If C_x is finite, then $d(x)$ is an odd positive integer that does not belong to $\{d(y) \mid y \in C_x\}$;
(2) If C_x is infinite, then $d(x) = c(\pi_x \restriction \sup(\mathrm{otp}(x))) \cdot 2$.

As the graph $(T(S), \{\{y,x\} \mid x \in T(S), y \in C_x\})$ is uncountably chromatic, we now pick a pair $y \sqsubset x$ of nodes in $T(S)$ such that $d(y) = d(x)$. Denote:

[4] Indeed, the immediate successor of a node $t \in T \restriction \mathrm{acc}(\omega_1)$ is π_x for $x := \mathrm{Im}(t) \cup \{\sup(\mathrm{Im}(t))\}$.

- $t := \pi_x \restriction \sup(\mathrm{otp}(x))$, and
- $s := \pi_y \restriction \sup(\mathrm{otp}(y))$.

As $d(x) = d(y)$, by the choice of d, C_x cannot be finite, so the only other option is that C_x is a cofinal subset of $x_\downarrow$ of order-type ω. In particular, $x_\downarrow$ cannot have a maximal element, and hence $\mathrm{otp}(x) = \alpha + 1$ for some $\alpha \in \mathrm{acc}(\omega_1)$. Therefore, π_x is an immediate successor of the above node t, so that $t \in T \restriction \mathrm{acc}(\omega_1)$ and $x_t = x$. It thus follows from the definition of A_t that $s \in A_t$.

Finally, as C_x is not finite, $d(x) = c(t) \cdot 2$. From $d(y) = d(x)$ being even, we then infer that $d(y) = c(s) \cdot 2$. Altogether, $c(s) = c(t)$, as sought. □

This completes the proof. □

3. Forcing highly chromatic Hajnal-Máté graphs

A *Hajnal-Máté graph* is a graph of the form $G = (\kappa, E)$, where κ is a cardinal, E is a subset of $[\kappa]^2$, and for every pair $\beta < \gamma$ of ordinals from κ, $\sup\{\alpha < \beta \mid \{\beta, \gamma\} \in E\} < \beta$. The existence of an uncountably chromatic Hajnal-Máté graph over ω_1 gives rise to a tree T and a ladder system $\vec{A}$ satisfying the hypotheses of Theorem 2.1 by identifying ω_1 with the streamlined tree $T := {}^{<\omega_1}1$.

In this section, we highlight a class of notions of forcing that inevitably add highly chromatic Hajnal-Máté graphs.

Definition 3.1. Let $\mathbb{P} = (P, \leq)$ denote a notion of forcing, and λ denote an infinite regular cardinal.

- $\mathbb{P}$ is ${}^{\lambda}\lambda$*-bounding* iff for every $g \in {}^{\lambda}\lambda \cap V^{\mathbb{P}}$, there exists some $f \in {}^{\lambda}\lambda \cap V$ such that $g(\alpha) \leq f(\alpha)$ for all $\alpha < \lambda$;
- $\mathbb{P}$ satisfies the λ^+*-stationary chain condition* (λ^+*-stationary-cc*, for short) iff for every sequence $\langle p_\delta \mid \delta < \lambda^+\rangle$ of conditions in $\mathbb{P}$ there are a club $D \subseteq \lambda^+$ and a regressive map $h : D \cap E^{\lambda^+}_{\lambda} \rightarrow \lambda^+$ such that for all $\gamma, \delta \in \mathrm{dom}(h)$, if $h(\gamma) = h(\delta)$, then p_γ and p_δ are compatible.

Theorem 3.2. *Suppose that λ is an infinite regular cardinal, and $\mathbb{P}$ is a λ^+-stationary-cc notion of forcing satisfying at least one of the following:*

1. *$\mathbb{P}$ preserves the regularity of λ, and is not ${}^{\lambda}\lambda$-bounding;*
2. *$\mathbb{P}$ forces that $\mathrm{cf}(\lambda) < |\lambda|$. In addition, $\mathrm{cf}(\mathrm{NS}_\lambda, \subseteq) = \lambda^+$;*
3. *In $V^{\mathbb{P}}$, there exists a cofinal subset $\Lambda \subseteq \lambda$ such that for every function $f \in {}^{\lambda}\lambda \cap V$, there exists some $\xi \in \Lambda$ with $f(\xi) < \min(\Lambda \setminus (\xi + 1))$.*

Then, in $V^{\mathbb{P}}$, there exists a sequence $\langle C_\delta \mid \delta \in E^{\lambda^+}_{\lambda}\rangle$ satisfying the following:

- *For every $\delta \in E^{\lambda^+}_{\lambda}$, C_δ is a club in δ of order-type λ;*
- *For every coloring $c : E^{\lambda^+}_{\lambda} \rightarrow \lambda$, there are $\gamma, \delta \in E^{\lambda^+}_{\lambda}$ such that $\gamma \in C_\delta$ and $c(\gamma) = c(\delta)$.*

Proof. By [BR19, Proposition 3.1], Clause (3) follows from Clauses (1) and (2), so hereafter, we shall assume Clause (3).

Work in V. Write $\Delta := E^{\lambda^+}_{\lambda}$. For each $\delta \in \Delta$, let $\pi_\delta : \lambda \rightarrow \delta$ denote the inverse collapse of some club in δ, and let $\psi_\delta : \lambda \leftrightarrow \delta$ be some bijection.

