# PROXY PRINCIPLES IN COMBINATORIAL SET THEORY 

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#### Abstract

The parameterized proxy principles were introduced by Brodsky and Rinot in a 2017 paper, as new foundations for the construction of $\kappa$-Souslin trees in a uniform way that does not depend on the nature of the (regular uncountable) cardinal $\kappa$. Since their introduction, these principles have facilitated construction of Souslin trees with complex combinations of features, and have enabled the discovery of completely new scenarios in which Souslin trees must exist. Furthermore, the proxy principles have found new applications beyond the construction of trees.

This paper opens with a comprehensive exposition of the proxy principles. We motivate their very definition, emphasizing the utility of each of the parameters and the consequent flexibility that they provide. We then survey the findings surrounding them, presenting a rich spectrum of unrelated models and configurations in which the proxy principles are known to hold, and showcasing a gallery of Souslin trees constructed from the principles.

The last two sections of the paper offer new results. In particular, for every positive integer $n$, we give a construction of a $\lambda^{+}$-Souslin tree all of whose $n$-derived trees are Souslin, but whose $(n+1)$-power is special.


## 1. Introduction

In his trailblazing paper analyzing the fine structure of the constructible hierarchy, appearing more than fifty years ago, Jensen proved [Jen72, Theorem 6.2] that in Gödel's constructible universe L, there exists a $\kappa$-Souslin tree for every regular uncountable cardinal $\kappa$ that is not weakly compact. Jensen's proof goes through two newly-minted combinatorial principles, diamond $(\diamond)$ and square ( $\square$ ), introduced in $\S 5-6$ of that paper. The isolation and formulation of these new axioms have made the combinatorial properties of L accessible to generations of set theorists, enabling combinatorial constructions of complicated objects and leading to the settling of open problems in fields including topology, measure theory, and group theory.

Let $\kappa$ denote a regular uncountable cardinal. Recall that a coherent $C$ sequence over $\kappa$ is a sequence $\left\langle C_{\alpha} \mid \alpha<\kappa\right\rangle$ such that, for every limit ordinal $\alpha<\kappa$ :

- $C_{\alpha}$ is a club subset of $\alpha$; and

[^0]- $C_{\alpha} \cap \bar{\alpha}=C_{\bar{\alpha}}$ for every $\bar{\alpha} \in \operatorname{acc}\left(C_{\alpha}\right) .{ }^{1}$

An easy way to obtain such a sequence is to fix at the outset some club $D$ in $\kappa$, and then let $C_{\alpha}:=D \cap \alpha$ for every $\alpha \in \operatorname{acc}(D)$ and $C_{\alpha}:=\alpha \backslash \sup (D \cap \alpha)$ for any other $\alpha$. More interesting, however, are principles asserting the existence of coherent $C$-sequences satisfying some non-triviality condition. For example:

- Jensen's square principle $\square_{\lambda}$ of $[J e n 72, \S 5.1]$ asserts the existence of a coherent $C$-sequence over $\lambda^{+},\left\langle C_{\alpha} \mid \alpha<\lambda^{+}\right\rangle$, such that otp $\left(C_{\alpha}\right) \leq \lambda$ for every $\alpha<\lambda^{+}$.
- For a stationary set $E \subseteq \operatorname{acc}(\kappa)$, the principle described in the conclusion of [Jen72, Theorem 6.1], which is commonly denoted $\square(E)$, asserts the existence of a coherent $C$-sequence over $\kappa,\left\langle C_{\alpha} \mid \alpha<\kappa\right\rangle$, that avoids $E$, meaning that $\operatorname{acc}\left(C_{\alpha}\right) \cap E=\emptyset$ for every $\alpha<\kappa$.
- Todorčević's square principle $\square(\kappa)$ [Tod87] asserts the existence of a coherent $C$-sequence over $\kappa,\left\langle C_{\alpha} \mid \alpha<\kappa\right\rangle$, that is unthreadable meaning that there is no club $D \subseteq \kappa$ such that $D \cap \alpha=C_{\alpha}$ for every $\alpha \in \operatorname{acc}(D)$.
In the decades ensuing since [Jen72], many variants of both diamond and square have appeared - strengthening, weakening, or adapting each one as needed to solve various combinatorial problems. ${ }^{2}$ Strong combinations of square and diamond such as Gray's principle $\nabla_{\lambda}$ from [Gra80] and its further strengthening $\nabla_{\lambda}^{+}$from [RS17] have appeared as well. Square principles are primarily concerned with coherence, whereas diamond principles are prediction principles, asserting that objects of size $\kappa$ can be predicted by means of their initial segments.

The construction of complicated combinatorial objects such as $\kappa$-Souslin trees requires both prediction and coherence. Classical constructions of Souslin trees have followed Jensen's lead in requiring the $\diamond$-sequence's predictions to occur in some nonreflecting stationary set $E$, which must then be avoided by the square sequence in order not to interfere with building higher levels of the tree. There are particular scenarios where the coherence requirements are transparent, such as for $\kappa=\aleph_{1}$ where $\square_{\aleph_{0}}$ holds trivially, and we thus find many classical constructions tailored to such cases alone.

Examining the classical literature, one sees that construction of a $\kappa$-Souslin tree with an additional property (such as complete or regressive; rigid or homogeneous; specializable or non-specializable; admitting an ascent path or omitting an ascending path; free or uniformly coherent) often depends on the nature of the cardinal $\kappa$ (be it a successor of a regular, a successor of

[^1]a singular, or an inaccessible - in some cases even depending on whether $\kappa$ is the successor of a singular cardinal of countable or of uncountable cofinality). To obtain the additional features, constructions include extensive bookkeeping, counters, timers, coding and decoding, whose particular nature makes it difficult to transfer the process from one type of cardinal to another.

What happens if we want to replace an axiom known to imply the existence of a $\kappa_{0}$-Souslin tree with strong properties by an axiom from which a plain $\kappa_{1}$-Souslin tree can be constructed? Do we have to revisit each scenario and tailor each of these particular constructions in order to derive a tree with strong properties?

The parameterized proxy principles were introduced by Brodsky and Rinot in [BR17a] with the goal of overcoming this problem by offering new foundations for constructing $\kappa$-Souslin trees for an arbitrary regular uncountable cardinal $\kappa$. So far, they have been used to construct $\kappa$-Souslin trees in [BR17a, BR17b, BR19b, BR19c, BR21, RYY23a, RYY23b, Yad23]. The core feature of the proxy principles is that the non-triviality of a square-like sequence is ensured by a hitting requirement - a weak form of prediction, to be explained in Section 2 - that is tailored for the desired construction, rather than by the classical non-triviality conditions which were not flexible enough to obtain the optimal conclusions in many cases. This tailoring enables uniform construction of $\kappa$-Souslin trees and other combinatorial objects, oblivious to the nature of $\kappa$.

By incorporating such a hitting feature into the square-like proxy principle, one can reduce the requirements on the $\diamond$-sequence: In [BR17a], $\chi$-complete $\kappa$-Souslin trees were constructed using $\diamond(\kappa)$ instead of $\diamond(E)$ for some non-reflecting stationary subset $E$ of $E_{\geq \chi}^{\kappa},{ }^{3}$ and in [BR21], $\diamond(\kappa)$ was further relaxed to the arithmetic hypothesis $\kappa^{<\kappa}=\kappa$.

Since their introduction, the proxy principles have found new applications beyond the construction of Souslin trees. In conjunction with $\diamond$, we observe the following:

- In [BR19a], these principles were used to construct distributive Aronszajn trees, as well as special trees with a non-special projection.
- In [Kru23], these principles were used to a construct a large pairwise far family of Aronszajn trees.
- In [Sha23], these principles were used to construct minimal non- $\sigma$ scattered linear orders.

Furthermore, as a result of incorporating the hitting feature into the square-like sequence, applications of the proxy principles in the absence of an arithmetic hypothesis, let alone a prediction principle, have emerged, as follows:

[^2]- In [LHR19], these principles were used to construct a highly chromatic graph all of whose smaller subgraphs are countably chromatic.
- In [IR23, Lemma 5.9], these principles were used to construct Ulamtype matrices.
- In [RS23, §5], these principles were used to construct a Dowker space whose square is still Dowker;
- In $[$ RZ23, $\S 7]$, these principles were used to construct a $C$-sequence suitable for conducting walks on ordinals. ${ }^{4}$
Alongside the wealth of applications of the proxy principles, we turn our attention to the obvious question: How do these new proxy principles compare to the classical combinatorial axioms? In [BR17a, BR21], a bridge to the classical foundations was built, establishing that all previously known $\diamond$-based constructions of $\kappa$-Souslin trees may be redirected through the new foundations. Concurrently, in [BR17a, LH17, Rin17, RS17, LHR19, BR19a, BR19b, BR19c, Rin19, BR21, Rin22], instances of the proxy principles have been shown to hold in many unrelated configurations, so that any conclusion derived from those instances is known to hold in completely new, unrelated scenarios. Significantly, in addition to scenarios conforming to the spirit of $\mathrm{V}=\mathrm{L}$, it is shown that some instances of the proxy principles may consistently hold above large cardinals, at a cardinal satisfying stationary reflection, or in models of strong forcing axioms such as Martin's Maximum. Furthermore, various notions of forcing inadvertently add instances of the proxy principles. Thus, any application of the proxy principle will automatically be known to hold in a rich spectrum of unrelated models.

Altogether, the proxy principles provide a successful disconnection between the combinatorial constructions and the study of the hypotheses themselves. This project thus has two independent tasks: Deriving rich applications of the proxy principles, and proving instances of the proxy principles in various scenarios.
1.1. This paper. The main goal of this paper is to make the proxy principles accessible to anyone with experience in combinatorial set theory. Until now, the various definitions and related results have been scattered throughout multiple lengthy papers, making it difficult for the interested researcher to adopt these principles as a starting point for deriving desired results. At this point, we believe that the proxy principles have attained a significant level of maturity, and we hope that by presenting this comprehensive exposition we can engage the reader and encourage them to join us in our adventure of applying the proxy principles to obtain optimal results.

We now present the breakdown of the current paper, as follows.
In Section 2, we give a simple example from infinite graph theory that motivates the very need for a parameterized proxy principle, and then patiently discuss each of the eight parameters of the proxy principle $\mathrm{P}(\ldots)$.

[^3]By the end of this section, the reader will hopefully agree that all of the parameters are quite natural indeed.

In Section 3, we gather configurations in which instances of the proxy principle hold, as established in the literature.

In Section 4, we list various types of Souslin trees that have been constructed using the proxy principles, indicate where each of these results may be found in the literature and what vector of parameters is known to be sufficient for the relevant construction.

The last two sections of this paper are dedicated to new proxy-based constructions of Souslin trees. In Section 5, we present a proxy-based construction of a large family of pairwise-Souslin trees. A sample corollary of the latter reads as follows.
Theorem A. Assuming $\mathrm{P}(\kappa, 2, \sqsubseteq, \kappa)$, if there exists a $\kappa$-Kurepa tree, then there exists a $\kappa$-Aronszajn tree $\mathbf{T}$ admitting $\kappa^{+}$-many $\kappa$-Souslin subtrees such that the product of any finitely many of them is again Souslin.

We shall also show that this result is optimal in the sense that the tree $\mathbf{T}$ itself cannot be $\kappa$-Souslin.

In Section 6, we present a proxy-based construction of a Souslin tree whose square is special. More generally:
Theorem B. For an infinite cardinal $\lambda$, assuming $\mathrm{P}_{\lambda}\left(\lambda^{+}, 2, \sqsubseteq, \lambda^{+}\right)$, for every positive integer $n$, there exists a $\lambda^{+}$-Souslin tree $\mathbf{T}$ satisfying the following:

- all n-derived trees of $\mathbf{T}$ are Souslin;
- the $(n+1)$-power of $\mathbf{T}$ is special.

Such an $\aleph_{1}$-tree (i.e., the case $\lambda=\omega$ ) was constructed by Abraham and Shelah in [AS93, §2] building on the approach from [DJ74, JJ74] of taking generics over countable models; hence the construction does not generalize to $\lambda$ singular. A construction for $\lambda$ singular (and $n=1$ ) was given by Abraham, Shelah and Solovay in [ASS87] exploiting the fact that $\square_{\lambda}$ for $\lambda$ singular may be witnessed by a $C$-sequence $\left\langle C_{\alpha} \mid \alpha<\lambda^{+}\right\rangle$with otp $\left(C_{\alpha}\right)<\lambda$ for all $\alpha<\lambda^{+}$. As such, it does not apply to $\lambda$ regular. The construction that will be given here is the first that works uniformly for $\lambda$ both regular and singular.
1.2. Notation and conventions. Throughout this paper, $\kappa$ denotes a regular uncountable cardinal, and $H_{\kappa}$ denotes the collection of all sets of hereditary cardinality less than $\kappa$. The Greek letters $\lambda, \Lambda, \mu, \nu, \chi, \theta, \vartheta$ will denote (possibly finite) cardinals, and $\alpha, \beta, \gamma, \delta, \epsilon, \varepsilon, \iota, \sigma, \varsigma, \xi$ will denote ordinals. The class of all infinite regular (resp. singular) cardinals is denoted by REG (resp. SING), and we write $\operatorname{Reg}(\kappa)$ for $\operatorname{REG} \cap \kappa$. For a set $X$, write $[X]^{\theta}$ for the collection of all subsets of $X$ of size $\theta$, and define $[X]^{<\theta}$ in a similar fashion.

In order to maintain the flow of the text, we decided not to pause to give the definitions of standard objects, giving them in footnotes, instead.

For the reader's benefit, an index of all of these definitions is provided on Page 44.

## 2. What are the proxy principles, anyway?

2.1. Motivation. Recall that a graph $\mathcal{G}$ is a pair $(V, E)$ where $E \subseteq[V]^{2}$, and that the chromatic number of $\mathcal{G}$, denoted $\operatorname{Chr}(\mathcal{G})$, is the least cardinal $\theta$ for which there exists a coloring $c: V \rightarrow \theta$ such that for every $\{x, y\} \in E$, $c(x) \neq c(y)$. We say that a graph $\mathcal{G}$ is countably chromatic iff $\operatorname{Chr}(\mathcal{G}) \leq \aleph_{0} ;$ otherwise, it is uncountably chromatic.

A Hajnal-Máté graph is a graph $\mathcal{G}=(V, E)$ in which $V=\omega_{1}$ and, for every $\alpha<\omega_{1}$, the set $A_{\alpha}:=\{\beta<\alpha \mid\{\alpha, \beta\} \in E\}$ is either finite or a cofinal subset of $\alpha$ of order-type $\omega$.

Martin's axiom at the level of $\aleph_{1}$ implies that all Hajnal-Máté graphs are countably chromatic (see [Fre84, Proposition 31G]), and the same assertion is also consistent with the continuum hypothesis (see [ADS78, Theorem 2.1]). So, what does it take for $\mathcal{G}$ to be uncountably chromatic? It can be verified that the following are equivalent:
(1) $\mathcal{G}$ is uncountably chromatic;
(2) For every sequence $\left\langle B_{n} \mid n<\omega\right\rangle$ of uncountable subsets of $\omega_{1}$, there exists some $\alpha<\omega_{1}$ such that $A_{\alpha}$ meets $B_{n}$ for every $n<\omega$;
(3) For every sequence $\left\langle B_{n} \mid n<\omega\right\rangle$ of uncountable subsets of $\omega_{1}$, there are stationarily many $\alpha<\omega_{1}$ such that $\sup \left(A_{\alpha} \cap B_{n}\right)=\alpha$ for every $n<\omega$.
The above connection between the hitting property of $\vec{A}=\left\langle A_{\alpha} \mid \alpha<\omega_{1}\right\rangle$ and the chromatic number of the associated graph generalizes, as follows. Let $\vec{A}=\left\langle A_{\alpha} \mid \alpha \in S\right\rangle$ be a ladder system over some subset $S$ of a regular uncountable cardinal $\kappa{ }^{5}$ Derive a graph $\mathcal{G}:=(\kappa, E)$ by letting $E:=\{\{\alpha, \beta\} \mid$ $\left.\alpha \in S, \beta \in A_{\alpha}\right\}$. Then, for every cardinal $\theta, \operatorname{Chr}(\mathcal{G})>\theta$ provided that for every sequence $\vec{B}=\left\langle B_{i} \mid i<\theta\right\rangle$ of cofinal subsets of $\kappa$, there exists some $\alpha \in S$ such that $\bigwedge_{i<\theta} A_{\alpha} \cap B_{i} \neq \emptyset$.

Now, what happens if one wants, say, a ladder-system graph on $\omega_{2}$ of chromatic number $\omega_{2}$ such that, in addition, all of its smaller subgraphs are countably chromatic? ${ }^{6}$ This quest for incompactness highlights a second feature that a ladder system may possess, namely, coherence. However, coherence properties are typically imposed upon ladder systems in which the $\alpha^{\text {th }}$ ladder is moreover a closed subset of $\alpha$; these are better known as $C$-sequences. For two sets of ordinals $x, y$, write $x \sqsubseteq y$ iff $x=y \cap \varepsilon$ for some ordinal $\varepsilon$. A $C$-sequence $\left\langle C_{\alpha} \mid \alpha<\kappa\right\rangle$ is coherent iff for all $\alpha<\kappa$ and $\bar{\alpha} \in \operatorname{acc}\left(C_{\alpha}\right)$, it is the case that $C_{\bar{\alpha}} \sqsubseteq C_{\alpha}$. In order to obtain a graph that

[^4]may satisfy the desired incompactness property, we impose an additional constraint on the pairs in $E$, as follows: ${ }^{7}$

Definition 2.1 (The $C$-sequence graph, [Rin15, LHR19]). Given a $C$-sequence $\vec{C}=\left\langle C_{\alpha} \mid \alpha<\kappa\right\rangle$, and a subset $G \subseteq \operatorname{acc}(\kappa)$, the graph $G(\vec{C})$ is the pair $(G, E)$, where

$$
E:=\left\{\{\alpha, \gamma\} \in[G]^{2} \mid \gamma \in C_{\alpha}, \min \left(C_{\gamma}\right)>\sup \left(C_{\alpha} \cap \gamma\right) \geq \min \left(C_{\alpha}\right)\right\} .
$$

Remark 2.2. For every pair $\gamma<\alpha$ of vertices that are adjacent in $G(\vec{C})$, it is the case that $\gamma$ is an element of $\operatorname{nacc}\left(C_{\alpha}\right) .^{8}$

Consider $G(\vec{C})$ for a given $C$-sequence $\vec{C}=\left\langle C_{\alpha} \mid \alpha<\kappa\right\rangle$ and subset $G \subseteq \operatorname{acc}(\kappa)$. For an ordinal $\delta<\kappa$, let $G(\vec{C}) \upharpoonright \delta$ denote the initial-segment graph $\left(G \cap \delta, E \cap[\delta]^{2}\right)$. The next fact tells us, in particular, that if $\vec{C}$ is coherent, then every proper initial segment of $G(\vec{C})$ is countably chromatic.

Fact 2.3 ([LHR19, Lemma 2.11(1)]). Let $\chi \in \operatorname{Reg}(\kappa)$. If $C_{\bar{\alpha}} \sqsubseteq C_{\alpha}$ for all $\alpha \in G$ and $\bar{\alpha} \in \operatorname{acc}\left(C_{\alpha}\right) \cap E_{\chi}^{\kappa}$, then $\operatorname{Chr}(G(\vec{C}) \upharpoonright \delta) \leq \chi$ for every $\delta<\kappa$.

Next, what does it take for the whole of $G(\vec{C})$ to have a large chromatic number? Based on what we saw earlier, one may guess that $\operatorname{Chr}(G(\vec{C}))>\theta$ provided that for every sequence $\vec{B}=\left\langle B_{i} \mid i<\theta\right\rangle$ of cofinal subsets of $\kappa$, there exists some $\alpha \in G$ such that $C_{\alpha}$ meets each of the $B_{i}$ 's. However, due to the particular nature of $E$ (recall Definition 2.1), here we would want $C_{\alpha}$ to meet each of the $B_{i}$ 's in two consecutive points. Specifically:
Fact 2.4 ([LHR19, Lemma 2.13]). For an infinite $\theta<\kappa, \operatorname{Chr}(G(\vec{C}))>\theta$, provided that for every sequence $\vec{B}=\left\langle B_{i} \mid i<\theta\right\rangle$ of cofinal subsets of $\kappa$, there exists an $\alpha \in G$ with $\min \left(C_{\alpha}\right) \geq \min \left(B_{0}\right)$ such that, for every $i<\theta$, there are $\beta, \gamma \in C_{\alpha} \cap B_{i}$ such that $\gamma=\min \left(C_{\alpha} \backslash(\beta+1)\right)$.

Altogether, to obtain a graph of the form $G(\vec{C})$ satisfying a desired incompactness property - large chromatic number for the whole graph, along with small chromatic number for its proper initial segments - it suffices to begin with a $C$-sequence $\vec{C}$ satisfying a coherence property as in the hypothesis of Fact 2.3 along with a hitting property as in the hypothesis of Fact 2.4. An axiom asserting the existence of a sequence satisfying a combination of coherence and hitting properties is what we call a proxy principle.
2.2. The proxy principles. In order to capture the considerations of the previous subsection while maintaining the flexibility to vary both the coherence and hitting features as needed to prove various desired results, one

[^5]would like to introduce a concise parameterized notation for the proxy principles. We would like to be able to express something along the following lines:

- There exists a system $\left\langle\mathcal{C}_{\alpha} \mid \alpha<\kappa\right\rangle$ with each $\mathcal{C}_{\alpha}$ a nonempty collection of closed cofinal subsets of $\alpha$;
- There is a prescribed bound for how many sets are there at each level, e.g., $\left|\mathcal{C}_{\alpha}\right|=1$ for every $\alpha<\kappa$ (as in the examples we have seen thus far), or more generally, for some fixed cardinal $\mu,\left|\mathcal{C}_{\alpha}\right|<\mu$ for every $\alpha<\kappa ;{ }^{9}$
- The elements of any level are compatible with the ones from below, i.e., there is a prescribed binary coherence relation $\mathcal{R}$ (such as $\sqsubseteq) ~$ such that, for every $\alpha<\kappa$, every $C \in \mathcal{C}_{\alpha}$, and every $\bar{\alpha} \in \operatorname{acc}(C)$, there exists a $D \in \mathcal{C}_{\bar{\alpha}}$ with $D \mathcal{R} C$;
- For some prescribed cardinal $\theta$, every family $\mathcal{B} \subseteq[\kappa]^{\kappa}$ of size $\theta$ gets "hit" at some level $\alpha$, i.e., each $C \in \mathcal{C}_{\alpha}$ meets each $B \in \mathcal{B}$. Looking at Fact 2.4, we may also want the $\alpha$ of interest to come from some prescribed set $G$, and we may want a meeting that is successful twice in a row, or, more generally, $\sigma$ many times in a row for some prescribed ordinal $\sigma$.
The above considerations lead us to the definition of the parameterized proxy principle. First, let us establish some notational conventions that we shall use throughout the rest of the paper:
- $\kappa$ is a regular uncountable cardinal;
- $\chi$ is an infinite regular cardinal $\leq \kappa$;
- $\nu$ and $\mu$ are cardinals such that $2 \leq \nu \leq \mu \leq \kappa^{+}$;
- $\mathcal{R}$ is a binary relation over $[\kappa]^{<\kappa}$;
- $\theta$ is a cardinal $\leq \kappa$;
- $\mathcal{S}$ is a nonempty collection of stationary subsets of $\kappa$;
- $\xi$ and $\sigma$ are ordinals $\leq \kappa$.

Definition 2.5 ([BR17a, BR21]). The proxy principle $\mathrm{P}_{\xi}^{-}(\kappa, \mu, \mathcal{R}, \theta, \mathcal{S}, \nu, \sigma)$ asserts the existence of a sequence $\overrightarrow{\mathcal{C}}=\left\langle\mathcal{C}_{\alpha} \mid \alpha<\kappa\right\rangle$ such that the following three requirements are satisfied:
(1) for every $\alpha<\kappa, \mathcal{C}_{\alpha}$ is a nonempty collection of less than $\mu$ many closed subsets $C$ of $\alpha$ with $\operatorname{ssup}(C)=\alpha$ and $\operatorname{otp}(C) \leq \xi ;$
(2) for all $\alpha<\kappa, C \in \mathcal{C}_{\alpha}$ and $\bar{\alpha} \in \operatorname{acc}(C)$, there is a $D \in \mathcal{C}_{\bar{\alpha}}$ such that $D \mathcal{R} C$;
(3) for every sequence $\left\langle B_{i} \mid i<\theta\right\rangle$ of cofinal subsets of $\kappa$, for every $S \in \mathcal{S}$, there exist stationarily many $\alpha \in S$ for which: ${ }^{10}$

[^6]- $\left|\mathcal{C}_{\alpha}\right|<\nu$, and
- for all $C \in \mathcal{C}_{\alpha}$ and $i<\min \{\alpha, \theta\}$ :

$$
\begin{equation*}
\sup \left\{\beta \in C \mid \operatorname{succ}_{\sigma}(C \backslash \beta) \subseteq B_{i}\right\}=\alpha \tag{*}
\end{equation*}
$$

Remark 2.6. $\operatorname{succ}_{\sigma}(D):=\{\delta \in \operatorname{nacc}(D) \mid 0<\operatorname{otp}(D \cap \delta) \leq \sigma\}$ is the set of the first $\sigma$ many successor elements of $D$, should they exist. In particular, for every $\beta \in C$ such that $\sup (\operatorname{otp}(C \backslash \beta)) \geq \sigma, \operatorname{succ}_{\sigma}(C \backslash \beta)$ is nothing but the next $\sigma$ many successor elements of $C$ above $\beta$. In the special case $\sigma=1$, requirement $(\star)$ above is equivalent to asserting that $\sup \left(\operatorname{nacc}(C) \cap B_{i}\right)=\alpha$.

