THE POWER OF TREES

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Abstract. We give two consistent constructions of trees T whose finite power T^{n+1} is sharply different from T^n:

- An $\aleph_1$-tree T whose interval topology X_T is perfectly normal, but $(X_T)^2$ is not even countably metacompact.
- For an inaccessible κ and a positive integer n, a κ-tree such that all of its n-derived trees are Souslin and all of its $(n+1)$-derived trees are special.

1. Introduction

This paper is a contribution to the study of features of structures that are not preserved by taking products. As a simple example, the Sorgenfrey line [Sor47] constitutes a normal topological space whose square is not normal. As a more substantial example, a space is *Dowker* iff it is normal, yet its product with the unit interval is not normal. The naming comes from Dowker's theorem [Dow51] that $X \times [0,1]$ is normal iff X is normal and countably metacompact. Recall that a topological space is *countably metacompact (cmc)* iff every countable open cover admits a point-finite open refinement. A property stronger than cmc is that of being *perfect*: a topological space is perfect iff all of its closed subsets are G_δ. An even stronger property, *perfectly normal*, is equivalent to the conjunction of perfect and normal.

Of special interest is whether a tree $\mathbf{T} = (T, <_T)$ equipped with the *interval topology*,[1] denoted $X_{\mathbf{T}}$, can satisfy the above properties. To compare, while there are ZFC examples of Dowker spaces [Rud72, Bal96, KS98, Koj26], a space of the form $X_{\mathbf{T}}$ can never be Dowker [Nyi97].

Any antichain of $\mathbf{T}$ is closed discrete in $X_{\mathbf{T}}$, so if $\mathbf{T}$ is a special $\aleph_1$-tree, then $X_{\mathbf{T}}$ is perfect. Nyikos (see [Fle80, Theorem 4.1]) proved that almost-Souslin $\aleph_1$-trees are cmc, and Hanazawa [Han83, Theorem 3] proved that $\mathbb{R}$-embeddable almost-Souslin $\aleph_1$-trees are perfect.[2] It follows that a non-cmc $\aleph_1$-tree must be quite unusual, as it can be neither special nor almost-Souslin. A construction of such a spectacular creature was given first by

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[1] The definition of the interval topology may be found in Section 2.

[2] Every perfect $\aleph_1$-tree is $\mathbb{R}$-embeddable and cmc. A Souslin tree cannot be $\mathbb{R}$-embeddable, hence the need to focus on finer notions.

Fleissner assuming $\diamondsuit^+(\omega_1)$ [Fle80, §3], and then by Hanazawa assuming $\diamondsuit(\omega_1)$ [Han83, Theorem 1].

The first main result of this paper is an extension of the above works to control not only $X_{\mathbf{T}}$, but also $(X_{\mathbf{T}})^2$, and in an independent way. There are quite a few Boolean combinations of what $X_{\mathbf{T}}$ and $(X_{\mathbf{T}})^2$ could satisfy, so we decided to focus on one particular combination that we feel demonstrates well how the construction machinery developed in [BRY25] can contribute to settling this kind of classical questions. Specifically, we establish (consistently) the non-productivity of cmc in the class of spaces determined by the interval topology on trees, as follows.

Theorem A. *Suppose that $\diamondsuit^*(\omega_1)$ holds. Then there exists an $\mathbb{R}$-embeddable almost-Souslin $\aleph_1$-tree $\mathbf{T}$ such that:*

- *$X_{\mathbf{T}}$ is perfectly normal;*
- *$(X_{\mathbf{T}})^2$ is not cmc.*

We now turn to describe the second main result of this paper, dealing with the nonproductivity of the property of being nonspecial in the class of trees.[3] The literature has several consistent examples of λ^+-Souslin trees whose squares are special (see [JJ74] or [Bil89, Proposition 1.4.15] for $\lambda := \aleph_0$, [ASS87, §4] for λ singular, and [BRY25, §6] for a uniform treatment for all λ). In this paper, we give the first consistent example of a Souslin tree of inaccessible height whose square is special. This is obtained by introducing a new instance $\mathrm{P}_<(\dots)$ of the proxy principle from [BR17], proving that this instance is consistent, and presenting a construction of the desired tree from it, as follows.