Next, let G be $\mathbb{P}$-generic over V, and work in $V[G]$. By Clause (3) and the proof of [BR19, Lemma 3.2], we may fix a club $\Lambda \subseteq \lambda$ of order-type $\mathrm{cf}(\lambda)$, such that for every function $f \in {}^{\lambda}\lambda \cap V$, $\sup\{\xi \in \Lambda \mid f(\xi) < \min(\Lambda \setminus (\xi + 1))\} = \lambda$.

Let $\delta \in \Delta$. Clearly, $B_\delta := \pi_\delta[\Lambda]$ is a club in δ of order-type $\mathrm{cf}(\lambda)$. Next, let C_δ be the ordinal closure below δ of the following set

$$B_\delta \cup \bigcup \{\psi_\delta[\alpha^+] \cap (\pi_\delta(\alpha), \pi_\delta(\alpha^+)) \mid \alpha \in \Lambda \ \&\ \alpha^+ = \min(\Lambda \setminus (\alpha+1))\}.$$

Note that, for every pair $\beta < \beta^+$ of successive elements of $\pi_\delta[\Lambda]$, $C_\delta \cap (\beta, \beta^+)$ is covered by the closure of $\psi_\delta[\mathrm{otp}(\Lambda \cap \beta^+)]$, which is a set of size $< \lambda$. Therefore, $\mathrm{otp}(C_\delta) \leq \lambda$.

Claim 3.2.1. *For every $\Gamma \in [\lambda^+]^{\lambda^+}$ from V, for every $\delta \in \mathrm{acc}^+(\Gamma) \cap \Delta$, it is the case that $\sup(C_\delta \cap \Gamma) = \delta$.*

Proof. Let $\Gamma \in [\lambda^+]^{\lambda^+}$ in V. Let $\delta \in \Delta \cap \mathrm{acc}^+(\Gamma)$ and $\epsilon < \delta$; we shall find $\gamma \in \Gamma \cap C_\delta$ above ϵ. As $\delta \in \mathrm{acc}^+(\Gamma)$, we may define a function $f_0 : \lambda \to \lambda$ via

$$f_0(\alpha) := \min\{\alpha' < \lambda \mid (\pi_\delta(\alpha), \pi_\delta(\alpha')) \cap \Gamma \neq \emptyset\}.$$

Then, we may define a function $f_1 : \lambda \to \lambda$ via:

$$f_1(\alpha) := \min\{i < \lambda \mid \psi_\delta(i) \in (\pi_\delta(\alpha), \pi_\delta(f_0(\alpha))) \cap \Gamma\}.$$

Define $f : \lambda \to \lambda$ via $f(\alpha) := \max\{f_0(\alpha), f_1(\alpha)\}$. As $\Gamma \in V$, the function f is in ${}^\lambda\lambda \cap V$, and hence $A := \{\xi \in \Lambda \mid f(\xi) < \min(\Lambda \setminus (\xi+1))\}$ is cofinal in λ. Pick a large enough $\alpha \in A$ such that $\pi_\delta(\alpha) \geq \epsilon$. Denote $\alpha^+ := \min(\Lambda \setminus (\alpha+1))$. Then $\alpha' := f_0(\alpha)$ and $i := f_1(\alpha)$ are both less than $< \alpha^+$. So

$$\psi_\delta(i) \in \psi_\delta[\alpha^+] \cap (\pi_\delta(\alpha), \pi_\delta(\alpha^+)) \cap \Gamma,$$

meaning that $\psi_\delta(i)$ is an element of $C_\delta \cap \Gamma$ above ϵ. $\square$

Work in V. Suppose that p is a condition forcing that $\dot{c}$ is a name for a function from Δ to λ. For each $\delta \in \Delta$, let p_δ be a condition extending p and deciding $\dot{c}(\delta)$ to be, say, τ_δ. Fix a club $D \subseteq \lambda^+$ and a regressive map $h : D \cap E^{\lambda^+}_\lambda \to \lambda^+$ such that for all $\gamma, \delta \in \mathrm{dom}(h)$, if $h(\gamma) = h(\delta)$ then p_γ and p_δ are compatible.

Find $(\tau, \eta) \in \lambda \times \lambda^+$ for which

$$\Gamma := \{\delta \in \Delta \cap D \mid \tau_\delta = \tau \ \&\ h(\delta) = \eta\}$$

is stationary. As $\mathrm{acc}^+(\Gamma)$ is a club (in V), Claim 3.2.1 provides us with a $\delta \in \Gamma$ such that $\sup(C_\delta \cap \Gamma) = \delta$. Pick $\gamma \in C_\delta \cap \Gamma$. As $h(\delta) = \eta = h(\gamma)$, we may pick some q extending p_δ and p_γ. Then, q is an extension of p forcing that $\gamma, \delta \in \Delta$ and $c(\gamma) = \tau = c(\delta)$. $\square$

Corollary 3.3. *If λ is a measurable cardinal, then in the forcing extension by Prikry forcing using a normal measure on λ, there exists a Hajnal-Máté graph over λ^+ of chromatic number λ^+.* $\square$

Corollary 3.4. *After forcing to add any number of Cohen reals, there is an uncountably chromatic Hajnal-Máté graph over ω_1.* $\square$

Putting the preceding together with Theorem 2.1, we obtain Theorem B:

Corollary 3.5. *After forcing to add any number of Cohen reals, there exists a zero-dimensional regular space X of size $\aleph_1$, of character $\mathfrak{b}$, satisfying $X \to (\mathrm{top}\ \omega+1)^1_\omega$, but not $X \to (\mathrm{top}\ \omega^2+1)^1_1$.* $\square$

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