One can consider the proxy principle's eight parameters together as a vector of parameters ( $\xi, \kappa, \mu, \mathcal{R}, \theta, \mathcal{S}, \nu, \sigma$ ), and then divide the vector's components into three groups, according to the clause of Definition 2.5 in which each parameter first appears. The first three parameters $\xi, \kappa, \mu$ are in Clause (1), which amounts to saying that $\overrightarrow{\mathcal{C}}$ is a $\xi$-bounded $\mathcal{C}$-sequence over $\kappa$ of width less than $\mu$. The fourth parameter $\mathcal{R}$ appears in Clause (2), which amounts to saying that $\overrightarrow{\mathcal{C}}$ is $\mathcal{R}$-coherent. The remaining parameters capture the hitting characteristics of $\overrightarrow{\mathcal{C}}: \theta$ tells us how many sets can be hit simultaneously, each element of $\mathcal{S}$ prescribes the location at which hitting must take place, $\nu$ forces the width to be locally small upon a successful hit, and $\sigma$ sets a minimum for the number of serial successful meets.

The special case $\xi=\kappa$ imposes no order-type restriction on $\overrightarrow{\mathcal{C}}$, in which case we can freely omit it, writing $\mathrm{P}^{-}(\kappa, \cdots)$ instead of $\mathrm{P}_{\xi}^{-}(\kappa, \cdots)$. A small $\xi$ is indeed stronger (see [Kön03, Lemma 3.13]), and imposing it enables stronger properties in the constructed object (see [Sha23]). In case $\kappa=$ $\lambda^{+}$is a successor cardinal, it is tempting to view $\xi:=\lambda$ as the ultimate requirement; however, there are scenarios in which a particular choice of $\xi$ with $\lambda<\xi<\lambda^{+}$turns out to be the optimal one (see [BR19c, §3.3]).

The special case $\mu=2$ implies that each $\mathcal{C}_{\alpha}$ is a singleton, say $\left\{C_{\alpha}\right\}$, in which case we identify the $\mathcal{C}$-sequence $\left\langle\left\{C_{\alpha}\right\} \mid \alpha<\kappa\right\rangle$ with the $C$-sequence $\left\langle C_{\alpha} \mid \alpha<\kappa\right\rangle$. On the other extreme is the case $\mu=\kappa^{+}$, which may seem pointless, but is nevertheless valuable when combined with a small value for $\nu$ (see [RZ23, Theorems 7.2 and 7.6]), or with an arithmetic hypothesis (see [IR24, Theorem 5.14]).

The basic coherence relation $\mathcal{R}$ is the end-extension relation, $\sqsubseteq$, introduced in Subsection 2.1, indicating that $\overrightarrow{\mathcal{C}}$ is a coherent $\mathcal{C}$-sequence. A close examination of proxy-based constructions reveals that full coherence is not always necessary, and the $\sqsubseteq$ relation can be weakened in several ways, as follows.

First, considering some $C \in \bigcup_{\alpha<\kappa} \mathcal{C}_{\alpha}$ and some $\bar{\alpha} \in \operatorname{acc}(C)$, it may be that all we require is for some $D \in \mathcal{C}_{\bar{\alpha}}$ to agree with $C$ at the final approach to $\bar{\alpha}$. If this is the case, then the construction will work just as well from a $\sqsubseteq^{*}$-coherent instance of the proxy principle, where $D \sqsubseteq^{*} C$ iff there is some $\varepsilon<\sup (D)$ such that $D \backslash \varepsilon \sqsubseteq C \backslash \varepsilon$.

In another direction, some proxy-based constructions can be designed to require genuine coherence only for some of the clubs in $\overrightarrow{\mathcal{C}}$, or only at some of their accumulation points. ${ }^{11}$ Indeed, there are contexts in which, for some infinite cardinal $\chi$, there is no need to require coherence for clubs of ordertype $<\chi$, or possibly, there is no need to require coherence at accumulation points of cofinality $<\chi$. Thus, in such cases we may weaken $\sqsubseteq$ to either $\chi \sqsubseteq$ or $\sqsubseteq_{\chi}$, where for a coherence relation $\mathcal{R}$ :

- $D_{\chi} \mathcal{R} C$ iff $((D \mathcal{R} C)$ or $(\operatorname{cf}(\sup (D))<\chi))$, and
- $D \mathcal{R}_{\chi} C$ iff $((D \mathcal{R} C)$ or $(\operatorname{otp}(C)<\chi$ and $\operatorname{nacc}(C)$ consists only of successor ordinals)). ${ }^{12}$
The significance of such a weakening is that unlike coherent square sequences that are typically refuted by reflection principles, ${ }^{13} \sqsubseteq_{\chi}$-coherent proxy principles are compatible with a gallery of reflection principles and provide an effective means of obtaining optimal incompactness results (see for instance [BR17a, Corollary 1.20] and [LHR19, Theorem A]).

Note that the extreme case $\mathcal{R}={ }_{\kappa} \sqsubseteq$ amounts to saying that no coherence is needed at all, and we call it the trivial coherence relation. In this case, every $\mathcal{C}_{\alpha}$ may be shrunk to a singleton, yielding a proxy sequence with $\mu=2$.

In yet another direction, there are circumstances in which it is helpful to indicate that $\overrightarrow{\mathcal{C}}$ avoids a particular class of ordinals $\Omega$, meaning that acc $(C) \cap$ $\Omega=\emptyset$ for every $C \in \bigcup_{\alpha<\kappa} \mathcal{C}_{\alpha}$. This requirement is indicated by prepending $\Omega$ as a superscript to the coherence relation $\mathcal{R}$, thereby strengthening it to ${ }^{\Omega} \mathcal{R} .{ }^{14}$ In the context of walks on ordinals, the utility of avoiding a stationary subset of $\kappa$ is demonstrated by [Tod07, Theorem 6.2.7] and [CL17, Lemma 6.7]. In general, if $\overrightarrow{\mathcal{C}}$ is a ${ }^{\Omega} \sqsubseteq$-coherent proxy sequence, then for any $\alpha \in \Omega$, one is free to shrink $\mathcal{C}_{\alpha}$ to a singleton, and this has important ramifications (see [BR19c, Lemma 3.8] and [BR21, Corollary 4.27]).

The weakest nontrivial value for the hitting parameters is $(\theta, \mathcal{S}, \nu, \sigma):=$ $(1,\{\kappa\}, \mu, 1)$; this minimal amount of hitting ensures that $\overrightarrow{\mathcal{C}}$ is unthreadable, ${ }^{15}$ implying in particular that $\kappa$ is not weakly compact (see [Tod87, Theorem 1.8]).

We have already seen the utility of a large $\theta$ in the previous subsection, and here the extreme case $\theta=\kappa$ is understood as a diagonal requirement,

[^7]where each $C$ in $\mathcal{C}_{\alpha}$ is required to hit each $B_{i}$ for all the $i$ 's that are smaller than $\alpha$. When one constructs a $\chi$-complete or a $\chi$-free $\kappa$-Souslin tree, it is natural to require $E_{\geq \chi}^{\kappa}$ to be in $\mathcal{S}$ (see [BR19b, Proposition 2.2] and [BR19c, Theorem 4.12]). An extreme case is requiring $\mathcal{S}$ to contain all stationary subsets of a given stationary $S^{*} \subseteq \kappa$, e.g., $S^{*}:=\{\alpha<\kappa \mid \operatorname{cf}(\alpha)=\operatorname{cf}(|\alpha|)\}$ as was done in [Kru23, Theorem 3.1]. An application of $\nu=2$ may be found in [BR21, Theorem 6.17], where the narrowness requirement at the hitting ordinals enables sealing potential automorphisms of a tree; ${ }^{16}$ another application of $\nu=2$ is given by Corollary 5.7 below. The utility of $\sigma>1$ is demonstrated by Fact 2.4 above. Applications of $\sigma=\omega$ may be found throughout [BR17b]. Nota bene that in some cases $\sigma=\omega$ implies the existence of a nonreflecting stationary subset of $E_{\omega}^{\kappa}$ (see [LH17, Theorem 4.1] for a primary scenario).
2.3. Monotonicity. The reader can verify that the proxy principle satisfies monotonicity properties with respect to most of its parameters, as follows:

- Any sequence witnessing $\mathrm{P}_{\xi}^{-}(\kappa, \mu, \mathcal{R}, \theta, \mathcal{S}, \nu, \sigma)$ remains a witness to the principle if any of $\xi, \mu$, or $\nu$ is increased; if $\theta$ or $\sigma$ is decreased; if $\mathcal{R}$ is weakened; if $\mathcal{S}$ is shrunk; or if any element of $\mathcal{S}$ is expanded.
- The case $\mathcal{R}=\sqsubseteq$ coincides with $\mathcal{R}={ }_{\chi}^{\Omega} \sqsubseteq$ for $(\Omega, \chi):=(\emptyset, \omega)$; increasing $\chi$ weakens each of the relations ${ }_{\chi}^{\Omega} \sqsubseteq,{ }_{\chi}^{\Omega} \sqsubseteq^{*},{ }^{\Omega} \sqsubseteq_{\chi}$, and ${ }^{\Omega} \sqsubseteq_{\chi}^{*}$, while expanding $\Omega$ strengthens the same relations.
- For $\mathcal{R} \in\left\{\sqsubseteq, \sqsubseteq^{*}\right\}$, any $\chi$, and any $\Omega, \mathrm{P}_{\xi}^{-}\left(\kappa, \mu,{ }^{\Omega} \mathcal{R}_{\chi}, \cdots\right)$ entails $\mathrm{P}_{\xi}^{-}\left(\kappa, \mu,{ }_{\chi}^{\Omega} \mathcal{R}, \cdots\right)$.
2.4. Simplifications. To make the parameterized proxy principle more accessible, a few of its main instances have been given abbreviations in the literature. For $S$ a stationary subset of $\kappa$, the abbreviations are as follows:
- $\boxtimes_{\xi}^{-}(S)$ denotes $\mathrm{P}_{\xi}^{-}(\kappa, 2, \sqsubseteq, 1,\{S\}, 2,1)$, i.e., the instance asserting the existence of a coherent $\xi$-bounded $C$-sequence with a minimal nontrivial hitting feature. Together with $\diamond(\kappa)$ this enables a very simple construction of a $\kappa$-Souslin tree (see [BR21, §2.6-2.7]).
- $\boxtimes_{\xi}^{*}(S)$ denotes $\mathrm{P}_{\xi}^{-}\left(\kappa, \kappa, \chi \sqsubseteq^{*}, 1,\{S\}, \kappa, 1\right)$, where $\chi:=\min \{\operatorname{cf}(\alpha) \mid$ $\alpha \in S \cap \operatorname{acc}(\kappa)\}$. This is a weakening of $\boxtimes_{\xi}^{-}(S)$ in the spirit of Jensen's $\square_{\xi}^{*}$ that is nonetheless sufficient for various constructions (see [BR19b, Proposition 2.2]). In case that $\kappa$ happens to be $(<\chi)$ closed, ${ }^{17}$ [BR21, Theorem 4.39] tells us that $\mathrm{P}_{\xi}^{-}\left(\kappa, \kappa, \chi \sqsubseteq^{*}, 1,\{S\}, \kappa, 1\right)$ is no weaker than $\mathrm{P}_{\xi}^{-}(\kappa, \kappa, \sqsubseteq, 1,\{S\}, \kappa, 1)$.

In the special case where $\kappa$ is the successor of a regular cardinal $\xi$ and $S \subseteq E_{\xi}^{\xi^{+}}$, the principle $\boxtimes_{\xi}^{*}(S)$ becomes $\mathrm{P}_{\xi}^{-}\left(\kappa, 2,{ }_{\kappa} \sqsubseteq, 1,\{S\}, 2,1\right)$ as stated in [Kru23, Definition 1.8]. This is because $\min \{\operatorname{cf}(\alpha) \mid \alpha \in$

[^8]$S\}=\xi$, and no club in a witness for $\boxtimes_{\xi}^{*}(S)$ has accumulation points of cofinality $\geq \xi$, so here $\chi \sqsubseteq^{*}$ coincides with the trivial coherence relation $\kappa \sqsubseteq$, thereby allowing $\mu$ to be shrunk to 2 .

- One may replace the stationary set $S$ by a collection $\mathcal{S}$ of stationary sets, and/or add an indication for the width $\mu$, e.g., writing $\boxtimes_{\xi}^{-}(\mathcal{S},<\mu)$ for $\mathrm{P}_{\xi}^{-}(\kappa, \mu, \sqsubseteq, 1, \mathcal{S}, \mu, 1)$.
Whenever possible and in order to reduce an unnecessary load, a convention for the omission of parameters has been established. As already mentioned earlier, if we omit the parameter $\xi$, then we mean that $\xi=\kappa$, i.e., $\mathrm{P}^{-}(\kappa, \cdots)$ stands for $\mathrm{P}_{\kappa}^{-}(\kappa, \cdots) .{ }^{18}$ Independently, we may omit a tail of parameters, as follows:
- If we omit $\sigma$, then $\sigma=$ " $<\omega$ ", which we will discuss shortly;
- If in addition we omit $\nu$, then $\nu=\mu$;
- If in addition we omit $\mathcal{S}$, then $\mathcal{S}=\{\kappa\}$;
- If in addition we omit $\theta$, then $\theta=1$.

We are left with discussing our choice for a default value of $\sigma$.
Initially, in [BR17a, p. 1953], the authors stated that the omission of $\sigma$ would amount to putting $\sigma=1$ as this seemed to be the weakest possible value that is still useful. Later on, it was realized that this was an illusion, since all the applications of $\mathrm{P}^{-}(\kappa, \ldots)$ at the time were in the context of $\diamond(\kappa)$, and in this context any instance $\mathrm{P}_{\xi}^{-}(\kappa, \cdots, 1)$ may be witnessed by a sequence simultaneously witnessing $\mathrm{P}_{\xi}^{-}(\kappa, \amalg, n)$ for all $n<\omega$; that is, $\diamond(\kappa)$ implies that $\mathrm{P}_{\xi}^{-}(\kappa, \ldots, 1)$ is no weaker than $\mathrm{P}_{\xi}^{-}(\kappa, \ldots,<\omega) .{ }^{19}$ Through the work surrounding Fact 2.4, one learns to appreciate the possibility of having $\mathrm{P}_{\xi}^{-}(\kappa, \ldots, \sigma)$ holding with $\sigma$ slightly greater than 1 . The hitting feature of Fact 2.4 - namely, requiring two consecutive meets of $C_{\alpha}$ with $B_{i}$, but not insisting that the smaller of the two be a non-accumulation point of $C_{\alpha}$ - may be expressed as something like " $\sigma=1 \frac{1}{2}$ ", but fortunately such an awkward notation can be avoided, as [BR21, Theorem 4.15] shows that $\left.\mathrm{P}_{\xi}^{-}\left(\kappa, \varpi, 1 \frac{1}{2}\right), \mathrm{P}_{\xi}^{-}(\kappa, \varpi,)^{-}\right)$, and $\mathrm{P}_{\xi}^{-}(\kappa, \varpi,<\omega)$ are all logically equivalent.

With the revised convention of setting a default value of $\sigma=$ " $<\omega$ " as in [BR21, Convention 4.18], a door was opened to proxy-based constructions of $\kappa$-Souslin trees using $\kappa^{<\kappa}=\kappa$ instead of $\diamond(\kappa)$ (see [BR21, Theorems 5.13 and 6.8$]$ ), and to the second batch of applications mentioned in the paper's introduction (see Page 3). ${ }^{20}$

Finally, in order to facilitate the use of the proxy principles in conjunction with other common hypotheses, we adopt the following:

- $\mathrm{P}_{\xi}(\kappa, \mu, \mathcal{R}, \theta, \mathcal{S}, \nu, \sigma)$ denotes the conjunction of $\mathrm{P}_{\xi}^{-}(\kappa, \mu, \mathcal{R}, \theta, \mathcal{S}, \nu, \sigma)$ and $\diamond(\kappa)$;

[^9]- $\mathrm{P}_{\xi}^{\bullet}(\kappa, \mu, \mathcal{R}, \theta, \mathcal{S}, \nu)$ denotes the conjunction of $\mathrm{P}_{\xi}^{-}(\kappa, \mu, \mathcal{R}, \theta, \mathcal{S}, \nu,<\omega)$ and $\kappa^{<\kappa}=\kappa .^{21}$
2.5. To bullet or not to bullet? In our applications of $\mathrm{P}_{\xi}(\kappa, \ldots)$, we shall prefer to use a more versatile version $\diamond\left(H_{\kappa}\right)$ of $\diamond(\kappa)$, as follows:

Fact 2.7 ([BR17a, Lemma 2.2]). $\diamond(\kappa)$ is equivalent to the principle $\diamond\left(H_{\kappa}\right)$ asserting the existence of a partition $\left\langle R_{i} \mid i<\kappa\right\rangle$ of $\kappa$ and a sequence $\left\langle\Omega_{\beta}\right|$ $\beta<\kappa\rangle$ of elements of $H_{\kappa}$ such that for all $i<\kappa, \Omega \subseteq H_{\kappa}$ and $p \in H_{\kappa^{+}}$, the following set is stationary in $\kappa$ :

$$
B_{i}(\Omega, p):=\left\{\beta \in R_{i} \mid \exists \mathcal{M} \prec H_{\kappa^{+}}\left(p \in \mathcal{M}, \beta=\kappa \cap \mathcal{M}, \Omega_{\beta}=\Omega \cap \mathcal{M}\right)\right\} .
$$

The way that the combination of $\mathrm{P}_{\xi}^{-}(\kappa, \ldots)$ and $\diamond\left(H_{\kappa}\right)$ is typically used in recursive constructions of length $\kappa$ is as follows: At limit stage $\alpha<\kappa$, one uses the ladders of $\mathcal{C}_{\alpha}$ in order to climb up and eventually determine the $\alpha^{\text {th }}$-level of the ultimate object. Clause (2) of Definition 2.5 ensures that this climbing procedure will not reach a dead end. Finally, Clause (3) of Definition 2.5 is invoked with sets $B_{i}$ that arise from an application of $\diamond\left(H_{\kappa}\right)$, namely $B_{i}:=B_{i}(\Omega, p)$ for an educated choice of $\Omega$ and $p$.

Looking at [BR21, Definition 5.9], we see that the principle $\mathrm{P}_{\xi}^{\bullet}(\kappa, \cdots)$ can be understood as a weakening of $\mathrm{P}_{\xi}^{-}(\kappa, \cdots) \wedge \diamond\left(H_{\kappa}\right)$ that is tailored to hit only sets $B_{i}$ of the above particular form.

Strictly speaking, $\mathrm{P}_{\xi}^{\bullet}(\kappa, \cdots)$ is weaker than $\mathrm{P}_{\xi}(\kappa, \cdots),{ }^{22}$ but due to the nature of our constructions (as roughly described above), we do not know of any application of $\mathrm{P}_{\xi}(\kappa, \cdots)$ that cannot be transformed into an application of the weaker principle $\mathrm{P}_{\xi}^{\bullet}(\kappa, \cdots)$. In particular, all the $\kappa$-trees constructed from $\mathrm{P}_{\xi}(\kappa, \cdots)$ in Sections 5 and 6 below, may as well be constructed assuming $\mathrm{P}_{\xi}^{\bullet}(\kappa, \cdots)$, instead.
2.6. For the adventurous readers. We mention that there are a few additional values that can be assigned to the parameters of the proxy principle. First, by letting $\xi:=<\lambda$, we mean that all ladders in the witnessing proxy sequence have order-type strictly less than $\lambda$. The fact that singular cardinals may admit a ladder system having small order-type everywhere was exploited by Shelah and his co-authors a long time ago (see, e.g., [BDS86, ASS87]). Second, by letting $\mu:=\infty$ we mean that $\left|\mathcal{C}_{\alpha}\right| \leq|\alpha|$ for every nonzero $\alpha<\kappa$. Third, if we write $<\theta$ instead of $\theta$, then we mean that the proxy sequence witnesses $\theta=\vartheta$ for all $\vartheta<\theta$ simultaneously. An analogous interpretation applies when writing $<\sigma$ instead of $\sigma$. Fourth, by letting

[^10]$\sigma:=<\infty$, we mean to replace the assertion of Equation ( $\star$ ) of Definition 2.5 by:
$$
\forall \sigma<\operatorname{otp}(C) \sup \left\{\beta \in C \mid \operatorname{succ}_{\sigma}(C \backslash \beta) \subseteq B_{i}\right\}=\alpha
$$

Coming back to $\mu$, we have a variation $\mu^{\text {ind }}$ that aids in constructions of $\kappa$-trees with a $\mu$-ascent path (see Definition 4.7 below). It reads as follows.
Definition 2.8 ( $[\mathrm{BR} 21, \S 4.6])$. The principle $\mathrm{P}_{\xi}^{-}\left(\kappa, \mu^{\text {ind }}, \sqsubseteq, \theta, \mathcal{S}\right)$ asserts the existence of a $\xi$-bounded $\mathcal{C}$-sequence $\left\langle\mathcal{C}_{\alpha} \mid \alpha<\kappa\right\rangle$ together with a sequence $\langle i(\alpha) \mid \alpha<\kappa\rangle$ of ordinals in $\mu$, such that:

- for every $\alpha<\kappa$, there exists a canonical enumeration $\left\langle C_{\alpha, i}\right| i(\alpha) \leq$ $i<\mu\rangle$ of $\mathcal{C}_{\alpha}$ satisfying that the sequence $\left\langle\operatorname{acc}\left(C_{\alpha, i}\right) \mid i(\alpha) \leq i<\mu\right\rangle$ is $\subseteq$-increasing with $\bigcup_{i \in[i(\alpha), \mu)} \operatorname{acc}\left(C_{\alpha, i}\right)=\operatorname{acc}(\alpha)$;
- for all $\alpha<\kappa, i \in[i(\alpha), \mu)$ and $\bar{\alpha} \in \operatorname{acc}\left(C_{\alpha, i}\right)$, it is the case that $i \geq i(\bar{\alpha})$ and $C_{\bar{\alpha}, i} \sqsubseteq C_{\alpha, i} ;$
- for every sequence $\left\langle B_{\tau} \mid \tau<\theta\right\rangle$ of cofinal subsets of $\kappa$, and every $S \in \mathcal{S}$, there are stationarily many $\alpha \in S$ such that for all $C \in \mathcal{C}_{\alpha}$ and $\tau<\min \{\alpha, \theta\}, \sup \left(\operatorname{nacc}(C) \cap B_{\tau}\right)=\alpha$.
In addition to coherence as alluded to in the introduction, another concept that is invisible at the level of $\kappa=\aleph_{1}$ is that of a proxy-respecting tree. This concept arises when one tries to construct a companion $\kappa$-tree $S$ for a given $\kappa$-tree $T$ in such a way that the product tree $S \otimes T$ becomes special [Yad23, §6] or Souslin [RYY23b, §5], or that the cofinal branches of $T$ would injectively embed into the automorphism group of $S$ [Yad23, §7], or that a designated reduced power of $S$ would contain a copy of $T$ [BR17b, §6]. The point is that if we were to construct the new $\kappa$-tree $S$ using an instance $\mathrm{P}^{-}(\cdots)$ of the proxy principle, then it will be useful to be able to interpret (even if artificially) the other $\kappa$-tree $T$ as an outcome of a similar application of $\mathrm{P}^{-}(\rightleftarrows)$. This leads to the following definition (that makes use of some concepts that are defined in Subsection 4.1 below).
Definition 2.9 ([BR17b]). A streamlined $\kappa$-tree $T$ is $\mathrm{P}_{\xi}^{-}(\kappa, \mu, \mathcal{R}, \theta, \mathcal{S}, \nu, \sigma)-$ respecting if there exists a subset $\S \subseteq \kappa$ and a sequence of mappings $\left\langle b^{C}\right.$ : $(T \upharpoonright C) \rightarrow{ }^{\alpha} H_{\kappa} \cup\{\emptyset\}\left|\alpha<\kappa, C \in \mathcal{C}_{\alpha}\right\rangle$ such that:
(1) for all $\alpha \in \S$ and $C \in \mathcal{C}_{\alpha}, T_{\alpha} \subseteq \operatorname{Im}\left(b^{C}\right)$;
(2) $\overrightarrow{\mathcal{C}}=\left\langle\mathcal{C}_{\alpha} \mid \alpha<\kappa\right\rangle$ witnesses $\mathrm{P}_{\xi}^{-}(\kappa, \mu, \mathcal{R}, \theta,\{S \cap \S \mid S \in \mathcal{S}\}, \nu, \sigma)$;
(3) for all sets $D \sqsubseteq C$ from $\overrightarrow{\mathcal{C}}$ and $x \in T \upharpoonright D, b^{D}(x)=b^{C}(x) \upharpoonright \sup (D)$.