Theorem B. *Suppose that κ is a regular uncountable cardinal.*

1. *There is a $<\kappa$-strategically-closed forcing $\mathbb{P}$ of size $\kappa^{<\kappa}$ such that the proxy principle $\mathrm{P}_<(\kappa, 2, \sqsubseteq, \kappa)$ holds in the generic extension by $\mathbb{P}$.*
2. *If $\mathrm{P}_<(\kappa, 2, \sqsubseteq, \kappa)$ holds, then for every positive integer n, there exists a κ-Souslin tree $\mathbf{T}$ such that:*
   - *all n-derived trees of $\mathbf{T}$ are Souslin;*
   - *all $(n+1)$-derived trees of $\mathbf{T}$ are special.*

To those familiar with C-sequences and proxy principles, we explain here the obstacle we had to overcome in order to obtain such a tree for an inaccessible cardinal κ. As listed in [BR19b, Fact 1], κ-trees are intimately connected with C-sequences over κ. In particular, it is a classical theorem going back to Jensen (see [Jen72, p. 283]) that there is a special λ^+-Aronszajn tree iff there is a C-sequence $\vec{C} = \langle C_\alpha \mid \alpha < \lambda^+ \rangle$ satisfying the following:[4]

(I) $\vec{C}$ is weakly coherent and $\mathrm{otp}(C_\alpha) \leq \lambda$ for every $\alpha < \lambda^+$.

[3] See Definition 4.1 and Remark 4.2.

[4] For undefined terms, see Definition 3.1.

Jensen's theorem was extended by Krueger [Kru13], who proved that for every regular uncountable cardinal κ, there is a special κ-Aronszajn tree iff there is a C-sequence $\vec{C} = \langle C_\alpha \mid \alpha < \kappa\rangle$ satisfying the following:[5]

(II) $\vec{C}$ is weakly coherent and $\mathrm{otp}(C_\alpha) < \alpha$ for club many $\alpha < \kappa$.

For our purpose, a sequence as in (II) is problematic, being in conflict with the fact that any witness $\vec{C}$ to a strong enough instance of the proxy principle must have stationarily many α's for which $|C_\alpha| = |\alpha|$,[6] which, at the level of an inaccessible, means that stationarily many $\alpha < \kappa$ must satisfy $\mathrm{otp}(C_\alpha) = \alpha$. The mitigation here comes from the work in [IR25], which pinpointed the impact of features of $\vec{C} = \langle C_\alpha \mid \alpha < \kappa\rangle$ on the corresponding canonical tree $T(\rho_0^{\vec{C}})$ arising from walks on ordinals. It turned out that $T(\rho_0^{\vec{C}})$ is a special κ-Aronszajn tree provided that the following holds:

(III) $\vec{C}$ is weakly coherent and club many $\alpha < \kappa$ satisfy $\mathrm{otp}(C_\beta \cap \alpha) < \alpha$ for every $\beta \in \kappa \setminus \{\alpha\}$.

Now, Clause (1) of Theorem B establishes that a strong form of (III) is compatible with having some thin stationary set of α's for which $|C_\alpha| = |\alpha|$, and Clause (2) shows that this subtle variation is indeed sufficient. In a similar way, Clause (1) solves a question of Shalev raised in the opening paragraph of [Sha24, §4.1]. We find it interesting that while walks on ordinals and the recursive method of [BR17] provide two unrelated techniques for canonically producing trees out of C-sequences, a discovery concerning one of the methods can lead to a corresponding discovery about the other one.

1.1. **Notation and conventions.** By an *inaccessible* we mean a regular uncountable limit cardinal. For a cardinal κ, we denote by H_κ the collection of all sets of hereditary cardinality less than κ (see Section IV.6 and Definition III.3.4 of [Kun80]).

1.2. **Organization of this paper.** In Section 2, we provide some preliminaries on the interval topology and we prove Theorem A. To make this section more accessible, we have ensured that it is self-contained, focuses on $\aleph_1$, assumes no familiarity with complicated background on tree constructions nor requires deep background in topology. The section is concluded with a short discussion on variations of Theorem A, also noting that an $\aleph_1$-tree $\mathbf{T}$ for which $(X_{\mathbf{T}})^2$ is not cmc cannot be obtained on the grounds of ZFC alone.