Typically, a $\kappa$-Souslin tree obtained from an instance $\mathrm{P}_{\xi}^{-}(\cdots)$ of the proxy principle using the so-called microscopic approach will be $\mathrm{P}_{\xi}^{-}(\varpi)-$ respecting. In addition, if $\kappa=\lambda^{+}$for an infinite regular cardinal $\lambda$, and $\mathrm{P}_{\lambda}^{-}\left(\kappa, \mu,{ }_{\lambda} \sqsubseteq, \theta,\left\{E_{\lambda}^{\kappa}\right\}, \nu, \sigma\right)$ holds, then every $\kappa$-tree is $\mathrm{P}_{\lambda}^{-}\left(\kappa, \mu,{ }_{\lambda} \sqsubseteq, \theta,\left\{E_{\lambda}^{\kappa}\right\}, \nu, \sigma\right)$ respecting. Indeed, $\diamond\left(\aleph_{1}\right)$ implies that all $\aleph_{1}$-trees are $\mathrm{P}_{\omega}^{-}\left(\aleph_{1}, 2, \sqsubseteq, \aleph_{1},\left\{\aleph_{1}\right\}\right)$ respecting. Unorthodox examples of respecting trees, including Kurepa trees and the trees recording characteristics of walks on ordinals, may be found in [RS17, §4].
2.7. What's next? As mentioned in the paper's Introduction, the proxy principles provide a disconnection between the combinatorial constructions and the study of the hypotheses themselves. This is a well-known approach and is no different from other axioms such as $\diamond, \square$ or the P-Ideal Dichotomy (PID). In all of these cases, by matching any application of the axiom with an appropriate configuration in which the axioms is known to hold, one obtains a conclusion of possible interest. Arguably, one factor determining the success of an axiom of this sort is the exact cut point at which the disconnection is introduced. We think that a good cut point is one in which the study of applications and configurations is equally wealthy. With this view, in the upcoming two sections we shall establish that the proxy is indeed a successful axiom by demonstrating the rich findings in the two independent sides of this project. At no point will we try to list all the resulting conclusions, as their number has order of magnitude equal to the product of the two. In particular, we leave to the reader the task of reconnecting applications and configurations as needed or for the joy of verifying that all classical $\diamond$-based Souslin-tree constructions can now be redirected through the proxy principle. ${ }^{23}$

## 3. Deriving instances of the proxy principle

In this section, we shall give three tables demonstrating that instances of the proxy principles hold in many different configurations. Here, for $S \subseteq \kappa$, $\mathrm{NS}_{\kappa}^{+} \upharpoonright S$ stands for the collection of all stationary subsets of $S$, and $\operatorname{Refl}(S)$ asserts that every stationary subset of $S$ reflects at some ordinal in $E_{>\omega}^{\kappa}$. We also write $\mathrm{CH}_{\lambda}$ for the assertion that $2^{\lambda}=\lambda^{+}$, and $\mathrm{CH}(\lambda)$ for the assertion that $2^{<\lambda}=\lambda$.

[^11]| Hypothesis | Instance of proxy obtained | Citation |
| :---: | :---: | :---: |
| \% $(S)$ for $\kappa=\sup (S)$ | $\mathrm{P}^{-}(\kappa, 2, \kappa \sqsubseteq, 1,\{S\}, 2, \kappa)$ | [BR17a, §5] |
| $\boldsymbol{¢}(S)$ for $\lambda^{+}=\sup (S)$ | $\mathrm{P}_{\lambda}^{-}\left(\lambda^{+}, 2, \lambda \sqsubseteq, 1,\{S\}, 2, \lambda^{+}\right)$ | [BR17a, Thm 5.1(1)] |
| $\boldsymbol{\%}\left(E_{\lambda}^{\lambda^{+}}\right)$for $\lambda \in$ REG | $\boxtimes_{\lambda}^{*}\left(E_{\lambda}^{\lambda^{+}}\right)$ | [BR17a, Thm 5.1(1)] |
| Q $(S)$ for $\omega_{1}=\sup (S)$ | $\mathrm{P}_{\omega}^{-}\left(\aleph_{1}, 2, \sqsubseteq, 1,\{S\}, 2, \omega_{1}\right)$ | [BR17a, Lem 3.3(1)] |
| $\diamond(S)$ for $\kappa=\sup (S)$ | $\mathrm{P}\left(\kappa, 2,{ }_{\kappa} \sqsubseteq, 1,\{S\}, 2, \kappa\right)$ | [BR17a, §5] |
| $\diamond(S)$ for $\lambda^{+}=\sup (S)$ | $\mathrm{P}_{\lambda}\left(\lambda^{+}, 2,{ }_{\lambda} \sqsubseteq, 1,\{S\}, 2, \lambda^{+}\right)$ | [BR17a, Thm 5.1(2)] |
| $\diamond(S)$ for $S \subseteq E_{\text {cf }(\lambda)}^{\lambda^{+}}, \lambda^{+}=\sup (S)$ | $\mathrm{P}_{\lambda}\left(\lambda^{+}, 2, \lambda \sqsubseteq, \lambda^{+},\left\{E_{\mathrm{cf}(\lambda)}^{\lambda^{+}}\right\}, 2,<\lambda\right)$ | [BR17a, Thm 5.6] |
| $\diamond(S)$ for $\omega_{1}=\sup (S)$ | $\mathrm{P}_{\omega}\left(\aleph_{1}, 2, \sqsubseteq, \aleph_{1},\{S\}, 2,<\omega\right)$ | [BR17a, Thm 3.7] |
| $\diamond(S)$ for $\omega_{1}=\sup (S)$ | $\mathrm{P}_{\omega^{2}}\left(\aleph_{1}, 2, \sqsubseteq, \aleph_{1},\{S\}, 2,<\omega^{2}\right)$ | [BR17a, Thm 3.6] |
| $\diamond^{*}\left(E_{\lambda}^{\lambda^{+}}\right)$for $\lambda \in$ REG | $\mathrm{P}_{\lambda}\left(\lambda^{+}, 2, \lambda \sqsubseteq, \lambda^{+}, \mathrm{NS}_{\lambda^{+}}^{+} \upharpoonright E_{\lambda}^{\lambda^{+}}, 2,<\infty\right)$ | [BR21, §4.4] |
| $\diamond(S) \wedge \neg \operatorname{Refl}(S)$ for $\sup (S)=\kappa$ inaccessible | $\mathrm{P}\left(\kappa, \kappa,{ }^{S} \sqsubseteq, 1,\{S\}, 2, \kappa\right)$ | [BR21, Thm 4.26(1)] |
| $\forall_{\lambda}$ for $\lambda \geq \aleph_{1}$ | $\mathrm{P}_{\lambda}\left(\lambda^{+}, 2, \sqsubseteq, \lambda^{+},\left\{E_{\mathrm{cf}(\lambda)}^{\lambda+}\right\}, 2,<\lambda\right)$ | [BR17a, Thm 3.6] |
| $\square_{\lambda} \wedge \mathrm{CH}_{\lambda}$ for $\lambda \geq \aleph_{1}$ | $\mathrm{P}_{\lambda}\left(\lambda^{+}, 2, \sqsubseteq,<\lambda,\left\{E_{\chi}^{\lambda^{+}}\right\}, 2,<\lambda\right)$ for all $\chi \in \operatorname{Reg}(\lambda)$ | [BR17a, Cor 3.9] |
| $\square_{\lambda} \wedge \mathrm{CH}_{\lambda}, \lambda \in \mathrm{REG} \backslash\left\{\aleph_{0}\right\}$ | $\mathrm{P}\left(\lambda^{+}, 2, \sqsubseteq^{*}, 1,\left\{E_{\lambda}^{\lambda^{+}}\right\}, 2,<\omega\right)$ | [BR17a, Cor 6.2(1)] |
| $\square\left(\lambda^{+}\right) \wedge \mathrm{CH}_{\lambda}$ with $\lambda=\lambda^{\aleph_{0}}$ or $\lambda \geq \beth_{\omega}$ | $\boxtimes^{-}\left(\lambda^{+}\right)$ | [Rin17, Cor 4.4, 4.7] |
| $\square\left(\lambda^{+}\right) \wedge \mathrm{CH}_{\lambda}$ with $\mathfrak{b} \leq \lambda<\aleph_{\omega}$ | $\boxtimes^{-}\left(\lambda^{+}\right)$ | [Rin22, Cor 5.12] |
| $\square\left(\lambda^{+},<\lambda\right) \wedge \mathrm{CH}_{\lambda} \wedge \mathrm{CH}(\lambda)$ with $\lambda \geq \aleph_{1}$ | $\boxtimes^{*}\left(\lambda^{+}\right)$ | [Rin19, Thm 3.5] |
| $\square\left(\lambda^{+}\right) \wedge$ GCH with $\lambda \geq \aleph_{1}$ | $\boxtimes^{-}\left(E_{\chi}^{\lambda^{+}}\right)$for all $\chi \in \operatorname{Reg}(\lambda)$ | [Rin17, Cor 4.5] |
| $\square\left(\lambda^{+}\right) \wedge \mathrm{CH}_{\lambda} \wedge \mathrm{CH}(\lambda)$ for $\lambda \in$ SING | $\mathrm{P}\left(\lambda^{+}, 2, \sqsubseteq, \lambda^{+},\left\{\lambda^{+}\right\}, 2,<\omega\right)$ | [BR19a, Cor 4.22] |
| $\square(E) \wedge \diamond(E), \kappa=\sup (E) \geq \aleph_{2}$ | $\mathrm{P}\left(\kappa, 2, \sqsubseteq^{*}, 1,\{S\}, 2,<\omega\right)$ for all $S \in \mathrm{NS}_{\kappa}^{+}$ | [BR21, Cor 4.19(2)] |
| $\mathrm{CH}_{\lambda} \wedge \mathrm{NS} \upharpoonright E_{\theta}^{\lambda}$ is saturated, $\lambda=\theta^{+}, \theta \in \mathrm{REG}$ | $\mathrm{P}\left(\lambda^{+}, 2,{ }_{\lambda} \sqsubseteq^{*}, \theta,\left\{E_{\lambda}^{\lambda^{+}}\right\}, 2, \theta\right)$ | [BR17a, Thm 6.4] |
| $\lambda \in \operatorname{REG} \backslash\left\{\aleph_{0}\right\} \wedge \mathrm{CH}(\lambda) \wedge \mathrm{CH}_{\lambda} \wedge \neg \operatorname{Refl}\left(E_{\neq \lambda}^{\lambda+}\right)$ | $\mathrm{P}_{\lambda}\left(\lambda^{+}, \lambda^{+}, \sqsubseteq,<\lambda,\left\{\lambda^{+}\right\}, 2,<\lambda\right)$ | [BR19c, Thm A] |
| $\lambda \in \operatorname{REG} \backslash\left\{\aleph_{0}\right\} \wedge \mathrm{CH}(\lambda) \wedge \mathrm{CH}_{\lambda} \wedge \neg \operatorname{Refl}\left(E_{\neq \lambda}^{\lambda+}\right)$ | $\mathrm{P}\left(\lambda^{+}, \lambda^{+}, \sqsubseteq^{*}, 1,\left\{E_{\lambda}^{\lambda^{+}}\right\}, 2,<\omega\right)$ | [BR19c, Thm A] |
| $\lambda \in \operatorname{SING} \wedge \mathrm{CH}(\lambda) \wedge \mathrm{CH}_{\lambda} \wedge \square_{\lambda}^{*} \wedge \neg \operatorname{Refl}\left(E_{\neq \operatorname{cf}(\lambda)}^{\lambda+}\right)$ | $\mathrm{P}_{\lambda^{2}}\left(\lambda^{+}, \lambda^{+}, \sqsubseteq, \lambda^{+},\left\{\lambda^{+}\right\}, 2,<\lambda\right)$ | [BR19c, Thm B] |
| $\mathrm{V}=\mathrm{L}, \kappa$ not weakly compact | $\mathrm{P}(\kappa, 2, \sqsubseteq, \kappa, \mathcal{S}, 2, \omega)$ | [BR17a, Cor 1.10(5)] |

Table 3.1. Instances of proxy derived from combinatorial hypotheses. In the last line, $\mathcal{S}$ stands for $\left\{E_{\geq \chi}^{\kappa} \mid \chi \in \operatorname{Reg}(\kappa) \wedge\right.$ $\kappa$ is $(<\chi)$-closed $\}$.

Remark 3.1. By [Rin17, Cororollary 4.13], $\boxtimes^{-}(\kappa) \wedge \diamond(\kappa)$ implies that $\mathrm{P}(\kappa, 2$, $\left.\sqsubseteq^{*}, 1,\{S\}\right)$ holds for every stationary $S \subseteq \kappa$.

| Properties of forcing | Properties of ground model | Instance of proxy obtained | Citation |
| :---: | :---: | :---: | :---: |
| $\operatorname{Add}(\lambda, 1)$ | $\begin{aligned} & \mathrm{CH}(\lambda) \\ & \mathrm{CH}(\lambda) \wedge \square_{\lambda} \\ & \mathrm{CH}(\lambda) \wedge \square_{\lambda} \wedge \mathrm{CH}_{\lambda} \\ & \mathrm{CH}(\lambda) \wedge \square_{\lambda} \wedge \mathrm{CH}_{\lambda} \wedge \lambda>\kappa_{0} \\ & \hline \end{aligned}$ | $\begin{aligned} & \hline \hline \mathrm{P}_{\lambda}^{-}\left(\lambda^{+}, 2, \lambda \sqsubseteq, \lambda^{+}, \mathrm{NS}_{\lambda^{+}}^{+} \mid E_{\lambda}^{\lambda^{+}}, 2,<\lambda\right) \\ & \mathrm{P}_{\lambda}^{-}\left(\lambda^{+}, 2, \sqsubseteq, \lambda^{+}, \mathrm{NS}_{\lambda^{+}}^{+} \mid E_{\lambda^{+}}^{\lambda^{+}}, 2,<\lambda\right) \\ & \mathrm{P}_{\lambda}^{\prime}\left(\lambda^{+}, 2, \sqsubseteq, \lambda^{+}, \mathrm{NS}_{\lambda^{+}} \mid E_{\lambda}^{\lambda^{+}}, 2,<\lambda\right) \\ & \mathrm{P}_{\lambda}\left(\lambda^{+}, 2, \sqsubseteq, \lambda^{+}, \mathrm{NS}_{\lambda^{+}}^{+} \mid E_{\lambda^{+}}^{\lambda^{+}}, 2,<\lambda\right) \\ & \hline \end{aligned}$ | $[$ RZ23, Cor 7.4] $[$ Rin15, Thm 2.3] $[$ BR21, Thm 6.1(11)] $[$ BR17a, Thm 4.2(2) $]$ |
| ( $<\lambda$ )-distributive, $\kappa$-cc, collapsing $\kappa$ to $\lambda^{+}$ | $\mathrm{CH}(\lambda)$ and $\kappa$ is strongly inaccessible $>\lambda$ | $\mathrm{P}_{\lambda}\left(\lambda^{+}, \infty, \sqsubseteq, \lambda^{+}, \mathrm{NS}_{\lambda^{+}}^{+} \upharpoonright E_{\lambda}^{\lambda^{+}}, 2,<\infty\right)$ | [BR19b, Prop 3.10] |
| $\lambda^{+}-\mathrm{cc}, \text { size } \leq \lambda^{+},$ <br> preserves regularity of $\lambda$, not ${ }^{\lambda} \lambda$-bounding | $\mathrm{CH}(\lambda) \wedge \diamond\left(\lambda^{+}\right)$ | $\mathrm{P}_{\lambda}\left(\lambda^{+}, \infty, \sqsubseteq, \lambda^{+}, \mathrm{NS}_{\lambda^{+}}^{+} \upharpoonright E_{\lambda}^{\lambda^{+}}, 2,<\infty\right)$ | [BR19b, Thm 3.4] |
| $\begin{aligned} & \hline \lambda^{+} \text {-cc, size } \leq \lambda^{+}, \\ & \text {forces cff }(\lambda)<\|\lambda\| \\ & \text { (e.g., Prikry, Magidor, Radin) } \end{aligned}$ | $\begin{aligned} & \mathrm{CH}(\lambda) \wedge \mathrm{CH}_{\lambda} \\ & \kappa=\lambda^{+} \geq \aleph_{2} \text { and } S=E_{\lambda}^{\kappa} \end{aligned}$ | $\mathrm{P}\left(\kappa, \infty, \sqsubseteq, \kappa, \mathrm{NS}_{\kappa}^{+} \mid S, 2,<\infty\right)$ | [BR19b, Thm 3.4] |
| Lévy-collapsing $\lambda$ to $\chi$ | $\begin{aligned} & \lambda^{<\chi}=\lambda>\chi \wedge \mathrm{CH}_{\lambda} \\ & \kappa=\lambda^{+} \text {and } S=E_{\lambda}^{\kappa} \end{aligned}$ | $\mathrm{P}_{\chi}\left(\kappa, \infty, \sqsubseteq, \kappa, \mathrm{NS}_{\kappa}^{+} \mid S, 2,<\infty\right)$ | [BR19b, Prop 3.9] |

Table 3.2. Instances of proxy obtained in forcing extensions; in all cases $\lambda, \chi$ stand for infinite regular cardinals, and the improvement from the parameter $\theta=1$ to $\theta=\kappa$ (or $\theta=\lambda^{+}$) is secured by [BR21, Lemma 4.32].

Remark 3.2. In [LHR19, §3.3], one can find, for a pair $\chi<\kappa$ of infinite regular cardinals, a $\chi$-directed-closed and $\kappa$-strategically-closed forcing poset for introducing a witness to $\mathrm{P}^{-}\left(\kappa, 2, \sqsubseteq_{\chi}, \kappa,\left(\mathrm{NS}_{\kappa}^{+}\right)^{V}, 2, \sigma\right)$.

| Features of model | Instance(s) of proxy satisfied | Citation |
| :---: | :---: | :---: |
| Martin's Maximum $\forall \lambda \in \operatorname{SING} \cap \operatorname{cof}(\omega) \neg \square_{\lambda}^{*}$ | $\forall \lambda \in$ SING $\mathrm{P}_{\lambda}\left(\lambda^{+}, 2, \sqsubseteq_{\aleph_{2}}, \lambda^{+},\left\{E_{\operatorname{cf}(\lambda)}^{\lambda^{+}}\right\}, 2,<\lambda\right)$ | [BR17a, Cor 1.20] |
| $\begin{aligned} & \forall \lambda \in \operatorname{REG} \backslash\left\{\aleph_{0}\right\} \neg \square_{\lambda, \aleph_{1}} \\ & \forall \lambda \in \operatorname{REG} \backslash\left\{\aleph_{0}\right\} \diamond\left(E_{\lambda}^{\lambda^{+}}\right) \end{aligned}$ | $\begin{aligned} & \forall \lambda \in \operatorname{REG} \backslash\left\{\aleph_{0}\right\} \\ & \mathrm{P}_{\lambda}\left(\lambda^{+}, 2, \lambda \sqsubseteq, \lambda^{+},\left\{E_{\lambda}^{\lambda^{+}}\right\}, 2,<\lambda\right) \end{aligned}$ |  |
| $\lambda$ supercompact, $\operatorname{Refl}\left(E_{<\lambda}^{\lambda^{+}}\right), \neg \diamond\left(E_{\lambda}^{\lambda^{+}}\right), \neg \square_{\lambda}$ | $\mathrm{P}_{\lambda}\left(\lambda^{+}, 2, \sqsubseteq_{\lambda}, \lambda^{+},\left\{E_{\lambda}^{\lambda^{+}}\right\}, 2,<\lambda\right)$ | [BR17a, Cor 1.24] |
| $\begin{aligned} & \chi \text { supercompact, } \\ & \lambda=\chi^{+\omega}, \neg \square_{\lambda}^{*} \end{aligned}$ | $\mathrm{P}_{\lambda}\left(\lambda^{+}, 2, \sqsubseteq_{\chi}, \lambda^{+},\left\{E_{\text {cf }(\lambda)}^{\lambda^{+}}\right\}, 2,<\lambda\right)$ | [BR17a, Cor 4.7] |
| $\operatorname{Refl}\left(\lambda^{+}\right)$for $\lambda=\aleph_{\omega}$ | $\begin{aligned} & \mathrm{P}_{\aleph_{\omega}}\left(\aleph_{\omega+1}, \aleph_{1}, \sqsubseteq, \aleph_{\omega+1},\left\{E_{\aleph_{n}}^{\aleph_{\omega+1}} \mid n<\omega\right\}\right) \\ & \text { and } \boxtimes^{-}\left(\mathrm{NS}_{\lambda^{+}}^{+}\right) \end{aligned}$ | [LH17, Thm 1.12] |
| $\operatorname{Refl}\left(E_{<\lambda}^{\lambda+}\right)$ for $\lambda:=\aleph_{1}$ | $\boxtimes^{-}\left(\mathrm{NS}^{+}+\right)$ | [LH17, Thm 1.12] |

TABLE 3.3. Five models in which $\square_{\lambda}$ fails, yet strong instances of the proxy principle at $\lambda^{+}$hold.

## 4. A gallery of Souslin-tree constructions

As described in the paper's Introduction, the original catalyst for the formulation of the proxy principles was the desire for a uniform combinatorial construction of $\kappa$-Souslin trees, and indeed the principles have served this purpose successfully. In this section we showcase the various Souslin trees that have been built using the proxy principles, and provide references to where these constructions can be found.
4.1. The basics. The reader is probably familiar with the abstract definition of a tree as a poset $\left(T,<_{T}\right)$ all of whose downward cones are wellordered. In this project, we opt to work with a particular form of trees which we call streamlined in which elements of the tree are (transfinite) sequences, and the tree-order is nothing but the initial-sequence ordering. This choice does not restrict our study, but it does make some of the considerations smoother. ${ }^{24}$ For instance, in such a tree $T$, the $\alpha^{\text {th }}$-level of $T$ coincides with the set $T_{\alpha}:=\{x \in T \mid \operatorname{dom}(x)=\alpha\}$, and we may likewise define $T \upharpoonright C$ to be $\{x \in T \mid \operatorname{dom}(x) \in C\}$.

Definition 4.1 ([BR21, Definition 2.3]). A streamlined tree is a subset $T \subseteq{ }^{<\kappa} H_{\kappa}$ for some cardinal $\kappa$ such that $T$ is downward-closed, that is, $\{t \upharpoonright \beta \mid \beta<\operatorname{dom}(t)\} \subseteq T$ for every $t \in T$. The height of $T$, denoted ht $(T)$, is the least ordinal $\alpha$ such that $T_{\alpha}=\emptyset$. The set of cofinal branches through $T$, denoted $\mathcal{B}(T)$, is the collection of all functions $f$ with $\operatorname{dom}(f)=\operatorname{ht}(T)$ such that $\{f \upharpoonright \beta \mid \beta<\operatorname{dom}(f)\} \subseteq T$.

[^12]Following [BR21, Convention 2.6], we shall identify a streamlined tree $T$ with the abstract tree ( $T, \subsetneq$ ).
Definition 4.2. (1) A streamlined $\kappa$-tree is a streamlined tree $T \subseteq$ ${ }^{<\kappa} H_{\kappa}$ such that $0<\left|T_{\alpha}\right|<\kappa$ for every $\alpha<\kappa$.
(2) A streamlined $\kappa$-Aronszajn tree is a streamlined $\kappa$-tree $T$ such that $\mathcal{B}(T)=\emptyset$.
(3) A streamlined $\kappa$-Kurepa tree is a streamlined $\kappa$-tree $T$ such that $|\mathcal{B}(T)|>\kappa .^{25}$
(4) A streamlined $\kappa$-Souslin tree is a streamlined $\kappa$-Aronszajn tree $T$ with no antichains of size $\kappa$, that is, for every $A \in[T]^{\kappa}$, there are distinct $s, t \in A$ that are comparable. ${ }^{26}$
Section 2 of [BR21], entitled "How to construct a Souslin tree the right way", offers a comprehensive exposition of the subject of constructing $\kappa$-Souslin trees and the challenges involved, culminating in Subsections 2.6-2.7 with a detailed description of a very simple construction of a $\kappa$-Souslin tree, proving the following basic result:
Fact 4.3 ([BR21, Proposition 2.18]). $\boxtimes^{-}(\kappa) \wedge \diamond(\kappa)$ implies the existence of a $\kappa$-Souslin tree.

In [BR17a, p. 1965], one finds a comparison between the classical nonsmooth approach to Souslin-tree construction (requiring nonreflecting stationary sets) and the modern approach using the proxy principles (enabling construction of a $\kappa$-Souslin tree in models where every stationary subset of $\kappa$ reflects). Further advantages of the proxy principles in the context of Souslin-tree construction are described in [BR21, §1].

By a slightly more elaborate construction, it is possible to weaken the hypotheses of Fact 4.3 considerably, as follows:

Fact $4.4([B R 21$, Corollary 6.7$]) . \mathrm{P}^{\bullet}\left(\kappa, \kappa, \sqsubseteq^{*}, 1,\{\kappa\}, \kappa\right)$ implies the existence of a $\kappa$-Souslin tree.

Recalling the meaning of $\boxtimes^{-}(\kappa)$ and $\mathrm{P}^{\bullet}(\ldots)$ as given in Subsection 2.4, we see that the main improvements of Fact 4.4 over Fact 4.3 consist of weakening the parameter $\mu$ from 2 to $\kappa$, as well as weakening the prediction principle $\diamond(\kappa)$ to the arithmetic hypothesis $\kappa^{<\kappa}=\kappa$.
4.2. Properties of trees. The literature is rich with additional properties that $\kappa$-trees may posses. Let us discuss some of them.

Definition 4.5. A streamlined tree $T$ of height $\kappa$ is said to be:

[^13]- normal iff for all $\alpha<\beta<\kappa$ and $x \in T_{\alpha}$, there is some $y \in T_{\beta}$ such that $x \subsetneq y ;{ }^{27}$
- binary iff $T \subseteq{ }^{<\kappa} 2$;
- $\varsigma$-splitting (for an ordinal $\varsigma<\kappa$ ) iff every node in $T$ admits at least $\varsigma$ many immediate successors; ${ }^{28}$
- splitting iff it is 2-splitting;
- prolific iff, for all $\alpha<\kappa$ and $x \in T_{\alpha},\left\{x^{\curvearrowright}\langle\iota\rangle \mid \iota<\max \{\omega, \alpha\}\right\} \subseteq$ $T_{\alpha+1}$;
- slim iff $\left|T_{\alpha}\right| \leq \max \left\{|\alpha|, \aleph_{0}\right\}$ for every ordinal $\alpha$;
- $\chi$-complete iff, for any $\subsetneq$-increasing sequence $\eta$, of length $<\chi$, of elements of $T$, the limit of the sequence, $\cup \operatorname{Im}(\eta)$, is also in $T$.
- full iff for every $\alpha \in \operatorname{acc}(\kappa),\left|\mathcal{B}(T \upharpoonright \alpha) \backslash T_{\alpha}\right| \leq 1$;
- rigid iff its only automorphism is the trivial one;
- $\chi$-coherent iff $|\{\alpha \in \operatorname{dom}(x) \cap \operatorname{dom}(y) \mid x(\alpha) \neq y(\alpha)\}|<\chi$ for all $x, y \in T$;
- coherent iff it is $\omega$-coherent;
- uniformly homogeneous iff for all $y \in T$ and $x \in T \upharpoonright \operatorname{dom}(y)$, the union of $x$ and $y \upharpoonright(\operatorname{dom}(y) \backslash \operatorname{dom}(x))$ (which is usually denoted by $x * y)$ is in $T$;
- uniformly coherent iff it is coherent and uniformly homogeneous;
- regressive iff there exists a map $\rho: T \rightarrow T$ satisfying the following:
- for every non-minimal $x \in T, \rho(x) \subsetneq x$;
- for all $\alpha \in \operatorname{acc}(\kappa)$ and $x \neq y$ from $T_{\alpha}$, either $(x, \rho(y))$ or $(\rho(x), y)$ is a pair of incomparable nodes;
- special iff there exists a map $\rho: T \rightarrow T$ satisfying the following:
- for every non-minimal $x \in T, \rho(x) \subsetneq x$;
- for every $y \in T, \rho^{-1}\{y\}$ is covered by less than $\kappa$ many antichains;
- specializable iff there exists a forcing extension with the same cardinal structure up to and including $\kappa$, in which $T$ is special.
- almost-Kurepa iff it is a $\kappa$-tree and $|\mathcal{B}(T)|>\kappa$ holds in the forcing extension by $(T, \supseteq)$.

Any $\kappa$-Aronszajn tree contains a $\kappa$-subtree that is normal. Inspecting the proofs of Facts 4.3 and 4.4, we observe that the constructed trees themselves are normal, and indeed all Souslin trees showcased here are normal. As a result of taking the simplest approach in the construction of Fact 4.3, the tree also satisfies the property of being club-regressive, as explored in [BR17a, Proposition 2.3].

[^14]Remark 4.6. Coherent trees are regressive; regressive trees are slim; slim trees are not $\aleph_{1}$-complete; a full splitting $\aleph_{2}$-tree is neither slim nor $\aleph_{1}{ }^{-}$ complete; the existence of a binary $\kappa$-Souslin tree is equivalent to the existence of a prolific $\kappa$-Souslin tree; prolific trees are $\omega$-splitting.

Definition 4.7 ([BR17b, Definition 1.2]). Suppose that $X$ is a streamlined $\kappa$-tree, and $\mathcal{F} \subseteq \mathcal{P}(\mu)$ for some cardinal $\mu$. An $(\mathcal{F}, X)$-ascent path through a streamlined $\kappa$-tree $T$ is a system $\vec{f}=\left\langle f_{x} \mid x \in X\right\rangle$ such that for all $x, y \in X$ :
(1) $f_{x}: \mu \rightarrow T_{\operatorname{dom}(x)}$ is a function;
(2) if $x \subsetneq y$, then $\left\{i<\mu \mid f_{x}(i) \subsetneq f_{y}(i)\right\} \in \mathcal{F}$;
(3) if $x \neq y$ and $\operatorname{dom}(x)=\operatorname{dom}(y)$, then $\left\{i<\mu \mid f_{x}(i) \neq f_{y}(i)\right\} \in \mathcal{F}$.

If $(X, \subsetneq)$ is isomorphic to $(\kappa, \in)$, then $\vec{f}$ is simply said to be an $\mathcal{F}$-ascent path. If in addition $\mathcal{F}$ is equal to $\mathcal{F}_{\mu}^{\mathrm{bd}}=\{Z \subseteq \mu \mid \sup (\mu \backslash Z)<\mu\}$, then $\vec{f}$ is said to be a $\mu$-ascent path.

Remark 4.8. A weakening of $\mu$-ascent path was isolated by Lücke in [Lüc17] and was named a $\mu$-ascending path. By [Lüc17, Corollary 1.7], a special $\lambda^{+}$-Aronszajn tree cannot admit a $\mu$-ascent path for $\mu<\operatorname{cf}(\lambda)$.

Notation 4.9 ( $i^{\text {th }}$-component). For every function $x: \alpha \rightarrow{ }^{\tau} H_{\kappa}$ and every $i<\tau$, we let $(x)_{i}: \alpha \rightarrow H_{\kappa}$ stand for $\langle x(\beta)(i) \mid \beta<\alpha\rangle$.

Definition 4.10 ([BR19c, Definition 4.4]). Suppose that $T \subseteq{ }^{<\kappa} H_{\kappa}$ is a streamlined tree, and $\tau$ is a nonzero ordinal.

- For a sequence $\vec{s}=\left\langle s_{i} \mid i<\tau\right\rangle$ of nodes of $T$, we let

$$
T(\vec{s}):=\left\{x \in{ }^{<\kappa}\left({ }^{\tau} H_{\kappa}\right) \mid \forall i<\tau\left[(x)_{i} \cup s_{i} \in T\right]\right\} ;
$$

- A $\tau$-derived tree of $T$ is a tree of the form $T(\vec{s})$ for some injective sequence $\vec{s}=\left\langle s_{i} \mid i<\tau\right\rangle$ of nodes of $T$ on which the map $i \mapsto \operatorname{dom}\left(s_{i}\right)$ is constant.

Definition 4.11. A streamlined $\kappa$-tree $T$ is $\chi$-free iff for every nonzero $\tau<\chi$, all the $\tau$-derived trees of $T$ are $\kappa$-Souslin. $T$ is free iff it is $\omega$-free.

Remark 4.12. A $\lambda$-free $\lambda^{+}$-tree is specializable; 3 -free and full trees are both rigid.

Definition 4.13 (The levels of vanishing branches, [RS23, RYY23b]). For a streamlined $\kappa$-tree $T$ :

- $V^{-}(T):=\left\{\alpha \in \operatorname{acc}(\kappa) \mid \mathcal{B}(T \upharpoonright \alpha) \neq T_{\alpha}\right\} ;$
- $V(T)$ denotes the set of all $\alpha \in \operatorname{acc}(\kappa)$ such that for every $x \in T \upharpoonright \alpha$, there exists $f \in \mathcal{B}(T \upharpoonright \alpha) \backslash T_{\alpha}$ with $x \subsetneq f$.

Remark 4.14. If $T$ is uniformly homogeneous, then $V^{-}(T)=V(T)$; if $T$ is uniformly coherent, then $V(T)=E_{\omega}^{\kappa}$; if $V(T)$ is cofinal in $\kappa$, then $T$ is normal; if $T$ is normal, splitting and full, then $V(T)$ is empty.
4.3. Slim vs. complete and binary vs. prolific. As demonstrated in [BR21, §6.1], there is an obvious way of transforming any proxy-based construction of a binary tree into a construction of a prolific tree, and vice versa. In addition, there are abstract translations as in the appendix of [BR17b].

Likewise, there is a transparent way of transforming any proxy-based construction of slim tree into a construction of a complete tree, and vice versa. This is demonstrated by the construction of a $\chi$-complete $\kappa$-Souslin tree from $\boxtimes^{-}\left(E_{\geq \chi}^{\kappa}\right) \wedge \diamond(\kappa)$, where $\kappa$ is $(<\chi)$-closed, in [BR17a, Proposition 2.4]. By taking some extra care in the construction, [BR19b, Proposition 2.2] shows that we can replace $\boxtimes^{-}\left(E_{\geq \chi}^{\kappa}\right)$ with the weaker instance $\boxtimes^{*}\left(E_{\geq \chi}^{\kappa}\right)$. As a rule of thumb, the construction of slim trees requires $\mu \leq \aleph_{1}$; on the other hand, for a $\chi$-complete tree we require $\kappa$ to be $(<\chi)$-closed and also require the parameter $\mathcal{S}$ to contain (some subset of) $E_{\geq \chi}^{\kappa}$.
4.4. The tables. We now turn to present a few tables summarizing various $\kappa$-Souslin trees constructed in the literature using instances of the proxy. Note that the monotonicity features of the proxy principles suggest an informal way of comparing two $\kappa$-trees $T$ and $S$ by viewing $T$ as 'weaker' than $S$ provided that $T$ can be obtained from a vector of parameters weaker than the one necessary for the construction of $S$. This informal understanding becomes more precise through the observation that the content of Remarks 4.6, 4.12 and 4.14 indeed corresponds with Subsection 2.3.

Throughout the tables in this section, $\chi$ stands for an infinite regular cardinal such that $\kappa$ is $(<\chi)$-closed. Also, in many of the cited references the trees are constructed from $\mathrm{P}(\cdots)$, but as explained in Subsection 2.5, such constructions can always be carried out from the weaker $\mathrm{P}^{\bullet}(\underset{ }{( } \cdot)$.

Table 4.1 presents $\kappa$-Souslin trees constructed from instances of the proxy principle with the strong values $\mu=\nu=2$ while maintaining the default value $\sigma=<\omega$.

|  | Citation | $\mathcal{R}$ | $\theta$ | $\mathcal{S}$ | Type of $\kappa$-Souslin tree |
| :---: | :---: | :---: | :---: | :---: | :---: |
| (1) | [RYY23b, Thm 3.7] | $\sqsubseteq^{*}$ | 1 | \{ $\kappa$ \} | $V^{-}(X) \subseteq V(T)$ |
| (2) | [BR17a, Prop 2.4] | $\sqsubseteq$ | 1 | $\left\{E_{\geq \chi}^{\kappa}\right\}$ | $\chi$-complete |
| (3) | [BR17a, Prop 2.3] | $\sqsubseteq$ | 1 | $\{\kappa\}$ | club-regressive |
| (4) | [BR17a, Prop 2.5] | $\sqsubseteq$ | $\kappa$ | \{ $\kappa$ \} | club-regressive, uniformly coherent |
| (5) | [BR17b, Thm 6.2] | $\sqsubseteq$ | $\kappa$ | $\left\{E_{\geq \chi}^{\kappa}\right\}$ | club-regressive, $\chi$-free |
| (6) | [Yad23, Thm 7.2] | $\sqsubseteq$ | $\kappa$ | $\left\{E_{\geq \chi}^{\kappa}\right\}$ | having $\|\mathcal{B}(X)\|$-many automorphisms |
| (7) | [Yad23, Thm 7.2] | $\sqsubseteq$ | $\kappa$ | $\left\{E_{\geq \chi}^{\kappa}\right\}$ | almost-Kurepa |
| (8) | [RYY23b, Thm 5.9] | $\eta \sqsubseteq$ | $\kappa$ | $\left\{E_{\geq \chi}^{\kappa}\right\}$ | uniformly homogeneous, $E_{\geq \chi}^{\kappa}$-regressive, $\chi$-complete, $\chi$-coherent, $X \otimes T$ is $\kappa$-Souslin and $\mathrm{P}^{-}\left(\kappa, 2, \eta \sqsubseteq, \kappa,\left\{E_{\geq \chi}^{\kappa}\right\}, 2\right)$-respecting |
| Table 4.1. $\kappa$-Souslin trees $T$ obtained from |  |  |  |  |  |
| $\mathrm{P}^{\bullet}(\kappa, 2, \mathcal{R}, \theta, \mathcal{S}, 2)$. In (1), $X$ stands for a streamlined |  |  |  |  |  |
| $\kappa$-tree. In (5), the original paper does not mention being |  |  |  |  |  |
| club-regressive, but this can easily be verified. In (6), |  |  |  |  |  |
| $X$ stands for a $\mathrm{P}^{-}(\cdots)$-respecting binary $\kappa$-tree. In (7), |  |  |  |  |  |
| we assume the existence of a $\mathrm{P}^{-}(\cdots)$-respecting binary |  |  |  |  |  |
|  |  |  |  |  |  |
| $\kappa$-Kurepa tree. In (8), $X$ stands for a $\mathrm{P}^{-}(\varpi)$-respecting |  |  |  |  |  |

Table 4.2 presents $\kappa$-Souslin trees with ascent paths constructed from instances of the proxy principle with $\mu=\nu=2$ together with the strong value $\sigma=\omega$. Here, $\mathcal{F}_{\theta}^{\eta}$ stands for $\{Z \subseteq \theta||\theta \backslash Z|<\eta\}$.

|  | Citation | $\mathcal{R}$ | $\theta$ | $\mathcal{S}$ | Type of $\kappa$-Souslin tree |
| :---: | :---: | :---: | :---: | :---: | :---: |
| (1) | [BR17b, Thm 4.1] | $\sqsubseteq$ | 1 | $\{\kappa\}$ | slim with $\mathcal{F}_{\omega}^{\omega}$-ascent |
| (2) | [BR17b, Thm 4.2] |  |  |  | slim with ( $\mathcal{F}_{\omega}^{\omega}, X$ )-ascent path |
| (3) | [BR17b, Thm 4.3] | $\sqsubseteq$ | 1 | $\left\{E_{\geq \chi}^{\kappa}\right\}$ | $\chi$-complete with ( $\left.\mathcal{F}_{\omega}^{\omega}, X\right)$-ascent path |
| (4) | [BR17b, Thm 5.1] | $\sqsubseteq_{\eta}$ | $\theta$ | $\{\kappa\}$ | slim with ( $\left.\mathcal{F}_{\theta}^{\eta}, X\right)$-ascent path |
| (5) | [BR17b, Thm 5.3] | $\sqsubseteq_{\eta}$ | $\theta$ | $\left\{E_{\geq \chi}^{\kappa}\right\}$ | $\chi$-complete with $\left(F_{\theta}^{n}, X\right)$-ascent path |
| (6) | [BR17b, Thm 6.3] | $\sqsubseteq_{\eta}$ | $\kappa$ | $\left\{E_{\geq \chi}^{\kappa}{ }^{\chi}\right\}$ | $\min \{\chi, \eta\}$-free, $\chi$-complete with an $\left(\mathcal{F}_{\theta}^{\eta}, X\right)$-ascent path |
| (7) | [BR17b, Thm 6.5] | $\sqsubseteq_{\eta}$ | $\kappa$ | $\left\{E_{\geq \chi}^{\kappa}\right\}$ | slim, $\chi$-free with $\left(\mathcal{F}_{\theta}^{\eta}, X\right)$-ascent path |
| TABLE 4.2. $\kappa$-Souslin trees obtained from $\mathrm{P}(\kappa, 2, \mathcal{R}, \theta, \mathcal{S}, 2, \omega)$, where $\eta \in \operatorname{Reg}(\kappa)$. In (2) and (4), $X$ stands for a slim streamlined $\kappa$-tree. In (3) and (5), X stands for a streamlined tree. In (4), (5) and (6), $\eta \leq \theta<\kappa$. In (6), $X$ stands for a $\mathrm{P}^{-}(\ldots)$-respecting $\kappa$-tree. In (7), $X$ stands for a slim $\mathrm{P}^{-}(\varpi)$-respecting $\kappa$-tree, and $\chi \leq \eta$. |  |  |  |  |  |
|  |  |  |  |  |  |
|  |  |  |  |  |  |
|  |  |  |  |  |  |

Table 4.3 presents $\kappa$-Souslin trees constructed from instances of the proxy principle where $\mu$ is weakened to $\kappa$ but $\nu$ maintains its strong value, 2 .

|  | Citation | $\mathcal{R}$ | $\theta$ | $\mathcal{S}$ | Type of $\kappa$-Souslin tree |
| :---: | :--- | :--- | :--- | :---: | :--- |
| $(1)$ | $[$ BR21, Thm 6.17] | $\chi \sqsubseteq$ | 1 | $\mathcal{S}^{*}$ | $\chi$-complete, rigid, <br> $\forall \Lambda<\lambda, T$ has no $\Lambda$-ascending path |
| $(2)$ | $[$ BR21, Thm 6.14] | $\chi \sqsubseteq^{*}$ | 1 | $\mathcal{S}^{*}$ | $\chi$-complete, <br> $\forall \Lambda<\lambda, T$ has no $\Lambda$-ascending path |
| $(3)$ | [RYY23b, Thm 3.7] | $\sqsubseteq$ | 1 | $\{\kappa\}$ | $V(T) \supseteq V(K) \cap E_{>\omega}^{\kappa}$ |
| $(4)$ | [RYY23b, Thm 4.4] | ${ }^{E} \sqsubseteq$ | 1 | $\left\{E \cup E_{>\chi}^{\kappa}\right\}$ | $V(T) \cap E_{\leq \chi}^{\kappa}=E=V^{-}(T) \cap E_{\leq \chi}^{\kappa}$ |
| (5) | [BR21, Thm 6.27] | $\chi \sqsubseteq$ | $\kappa$ | $\left\{E_{\geq \chi}^{\kappa}\right\}$ | $\chi$-complete, $\chi$-free |
| (6) | [RYY23b, Thm 4.3] | ${ }^{S} \sqsubseteq$ | 1 | $\{\kappa\}$ | $V(T) \supseteq S$ |
| (7) | [RYY23b, Thm 4.8] | ${ }^{S} \sqsubseteq$ | 1 | $\{S\}$ | $V(T)=S=V^{-}(T)$ |

Table 4.3. $\kappa$-Souslin trees $T$ obtained from $\mathrm{P}^{\bullet}(\kappa, \kappa, \mathcal{R}, \theta, \mathcal{S}, 2)$. In (1) and (2), $\lambda<\kappa$ is an infinite cardinal, and $\mathcal{S}^{*}:=\left\{E_{\geq \chi}^{\kappa} \cap E_{>\Lambda}^{\kappa} \mid \Lambda<\lambda\right\}$. In (4), $E \subseteq \operatorname{acc}(\kappa) \cap E_{\leq \chi}^{\kappa}$ is stationary. In (6) and (7), $S \subseteq \operatorname{acc}(\kappa)$.
In (7), we also assume that $\kappa$ is strongly inaccessible.

Table 4.4 presents $\kappa$-Souslin trees constructed from instances of the proxy principle where both $\mu$ and $\nu$ are relaxed to $\kappa$.

| Citation | $\mu$ | $\mathcal{R}$ | $\theta$ | Type of $\kappa$-Souslin tree |
| :--- | :---: | :---: | :---: | :--- |
| [BR21, Thm 6.11] | $\mu^{\text {ind }}$ | $\sqsubseteq$ | 1 | $\chi$-complete with a $\mu$-ascent path |
| [BR21, Thm 6.8] | $\kappa$ | $\chi \sqsubseteq^{*}$ | 1 | $\chi$-complete |
| [BR21, Thm 6.32] | $\kappa$ | $\chi \sqsubseteq$ | $\kappa$ | $\chi$-complete, uniformly homogeneous |

Table 4.4. $\kappa$-Souslin trees obtained from $\mathrm{P}^{\bullet}\left(\kappa, \kappa, \mathcal{R}, \theta,\left\{E_{\geq \chi}^{\kappa}\right\}, \kappa\right)$.

There are a few more tree constructions that do not fit the above tables. We list them below and refer the reader to the original papers for any missing definitions.

Fact 4.15 ([BR17b, Theorem 6.1]). Suppose that $\mathrm{P}\left(\kappa, 2, \sqsubseteq, \kappa,\left\{E_{\geq \chi}^{\kappa}\right\}, 2, \omega\right)$ holds, and $\kappa$ is ( $<\chi$ )-closed. For every infinite cardinal $\theta$ such that $\theta^{+}<\chi$, there exists a prolific slim $\left(\chi, \theta^{+}\right)$-free $\kappa$-Souslin tree with an injective $\mathcal{F}_{\theta}^{\mathrm{fin}}-$ ascent path.
Fact 4.16 ([BR17b, Theorem 6.4]). Suppose that $\operatorname{cf}(\nu)=\nu<\theta^{+}<\chi<\kappa$ are infinite cardinals, $\kappa$ is $(<\chi)$-closed, and $\mathrm{P}\left(\kappa, 2, \sqsubseteq, \kappa,\left\{E_{\geq \chi}^{\kappa}\right\}, 2, \omega\right)$ holds. Then there exists a prolific $\nu$-free, $\left(\chi, \theta^{+}\right)$-free, $\chi$-complete $\kappa$-Souslin tree with an injective $\mathcal{F}_{\theta}^{\nu}$-ascent path.
Fact 4.17 ([BR17b, Theorem 6.6]). Suppose that $\operatorname{cf}(\nu)=\nu<\theta^{+}<\chi<\kappa$ are infinite cardinals, $\kappa$ is $(<\chi)$-closed, and $\mathrm{P}\left(\kappa, 2, \sqsubseteq, \kappa,\left\{E_{\geq \chi}^{\kappa}\right\}, 2, \omega\right)$ holds. Then there exists a prolific slim $\nu$-free, $\left(\chi, \theta^{+}\right)$-free, $\kappa$-Souslin tree with an injective $\mathcal{F}_{\theta}^{\nu}$-ascent path.

Fact $4.18([\operatorname{Yad} 23$, Theorem 7.5]). Suppose that:

- $S \subseteq E_{\mathrm{cf}(\theta)}^{\kappa}$ is a stationary subset of $\kappa$;
- $\mathrm{P}^{-}\left(\kappa, \kappa,{ }^{S} \sqsubseteq, 1,\{S\}, 2\right)$ holds and is witnessed by a sequence $\left\langle\mathcal{C}_{\alpha}\right|$ $\alpha<\kappa\rangle$ such that $B:=\left\{\alpha \in \operatorname{acc}(\kappa)| | \mathcal{C}_{\alpha} \mid=1\right\}$ covers $E_{>\operatorname{cf}(\theta)}^{\kappa}$, and $\operatorname{acc}\left(\bigcup \mathcal{C}_{\alpha}\right) \subseteq B$ for every $\alpha \in B ;$
- $\kappa^{<\kappa}=\kappa$.

Then there exists a $\kappa$-Souslin tree with a $\theta$-ascent path.
Fact 4.19 ([RYY23a, Theorem 4.4]). Suppose that:

- $\kappa$ is a strongly inaccessible cardinal;
- $S \subseteq E_{>\omega}^{\kappa}$ is stationary, and $\diamond_{S}^{*}(\kappa$-trees $)$ holds;
- $\mathrm{P}^{-}(\kappa, 2, \sqsubseteq, 1,\{S\})$ holds.

Then there is a family $\mathcal{T}$ of $2^{\kappa}$ many streamlined, normal, binary, splitting, full $\kappa$-trees such that $\otimes \mathcal{T}^{\prime}$ is $\kappa$-Souslin for every nonempty $\mathcal{T}^{\prime} \in[\mathcal{T}]^{<\kappa}$.

Fact 4.20 ([RYY23a, Theorem 5.1]). Suppose that:

- $\kappa=\lambda^{+}=2^{\lambda}$ for $\lambda$ a regular uncountable cardinal;
- $\square_{\lambda}^{B}$ and $\diamond(\lambda)$ both hold;
- $\mathrm{P}^{-}\left(\kappa, 2, \sqsubseteq_{\lambda}, \kappa,\left\{E_{\lambda}^{\kappa}\right\}\right)$ holds.

Then there is a family $\mathcal{T}$ of $2^{\kappa}$ many streamlined, normal, binary, splitting, full $\kappa$-trees such that $\otimes \mathcal{T}^{\prime}$ is $\kappa$-Souslin for every nonempty $\mathcal{T}^{\prime} \in[\mathcal{T}]^{<\lambda}$.

## 5. A large family of Souslin trees

In [Zak81], Zakrzewski constructed from $\diamond\left(\aleph_{1}\right)$ a family of $2^{\aleph_{1}}$ many $\aleph_{1}$-Souslin trees such that the product of any finitely (nonzero) many of them is again Souslin. The main result of this section (Theorem 5.5 below) generalizes Zakrzewski's theorem in various ways. First, let us recall the definition of product of trees.

Definition 5.1. For a sequence of streamlined trees $\left\langle T^{j} \mid j<\tau\right\rangle$, the product tree $\bigotimes_{j<\tau} T^{j}$ is defined to be the poset $\mathbf{T}=\left(T,<_{T}\right)$, where:

- $T:=\left\{\vec{x} \in \prod_{j<\tau} T^{j} \mid j \mapsto \operatorname{dom}(\vec{x}(j))\right.$ is constant $\}$, and
- $\vec{x}<_{T} \vec{y}$ iff $\vec{x}(j) \subsetneq \vec{y}(j)$ for every $j<\tau$.

Remark 5.2. The tree $\mathbf{T}$ is easily seen to be isomorphic to a streamlined tree via Notation 4.9, but in order to ease on the reader, we stick here to the classical representation of products.

Second, to motivate Theorem 5.5, we state a sample corollary that does not mention products. In what follows, two $\kappa$-trees $S, T$ are almost-disjoint iff their intersection has size less than $\kappa$.

Corollary 5.3. If $\diamond^{+}\left(\aleph_{1}\right)$ holds, then there exists a streamlined $\aleph_{1}$-Aronszajn tree $T$ admitting $\aleph_{2}$-many pairwise almost-disjoint $\aleph_{1}$-Souslin subtrees.