In Section 3, we introduce the proxy principle $\mathrm{P}_<(\kappa, 2, \sqsubseteq, \kappa)$ and prove its consistency, as promised in Clause (1) of Theorem B.

In Section 4, we give our application of the new proxy principle, as promised in Clause (2) of Theorem B.

[5]Strictly speaking, [Kru13, Definition 1.1] is concerned with the existence of a C-sequence *over a club* in κ. Extending it to get a C-sequence as in (II) is a trivial task.

[6]See the argument of [BR19a, Remarks 3.16(1)].

2. Theorem A

To make this section accessible to a wide audience, we avoid unnecessary abstractions throughout. In particular, we write Λ for the set of infinite countable limit ordinals, and we agree here on the following concrete implementation of trees, as follows.[7]

Definition 2.1. A *tree* is a subset T of ${}^{<\omega_1}w$, for some countable set w, that is downward closed, i.e., for every $t \in T$ and every $\alpha < \mathrm{dom}(t)$, $t \restriction \alpha$ is as well in T. $\mathrm{ht}(T)$ stands for the least $\alpha \leq \omega_1$ for which $T_\alpha := T \cap {}^{\alpha}w$ is empty.

Every two comparable nodes $x \subsetneq y$ of a tree T give rise to an interval as follows: $(x, y] := \{z \in T \mid x \subsetneq z \subseteq y\}$. The *interval topology* on T has, as its basic open sets, the intervals $(x, y]$ for all $x, y \in T$ with $x \subsetneq y$, as well as the singleton $\{\emptyset\}$. We denote by X_T the outcome topological space. Note that for every $t \in T$, t is an isolated point of X_T iff $\mathrm{dom}(t) \notin \Lambda$. Also note that for every nonisolated $y \in T$, we have $\bigcup\{x \in T \mid x \subsetneq y\} = y$, and hence X_T is a Hausdorff space.

Definition 2.2.
(1) A tree T is an $\aleph_1$*-tree* iff $\mathrm{ht}(T) = \omega_1$, and T_α is countable for every $\alpha < \omega_1$.
(2) A subset A of a tree T is an *antichain* iff for every pair $x \subseteq y$ of nodes from A, we have $x = y$.
(3) An $\aleph_1$-tree T is *almost-Souslin* iff for every antichain $A \subseteq T$, the set $\{\mathrm{dom}(x) \mid x \in A\}$ is nonstationary in ω_1.
(4) A tree T is $\mathbb{R}$*-embeddable* iff there exists a map $c : T \to \mathbb{R}$ such that for every pair $x \subsetneq y$ of nodes of T, $c(x) < c(y)$.

In [DS79, Theorem 4.4], Devlin and Shelah constructed from $\diamondsuit^*(\omega_1)$ an $\mathbb{R}$-embeddable almost-Souslin $\aleph_1$-tree T for which X_T is not normal. In [Han80, Theorem 6], Hanazawa constructed from $\diamondsuit^*(\omega_1)$ an $\mathbb{R}$-embeddable almost-Souslin $\aleph_1$-tree T for which X_T is normal.

Fact 2.3 ([DS79, Theorems 3.3 and 4.1])**.** *Suppose that T is an $\aleph_1$-tree.*
(1) *T is almost-Souslin iff X_T is* collectionwise Hausdorff*, that is, iff for every closed discrete $A \subseteq X_T$, there exists a pairwise disjoint system $\langle O_x \mid x \in A\rangle$ such that, for every $x \in A$, O_x is an open neighborhood of x.*
(2) *The following condition on T, called* property γ*, which clearly implies that T is almost-Souslin, also implies that X_T is normal: for every antichain $A \subseteq T$, there exists a club $D \subseteq \omega_1$ such that $\bigcup_{\alpha \in \omega_1 \setminus D} T_\alpha$ contains a closed neighborhood of A.*

Remark 2.4. The question of whether X_T for a given $\aleph_1$-tree T is normal goes back to Jones' work [Jon66] on *Moore spaces* [Moo62]. The proof of

[7] To clarify, if T is a tree in the sense of Definition 2.1, then $\mathbf{T} := (T, \subseteq)$ is a tree in the abstract sense, and the height of any node $x \in T$ is nothing but $\mathrm{dom}(x)$.