Proof. Recall that $\diamond^{+}\left(\aleph_{1}\right)$ entails the existence of an $\aleph_{1}$-Kurepa tree. In addition, by [BR17a, Corollary 1.10], $\diamond\left(\aleph_{1}\right)$ implies $\mathrm{P}\left(\aleph_{1}, 2, \sqsubseteq, \aleph_{1}\right)$. Thus, by Corollary 5.7 below (using $\kappa=\aleph_{1}$ and $\mho=\aleph_{2}$ ), there exists an $\aleph_{1-}$ Aronszajn admitting $\aleph_{2}$-many $\aleph_{1}$-subtrees such that the product of any two of them is Souslin. Finally, it is a classical theorem of Kurepa [Kur52] that the square of a $\kappa$-tree cannot be a $\kappa$-Souslin tree; therefore, if $S$ and $T$ are $\kappa$-trees whose product is $\kappa$-Souslin, then there exists no $\kappa$-tree $R$ that is embeddable in both $S$ and $T$, so that in particular, $|S \cap T|<\kappa$.

Looking at the preceding corollary, one may wonder whether the conclusion be strengthened to make the ultimate tree Souslin, as well. The next proposition shows that this is impossible.
Proposition 5.4. If $\left\langle T^{\eta} \mid \eta<\kappa\right\rangle$ is a pairwise almost-disjoint sequence of streamlined $\kappa$-subtrees of a given $\kappa$-tree $T$, then $T$ is not a $\kappa$-Souslin tree.
Proof. We commence with a well-known fact.
Claim 5.4.1. Suppose that $S$ is a streamlined $\kappa$-subtree of a streamlined $\kappa$-Souslin tree $T$. Then there exists some $s \in S$ such that $\{t \in T \mid s \subseteq t\}$ is a subset of $S$.
Proof. Suppose not. In particular, for every $\alpha<\kappa$, we may find a pair $\left(s_{\alpha}, t_{\alpha}\right)$ such that:

- $s_{\alpha} \in S_{\alpha}$;
- $t_{\alpha} \in T \backslash S$ with $s_{\alpha} \subseteq t_{\alpha}$.

Fix a sparse enough set $A \in[\kappa]^{\kappa}$ such that for every pair $\alpha<\beta$ of ordinals from $A$, $\operatorname{dom}\left(t_{\alpha}\right)<\beta$. Since $T$ is Souslin, we may pick a pair $\alpha<\beta$ of ordinals from $A$ such that $t_{\alpha} \subseteq t_{\beta}$. As also $s_{\beta} \subseteq t_{\beta}$ and $\operatorname{dom}\left(t_{\alpha}\right)<\operatorname{dom}\left(s_{\beta}\right)$, it follows that $t_{\alpha} \subseteq s_{\beta}$. But $s_{\beta}$ belongs to the streamlined tree $S$, which must mean that $t_{\alpha} \in S$, contradicting the choice of $t_{\alpha}$.

Towards a contradiction, suppose that $T$ is a $\kappa$-Souslin tree. It follows from the claim that for every $\eta<\kappa$, we may pick some $s_{\eta} \in T$ such that $\left\{t \in T \mid s_{\eta} \subseteq t\right\}$ is a subset of $T^{\eta}$. As $T$ is a $\kappa$-Souslin tree, the set $N:=\{s \in T| |\{t \in T \mid s \subseteq t\} \mid<\kappa\}$ has size less than $\kappa$. Thus since $T$ is a $\kappa$-Souslin tree, we may then find $\eta<\rho<\kappa$ such that $s_{\eta} \subseteq s_{\rho}$ and $s_{\rho} \notin N$. So $\left\{t \in T \mid s_{\rho} \subseteq t\right\}$ is a subset of $T^{\eta} \cap T^{\rho}$, contradicting the fact that $T^{\eta}$ and $T^{\rho}$ are almost-disjoint.

We now arrive at the main technical result of this section. To recover Zakrzewski's theorem, consider the case $\kappa:=\aleph_{1}, K:={ }^{<\kappa} 2$ and $\mathcal{S}:=\{\kappa\}$.
Theorem 5.5. Suppose:

- $K \subseteq{ }^{<\kappa} H_{\kappa}$ is a normal streamlined tree of height $\kappa$;
- $\mathcal{S}$ is a nonempty collection of stationary subsets of $\kappa$;
- $\mathrm{P}(\kappa, \kappa, \sqsubseteq, \kappa, \mathcal{S}, 2)$ holds.

Then there exists a sequence $\left\langle T^{\eta} \mid \eta \in \mathcal{B}(K)\right\rangle$ of prolific normal streamlined $\kappa$-Souslin trees satisfying all of the following:
(1) For every nonzero cardinal $\tau$ such that $\kappa$ is $\tau$-closed and such that there exists $\S \in \mathcal{S}$ for which $\S \backslash E_{>\tau}^{\kappa}$ is nonstationary, for every injective sequence $\left\langle\eta_{j} \mid j<\tau\right\rangle$ of elements of $\mathcal{B}(K)$, the product tree $\bigotimes_{j<\tau} T^{\eta_{j}}$ is again $\kappa$-Souslin;
(2) The union $T:=\bigcup\left\{T^{\eta} \mid \eta \in \mathcal{B}(K)\right\}$ of these trees has no $\kappa$-branches;
(3) If $K$ is a $\kappa$-tree, then $T$ is a $\kappa$-Aronszajn tree.

Proof. Fix a sequence $\overrightarrow{\mathcal{C}}=\left\langle\mathcal{C}_{\alpha} \mid \alpha<\kappa\right\rangle$ witnessing $\mathrm{P}^{-}(\kappa, \kappa, \sqsubseteq, \kappa, \mathcal{S}, 2)$. Without loss of generality, $0 \in \bigcap_{0<\alpha<\kappa} \bigcap \mathcal{C}_{\alpha}$. As $\diamond(\kappa)$ holds, fix sequences $\left\langle\Omega_{\beta} \mid \beta<\kappa\right\rangle$ and $\left\langle R_{i} \mid i<\kappa\right\rangle$ together witnessing $\diamond\left(H_{\kappa}\right)$, as in Fact 2.7. Let $\pi: \kappa \rightarrow \kappa$ be such that $\beta \in R_{\pi(\beta)}$ for all $\beta<\kappa$. Let $\triangleleft$ be some wellordering of $H_{\kappa}$ of order-type $\kappa$, and let $\phi: \kappa \leftrightarrow H_{\kappa}$ witness the isomorphism $(\kappa, \in) \cong\left(H_{\kappa}, \triangleleft\right)$. Put $\psi:=\phi \circ \pi$.

We shall construct a sequence $\left\langle L^{\eta} \mid \eta \in K\right\rangle$ such that, for all $\alpha<\kappa$ and $\eta \in K_{\alpha}$ :
(i) $L^{\eta} \in\left[{ }^{\alpha} \kappa\right]^{<\kappa}$;
(ii) for every $\beta<\alpha, L^{\eta \upharpoonright \beta}=\left\{t \upharpoonright \beta \mid t \in L^{\eta}\right\}$.

By convention, for every $\alpha \in \operatorname{acc}(\kappa+1)$ such that $\left\langle L^{\eta} \mid \eta \in K \upharpoonright \alpha\right\rangle$ has already been defined, and for every $\eta \in K_{\alpha}$, we shall let $T^{\eta}:=\bigcup_{\beta<\alpha} L^{\eta\lceil\beta}$, so that $T^{\eta}$ is a tree of height $\alpha$ whose $\beta^{\text {th }}$ level is $L^{\eta \upharpoonright \beta}$ for all $\beta<\alpha$.

The construction of the sequence $\left\langle L^{\eta} \mid \eta \in K\right\rangle$ is by recursion on $\operatorname{dom}(\eta)$. We start by letting $L^{\emptyset}:=\{\emptyset\}$. Next, for every $\alpha<\kappa$ such that $\left\langle L^{\eta} \mid \eta \in K_{\alpha}\right\rangle$ has already been defined, for every $\eta \in K_{\alpha+1}$, we let

$$
L^{\eta}:=\left\{t^{\curvearrowright}\langle\iota\rangle \mid t \in L^{\eta\lceil\alpha}, \iota<\max \{\omega, \alpha\}\right\} .
$$

Suppose now that $\alpha \in \operatorname{acc}(\kappa)$ is such that $\left\langle L^{\eta} \mid \eta \in K \upharpoonright \alpha\right\rangle$ has already been defined. For each $C \in \mathcal{C}_{\alpha}$, we shall define a matrix

$$
\mathbb{B}^{C}=\left\langle b_{x}^{\alpha, C \cap \beta, \eta} \mid \beta \in C, \eta \in K_{\beta}, x \in T^{\eta} \upharpoonright C \cap(\beta+1)\right\rangle
$$

ensuring that $x \subseteq b_{x}^{\alpha, \bar{D}, \bar{\eta}} \subseteq b_{x}^{\alpha, D, \eta} \in L^{\eta}$ whenever $\bar{\eta} \subseteq \eta .^{29}$ Then, for all $C \in$ $\mathcal{C}_{\alpha}, \eta \in K_{\alpha}$ and $\bar{x} \in T^{\eta} \upharpoonright \bar{C}$, it will follow that $\mathbf{b}_{x}^{C, \eta}:=\bigcup_{\beta \in C \backslash \operatorname{dom}(x)} b_{x}^{\alpha, C \cap \beta, \eta \upharpoonright \beta}$ is an element of $\mathcal{B}\left(T^{\eta}\right)$ extending $x$, and we shall let

$$
L^{\eta}:=\left\{\mathbf{b}_{x}^{C, \eta} \mid C \in \mathcal{C}_{\alpha}, x \in T^{\eta} \upharpoonright C\right\}
$$

Let $C \in \mathcal{C}_{\alpha}$. We now turn to define the components of the matrix $\mathbb{B}^{C}$ by recursion on $\beta \in C$. So suppose that $\beta \in C$ is such that

$$
\mathbb{B}_{<\beta}^{C}:=\left\langle b_{x}^{\alpha, C \cap \bar{\beta}, \eta} \mid \bar{\beta} \in C \cap \beta, \eta \in K_{\bar{\beta}}, x \in T^{\eta} \upharpoonright C \cap(\bar{\beta}+1)\right\rangle
$$

has already been defined.

- For all $\eta \in K_{\beta}$ and $x \in T^{\eta}$ such that $\operatorname{dom}(x)=\beta$, let $b_{x}^{\alpha, C \cap \beta, \eta}:=x$.
- For all $\eta \in K_{\beta}$ and $x \in T^{\eta}$ such that $\operatorname{dom}(x)<\beta$, there are two main cases to consider:

[^15]$\rightarrow$ Suppose that $\beta \in \operatorname{nacc}(C)$ and denote $\beta^{-}:=\sup (C \cap \beta)$.
$\Rightarrow$ If $\beta \in \operatorname{acc}(\kappa)$ and there exists a nonzero cardinal $\tau$ such that all of the following hold:
(1) There exists a sequence $\left\langle\eta_{j} \mid j<\tau\right\rangle$ of elements of $K_{\beta}$, and a maximal antichain $A$ in the product tree $\bigotimes_{j<\tau} T^{\eta_{j}}$ such that $\Omega_{\beta}=\left\{\left(\left\langle\eta_{j} \upharpoonright \epsilon\right|\right.\right.$ $\left.\left.j<\tau\rangle, A \cap \tau\left({ }^{\epsilon} \kappa\right)\right) \mid \epsilon<\beta\right\} ;{ }^{30}$
(2) $\psi(\beta)$ is a sequence $\left\langle x_{j} \mid j<\tau\right\rangle$ such that $x_{j} \in T^{\eta_{j} \upharpoonright \beta^{-}} \upharpoonright\left(C \cap \beta^{-}\right)$for every $j<\tau$;
(3) There exists a unique $j<\tau$ such that $\eta_{j}=\eta$ and $x_{j}=x$.

In this case, by Clauses (1) and (2), the following set is nonempty

$$
Q^{C, \beta}:=\left\{\vec{t} \in \prod_{j<\tau} L^{\eta_{j}} \mid \exists \vec{s} \in A \forall j<\tau\left[\left(\vec{s}(j) \cup b_{x_{j}}^{\alpha, C \cap \beta^{-}, \eta_{j} \upharpoonright \beta^{-}}\right) \subseteq \vec{t}(j)\right]\right\},
$$

so we let $\vec{t}:=\min \left(Q^{C, \beta}, \triangleleft\right)$, and then we let $b_{x}^{\alpha, C \cap \beta, \eta}:=\vec{t}(j)$ for the unique index $j$ of Clause (3). It follows that $b_{x}^{\alpha, C \cap \beta^{-}, \eta\left\lceil\beta^{-}\right.} \subseteq \vec{t}(j)=b_{x}^{\alpha, C \cap \beta, \eta}$.
$\xrightarrow{\gg}$ Otherwise, let $b_{x}^{\alpha, C \cap \beta, \eta}$ be the $\triangleleft$-least element of $L^{\eta} \backslash\left\{\Omega_{\beta}\right\}$ extending $b_{x}^{\alpha, C \cap \beta^{-}, \eta \mid \beta^{-}}$. As our trees thus far are normal and splitting (in fact, prolific), this is well-defined.

- Suppose that $\beta \in \operatorname{acc}(C)$. Then we define $b_{x}^{\alpha, C \cap \beta, \eta}:=\bigcup\left\{b_{x}^{\alpha, C \cap \bar{\beta}, \eta \backslash \bar{\beta}} \mid\right.$ $\bar{\beta} \in C \cap \beta \backslash \operatorname{dom}(x)\}$. We must show that the latter belongs to $L^{\eta}$. Since $\overrightarrow{\mathcal{C}}$ is coherent and $\beta \in \operatorname{acc}(C)$, it is the case that $C \cap \beta \in \mathcal{C}_{\beta}$, so, by ( $\star$ ), it suffices to prove that $b_{x}^{\alpha, C \cap \beta, \eta}=\mathbf{b}_{x}^{C \cap \beta, \eta}$. Proving the latter amounts to showing that $b_{x}^{\alpha, C \cap \delta, \eta \mid \delta}=b_{x}^{\beta, C \cap \delta, \eta \mid \delta}$ for all $\delta \in C \cap \beta \backslash \operatorname{dom}(x)$. This is taken care of by the following claim.

Claim 5.5.1. $\mathbb{B}_{<\beta}^{C}=\mathbb{B}^{C \cap \beta}$. That is, the following matrices coincide:

- $\left\langle b_{y}^{\alpha, C \cap \bar{\beta}, \rho} \mid \bar{\beta} \in C \cap \beta, \rho \in K_{\bar{\beta}}, y \in T^{\rho} \upharpoonright C \cap(\bar{\beta}+1)\right\rangle ;$
- $\left\langle b_{y}^{\beta, C \cap \bar{\beta}, \rho} \mid \bar{\beta} \in C \cap \beta, \rho \in K_{\bar{\beta}}, y \in T^{\rho} \upharpoonright C \cap(\bar{\beta}+1)\right\rangle$.

Proof. For the scope of this proof we denote $C \cap \beta$ by $D$. Now, by induction on $\delta \in D$, we prove that
$\left\langle b_{y}^{\alpha, D \cap \delta, \rho} \mid \rho \in K_{\delta}, y \in T^{\rho} \backslash D \cap(\delta+1)\right\rangle=\left\langle b_{y}^{\beta, D \cap \delta, \rho} \mid \rho \in K_{\delta}, y \in T^{\rho} \backslash D \cap(\delta+1)\right\rangle$.
The base case $\delta=\min (D)=0$ is immediate since $b_{\emptyset}^{\alpha, \emptyset}=\emptyset=b_{\emptyset}^{\beta, \emptyset}$. The limit case $\delta \in \operatorname{acc}(D)$ follows from the continuity of the matrices under discussion as remarked in Footnote 29, with the exception of those $y$ 's such that $\operatorname{dom}(y)=\delta$, but in this case, $b_{y}^{\alpha, D \cap \delta, \rho}=y=b_{y}^{\beta, D \cap \delta, \rho}$ for all $\rho \in K_{\delta}$.

Finally, assuming that $\delta^{-}<\delta$ are two successive elements of $D$ such that

$$
\begin{aligned}
& \left\langle b_{y}^{\alpha, D \cap \delta^{-}, \rho}\right| \rho \in K_{\delta^{-}}, y \in T^{\rho}\left|D \cap\left(\delta^{-}+1\right)\right\rangle \\
= & \left\langle b_{y}^{\beta, D \cap \delta^{-}, \rho}\right| \rho \in K_{\delta^{-}}, y \in T^{\rho}\left|D \cap\left(\delta^{-}+1\right)\right\rangle,
\end{aligned}
$$

[^16]we argue as follows. Given $\zeta \in K_{\delta}$ and $z \in T^{\zeta} \upharpoonright D \cap(\delta+1)$, there are a few possible options. If $\operatorname{dom}(z)=\delta$, then $b_{z}^{\alpha, D \cap \delta, \zeta}=z=b_{z}^{\beta, D \cap \delta, \zeta}$, and we are done. If $\operatorname{dom}(z)<\delta$, then $\operatorname{dom}(z) \leq \delta^{-}$and, by the above construction, for every $\gamma \in\{\alpha, \beta\}$, the value of $b_{z}^{\gamma, \bar{D} \cap \delta, \zeta}$ is completely determined by $\delta$, $\left\langle L^{\rho} \mid \rho \in K \upharpoonright(\delta+1)\right\rangle, \Omega_{\delta}, D, \psi(\delta), \zeta, x$, and $\left\langle b_{y}^{\gamma, D \cap \delta^{-}, \rho}\right| \rho \in K_{\delta^{-}}, y \in$ $\left.T^{\rho} \upharpoonright\left(D \cap \delta^{-}\right)\right\rangle$in such a way that our inductive assumptions imply that $b_{z}^{\alpha, D \cap \delta, \zeta}=b_{z}^{\beta, D \cap \delta, \zeta}$.

This completes the definition of the matrix $\mathbb{B}^{C}$, from which we derive $\mathbf{b}_{x}^{C, \eta}:=\bigcup_{\beta \in C \backslash \operatorname{dom}(x)} b_{x}^{\alpha, C \cap \beta, \eta \upharpoonright \beta}$ for all $\eta \in K_{\alpha}$ and $x \in T^{\eta} \upharpoonright C$. Finally, we define $L^{\eta}$ as per $(\star)$.

Claim 5.5.2. For all $\eta \in K_{\alpha}, C \in \mathcal{C}_{\alpha}$, and $t \in\left\{\mathbf{b}_{x}^{C, \eta} \mid x \in T^{\eta} \upharpoonright C\right\}$, there exists a tail of $\varepsilon \in C$ such that $t=\mathbf{b}_{t \upharpoonright \varepsilon}^{C, \eta}$.
Proof. This follows from the canonical nature of the construction, and the analysis is similar to the proof of Claim 5.5.1. We leave it to the reader.

At the end of the above process, for every $\eta \in \mathcal{B}(K)$, we have obtained a streamlined prolific $\kappa$-tree $T^{\eta}:=\bigcup_{\alpha<\kappa} L^{\eta \upharpoonright \alpha}$ whose $\alpha^{\text {th }}$ level is $L^{\eta \upharpoonright \alpha}$.

Claim 5.5.3. Suppose:

- $\tau$ is nonzero cardinal such that $\kappa$ is $\tau$-closed;
- $\S \in \mathcal{S}$ is such that $\S \backslash E_{>\tau}^{\kappa}$ is nonstationary;
- $\left\langle\eta_{j} \mid j<\tau\right\rangle$ is an injective sequence of elements of $\mathcal{B}(K)$.

The product tree $\bigotimes_{j<\tau} T^{\eta_{j}}$ is a $\kappa$-Souslin tree.
Proof. For the sake of this proof, denote $\bigotimes_{j<\tau} T^{\eta_{j}}$ by $\mathbf{T}=\left(T,<_{T}\right)$. As $\kappa$ is $\tau$-closed, $\mathbf{T}$ is a (splitting, normal) $\kappa$-tree. Thus, to show that it is a $\kappa$-Souslin tree, it suffices to establish that it has no antichains of size $\kappa$. To this end, let $A$ be a maximal antichain in $\mathbf{T}$.

Set $\Omega:=\left\{\left(\left\langle\eta_{j} \upharpoonright \epsilon \mid j<\tau\right\rangle, A \cap \tau\left({ }^{\epsilon} \kappa\right)\right) \mid \epsilon<\kappa\right\}$. As an application of $\diamond\left(H_{\kappa}\right)$, using the parameter $p:=\left\{\phi, A, \Omega,\left\langle T^{\eta_{j}} \mid j<\tau\right\rangle\right\}$, we get that for every $i<\kappa$, the following set is cofinal (in fact, stationary) in $\kappa$ :

$$
B_{i}:=\left\{\beta \in R_{i} \cap \operatorname{acc}(\kappa) \mid \exists \mathcal{M} \prec H_{\kappa^{+}}\left(p \in \mathcal{M}, \beta=\kappa \cap \mathcal{M}, \Omega_{\beta}=\Omega \cap \mathcal{M}\right)\right\}
$$

Note that, for every $\beta \in \bigcup_{i<\kappa} B_{i}$, it is the case that $T \upharpoonright \beta \subseteq \phi[\beta]$.
Fix a large enough $\delta<\kappa$ for which the map $j \mapsto \eta_{j} \upharpoonright \delta$ is injective over $\tau$. By the choice of $\overrightarrow{\mathcal{C}}$, we may now find an ordinal $\alpha \in \S \cap E_{>\tau}^{\kappa}$ above $\delta$ such that $\mathcal{C}_{\alpha}$ is a singleton, say $\mathcal{C}_{\alpha}=\left\{C_{\alpha}\right\}$, and, for all $i<\alpha$,

$$
\sup \left(\operatorname{nacc}\left(C_{\alpha}\right) \cap B_{i}\right)=\alpha
$$

In particular, $T \upharpoonright \alpha \subseteq \phi[\alpha]$. Set $\bar{\eta}_{j}:=\eta_{j} \upharpoonright \alpha$ for each $j<\tau$, and note that $T \upharpoonright \alpha=\bigotimes_{j<\tau} T^{\bar{\eta}_{j}}$.
Subclaim 5.5.3.1. $A \subseteq T \upharpoonright \alpha$. In particular, $|A|<\kappa$.

Proof. It suffices to show that every element of $T_{\alpha}$ extends some element of the antichain $A$. To this end, let $\vec{y}=\left\langle y_{j} \mid j<\tau\right\rangle$ be an arbitrary element of $T_{\alpha}$. By $(\star)$, for each $j<\tau$, we may find some $x_{j} \in T^{\bar{\eta}_{j}} \upharpoonright C_{\alpha}$ such that $y_{j}=\mathbf{b}_{x_{j}}^{C_{\alpha}, \bar{\eta}_{j}}$. By Claim 5.5.2 and the fact that $\alpha \in E_{>\tau}^{\kappa}$, we may assume the existence of a large enough $\gamma \in C_{\alpha} \backslash(\delta+1)$ such that $\operatorname{dom}\left(x_{j}\right)=\gamma$ for all $j<\tau$. In particular, $\vec{x}:=\left\langle x_{j} \mid j<\tau\right\rangle$ is an element of $T \upharpoonright \alpha \subseteq \phi[\alpha]$. Fix some $i<\alpha$ such that $\phi(i)=\vec{x}$, and then pick a large enough $\beta \in \operatorname{nacc}\left(C_{\alpha}\right) \cap B_{i}$ for which $\beta^{-}:=\sup \left(C_{\alpha} \cap \beta\right)$ is bigger than $\gamma$. Note that $\psi(\beta)=\phi(\pi(\beta))=\phi(i)=\vec{x},\left\langle\bar{\eta}_{j} \upharpoonright \beta \mid j<\tau\right\rangle$ is an injective sequence, and

$$
\delta<\gamma<\beta^{-}<\beta<\alpha .
$$

Let $\mathcal{M} \prec H_{\kappa^{+}}$be a witness for $\beta$ being in $B_{i}$. Clearly,

- $T \cap \mathcal{M}=T \upharpoonright \beta=\bigotimes_{j<\tau} T^{\bar{\eta}_{j} \upharpoonright \beta}$,
- $A \cap \mathcal{M}=A \cap(T \upharpoonright \beta)$ is a maximal antichain in $T \upharpoonright \beta$, and
- $\Omega_{\beta}=\Omega \cap \mathcal{M}=\left\{\left(\left\langle\eta_{j} \upharpoonright \epsilon \mid j<\tau\right\rangle, A \cap^{\tau}\left({ }^{\epsilon} \kappa\right)\right) \mid \epsilon<\beta\right\}$.

It thus follows that for every $j<\tau, b_{x_{j}}^{\alpha, C_{\alpha} \cap \beta, \bar{\eta}_{j} \upharpoonright \beta}=\vec{t}(j)$, where $\vec{t}=$ $\min \left(Q^{C_{\alpha}, \beta}, \triangleleft\right)$. In particular, we may fix some $\vec{s} \in A$ such that, for every $j<\tau$,

$$
\left(\vec{s}(j) \cup b_{x_{j}}^{\alpha, C_{\alpha} \cap \beta^{-}, \bar{\eta}_{j} \backslash \beta^{-}}\right) \subseteq \vec{t}(j)=b_{x_{j}}^{\alpha, C_{\alpha} \cap \beta, \bar{\eta}_{j} \upharpoonright \beta} \subseteq \mathbf{b}_{x_{j}}^{C_{\alpha}, \bar{\eta}_{j}}=y_{j} .
$$

So $\vec{s}<_{T} \vec{y}$. As $\vec{s}$ is an element of $A$, we are done.
This completes the proof.
As a final step, we consider the tree $T:=\bigcup\left\{T^{\eta} \mid \eta \in \mathcal{B}(K)\right\}$. Evidently, the $\alpha^{\text {th }}$ level of $T$ is the union of $\left|K_{\alpha}\right|$ many sets of size less than $\kappa$. Thus, if $K$ is a $\kappa$-tree, then so is $T$.