[DS79, Theorem 4.2] shows that if $\diamondsuit(S)$ holds for every stationary $S \subseteq \omega_1$, then an $\aleph_1$-tree T satisfies property γ iff X_T is normal.

Definition 2.5 (Two types of square)**.** Let T be a tree.

- $(X_T)^2$ stands for the topological product space $X_T \times X_T$;
- T^2 stands for the set $\{(x_0, x_1) \in T \times T \mid \mathrm{dom}(x_0) = \mathrm{dom}(x_1)\}$, which we typically equip with the ordering $\subseteq^2$ defined via $(x_0, x_1) \subseteq^2 (y_0, y_1)$ iff $x_0 \subseteq y_0$ and $x_1 \subseteq y_1$.

Remark 2.6. The poset $(T^2, \subseteq^2)$ is a tree in the abstract sense, and the height of a node $(x_0, x_1) \in T^2$ is nothing but $\mathrm{dom}(x_0)$. A subset A of T^2 is a $\subseteq^2$-*antichain* iff for every pair $x \subseteq^2 y$ of nodes from A, we have $x = y$.

The next fact is standard. We include a proof for completeness.

Proposition 2.7. (1) *T^2 is a closed subspace of $(X_T)^2$;*
(2) *Every $\subseteq^2$-antichain in T^2 is a closed discrete subspace of $(X_T)^2$.*

Proof. (1) Given $(x_0, x_1) \in (X_T)^2 \setminus T^2$, we shall find an open neighborhood of (x_0, x_1) disjoint from T^2. Fix $i < 2$ such that $\mathrm{dom}(x_i) < \mathrm{dom}(x_{1-i})$. Set $I_i := [\emptyset, x_i]$,[8] and $I_{1-i} := (x_{1-i} \restriction \mathrm{dom}(x_i), x_{1-i}]$. Then $U := I_0 \times I_1$ is an open neighborhood as sought.

(2) Given a $\subseteq^2$-antichain $A \subseteq T^2$, to show it is closed discrete, let $(x_0, x_1) \in (X_T)^2$ be given, and we shall find an open neighborhood U of (x_0, x_1) such that $A \cap U \subseteq \{(x_0, x_1)\}$. By Clause (1), we may assume that $(x_0, x_1) \in T^2$. Thus, consider the set $A' := \{(y_0, y_1) \in A \mid y_0 \subsetneq x_0 \ \& \ y_1 \subsetneq x_1\}$. Every two pairs in A' are compatible elements of $(T^2, \subseteq^2)$. But A' is a subset of a $\subseteq^2$-antichain, and hence there are only two possibilities:

▶ A' is empty. In this case, $U := \prod_{i<2} [\emptyset, x_i]$ is an open neighborhood as sought.

▶ A' is a singleton, say, $A' = \{(y_0, y_1)\}$. In this case, $U := \prod_{i<2} (y_i, x_i]$ is an open neighborhood as sought. □

2.1. Trees embeddable to the reals.

For the scope of Section 2, we define a map $c : {}^{<\omega_1}\mathbb{Q} \to \mathbb{R} \cup \{\infty\}$ via

$$c(x) := \begin{cases} 0, & \text{if } x = \emptyset; \\ \sup(\mathrm{Im}(x)), & \text{otherwise.} \end{cases}$$

Definition 2.8. $\mathcal{T}$ denotes the collection of all $x \in {}^{<\omega_1}\mathbb{Q}$ for which $\langle c(x \restriction \beta) \mid \beta \leq \mathrm{dom}(x) \rangle$ is a strictly increasing sequence of real numbers.

The following facts are readily checked:

Proposition 2.9. *(1) $\mathcal{T}$ is a tree (in the sense of Definition 2.1).*
(2) For every $x \in \mathcal{T}$ with $\mathrm{dom}(x) = \alpha + 1$ a successor, $c(x) = x(\alpha)$.
(3) $c \restriction \mathcal{T}$ is a strictly increasing map from $(\mathcal{T}, \subseteq)$ to $(\mathbb{R}, \leq)$.