Claim 5.5.4. $T$ has no $\kappa$-branches.
Proof. Towards a contradiction, suppose that $f \in \mathcal{B}(T)$. Fix an $i<\alpha$ such that $\phi(i)=\emptyset$. As an application of $\diamond\left(H_{\kappa}\right)$, we get that the following set is cofinal in $\kappa$ :

$$
B_{i}:=\left\{\beta \in R_{i} \mid f \upharpoonright \beta=\Omega_{\beta}\right\} .
$$

By the choice of $\overrightarrow{\mathcal{C}}$, we may now find an ordinal $\alpha \in \operatorname{acc}(\kappa)$ such that $\mathcal{C}_{\alpha}$ is a singleton, say $\mathcal{C}_{\alpha}=\left\{C_{\alpha}\right\}$, and

$$
\sup \left(\operatorname{nacc}\left(C_{\alpha}\right) \cap B_{i}\right)=\alpha
$$

Recalling $(\star)$, fix $\eta \in K_{\alpha}$ and $x \in T^{\eta} \upharpoonright C_{\alpha}$ such that $f \upharpoonright \alpha=\mathbf{b}_{x}^{C_{\alpha}, \eta}$. Pick a large enough $\beta \in \operatorname{nacc}\left(C_{\alpha}\right) \cap B_{i}$ for which $\beta^{-}:=\sup \left(C_{\alpha} \cap \beta\right)$ is bigger than $\operatorname{dom}(x)$. As $\psi(\beta)=\phi(\pi(\beta))=\phi(i)=\emptyset$, it is the case that $b_{x}^{\alpha, C_{\alpha} \cap \beta, \eta \dagger \beta}$ is an element of $L^{\eta\lceil\beta} \backslash\left\{\Omega_{\beta}\right\}$. In particular, $\mathbf{b}_{x}^{C_{\alpha}, \eta} \upharpoonright \beta \neq \Omega_{\beta}$, contradicting the fact that $\mathbf{b}_{x}^{C_{\alpha}, \eta}=f \upharpoonright \alpha$ and $\Omega_{\beta}=f \upharpoonright \beta$.

This completes the proof.

Remark 5.6. It is tedious yet not impossible to verify that for every nonzero cardinal $\tau$ such that $\kappa$ is $\tau$-closed and such that there exists $\S \in \mathcal{S}$ for which $\S \backslash E_{>\tau}^{\kappa}$ is nonstationary, for every injective sequence $\left\langle\eta_{j} \mid j<\tau\right\rangle$ of elements of $\mathcal{B}(K)$, not only that the product tree $\bigotimes_{j<\tau} T^{\eta_{j}}$ is $\kappa$-Souslin, but in fact, all of its $\tau$-derived trees are $\kappa$-Souslin. In particular, for every $\eta \in \mathcal{B}(K)$, $T^{\eta}$ is a free $\kappa$-Souslin tree.

We now arrive at the following strong form of Theorem A:
Corollary 5.7. Suppose that $\mathrm{P}(\kappa, \kappa, \sqsubseteq, \kappa,\{\kappa\}, 2)$ holds. For every infinite cardinal $\mho$, if there exists a $\kappa$-tree with $\mho$-many branches, then there exists a binary $\kappa$-Aronszajn tree admitting $\mho$-many $\kappa$-Souslin subtrees such that the product of any finitely (nonzero) many of them is again Souslin.

Proof. By a standard fact (see [BR21, Lemma 2.5]), if there exists a $\kappa$-tree with $\mho$-many branches, then there exists one $K$ that is streamlined. Now, appeal to Theorem 5.5 with $K$ and $\mathcal{S}:=\{\kappa\}$, bearing in mind Subsection 4.3.

## 6. A free Souslin tree with a special power

Throughout this section, $\kappa=\lambda^{+}$for a fixed infinite cardinal $\lambda$, and $\triangleleft$ stands for some well-ordering of $H_{\kappa}$. Recall that $\mathbb{Q}_{\lambda}:={ }^{<\omega} \lambda \backslash\{\emptyset\}$, where $q<\mathbb{Q}_{\boldsymbol{\lambda}} p$ iff either $p \subseteq q$ or $q(n)<p(n)$ for the least $n<\omega$ such that $q(n) \neq p(n)$. It is easy to see that $\mathbb{Q}_{\lambda}$ has no first or last elements, that in-between any two elements of $\mathbb{Q}_{\lambda}$ there are $\lambda$-many elements, and that every subset of $\mathbb{Q}_{\lambda}$ of size less than $\operatorname{cf}(\lambda)$ has an upper bound. These and other properties of $\mathbb{Q}_{\lambda}$ and its connection to trees are surveyed in [HS16, Section 3].

Definition 6.1. A tree $T$ of height $\lambda^{+}$is special iff it may be covered by $\lambda$ many antichains.

Note that the preceding definition of special trees at successor cardinals coincides with the general definition given in Subsection 4.2, following [Tod07, Definition 6.1.1], ${ }^{31}$ and that every special $\lambda^{+}$-tree is a $\lambda^{+}$-Aronszajn tree that is not Souslin. The existence of a special $\lambda^{+}$-tree is equivalent to the existence of a $\lambda^{+}$-tree $T$ admitting an order-preserving map $f: T \rightarrow \mathbb{Q}_{\lambda}$, i.e., $f$ is strictly increasing over the chains of $T$.

Definition 6.2. For a streamlined tree $T$ and a nonzero cardinal $\chi$ :

- Denote $T^{\chi}:=\left\{\vec{x} \in \bigcup_{\alpha<\kappa}{ }^{\chi} T_{\alpha} \mid \vec{x}\right.$ is injective $\}$. The ordering $<_{T \chi}$ of $T^{\chi}$ is defined as follows: ${ }^{32}$

$$
\vec{x}<_{T \chi} \vec{y} \Longleftrightarrow \bigwedge_{i<\chi} \vec{x}(i) \subseteq \vec{y}(i) .
$$

[^17]- For $\vec{x} \in T^{\chi}, \Delta(\vec{x})$ stands for the least $\delta<\kappa$ such that $\langle\vec{x}(i) \upharpoonright(\delta+1)|$ $i\langle\chi\rangle$ is injective.
Motivated by Notation 4.9, for every $y \in{ }^{\tau}\left({ }^{\alpha} H_{\kappa}\right)$, we let $\pitchfork(y)$ denote the unique function from $\alpha$ to ${ }^{\tau} H_{\kappa}$ to satisfy $(\pitchfork(y))_{i}=y(i)$ for every $i<\tau$.

Notation 6.3. For every $T \in H_{\kappa}$, denote $\beta(T):=0$ unless there is $\beta<\kappa$ such that $T \subseteq{ }^{\leq \beta} H_{\kappa}$ and $T \nsubseteq{ }^{<\beta} H_{\kappa}$, in which case, we let $\beta(T):=\beta$ for this unique $\beta$.

We collect here a couple of actions from [BR21, §6.2] which we will be used in the upcoming construction. The readers can verify to themselves that additional actions from the same reference can be incorporated into the upcoming proof.

Definition 6.4. (1) The default extension function, extend : $\left(H_{\kappa}\right)^{2} \rightarrow$ $H_{\kappa}$, is defined as follows. Let extend $(x, T):=x$, unless

$$
Q:=\left\{z \in T_{\beta(T)} \mid x \subseteq z\right\}
$$

is nonempty, ${ }^{33}$ in which case, we let extend $(x, T):=\min \left(Q, \triangleleft_{\kappa}\right)$.
(2) The function for sealing antichains, anti : $\left(H_{\kappa}\right)^{3} \rightarrow H_{\kappa}$, is defined as follows. Let anti $(x, T, \mho):=\operatorname{extend}(x, T)$, unless

$$
Q:=\left\{z \in T_{\beta(T)} \mid \exists y \in \mho(x \cup y \subseteq z)\right\}
$$

is nonempty, in which case, we let anti $(x, T, \mho):=\min \left(Q, \triangleleft_{\kappa}\right)$.
The following is obvious.
Lemma 6.5. Suppose $T, \mho, b \in H_{\kappa}$, where $T$ is a normal subtree of $\leq \beta(T) H_{\kappa}$. For every $x \in T$, anti $(x, T, \mho)$ is an element of $T_{\beta(T)}$ extending $x$.

Hereafter, $\chi$ denotes some cardinal in $[2, \omega]$. The next batch of definitions is motivated by the Abraham-Shelah-Solovay construction from [ASS87, §4]. Note, however, that the approach taken here is eventually quite different than the one from [ASS87], since it works uniformly for both $\lambda$ singular and regular cardinals.
Definition 6.6. Define three maps $\varphi_{0}, \varphi_{1}, \varphi_{2}: \mathbb{Q}_{\lambda} \rightarrow \mathbb{Q}_{\lambda}$. For every $q \in \mathbb{Q}_{\lambda}$, the definitions of $\varphi_{0}(q), \varphi_{1}(q), \varphi_{2}(q)$ depend on whether $q$ has length 1 or more, as follows:

$$
\begin{aligned}
& \text { - } \varphi_{0}(q):= \begin{cases}\langle\xi+1,0,0\rangle, & \text { if } q=\langle\xi\rangle ; \\
p^{\wedge}\langle\xi+1,0\rangle, & \text { if } q=p^{\wedge}\langle\xi\rangle .\end{cases} \\
& \text { - } \varphi_{1}(q):= \begin{cases}\langle\xi+1,0\rangle, & \text { if } q=\langle\xi\rangle ; \\
p^{\wedge}\langle\xi+1\rangle, & \text { if } q=p^{\wedge}\langle\xi\rangle .\end{cases} \\
& \text { - } \varphi_{2}(q):= \begin{cases}\langle\xi+1\rangle, & \text { if } q=\langle\xi\rangle ; \\
p, & \text { if } q=p^{\wedge}\langle\xi\rangle .\end{cases}
\end{aligned}
$$

[^18]Remark 6.7. For every $q \in \mathbb{Q}_{\lambda}$ :

- $q<_{\mathbb{Q}_{\lambda}} \varphi_{0}(q)<\mathbb{Q}_{\lambda} \varphi_{1}(q)<_{\mathbb{Q}_{\lambda}} \varphi_{2}(q) ;$
- $\varphi_{2}(q)=\varphi_{2}\left(\varphi_{1}(q)\right)$;
- $\varphi_{1}(q)=\varphi_{2}\left(\varphi_{0}(q)\right)$.

Definition 6.8 (Elevators). For a streamlined tree $T$, a map $f: T^{\chi} \rightarrow \mathbb{Q}_{\lambda}$, two maps $\varphi, \psi: \mathbb{Q}_{\lambda} \rightarrow \mathbb{Q}_{\lambda}$, and ordinals $\beta<\alpha$, we say that a function $e: T_{\beta} \rightarrow T_{\alpha}$ is a $(\varphi, \psi)$-elevator iff the two hold:
(1) $y \subseteq e(y)$ for every $y \in T_{\beta}$, and
(2) for every $\left\langle y_{i} \mid i<\chi\right\rangle \in\left(T_{\beta}\right)^{\chi}$,

$$
(\varphi \circ f)\left(\left\langle e\left(y_{i}\right) \mid i<\chi\right\rangle\right)=(\psi \circ f)\left(\left\langle y_{i} \mid i<\chi\right\rangle\right) .
$$

A map $e$ satisfying just Clause (1) will be simply referred to as an elevator.
Definition 6.9 (Coordination). Let $T$ be a streamlined tree, and let $f$ be a function from $T^{\chi}$ to $\mathbb{Q}_{\lambda}$. For a pair of ordinals $\beta<\alpha$, we say that $T_{\beta}$ and $T_{\alpha}$ are coordinated (with respect to $f$ ) iff for all $n<\chi$ and $\left\langle z_{j} \mid j<n\right\rangle \in\left(T_{\alpha}\right)^{n}$ such that $\Delta\left(\left\langle z_{j} \mid j<n\right\rangle\right)<\beta$, the following three hold:
(i) there exists a $\left(\varphi_{2}, \varphi_{1}\right)$-elevator $e_{1}: T_{\beta} \rightarrow T_{\alpha}$ such that $e_{1}\left(z_{j} \upharpoonright \beta\right)=z_{j}$ for all $j<n$;
(ii) there exists a $\left(\varphi_{2}, \varphi_{2}\right)$-elevator $e_{2}: T_{\beta} \rightarrow T_{\alpha}$ such that $e_{2}\left(z_{j} \upharpoonright \beta\right)=z_{j}$ for all $j<n$;
(iii) if $\alpha=\beta+1$, then for every $m<3$, there exists an (id, $\varphi_{m}$ )-elevator $e: T_{\beta} \rightarrow T_{\alpha}$ such that $e\left(z_{j} \upharpoonright \beta\right)=z_{j}$ for all $j<n$.

We are now ready to prove the main result of this section, which also yields Theorem B.

Theorem 6.10. Suppose that $\mathrm{P}_{\lambda}\left(\lambda^{+}, 2, \sqsubseteq, \lambda^{+}\right)$holds. Let $\chi \in[2, \omega]$ with $\chi<\operatorname{cf}(\lambda)$. Then there exists a $\chi$-free, $\lambda$-splitting, club-regressive, streamlined $\lambda^{+}$-Souslin tree $T$ such that $T^{\chi}$ is special.

Proof. Recall that $\kappa=\lambda^{+}$. Fix $\vec{C}=\left\langle C_{\alpha} \mid \alpha<\kappa\right\rangle$ witnessing $\mathrm{P}_{\lambda}^{-}(\kappa, 2, \sqsubseteq, \kappa)$. For every $\alpha \in \operatorname{acc}(\kappa)$, let

$$
D_{\alpha}:=\{0\} \cup\left\{\eta+1 \mid \eta \in \operatorname{nacc}\left(C_{\alpha}\right)\right\} \cup \operatorname{acc}\left(C_{\alpha}\right),
$$

so that $D_{\alpha}$ is a club in $\alpha$ for which $\operatorname{nacc}\left(D_{\alpha}\right) \subseteq \operatorname{nacc}(\alpha)$. Evidently, $\vec{D}=$ $\left\langle D_{\alpha} \mid \alpha \in \operatorname{acc}(\kappa)\right\rangle$ is yet another $\sqsubseteq$-coherent $C$-sequence.

As $\diamond(\kappa)$ holds, fix sequences $\left\langle\Omega_{\beta} \mid \beta<\kappa\right\rangle$ and $\left\langle R_{i} \mid i<\kappa\right\rangle$ together witnessing $\diamond\left(H_{\kappa}\right)$, as in Fact 2.7. Let $\pi: \kappa \rightarrow \kappa$ be such that $\beta \in R_{\pi(\beta)}$ for all $\beta<\kappa$. Let $\triangleleft$ be some well-ordering of $H_{\kappa}$ of order-type $\kappa$, and let $\phi: \kappa \leftrightarrow H_{\kappa}$ witness the isomorphism $(\kappa, \in) \cong\left(H_{\kappa}, \triangleleft\right)$. Put $\psi:=\phi \circ \pi$.

By recursion on $\alpha<\kappa$, we shall construct $\left\langle\left(T_{\alpha}, f_{\alpha}\right) \mid \alpha<\kappa\right\rangle$ such that $T_{\alpha}$ will end up being the $\alpha^{\text {th }}$-level of the ultimate tree $T$, and $f_{\alpha}:\left(T_{\alpha}\right)^{\chi} \rightarrow \mathbb{Q}_{\lambda}$ will form the $\alpha^{\text {th }}$-level of the specializing map of $T^{\chi}$. We shall also make sure that for all $\beta<\alpha<\kappa, T_{\beta}$ and $T_{\alpha}$ be coordinated.

By convention, for every $\alpha<\kappa$ such that $\left\langle\left(T_{\beta}, f_{\beta}\right) \mid \beta<\kappa\right\rangle$ has already been defined, for every $C \subseteq \alpha$, we shall let $T \upharpoonright C:=\bigcup_{\beta \in C} T_{\beta}$.

The recursion starts by setting $T_{0}:=\{\emptyset\}$ and letting $f_{0}$ be the empty function. Next, given $\alpha<\kappa$ such that ( $T_{\alpha}, f_{\alpha}$ ) has already been successfully defined, set $T_{\alpha+1}:=\left\{t^{\imath}\langle\iota\rangle \mid t \in T_{\alpha}, \iota<\lambda\right\}$. Before we can define $f_{\alpha+1}$ : $\left(T_{\alpha+1}\right)^{\chi} \rightarrow \mathbb{Q}_{\lambda}$, we shall need the following claim.

Claim 6.10.1. Let $S_{\alpha+1}:=\left\{s \in{ }^{<\chi} T_{\alpha+1} \mid s\right.$ is injective and $\left.\Delta(s)<\alpha\right\}$. Then there exists a matrix $\left\langle A_{s, m} \mid s \in S_{\alpha+1}, m<3\right\rangle$ such that, for every $(s, m) \in S_{\alpha+1} \times 3$, the following three hold:
(1) $\operatorname{Im}(s) \subseteq A_{s, m} \subseteq T_{\alpha+1}$;
(2) for every $y \in T_{\alpha}$, there exists a unique $z \in A_{s, m}$ extending $y$;
(3) for every $\left(s^{\prime}, m^{\prime}\right) \in S_{\alpha+1} \times 3$, if $(s, m) \neq\left(s^{\prime}, m^{\prime}\right)$, then $\mid A_{s, m} \cap$ $A_{s^{\prime}, m^{\prime}} \mid<\chi$.

Proof. Let $\left\langle s_{\gamma} \mid \gamma<\lambda\right\rangle$ enumerate $S_{\alpha+1}$. We define $\left\langle A_{s_{\gamma}, m} \mid \gamma<\lambda, m<3\right\rangle$ by recursion on $\left(\lambda \times 3,<_{\text {lex }}\right)$. We start by letting:

$$
A_{s_{0}, 0}:=\operatorname{Im}\left(s_{0}\right) \cup\left\{y^{\frown}\langle 0\rangle \mid y \in T_{\alpha} \backslash\left\{(z \upharpoonright \alpha) \mid z \in \operatorname{Im}\left(s_{0}\right)\right\}\right\} .
$$

Next, given $(\gamma, m) \in \lambda \times 3$ such that $A_{s_{\beta}, i}$ has already been defined for all $(\beta, i)<_{\text {lex }}(\gamma, m)$, let $A_{s_{\gamma}, m}$ be the following set:
$\operatorname{Im}\left(s_{\gamma}\right) \cup\left\{\begin{array}{l|l}y^{\curvearrowright}\left\langle\iota_{y}\right\rangle & \begin{array}{l}y \in T_{\alpha} \backslash\left\{(z \upharpoonright \alpha) \mid z \in \operatorname{Im}\left(s_{\beta}\right)\right\} \& \\ \iota_{y}:=\min \left\{\iota<\lambda \mid \forall(\beta, i)<_{\text {lex }}(\gamma, m)\left[y^{\sim}\langle\iota\rangle \notin A_{s_{\beta}, i}\right]\right.\end{array}\end{array}\right\}$.
It is clear that the above construction takes care of Clauses (1) and (2). To verify Clause (3), fix a pair $(\beta, i)<_{\text {lex }}(\gamma, m)$ of elements of $\lambda \times 3$. If $\left|A_{s_{\beta}, i} \cap A_{s_{\gamma}, m}\right| \geq \chi$, then we may fix $z \in A_{s_{\beta}, i} \cap A_{s_{\gamma}, m} \backslash \operatorname{Im}\left(s_{\gamma}\right)$. It follows that $z=y^{\frown}\langle\iota\rangle$ for some $y \in T_{\alpha}$ and $\iota<\lambda$ such that, in particular, $y^{\curvearrowleft}\langle\iota\rangle \notin A_{s_{\beta}, i}$. This is a contradiction.

Fix a matrix $\left\langle A_{s, m} \mid s \in S_{\alpha+1}, m<3\right\rangle$ as in the preceding claim. To define $f_{\alpha+1}:\left(T_{\alpha+1}\right)^{\chi} \rightarrow \mathbb{Q}_{\lambda}$, let $\left\langle w_{i} \mid i<\chi\right\rangle \in\left(T_{\alpha+1}\right)^{\chi}$. There are three cases to consider:

- If $\Delta\left(\left\langle w_{i} \mid i<\chi\right\rangle\right)=\alpha$, then let $f_{\alpha+1}\left(\left\langle w_{i} \mid i<\chi\right\rangle\right):=\langle 0\rangle$.
- If $\Delta\left(\left\langle w_{i} \mid i<\chi\right\rangle\right)<\alpha$ and $\left\{w_{i} \mid i<\chi\right\} \subseteq A_{s, m}$ for some $(s, m) \in$ $S_{\alpha+1} \times 3$, then by Clause (3) of the above claim, the pair $(s, m)$ is unique, so we let

$$
f_{\alpha+1}\left(\left\langle w_{i} \mid i<\chi\right\rangle\right):=\left(\varphi_{m} \circ f_{\alpha}\right)\left(\left\langle w_{i} \upharpoonright \alpha \mid i<\chi\right\rangle\right) .
$$

- Otherwise, since $\mathbb{Q}_{\lambda}$ has no maximal elements, let $f_{\alpha+1}\left(\left\langle w_{i} \mid i<\chi\right\rangle\right)$ be some $q \in \mathbb{Q}_{\lambda}$ which is bigger than $f_{\alpha}\left(\left\langle w_{i} \upharpoonright \alpha \mid i<\chi\right\rangle\right)$.

Altogether, for every $\left\langle w_{i} \mid i<\chi\right\rangle \in\left(T_{\alpha+1}\right)^{\chi}$ such that $\Delta\left(\left\langle w_{i} \mid i<\chi\right\rangle\right)<$ $\alpha, f_{\alpha}\left(\left\langle w_{i} \upharpoonright \alpha \mid i<\chi\right\rangle\right)<\mathbb{Q}_{\lambda} f_{\alpha+1}\left(\left\langle w_{i} \mid i<\chi\right\rangle\right)$.

Claim 6.10.2. (1) $T_{\alpha}$ and $T_{\alpha+1}$ are coordinated;
(2) For every $\beta<\alpha, T_{\beta}$ and $T_{\alpha+1}$ are coordinated.

Proof. (1) Let $n<\chi$ and $\left\langle z_{j} \mid j<n\right\rangle \in\left(T_{\alpha+1}\right)^{n}$ with $\Delta\left(\left\langle z_{j} \mid j<n\right\rangle\right)<\alpha$. In particular, $s:=\left\langle z_{j} \mid j<n\right\rangle$ is in $S_{\alpha+1}$. We go over the clauses of Definition 6.9 in reverse order:
(iii) Given $m<3$, define an elevator $e: T_{\alpha} \rightarrow T_{\alpha+1}$ by letting for every $y \in T_{\alpha}, e(y)$ be the unique $z \in A_{s, m}$ extending $y$. As $\operatorname{Im}(s)=\left\{z_{j} \mid\right.$ $j<n\} \subseteq A_{s, m}$, for every $j<n, e\left(z_{j} \upharpoonright \alpha\right)$ must be $z_{j}$. We claim that $e$ is an (id, $\varphi_{m}$ )-elevator. Indeed, for any $\left\langle y_{i} \mid i<\chi\right\rangle \in\left(T_{\alpha}\right)^{\chi}$, $\left\{e\left(y_{i}\right) \mid i<\chi\right\} \subseteq A_{s, m}$, so by the definition of $f_{\alpha+1}$ :

$$
\left.f_{\alpha+1}\left(\left\langle e\left(y_{i}\right) \mid i<\chi\right\rangle\right)=\left(\varphi_{m} \circ f_{\alpha}\right)\left(\left\langle y_{i} \mid i<\chi\right\rangle\right)\right) .
$$

(ii) By Clause (iii), we may fix an (id, $\varphi_{1}$ )-elevator $e_{2}: T_{\alpha} \rightarrow T_{\alpha+1}$ such that $e_{2}\left(z_{j} \upharpoonright \alpha\right)=z_{j}$ for all $j<n$. By Remark 6.7, it is also a $\left(\varphi_{2}, \varphi_{2}\right)$-elevator.
(i) By Clause (iii), we may fix an (id, $\varphi_{0}$ )-elevator $e_{1}: T_{\alpha} \rightarrow T_{\alpha+1}$ such that $e_{1}\left(z_{j} \upharpoonright \alpha\right)=z_{j}$ for all $j<n$. By Remark 6.7 , it is also a $\left(\varphi_{2}, \varphi_{1}\right)$-elevator.
(2) Fix $\beta<\alpha$. Let $n<\chi$ and $\left\langle z_{j} \mid j<n\right\rangle \in\left(T_{\alpha+1}\right)^{n}$ with $\Delta\left(\left\langle z_{j}\right|\right.$ $j<n\rangle)<\beta$. We go over the clauses of Definition 6.9:
(i) By Clause (1) of this claim, fix a $\left(\varphi_{2}, \varphi_{2}\right)$-elevator $e: T_{\alpha} \rightarrow T_{\alpha+1}$ such that $e\left(z_{j} \upharpoonright \alpha\right)=z_{j}$ for all $j<n$. In addition, as $T_{\beta}$ and $T_{\alpha}$ are coordinated, fix a $\left(\varphi_{2}, \varphi_{1}\right)$-elevator $e_{1}: T_{\beta} \rightarrow T_{\alpha}$ such that $e_{1}\left(z_{j} \upharpoonright \beta\right)=z_{j} \upharpoonright \alpha$ for all $j<n$. Set $E_{1}:=e \circ e_{1}$. Then $E_{1}\left(z_{j} \upharpoonright \beta\right)=$ $e\left(e_{1}\left(z_{j} \upharpoonright \beta\right)\right)=e\left(z_{j} \upharpoonright \alpha\right)=z_{j}$ for all $j<n$. In addition, for all $y \in T_{\beta}$, $y \subseteq e_{1}(y) \subseteq e\left(e_{1}(y)\right)=E_{1}(y)$. Finally, for every $\left\langle y_{i} \mid i<\chi\right\rangle \in\left(T_{\beta}\right)^{\chi}$, $\left(\varphi_{2} \circ f_{\alpha+1}\right)\left(\left\langle E_{1}\left(y_{i}\right) \mid i<\chi\right\rangle\right)=\left(\varphi_{2} \circ f_{\alpha+1}\right)\left(\left\langle e\left(e_{1}\left(y_{i}\right)\right) \mid i<\chi\right\rangle\right)$ $=\left(\varphi_{2} \circ f_{\alpha}\right)\left(\left\langle e_{1}\left(y_{i}\right) \mid i<\chi\right\rangle\right)$ $=\left(\varphi_{1} \circ f_{\beta}\right)\left(\left\langle y_{i} \mid i<\chi\right\rangle\right)$.
So, $E_{1}$ is a $\left(\varphi_{2}, \varphi_{1}\right)$-elevator as sought.
(ii) Replace 1 by 2 throughout the above proof.
(iii) This case is irrelevant.