[8] For $x \in T$, write $[\emptyset, x] := \{\emptyset\} \cup (\emptyset, x]$, which is an open neighborhood of x in X_T.

(4) For every strictly $\subseteq$-increasing sequence $\langle t_n \mid n < \omega\rangle$ of elements of $\mathcal{T}$, if $\sup\{c(t_n) \mid n < \omega\} \neq \infty$, then the unique limit of the sequence, $t := \bigcup\{t_n \mid n < \omega\}$, is also in $\mathcal{T}$, and $c(t) = \sup\{c(t_n) \mid n < \omega\}$. □

Corollary 2.10. *Every tree $T \subseteq \mathcal{T}$ is $\mathbb{R}$-embeddable.*[9] □

Definition 2.11. For all $x \in \mathcal{T}$ and $q \in \mathbb{Q}$, denote

$$U(x,q) := \{y \in \mathcal{T} \mid x \subsetneq y \ \&\ c(y) < c(x) + q\}.$$

Evidently, for every tree $T \subseteq \mathcal{T}$, for all $x \in T$ and $q \in \mathbb{Q}$, the set $U(x,q) \cap T$ is open in X_T.

2.2. **Elevators and friends.** We now introduce a few key concepts that will aid in our upcoming construction.

Definition 2.12 (Tree-coarsening)**.** For a tree T, we say that a partial ordering $\trianglelefteq$ on T^2 is a *tree-coarsening* of $\subseteq^2$ iff for all $(x_0,x_1),(y_0,y_1) \in T^2$:

- if $(x_0,x_1) \trianglelefteq (y_0,y_1)$, then $(x_0,x_1) \subseteq^2 (y_0,y_1)$;
- if $(x_0,x_1) \trianglelefteq (y_0,y_1)$, then $(x_1,x_0) \trianglelefteq (y_1,y_0)$; and
- if $(x_0,x_1) \trianglelefteq (y_0,y_1)$, then for every β with $\operatorname{dom}(x_0) < \beta < \operatorname{dom}(y_0)$, $(x_0,x_1) \trianglelefteq (y_0 \restriction \beta, y_1 \restriction \beta) \trianglelefteq (y_0,y_1)$.

Definition 2.13 (q-elevator)**.** For a tree $T \subseteq \mathcal{T}$, a tree-coarsening $\trianglelefteq$ of $\subseteq^2$, ordinals $\beta < \alpha < \operatorname{ht}(T)$, and $q \in \mathbb{Q}$, a function e from a subset of T_β to T_α is said to be a *q-elevator* (with respect to $\trianglelefteq$) iff:

- for every $x \in \operatorname{dom}(e)$, $e(x) \in U(x,q)$, and
- for every $(x_0,x_1) \in (\operatorname{dom}(e))^2$, $(x_0,x_1) \trianglelefteq (e(x_0),e(x_1))$.

Remark 2.14. The composition of a p-elevator with a q-elevator is an r-elevator for every $r \geq p+q$.

Notation 2.15. $\bar{\mathbb{Q}}$ stands for $\mathbb{Q} \cap (0,1)$.

Definition 2.16 (Coordination)**.** For a tree $T \subseteq \mathcal{T}$, a tree-coarsening $\trianglelefteq$ of $\subseteq^2$, and ordinals $\beta < \alpha < \operatorname{ht}(T)$, we say that T_β and T_α are *coordinated* (with respect to $\trianglelefteq$) iff for every $q \in \bar{\mathbb{Q}}$, the following three hold:

(0) For every finite $W \subseteq T_\alpha$, there exists a q-elevator $e : T_\beta \to T_\alpha$ such that $\operatorname{Im}(e) \cap W = \emptyset$.

(1) For every $x \in T_\beta$, every $y \in U(x,q) \cap T_\alpha$ and every finite set $W \subseteq T_\alpha \setminus \{y\}$, there exists a q-elevator $e : T_\beta \to T_\alpha$ such that:
  - $e(x) = y$, and
  - $\operatorname{Im}(e) \cap W = \emptyset$.