Next, suppose that we have reached an $\alpha \in \operatorname{acc}(\kappa)$ such that $\left\langle\left(T_{\beta}, f_{\beta}\right)\right|$ $\beta<\alpha\rangle$ has already been successfully defined. For each $x \in T \upharpoonright D_{\alpha}$, we shall define a branch $\mathbf{b}_{x}^{\alpha}$ through $\bigcup_{\beta<\alpha} T_{\beta}$, and then let

$$
T_{\alpha}:=\left\{\mathbf{b}_{x}^{\alpha} \mid x \in T \upharpoonright D_{\alpha}\right\}
$$

The branch $\mathbf{b}_{x}^{\alpha}$ will be obtained as the limit $\bigcup \operatorname{Im}\left(b_{x}^{\alpha}\right)$ of a sequence $\left\langle b_{x}^{\alpha}(\beta) \mid \beta \in D_{\alpha} \backslash \operatorname{dom}(x)\right\rangle$ of nodes such that:

- for every $\beta \in D_{\alpha} \backslash \operatorname{dom}(x), b_{x}^{\alpha}(\beta) \in T_{\beta}$;
- for every pair $\beta<\beta^{\prime}$ of ordinals in $D_{\alpha} \backslash \operatorname{dom}(x), x \subseteq b_{x}^{\alpha}(\beta) \subseteq b_{x}^{\alpha}\left(\beta^{\prime}\right)$;
- for every $\beta \in \operatorname{acc}\left(D_{\alpha} \backslash \operatorname{dom}(x)\right), b_{x}^{\alpha}(\beta)=\bigcup\left(\operatorname{Im}\left(b_{x}^{\alpha} \upharpoonright \beta\right)\right)$.

The construction is by recursion on $\beta \in D_{\alpha}$.

- For $\beta:=\min \left(D_{\alpha}\right)$, it is the case that $\beta=0$ and so $T_{\beta}=\{x\}$ for $x:=\emptyset$. Therefore, we set $b_{x}^{\alpha}(\beta):=x$.
- Suppose that we are given a nonzero $\beta \in \operatorname{nacc}\left(D_{\alpha}\right)$ such that $b_{x}^{\alpha} \upharpoonright \beta$ has already been defined for all $x \in T \upharpoonright\left(D_{\alpha} \cap \beta\right)$. We need to define $b_{x}^{\alpha}(\beta)$ for all $x \in T \upharpoonright\left(D_{\alpha} \cap(\beta+1)\right)$. For every $x \in T_{\beta}$, we just let $b_{x}^{\alpha}(\operatorname{dom}(x)):=x$. Our next task is defining $b_{x}^{\alpha}(\beta)$ for $x \in T \upharpoonright\left(D_{\alpha} \cap \beta\right)$. To this end, we introduce the following pieces of notation.

Notation 6.10.1. Denote $\beta^{-}:=\sup \left(D_{\alpha} \cap \beta\right)$ and note that by the definition of $D_{\alpha}$, it is the case that $\beta=\eta+1$ for a unique $\eta \in C_{\alpha} \backslash \beta^{-}$. Now, let $\mathcal{E}_{\beta}^{\alpha}$ denote the collection of all (id, $\varphi_{1}$ )-elevators $e: T_{\beta^{-}} \rightarrow T_{\beta}$ to satisfy that if there exists an $n<\chi$ such that $\psi(\eta) \in\left(T \upharpoonright\left(D_{\alpha} \cap \beta^{-}\right)\right)^{n}$, then letting $\bar{T}:=(T \upharpoonright \beta+1)(\psi(\eta))$ (using Definition 4.10), for every $j<n$,

$$
e\left(b_{\psi(\eta)(j)}^{\alpha}\left(\beta^{-}\right)\right)=\left(\operatorname{anti}\left(\pitchfork\left(\left\langle b_{\psi(\eta)(j)}^{\alpha}\left(\beta^{-}\right) \mid j<n\right\rangle\right), \bar{T}, \Omega_{\eta}\right)\right)_{j} .
$$

Note that the definitions of $\beta^{-}, \eta$ and $\mathcal{E}_{\beta}^{\alpha}$ are all determined by no more than the following objects:

- $\left\langle\left(T_{\gamma}, f_{\gamma}\right) \mid \gamma \leq \beta\right\rangle$,
- $\psi(\eta)$,
- $D_{\alpha} \cap \beta$, and possibly also on
- $\left\langle b_{\psi(\eta)(j)}^{\alpha}\left(\beta^{-}\right) \mid j<n\right\rangle$ and $\Omega_{\eta}$.

Claim 6.10.3. $\mathcal{E}_{\beta}^{\alpha}$ is nonempty.
Proof. If there exists some $n<\chi$ such that $\psi(\eta) \in\left(T \upharpoonright\left(D_{\alpha} \cap \beta^{-}\right)\right)^{n}$, then let

$$
z:=\operatorname{anti}\left(\pitchfork\left(\left\langle b_{\psi(\eta)(j)}^{\alpha}\left(\beta^{-}\right) \mid j<n\right\rangle\right), \bar{T}, \Omega_{\eta}\right)
$$

Otherwise, just set $(n, z):=(0, \emptyset)$.
Next, as $T_{\beta^{-}}$and $T_{\eta}$ are coordinated, we may fix a $\left(\varphi_{2}, \varphi_{1}\right)$-elevator $e_{1}$ : $T_{\beta^{-}} \rightarrow T_{\eta}$ such that $e_{1}\left(b_{\psi(\eta)(j)}^{\alpha}\left(\beta^{-}\right)\right)=(z)_{j} \upharpoonright \eta$ for all $j<n$. As $T_{\eta}$ and $T_{\beta}$ are coordinated, and $\beta=\eta+1$, we may also fix an (id, $\varphi_{2}$ )-elevator $e_{2}: T_{\eta} \rightarrow T_{\beta}$ such that $e_{2}\left((z)_{j} \upharpoonright \eta\right)=(z)_{j}$ for all $j<n$. Set $e:=e_{2} \circ e_{1}$. Clearly, $e\left(b_{\psi(\eta)(j)}^{\alpha}\left(\beta^{-}\right)\right)=(z)_{j}$ for all $j<n$. In addition, for every $\left\langle y_{i}\right|$ $i<\chi\rangle \in\left(T_{\beta^{-}}\right)^{\chi}$,

$$
\begin{aligned}
f_{\beta}\left(\left\langle e\left(y_{i}\right) \mid i<\chi\right\rangle\right) & =f_{\beta}\left(\left\langle e_{2}\left(e_{1}\left(y_{i}\right)\right) \mid i<\chi\right\rangle\right) \\
& =\left(\varphi_{2} \circ f_{\eta}\right)\left(\left\langle e_{1}\left(y_{i}\right) \mid i<\chi\right\rangle\right) \\
& =\left(\varphi_{1} \circ f_{\beta^{-}}\right)\left(\left\langle y_{i} \mid i<\chi\right\rangle\right) .
\end{aligned}
$$

So, $e$ demonstrates that $\mathcal{E}_{\beta}^{\alpha}$ is nonempty.
Let $e:=\min \left(\mathcal{E}_{\beta}^{\alpha}, \triangleleft\right)$, and then define $b_{x}^{\alpha}(\beta):=e\left(b_{x}^{\alpha}\left(\beta^{-}\right)\right)$for every $x \in$ $T \upharpoonright\left(D_{\alpha} \cap \beta\right)$.

- Suppose that we are given a $\beta \in \operatorname{acc}\left(D_{\alpha}\right)$ such that $b_{x}^{\alpha} \upharpoonright \beta$ has already been defined for all $x \in T \upharpoonright\left(D_{\alpha} \cap \beta\right)$. For every $x \in T \upharpoonright\left(D_{\alpha} \cap(\beta+1)\right)$, we let

$$
b_{x}^{\alpha}(\beta):= \begin{cases}x, & \text { if } x \in T_{\beta} ; \\ \bigcup \operatorname{Im}\left(b_{x}^{\alpha} \upharpoonright \beta\right), & \text { if } x \in T \upharpoonright\left(D_{\alpha} \cap \beta\right) .\end{cases}
$$

In order to argue that $b_{x}^{\alpha}(\beta)$ is indeed an element of $T_{\beta}$, it suffices to prove the following claim.

Claim 6.10.4. For every $x \in T \upharpoonright\left(D_{\alpha} \cap \beta\right)$, $b_{x}^{\alpha}(\beta)=\mathbf{b}_{x}^{\beta}$.
Proof. It suffices to show that for every $x \in T \upharpoonright\left(D_{\alpha} \cap \beta\right)$, for every $\gamma \in\left(D_{\alpha} \cap\right.$ $\beta) \backslash \operatorname{dom}(x)$, it is the case that $b_{x}^{\alpha}(\gamma)=\mathbf{b}_{x}^{\beta} \upharpoonright \gamma$. Recalling that $\beta \in \operatorname{acc}\left(D_{\alpha}\right)$ and that $\vec{D}$ is coherent, we infer that $D_{\alpha} \cap \beta=D_{\beta}$. Thus, it suffices to show that for every $x \in T \upharpoonright D_{\beta}$, for every $\gamma \in D_{\beta} \backslash \operatorname{dom}(x)$, it is the case that $b_{x}^{\alpha}(\gamma)=b_{x}^{\beta}(\gamma)$. The proof is by induction, as follows:

- The base case $\gamma=0$ is obvious, as $b_{\emptyset}^{\alpha}(0)=\emptyset=b_{\emptyset}^{\beta}(0)$.
- Suppose that $\gamma^{-}<\gamma$ are successive elements of $D_{\beta}$ and that, for every $x \in\left(D_{\beta} \cap \gamma\right), b_{x}^{\alpha}\left(\gamma^{-}\right)=b_{x}^{\beta}\left(\gamma^{-}\right)$. By our construction, it is the case that $b_{x}^{\alpha}(\gamma):=e_{\alpha}\left(b_{x}^{\alpha}\left(\gamma^{-}\right)\right)$for $e_{\alpha}:=\min \left(\mathcal{E}_{\gamma}^{\alpha}, \triangleleft\right)$ and that $b_{x}^{\beta}(\gamma):=e_{\beta}\left(b_{x}^{\beta}\left(\gamma^{-}\right)\right)$for $e_{\beta}:=\min \left(\mathcal{E}_{\gamma}^{\beta}, \triangleleft\right)$. Reading the comment right after Definition 6.10.1, it is clear that in this case $\mathcal{E}_{\gamma}^{\alpha}=\mathcal{E}_{\gamma}^{\beta}$, and so $e_{\alpha}=e_{\beta}$ and $b_{x}^{\alpha}(\gamma)=b_{x}^{\beta}(\gamma)$.
- For $\gamma \in \operatorname{acc}\left(D_{\beta}\right)$ such that the two sequences agree up to $\gamma$, it is the case that they have the same unique limit.

We are done defining $\left\langle\mathbf{b}_{x}^{\alpha} \mid x \in T \upharpoonright D_{\alpha}\right\rangle$, and so we define $T_{\alpha}$ as per ( $\star$ ).
Claim 6.10.5. For every $x \in T \upharpoonright D_{\alpha}$, for every $\gamma \in D_{\alpha} \backslash \operatorname{dom}(x), \mathbf{b}_{x}^{\alpha}=$ $\mathbf{b}_{b_{x}^{\alpha}(\gamma)}^{\alpha}$.

Proof. By the canonical nature of the above construction.
Now, to define $f_{\alpha}:\left(T_{\alpha}\right)^{\chi} \rightarrow \mathbb{Q}_{\lambda}$, let $\left\langle w_{i} \mid i<\chi\right\rangle \in\left(T_{\alpha}\right)^{\chi}$ be arbitrary. For each $i<\chi$, find $x_{i} \in T \upharpoonright D_{\alpha}$ of minimal height such that $\mathbf{b}_{x_{i}}^{\alpha}=w_{i}$. Set $\gamma:=\sup \left\{\operatorname{dom}\left(x_{i}\right) \mid i<\chi\right\}$. There are two main cases to consider:

- If $\gamma=\alpha$, then $\operatorname{cf}(\alpha) \leq \chi$. Now, there are two subcases here:
- If there exists some $\epsilon<\alpha$ such that $\left\langle w_{i} \upharpoonright \epsilon \mid i<\chi\right\rangle$ is injective, then fix such an $\epsilon$, and since $\operatorname{cf}(\alpha) \leq \chi<\operatorname{cf}(\lambda)$, the set $F:=$ $\left\{f_{\delta}\left(\left\langle x_{i}\right| \delta|i<\chi\rangle \mid \epsilon<\delta<\alpha\right\}\right.$ is bounded in $\mathbb{Q}_{\lambda}$; fix $q$ to be some bound of $F$, and let $f_{\alpha}\left(\left\langle w_{i} \mid i<\chi\right\rangle\right):=q$.
- If there exists no such $\epsilon<\alpha$, then it is harmless to just let $f_{\alpha}\left(\left\langle w_{i} \mid i<\chi\right\rangle\right):=\langle 0\rangle$.
- If $\gamma<\alpha$, then denote $\left(\varphi_{1} \circ f_{\gamma}\right)\left(\left\langle w_{i}\right| \gamma|i<\chi\rangle\right)$ by $p^{\wedge}\langle\xi\rangle$. Now, again there are two subcases:
- If $\operatorname{otp}\left(D_{\alpha}\right)=\lambda$, then let $f_{\alpha}\left(\left\langle w_{i} \mid i<\chi\right\rangle\right):=p$.
$\rightarrow$ If $\operatorname{otp}\left(D_{\alpha}\right)<\lambda$, then let $f_{\alpha}\left(\left\langle w_{i} \mid i<\chi\right\rangle\right):=p^{\curvearrowleft}\langle\xi+\sigma\rangle$ for the least $\sigma<\lambda$ such that $\operatorname{otp}\left(D_{\alpha} \cap \gamma\right)+1+\sigma=\operatorname{otp}\left(D_{\alpha}\right)$.
This completes the definition of the function $f_{\alpha}$. We must verify that $\bigcup_{\beta \leq \alpha} f_{\beta}$ is order-preserving, and that $T_{\beta}$ and $T_{\alpha}$ are coordinated for every $\beta<\alpha$. It will be easier to show once we have established the following claim.

Claim 6.10.6. Let $\left\langle w_{i} \mid i<\chi\right\rangle \in\left(T_{\alpha}\right)^{\chi}$. For each $i<\chi$, find $x_{i} \in T \upharpoonright D_{\alpha}$ of minimal height such that $\mathbf{b}_{x_{i}}^{\alpha}=w_{i}$. Suppose that $\gamma:=\sup \left\{\operatorname{dom}\left(x_{i}\right) \mid i<\chi\right\}$ is smaller than $\alpha$, and let $p^{\wedge}\langle\xi\rangle$ denote $\left(\varphi_{1} \circ f_{\gamma}\right)\left(\left\langle w_{i} \upharpoonright \gamma \mid i<\chi\right\rangle\right)$. Then:
(1) For every $\beta \in D_{\alpha} \backslash \gamma$,

$$
\left(\varphi_{2} \circ f_{\beta}\right)\left(\left\langle w_{i} \upharpoonright \beta \mid i<\chi\right\rangle\right)=p
$$

(2) For every $\beta \in D_{\alpha} \backslash(\gamma+1)$,

$$
f_{\beta}\left(\left\langle w_{i} \upharpoonright \beta \mid i<\chi\right\rangle\right)=p^{\curvearrowleft}\langle\xi+\sigma\rangle,
$$

for the least $\sigma$ such that $\operatorname{otp}\left(D_{\alpha} \cap \gamma\right)+1+\sigma=\operatorname{otp}\left(D_{\alpha} \cap \beta\right)$.
Proof. (1) The conclusion for $\beta>\gamma$ follows from Clause (2) below. As for $\beta=\gamma$, note that by Remark 6.7,
$\left(\varphi_{2} \circ f_{\beta}\right)\left(\left\langle w_{i} \upharpoonright \beta \mid i<\chi\right\rangle\right)=\left(\varphi_{2} \circ \varphi_{1} \circ f_{\beta}\right)\left(\left\langle w_{i} \upharpoonright \beta \mid i<\chi\right\rangle\right)=\varphi_{2}\left(p^{\frown}\langle\xi\rangle\right)=p$.
(2) For each $i<\chi$, write $y_{i}:=w_{i} \upharpoonright \gamma$, and note that $\mathbf{b}_{y_{i}}^{\alpha}=w_{i}$ by Claim 6.10.5. We now prove the claim by induction on $\beta \in D_{\alpha} \backslash(\gamma+1)$ :

Base: Suppose $\beta=\min \left(D_{\alpha} \backslash(\gamma+1)\right)$, so that $\sigma:=0$ satisfies otp $\left(D_{\alpha} \cap\right.$ $\gamma)+1+\sigma=\operatorname{otp}\left(D_{\alpha} \cap \beta\right)$. Recalling the construction, there exists some (id, $\varphi_{1}$ )-elevator $e: T_{\gamma} \rightarrow T_{\beta}$ (coming from $\mathcal{E}_{\beta}^{\alpha}$ ) such that $b_{x}^{\alpha}(\beta)=e\left(b_{x}^{\alpha}(\gamma)\right)$ for every $x \in T_{\gamma}$. Thus,

$$
\begin{aligned}
f_{\beta}\left(\left\langle b_{y_{i}}^{\alpha}(\beta) \mid i<\chi\right\rangle\right) & =\left(\varphi_{1} \circ f_{\gamma}\right)\left(\left\langle b_{y_{i}}^{\alpha}(\gamma) \mid i<\chi\right\rangle\right) \\
& =\left(\varphi_{1} \circ f_{\gamma}\right)\left(\left\langle y_{i} \mid i<\chi\right\rangle\right) \\
& =\left(\varphi_{1} \circ f_{\gamma}\right)\left(\left\langle w_{i} \upharpoonright \gamma \mid i<\chi\right\rangle\right) \\
& =p^{\wedge}\langle\xi\rangle \\
& =p^{\complement}\langle\xi+\sigma\rangle
\end{aligned}
$$

since $\sigma=0$.
Successor step: Suppose $\beta \in \operatorname{nacc}\left(D_{\alpha}\right)$ is such that $\beta^{-}:=\sup \left(D_{\alpha} \cap\right.$ $\beta$ ) is bigger than $\gamma$ and satisfies

$$
f_{\beta^{-}}\left(\left\langle b_{y_{i}}^{\alpha}\left(\beta^{-}\right) \mid i<\chi\right\rangle\right)=p^{\curvearrowleft}\langle\xi+\sigma\rangle,
$$

for the least $\sigma$ such that $\operatorname{otp}\left(D_{\alpha} \cap \gamma\right)+1+\sigma=\operatorname{otp}\left(D_{\alpha} \cap \beta^{-}\right)$. Recalling the construction, there exists some (id, $\varphi_{1}$ )-elevator $e: T_{\beta^{-}} \rightarrow T_{\beta}$ such that $b_{x}^{\alpha}(\beta)=e\left(b_{x}^{\alpha}\left(\beta^{-}\right)\right)$for every $x \in T_{\beta^{-}}$. Thus,

$$
\begin{aligned}
f_{\beta}\left(\left\langle b_{y_{i}}^{\alpha}(\beta) \mid i<\chi\right\rangle\right) & =\left(\varphi_{1} \circ f_{\beta^{-}}\right)\left(\left\langle b_{y_{i}}^{\alpha}\left(\beta^{-}\right) \mid i<\chi\right\rangle\right) \\
& =\varphi_{1}\left(p^{\wedge}\langle\xi+\sigma\rangle\right) \\
& =p^{\curvearrowleft}\langle\xi+\sigma+1\rangle,
\end{aligned}
$$

as sought.
Limit step: Suppose $\beta \in \operatorname{acc}\left(D_{\alpha} \backslash \gamma\right)$. By Claim 6.10.4, $\left\langle b_{y_{i}}^{\alpha}(\beta)\right|$ $i<\chi\rangle=\left\langle\mathbf{b}_{y_{i}}^{\beta} \mid i<\chi\right\rangle$. By the coherence of $\vec{D}$ and since in particular $\operatorname{otp}\left(D_{\beta}\right)=\operatorname{otp}\left(D_{\alpha} \cap \beta\right)<\operatorname{otp}\left(D_{\alpha}\right) \leq \lambda$, the definition of $f_{\beta}\left(\left\langle w_{i} \upharpoonright \beta\right|\right.$ $i<\chi\rangle$ ) goes through the following considerations. For each $i<\chi$,
find $\bar{x}_{i} \in T \upharpoonright D_{\beta}$ of minimal height such that $\mathbf{b}_{\bar{x}_{i}}^{\beta}=w_{i} \upharpoonright \beta$, and then set $\bar{\gamma}:=\sup \left\{\operatorname{dom}\left(\bar{x}_{i}\right) \mid i<\chi\right\}$. As $\mathbf{b}_{y_{i}}^{\alpha} \upharpoonright \beta=w_{i} \upharpoonright \beta$ for every $i<\chi$, we infer that $\bar{\gamma} \leq \gamma$. To see that also $\bar{\gamma} \geq \gamma$, note that for every $i<\chi$, by Claim 6.10.4, $\mathbf{b}_{\bar{x}_{i}}^{\alpha} \upharpoonright \beta=\mathbf{b}_{\bar{x}_{i}}^{\beta}$, and hence $\mathbf{b}_{\bar{x}_{i}}^{\alpha} \upharpoonright \gamma=\mathbf{b}_{\bar{x}_{i}}^{\beta} \upharpoonright \gamma=w_{i} \upharpoonright \gamma=y_{i}$, and then Claim 6.10.5 implies that $\mathbf{b}_{\bar{x}_{i}}^{\alpha}=\mathbf{b}_{y_{i}}^{\alpha}=w_{i}$.

Now, since $\bar{\gamma}=\gamma$, it is the case that $f_{\beta}\left(\left\langle w_{i} \upharpoonright \beta \mid i<\chi\right\rangle\right):=p^{\wedge}\langle\xi+\sigma\rangle$ for the least $\sigma<\lambda$ such that $\operatorname{otp}\left(D_{\beta} \cap \gamma\right)+1+\sigma=\operatorname{otp}\left(D_{\beta}\right)$. But $D_{\beta}=D_{\alpha} \cap \beta$ and hence $\sigma$ is the least to satisfy $\operatorname{otp}\left(D_{\alpha} \cap \gamma\right)+1+\sigma=$ $\operatorname{otp}\left(D_{\alpha} \cap \beta\right)$.

Claim 6.10.7. Let $\left\langle w_{i} \mid i<\chi\right\rangle \in\left(T_{\alpha}\right)^{\chi}$. Then $f_{\beta}\left(\left\langle w_{i} \upharpoonright \beta \mid i<\chi\right\rangle\right)<\mathbb{Q}_{\lambda}$ $f_{\alpha}\left(\left\langle w_{i} \mid i<\chi\right\rangle\right)$ for every $\beta \in\left[\Delta\left(\left\langle w_{i} \mid i<\chi\right\rangle\right), \alpha\right)$.
Proof. By the induction hypothesis on $\left\langle\left(T_{\beta}, f_{\beta}\right) \mid \beta<\alpha\right\rangle$, to show that $f_{\beta}\left(\left\langle w_{i} \upharpoonright \beta \mid i<\chi\right\rangle\right)<\mathbb{Q}_{\lambda} f_{\alpha}\left(\left\langle w_{i} \mid i<\chi\right\rangle\right)$ for a tail of $\beta<\alpha$, it suffices to prove that this is the case for cofinally many $\beta<\alpha$. For each $i<\chi$, find $x_{i} \in T \upharpoonright D_{\alpha}$ of minimal height such that $\mathbf{b}_{x_{i}}^{\alpha}=w_{i}$. Set $\gamma:=\sup \left\{\operatorname{dom}\left(x_{i}\right) \mid i<\chi\right\}$, so that $\gamma \in D_{\alpha} \cup\{\alpha\}$. By the definition of $f_{\alpha}$, we may avoid trivialities and assume that $\gamma<\alpha$. In this case, we let $p^{\curvearrowleft}\langle\xi\rangle$ denote $\left(\varphi_{1} \circ f_{\gamma}\right)\left(\left\langle w_{i} \upharpoonright \gamma\right|\right.$ $i<\chi\rangle$ ), and observe that by Claim 6.10.6(2), it suffices to prove that for every $\sigma$ such that $\operatorname{otp}\left(D_{\alpha} \cap \gamma\right)+1+\sigma<\operatorname{otp}\left(D_{\alpha}\right)$,

$$
p^{\curvearrowleft}\langle\xi+\sigma\rangle<_{\mathbb{Q}_{\lambda}} f_{\alpha}\left(\left\langle w_{i} \mid i<\chi\right\rangle\right) .
$$

However, the definition of $f_{\alpha}$ makes it clear that this is indeed the case.
Claim 6.10.8. Let $\epsilon<\alpha$. Then $T_{\epsilon}$ and $T_{\alpha}$ are coordinated.
Proof. Let $n<\chi$ and $\left\langle z_{j} \mid j<n\right\rangle \in\left(T_{\alpha}\right)^{n}$ with $\Delta\left(\left\langle z_{j} \mid j<n\right\rangle\right)<\epsilon$. Before going over the clauses of Definition 6.9, let us first establish the following crucial subclaim.

Subclaim 6.10.8.1. There are $\beta \in(\epsilon, \alpha)$ and $a\left(\varphi_{2}, \varphi_{2}\right)$-elevator $e: T_{\beta} \rightarrow$ $T_{\alpha}$ such that $\left\langle e\left(z_{j} \upharpoonright \beta\right) \mid j<n\right\rangle=\left\langle z_{j} \mid j<n\right\rangle$.

Proof. For each $j<n$, fix $\bar{x}_{j} \in T \upharpoonright D_{\alpha}$ of minimal height such that $z_{j}=\mathbf{b}_{\bar{x}_{j}}^{\alpha}$. As $n$ is finite, we may let $\beta:=\min \left(D_{\alpha} \backslash \max \left\{\operatorname{dom}\left(\bar{x}_{j}\right), \epsilon+1 \mid j<n\right\}\right)$.

- If $\operatorname{otp}\left(D_{\alpha}\right)<\lambda$, then define an elevator $e: T_{\beta} \rightarrow T_{\alpha}$ via $e(y):=\mathbf{b}_{y}^{\alpha}$. By Claim 6.10.5, $e\left(z_{j} \upharpoonright \beta\right)=z_{j}$ for every $j<n$.

To see that $e$ is a $\left(\varphi_{2}, \varphi_{2}\right)$-elevator, let $\left\langle y_{i} \mid i<\chi\right\rangle \in\left(T_{\beta}\right)^{\chi}$. For each $i<\chi$, denote $w_{i}:=e\left(y_{i}\right)$, and find $x_{i} \in T \upharpoonright D_{\alpha}$ of minimal height such that $\mathbf{b}_{x_{i}}^{\alpha}=w_{i}$. As $w_{i}=\mathbf{b}_{y_{i}}^{\alpha}$, Claim 6.10.5 implies that $\operatorname{dom}\left(x_{i}\right) \leq \operatorname{dom}\left(y_{i}\right)$, so that $\gamma:=\sup \left\{\operatorname{dom}\left(x_{i}\right) \mid i<\chi\right\}$ satisfies $\gamma \leq \beta<\alpha$. Let $p^{\wedge}\langle\xi\rangle$ denote $\left(\varphi_{1} \circ f_{\gamma}\right)\left(\left\langle w_{i} \upharpoonright \gamma \mid i<\chi\right\rangle\right)$.

On one hand, since $\beta \in D_{\alpha} \backslash \gamma$, Claim 6.10.6(1) asserts that $\left(\varphi_{2} \circ f_{\beta}\right)\left(\left\langle w_{i}\right\rceil\right.$ $\beta|i<\chi\rangle)=p$. On the other hand, by the definition of $f_{\alpha}$, it is the case that $f_{\alpha}\left(\left\langle w_{i} \mid i<\chi\right\rangle\right)=p^{\curvearrowleft}\langle\xi+\sigma\rangle$ for some $\sigma<\lambda$, and hence $\left(\varphi_{2} \circ f_{\alpha}\right)\left(\left\langle w_{i}\right|\right.$ $i<\chi\rangle)=p$. Altogether, $\left(\varphi_{2} \circ f_{\alpha}\right)\left(\left\langle w_{i} \mid i<\chi\right\rangle\right)=\left(\varphi_{2} \circ f_{\beta}\right)\left(\left\langle w_{i} \upharpoonright \beta \mid i<\chi\right\rangle\right)$.

- If $\operatorname{otp}\left(D_{\alpha}\right)=\lambda$, then as $T_{\beta}$ and $T_{\beta+1}$ are coordinated, let us fix an (id, $\varphi_{0}$ )-elevator $e_{0}: T_{\beta} \rightarrow T_{\beta+1}$ such that $e_{0}\left(z_{j} \upharpoonright \beta\right)=z_{j} \upharpoonright(\beta+1)$ for every $j<n$. Set $\delta:=\min \left(D_{\alpha} \backslash(\beta+2)\right)$. As $T_{\beta+1}$ and $T_{\delta}$ are coordinated, fix a $\left(\varphi_{2}, \varphi_{2}\right)$-elevator $e_{2}: T_{\beta+1} \rightarrow T_{\delta}$ such that $e_{2}\left(z_{j} \upharpoonright(\beta+1)\right)=z_{j} \upharpoonright \delta$ for every $j<n$. Finally, define an elevator $e: T_{\beta} \rightarrow T_{\alpha}$ via $e(y):=\mathbf{b}_{e_{2}\left(e_{0}(y)\right)}^{\alpha}$. By Claim 6.10.5, $e\left(z_{j} \upharpoonright \beta\right)=z_{j}$ for every $j<n$.

To see that $e$ is a $\left(\varphi_{2}, \varphi_{2}\right)$-elevator, let $\left\langle y_{i} \mid i<\chi\right\rangle \in\left(T_{\beta}\right)^{\chi}$. For each $i<\chi$, denote $w_{i}:=e\left(y_{i}\right)$ and find $x_{i} \in T \backslash D_{\alpha}$ of minimal height such that $\mathbf{b}_{x_{i}}^{\alpha}=w_{i}$. As $w_{i}=\mathbf{b}_{e_{2}\left(e_{0}\left(y_{i}\right)\right)}^{\alpha}$, Claim 6.10.5 implies that $\operatorname{dom}\left(x_{i}\right) \leq \operatorname{dom}\left(e_{2}\left(e_{0}\left(y_{i}\right)\right)\right)$, so that $\gamma:=\sup \left\{\operatorname{dom}\left(x_{i}\right) \mid i<\chi\right\}$ satisfies $\gamma \leq \delta<\alpha$.

Now, by the definition of $f_{\alpha}$, letting $p^{\curvearrowleft}\langle\xi\rangle$ denote $\left(\varphi_{1} \circ f_{\gamma}\right)\left(\left\langle w_{i}\right| \gamma|i<\chi\rangle\right)$, it is the case that $f_{\alpha}\left(\left\langle w_{i} \mid i<\chi\right\rangle\right)=p$. In addition, since $\delta \in D_{\alpha} \backslash \gamma$, Claim 6.10.6(1) asserts that $\left(\varphi_{2} \circ f_{\delta}\right)\left(\left\langle w_{i} \upharpoonright \delta \mid i<\chi\right\rangle\right)=p$.

By Remark 6.7 and the choice of $e_{0}$ and $e_{2}$ :

$$
\begin{aligned}
\left(\varphi_{2} \circ f_{\beta}\right)\left(\left\langle w_{i} \upharpoonright \beta \mid i<\chi\right\rangle\right) & =\left(\varphi_{2} \circ \varphi_{2} \circ \varphi_{0} \circ f_{\beta}\right)\left(\left\langle w_{i} \upharpoonright \beta \mid i<\chi\right\rangle\right) \\
& =\left(\varphi_{2} \circ \varphi_{2} \circ f_{\beta+1}\right)\left(\left\langle e_{0}\left(w_{i} \upharpoonright \beta\right) \mid i<\chi\right\rangle\right) \\
& =\left(\varphi_{2} \circ \varphi_{2} \circ f_{\beta+1}\right)\left(\left\langle w_{i} \upharpoonright \beta+1 \mid i<\chi\right\rangle\right) \\
& =\left(\varphi_{2} \circ \varphi_{2} \circ f_{\delta}\right)\left(\left\langle e_{2}\left(w_{i} \upharpoonright \beta+1\right)\right)|i<\chi\rangle\right) \\
& =\left(\varphi_{2} \circ \varphi_{2} \circ f_{\delta}\right)\left(\left\langle w_{i} \upharpoonright \delta \mid i<\chi\right\rangle\right) \\
& =\varphi_{2}(p)=\left(\varphi_{2} \circ f_{\alpha}\right)\left(\left\langle w_{i} \mid i<\chi\right\rangle\right),
\end{aligned}
$$

as sought.
Let $\beta$ and $e: T_{\beta} \rightarrow T_{\alpha}$ be given by the subclaim. We now go over the clauses of Definition 6.9:
(i) By the induction hypothesis thus far, $T_{\epsilon}$ and $T_{\beta}$ are coordinated, so we may fix a $\left(\varphi_{2}, \varphi_{1}\right)$-elevator $e_{1}: T_{\epsilon} \rightarrow T_{\beta}$ such that $e_{1}\left(z_{j} \upharpoonright \epsilon\right)=z_{j} \upharpoonright \beta$ for all $j<n$. Set $E_{1}:=e \circ e_{1}$. Then $E_{1}\left(z_{j} \upharpoonright \epsilon\right)=e\left(e_{1}\left(z_{j} \upharpoonright \epsilon\right)\right)=$ $e\left(z_{j} \upharpoonright \beta\right)=z_{j}$ for all $j<n$. In addition, for every $y \in T_{\epsilon}, y \subseteq e_{1}(y) \subseteq$ $e\left(e_{1}(y)\right)=E_{1}(y)$. Finally, for every $\left\langle y_{i} \mid i<\chi\right\rangle \in\left(T_{\epsilon}\right)^{\chi}$,

$$
\begin{aligned}
\left(\varphi_{2} \circ f_{\alpha}\right)\left(\left\langle E_{1}\left(y_{i}\right) \mid i<\chi\right\rangle\right) & =\left(\varphi_{2} \circ f_{\alpha}\right)\left(\left\langle e\left(e_{1}\left(y_{i}\right)\right) \mid i<\chi\right\rangle\right) \\
& =\left(\varphi_{2} \circ f_{\beta}\right)\left(\left\langle e_{1}\left(y_{i}\right) \mid i<\chi\right\rangle\right) \\
& =\left(\varphi_{1} \circ f_{\epsilon}\right)\left(\left\langle y_{i} \mid i<\chi\right\rangle\right) .
\end{aligned}
$$

So, $E_{1}$ is a $\left(\varphi_{2}, \varphi_{1}\right)$-elevator as sought.
(ii) Replace 1 by 2 throughout the above proof.
(iii) $\alpha$ is a limit ordinal, so the requirement is satisfied vacuously.

At the end of the above process, we have obtained a $\kappa$-tree $T:=\bigcup_{\alpha<\kappa} T_{\alpha}$ whose $\chi$-power is special, as witnessed by $f:=\bigcup_{\alpha<\kappa} f_{\alpha}$. The proof that $T$ is club-regressive is identical to that of [BR17a, Claim 2.3.4], thus we are left with proving that $T$ is a $\chi$-free. To this end, let $\vec{s}=\left\langle s_{j} \mid j<n\right\rangle \in T^{n}$ be given for some nonzero $n<\chi$, and suppose that $A$ is a maximal antichain
in $T(\vec{s})$. Let $\epsilon$ denote the unique element of $\left\{\operatorname{dom}\left(s_{j}\right) \mid j<n\right\}$. Consider the club $E:=\{\alpha \in \operatorname{acc}(\kappa \backslash \epsilon) \mid T \upharpoonright \alpha \subseteq \phi[\alpha]\}$.

Using $\diamond\left(H_{\kappa}\right)$, for each $i<\kappa$, the following set is stationary in $\kappa$ :
$B_{i}:=\left\{\eta \in R_{i} \mid A \cap(T(\vec{s}) \upharpoonright \eta)=\Omega_{\eta}\right.$ is a maximal antichain in $\left.T(\vec{s}) \upharpoonright \eta\right\}$.
Finally, using the hitting feature of the proxy sequence, pick some $\alpha \in E$ such that, for all $i<\alpha$,

$$
\sup \left(\operatorname{nacc}\left(C_{\alpha}\right) \cap B_{i}\right)=\alpha
$$

Claim 6.10.9. $A \subseteq T(\vec{S}) \upharpoonright \alpha$. In particular, $|A|<\kappa$.
Proof. Let $w$ be an arbitrary element of the $\alpha^{\text {th }}$ level of $T(\vec{S})$, and we shall show that it extends an element of $A$. Recalling $(\star)$, for each $j<n$, we may fix some $x_{j} \in T \upharpoonright D_{\alpha}$ such that $(w)_{j}=\mathbf{b}_{x_{j}}^{\alpha}$. As $n$ is finite, by Claim 6.10.5, we may assume the existence of some $\gamma \in D_{\alpha} \backslash \epsilon$ such that $s_{j} \subseteq x_{j}$ and $\operatorname{dom}\left(x_{j}\right)=\gamma$ for all $j<n$. In particular, the trees $T(\vec{S})$ and $\bar{T}:=T\left(\left\langle x_{j}\right|\right.$ $j<n\rangle$ ) agree on all levels $\geq \gamma$.

Next, as $\alpha \in E$, we may find some $i<\alpha$ with $\phi(i)=\left\langle x_{j} \mid j<n\right\rangle$. Pick a large enough $\eta \in \operatorname{nacc}\left(C_{\alpha}\right) \cap B_{i}$ such that $\sup \left(D_{\alpha} \cap \eta\right)>\gamma$. Denote $\beta:=\eta+1$ and $\beta^{-}:=\sup \left(D_{\alpha} \cap \beta\right)$, so that $\gamma<\beta^{-} \leq \eta<\beta$ with $\beta \in D_{\alpha}$.

Recalling Notation 6.10.1, let $e:=\min \left(\mathcal{E}_{\beta}^{\alpha}, \triangleleft\right)$, so that $b_{x_{j}}^{\alpha}(\beta)=e\left(b_{x_{j}}^{\alpha}\left(\beta^{-}\right)\right)$ for every $j<n$. As $\psi(\eta)=\phi(\pi(\eta))=\phi(i)=\left\langle x_{j} \mid j<n\right\rangle$ and the latter is indeed an element of $\left(T \upharpoonright\left(D_{\alpha} \cap \beta^{-}\right)\right)^{n}$, we get that for every $j<n$,

$$
e\left(b_{x_{j}}^{\alpha}\left(\beta^{-}\right)\right)=\left(\operatorname{anti}\left(\pitchfork\left(\left\langle b_{x_{j}}^{\alpha}\left(\beta^{-}\right) \mid j<n\right\rangle\right), \bar{T} \upharpoonright \beta+1, \Omega_{\eta}\right)\right)_{j} .
$$

By the choice of $\eta, A \cap(T(\vec{s}) \upharpoonright \eta)=\Omega_{\eta}$ is a maximal antichain in $T(\vec{s}) \upharpoonright \eta$, and since $T(\vec{s}) \upharpoonright\left[\beta^{-}, \eta\right)=\bar{T} \upharpoonright\left[\beta^{-}, \eta\right)$, the following set is nonempty:

$$
Q:=\left\{z \in \bar{T}_{\beta} \mid \exists y \in \Omega_{\eta}\left(\pitchfork\left(\left\langle b_{x_{j}}^{\alpha}\left(\beta^{-}\right) \mid j<n\right\rangle\right) \cup y \subseteq z\right)\right\} .
$$

Denote $z:=\min \left(Q, \triangleleft_{\kappa}\right)$, and let $y \in \Omega_{\eta}$ be a witness for $z \in Q$. Recalling Definition 6.4, this means that for every $j<n$,

$$
(w)_{j} \upharpoonright \beta=b_{x_{j}}^{\alpha}(\beta)=e\left(b_{x_{j}}^{\alpha}\left(\beta^{-}\right)\right)=(z)_{j},
$$

and hence $y \subseteq z \subseteq w$. As $y \in \Omega_{\eta} \subseteq A$, we infer that $w$ indeed extends an element of $A$.

This completes the proof.
Corollary 6.11. Suppose that $\lambda$ is a singular cardinal such that $\square_{\lambda}$ and $2^{\lambda}=\lambda^{+}$both hold. Then for every positive integer $n$, there exists a $\lambda^{+}$Souslin streamlined tree $T$ satisfying the two:

- all $n$-derived trees of $T$ are Souslin;
- the $(n+1)$-power of $T$ is special.

Proof. By [BR17a, Corollary 3.10], for a singular cardinal $\lambda, \mathrm{P}_{\lambda}\left(\lambda^{+}, 2, \sqsubseteq, \lambda^{+}\right)$ is equivalent to the conjunction of $\square_{\lambda}$ and $2^{\lambda}=\lambda^{+}$. Now, appeal to Theorem 6.10.

It is not hard to see that assuming $\lambda=\lambda^{<\lambda}$, every $\lambda^{+}$-tree whose square is special is in particular specializable. We do not know of an example of a specializable $\lambda^{+}$-Souslin tree for $\lambda$ singular, and so we ask whether the tree given by Corollary 6.11 is (or can be tweaked to be) specializable.

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[^1]:    ${ }^{1}$ For any set $C$ of ordinals, $\operatorname{acc}(C)$ stands for the set $\{\beta \in C \mid \sup (C \cap \beta)=\beta>0\}$ of its accumulation points. In particular, for an ordinal $\gamma, \operatorname{acc}(\gamma)$ stands for the set of non-zero limit ordinals below $\gamma$.
    ${ }^{2}$ Variants of square are surveyed in [ML12]. Variants of diamond are surveyed in [Rin11], including the close connection between diamond principles and cardinal arithmetic.

[^2]:    ${ }^{3} E_{\geq \chi}^{\kappa}$ denotes the set $\{\alpha<\kappa \mid \operatorname{cf}(\alpha) \geq \chi\}$. The sets $E_{>\chi}^{\kappa}, E_{\chi}^{\kappa}, E_{\neq \chi}^{\kappa}, E_{<\chi}^{\kappa}$, and $E_{\leq \chi}^{\kappa}$ are defined analogously.

[^3]:    ${ }^{4}$ By walking along the outcome $C$-sequence, the extreme instance $\operatorname{Pr}_{1}(\kappa, \kappa, \kappa, \kappa)$ of Shelah's strong coloring principle was shown to be consistent.

[^4]:    ${ }^{5}$ This means that for every $\alpha \in S, A_{\alpha}$ is a subset of $\alpha$ and there is no $\beta<\alpha$ such that $A_{\alpha} \subseteq \beta$.
    ${ }^{6}$ This is a nontrivial requirement. By [FL88], it is consistent that every graph of size and chromatic number $\omega_{2}$ has a subgraph of size and chromatic number $\omega_{1}$.

[^5]:    ${ }^{7}$ See [LHR19, Remark 2.6] for the history of Definition 2.1 and its connection to the Hajnal-Máté graphs.
    ${ }^{8}$ For any set $C$ of ordinals, nacc $(C)$ denotes the set $C \backslash \operatorname{acc}(C)$ of its non-accumulation points.

[^6]:    ${ }^{9}$ Compare this with Jensen's weak square principle $\square_{\lambda}^{*}$ [Jen72, §5.1] and Schimmerling's generalization $\square_{\lambda, \mu}$ [Sch95, §5]. The utility of multi-ladder systems, i.e., systems in which there is more than one ladder assigned to each level, is demonstrated in [RS23, Theorem 4.13].
    ${ }^{10}$ If $\min \{\theta, \sigma\}>0$, then merely requiring that "there exists a nonzero $\alpha \in S$ " has an equivalent effect. See the proof of [Dev79, Theorem 4.3].

[^7]:    ${ }^{11}$ Recall Fact 2.3.
    ${ }^{12}$ The condition "nacc $(C)$ consists only of successor ordinals" indicates that the club $C$ may be a "dummy club" that is not part of the genuine coherence structure of $\overrightarrow{\mathcal{C}}$. Thus, any construction from $\overrightarrow{\mathcal{C}}$ should ensure that the hitting does not occur at such a $C$.
    ${ }^{13}$ Such as large cardinals [CFM03, Theorem 4.1],[Fri06, Proposition 8], strong forcing axioms [CM11, Theorem 1.2] and simultaneous reflection of stationary sets [HLH17, §2].
    ${ }^{14}$ By convention, this superscript-prepending is understood as being applied last, so that, for instance, ${ }^{\Omega} \sqsubseteq_{\chi}$ is to be parsed as ${ }^{\Omega}\left(\sqsubseteq_{\chi}\right)$.
    ${ }^{15}$ That is, for every club $D \subseteq \kappa$, there is an $\alpha \in \operatorname{acc}(D)$ such that $D \cap \alpha \notin \mathcal{C}_{\alpha}$. This is easily seen by applying Definition $2.5(3)$ to the cofinal set $B_{0}:=\operatorname{acc}(D)$, as in the proof of [BR17a, Lemma 3.2]. This proof also highlights the necessity for successor (i.e., non-accumulation) points of $C$ in ( $\star$ ) of Definition 2.5(3).

[^8]:    ${ }^{16}$ In fact, it is open whether a $\kappa$-Souslin tree constructed from an instance of the proxy principle having $\nu>2$ can be secured to be rigid on a club.
    ${ }^{17}$ A cardinal $\kappa$ is $(<\chi)$-closed iff $\lambda^{<\chi}<\kappa$ for every $\lambda<\kappa$.

[^9]:    ${ }^{18}$ Likewise $\boxtimes^{-}(S), \boxtimes^{-}(\mathcal{S}), \boxtimes^{*}(S)$ and $\boxtimes^{*}(\mathcal{S})$ stand for $\boxtimes_{\kappa}^{-}(S), \ldots, \boxtimes_{\kappa}^{*}(\mathcal{S})$, respectively.
    ${ }^{19}$ See [BR21, Theorem 4.16(1)] for a stronger result.
    ${ }^{20}$ Strictly speaking, $\sigma=1$ suffices for the construction of the said Ulam-type matrix.

[^10]:    ${ }^{21}$ See Definition 5.9 and Corollary 5.14 of [BR21]. Note that here the value of $\sigma$ is hardcoded to be $<\omega$; this is because the obvious generalization to $\sigma \geq \omega$ (together with $\mu<\kappa$ ) will already imply that $\diamond(\kappa)$ holds (see [BR21, Proposition 5.17]).
    ${ }^{22} \mathrm{P}_{\omega}^{\bullet}\left(\omega_{1}, 2, \sqsubseteq, \omega_{1}\right)$ holds after adding a single Cohen real to a model of CH (see the next section). If $\diamond\left(\omega_{1}\right)$ failed in the ground model, then by [Kun80, Exercise VII.H9] it remains failing in the extension, so that $\mathrm{P}_{\omega}\left(\omega_{1}, \ldots\right)$ will fail in the extension.

[^11]:    ${ }^{23}$ Bear in mind Subsection 2.3.

[^12]:    ${ }^{24}$ See $[B R 21, \S 2.3]$ for a comparison of streamlined trees with abstract trees, highlighting the properties of streamlined trees, the advantages of constructing them, and the fact that we lose no generality by restricting our attention to them.

[^13]:    ${ }^{25}$ We follow the definition given in [Kun80, Definition 5.16$]$, which is satisfactory for our purposes. In some contexts, it may be useful to impose that a $\kappa$-Kurepa tree is slim (see Definition 4.5 below), in order to exclude trivial examples such as the full binary tree of height a strongly inaccessible cardinal (see [Dev84, p. 317]).
    ${ }^{26}$ For $s, t \in T$, we say that $s$ and $t$ are comparable iff $s \subseteq t$ or $t \subseteq s$; otherwise they are incomparable. An antichain $A$ in $T$ is a subset $A \subseteq T$ such that for all $s, t \in T$, if $s \neq t$ then $s$ and $t$ are incomparable.

[^14]:    ${ }^{27}$ Trees with this property are also called well-pruned; see [Kun80, Definition 5.10].
    ${ }^{28}$ For two nodes $x, y$ in a streamlined tree $T$, we say that $y$ is an immediate successor of $x$ iff $x \subsetneq y$ and $\operatorname{dom}(y)=\operatorname{dom}(x)+1$.

[^15]:    ${ }^{29}$ This also implies that the matrix is continuous, i.e., for $\beta \in \operatorname{acc}\left(C_{\alpha}\right) \eta \in K_{\beta}$ and $x \in T^{\eta} \upharpoonright\left(C_{\alpha} \cap \beta\right)$, it is the case that $b_{x}^{\alpha, \eta}=\bigcup\left\{b_{x}^{\alpha, \eta \upharpoonright \bar{\beta}} \mid \bar{\beta} \in C_{\alpha} \cap \beta \backslash \operatorname{dom}(x)\right\}$.

[^16]:    ${ }^{30}$ As $\beta \in \operatorname{acc}(\kappa)$, it is the case that $\tau,\left\langle\eta_{j} \mid j<\tau\right\rangle$ and $A$ are uniquely determined by $\Omega_{\beta}$.

[^17]:    ${ }^{31}$ For a proof of this equivalence, see [Tod85, Theorem 14] or [Bro14, Theorem 16].
    ${ }^{32}$ If $T$ is a $\kappa$-tree and $\kappa$ is $\chi$-closed, then $\left(T^{\chi},<_{T \chi}\right)$ is again a $\kappa$-tree.

[^18]:    ${ }^{33}$ Recall that $T_{\beta}$ stands for $\{x \in T \mid \operatorname{dom}(x)=\beta\}$.