(2) For every pair $(x_0,x_1) \in T_\beta \times T_\beta$ with $x_0 \neq x_1$, for every pair $(y_0,y_1) \in (U(x_0,q) \cap T_\alpha) \times (U(x_1,q) \cap T_\alpha)$ such that $(x_0,x_1) \trianglelefteq (y_0,y_1)$, and every finite set $W \subseteq T_\alpha \setminus \{y_0,y_1\}$, there exists a q-elevator $e : T_\beta \to T_\alpha$ such that:

[9] The poset $(\mathcal{T},\subseteq)$ is essentially the same as the poset $(\sigma\mathbb{Q},\preceq)$ from [Tod84, p. 245], in the sense of possessing a universal feature for $\mathbb{R}$-embeddable trees as in [KM11, Proposition 3].

- $e(x_0) = y_0$,
- $e(x_1) = y_1$, and
- $\mathrm{Im}(e) \cap W = \emptyset$.

Remark 2.17. It follows from Clause (0) above that for all $x \in T_\beta$ and $q \in \mathbb{Q}$, there is a y in $U(x, q) \cap T_\alpha$.

Lemma 2.18. *Suppose:*

- $T \subseteq \mathcal{T}$ *is a tree;*
- $\trianglelefteq$ *is a tree-coarsening of* $\subseteq^2$*;*
- $\gamma < \beta < \alpha < \mathrm{ht}(T)$ *are ordinals;*
- T_γ *and* T_β *are coordinated, and* T_β *and* T_α *are coordinated;*
- *for all* $(x_0, x_1) \in T_\beta \times T_\beta$ *and* $(y_0, y_1) \in T_\alpha \times T_\alpha$*,* $(x_0, x_1) \trianglelefteq (y_0, y_1)$ *iff* $(x_0, x_1) \subseteq^2 (y_0, y_1)$*.*

Then T_γ *and* T_α *are coordinated.*

Proof. Let $q \in \bar{\mathbb{Q}}$. We go over the clauses of Definition 2.16, keeping in mind Remark 2.14:

(0) Consider a given finite $W \subseteq T_\alpha$. As T_γ and T_β are coordinated, we may fix a $\frac{q}{2}$-elevator $e_0 : T_\gamma \to T_\beta$. As T_β and T_α are coordinated, we may fix a $\frac{q}{2}$-elevator $e_1 : T_\beta \to T_\alpha \setminus W$. Then $e := e_1 \circ e_0$ is a q-elevator from T_γ to T_α satisfying that $\mathrm{Im}(e) \cap W = \emptyset$.

(1) Let $x \in T_\gamma$, $z \in U(x, q) \cap T_\alpha$ and a finite set $W \subseteq T_\alpha \setminus \{z\}$ be given; we need to find a q-elevator $e : T_\gamma \to T_\alpha \setminus W$ such that $e(x) = z$.

As $c(z) - c(x) < q$, we may find some $\varepsilon \in \bar{\mathbb{Q}}$ such that

(i) $c(z) - c(x) + 2\varepsilon < q$.

Set $y := z \restriction \beta$, and then pick $q_0, q_1 \in \bar{\mathbb{Q}}$ such that

(ii) $c(y) - c(x) < q_0 < c(y) - c(x) + \varepsilon$, and

(iii) $c(z) - c(y) < q_1 < c(z) - c(y) + \varepsilon$.

As $y \in U(x, q_0) \cap T_\beta$, we may fix a q_0-elevator $e_0 : T_\gamma \to T_\beta$ such that $e_0(x) = y$. As $z \in U(y, q_1) \cap T_\alpha$ and $W \subseteq T_\alpha \setminus \{z\}$, we may fix a q_1-elevator $e_1 : T_\beta \to T_\alpha \setminus W$ such that $e_1(y) = z$. By (i)–(iii), $q_0 + q_1 < q$, and hence $e := e_1 \circ e_0$ is altogether a q-elevator as sought.

(2) Let $\{x_0, x_1\} \in [T_\gamma]^2$, $(z_0, z_1) \in (U(x_0, q) \cap T_\alpha) \times (U(x_1, q) \cap T_\alpha)$ such that $(x_0, x_1) \trianglelefteq (z_0, z_1)$, and a finite set $W \subseteq T_\alpha \setminus \{z_0, z_1\}$ be given; we need to find a q-elevator $e : T_\gamma \to T_\alpha \setminus W$ such that $e(x_0) = z_0$ and $e(x_1) = z_1$.

Set $y_0 := z_0 \restriction \beta$ and $y_1 := z_1 \restriction \beta$. Fix a large enough $q_0 \in \mathbb{Q} \cap (0, q)$ such that $(y_0, y_1) \in (U(x_0, q_0) \cap T_\beta) \times (U(x_1, q_0) \cap T_\beta)$. By Definition 2.12, $(x_0, x_1) \trianglelefteq (y_0, y_1)$. Thus, as T_γ and T_β are coordinated, we may fix a q_0-elevator $e_0 : T_\gamma \to T_\beta$ such that $e_0(x_j) = y_j$ for every $j < 2$. Set $q_1 := q - q_0$. As T_β and T_α are coordinated, we may fix a q_1-elevator $e_1 : T_\beta \to T_\alpha \setminus W$. It is clear that $e_1 \circ e_0$ is a q-elevator whose image is disjoint from W, but we did not secure that x_j goes to z_j for every $j < 2$. To this end, using the fact that

$\{z_0, z_1\} \cap W = \emptyset$, we define a map $e : T_\gamma \to T_\alpha \setminus W$ via

$$e(t) := \begin{cases} z_0, & \text{if } t = x_0; \\ z_1, & \text{if } t = x_1; \\ e_1(e_0(t)), & \text{otherwise.} \end{cases}$$

It is evident that $e(t) \in U(t, q)$ for every $t \in T_\gamma$. To see that the second bullet point of Definition 2.13 holds as well, let $(t_0, t_1) \in T_\gamma \times T_\gamma$ be given. A moment's reflection makes it clear that for every $t \in T_\gamma$, $e(t) \restriction \beta = e_0(t)$. Therefore, $(t_0, t_1) \trianglelefteq (e(t_0) \restriction \beta, e(t_1) \restriction \beta)$. In addition, $(e(t_0) \restriction \beta, e(t_1) \restriction \beta) \subseteq^2 (e(t_0), e(t_1))$ holds trivially. Then, by the hypotheses, furthermore, $(e(t_0) \restriction \beta, e(t_1) \restriction \beta) \trianglelefteq (e(t_0), e(t_1))$. Altogether, $(t_0, t_1) \trianglelefteq (e(t_0), e(t_1))$. □

2.3. **The construction.** Our upcoming applications of the principle $\diamondsuit^*(\omega_1)$ and its consequence $\diamondsuit(\omega_1)$ are encapsulated by the following two easy facts. The first derives a particular ladder system, and the second is a more versatile formulation of $\diamondsuit(\omega_1)$ motivated by the fact that $\mathcal{T}$ is a subset of H_{ω_1}. Here, instead of predicting initial segments of given subsets of ω_1, we predict the extent seen by a countable elementary submodel of given subsets of H_{ω_1}.

Fact 2.19 (special case of [BR21, Theorem 4.35]). *$\diamondsuit^*(\omega_1)$ implies that there is a sequence $\vec{C} = \langle C_\alpha \mid \alpha < \omega_1 \rangle$ satisfying the following two:*

- *for every $\alpha \in \Lambda$, C_α is a cofinal subset of α of order-type ω;*
- *for every uncountable $B \subseteq \omega_1$, there are club many $\alpha < \omega_1$ such that $\sup(C_\alpha \cap B) = \alpha$.*

Fact 2.20 (special case of [BR17, Lemma 2.2]). *$\diamondsuit(\omega_1)$ is equivalent to the existence of a partition $\langle R_i \mid i < \omega_1 \rangle$ of ω_1 and a sequence $\langle \Omega_\beta \mid \beta < \omega_1 \rangle$ of countable sets such that for all $p \in H_{\omega_2}$, $i < \omega_1$, and $\Omega \subseteq H_{\omega_1}$, there exists a countable elementary submodel $\mathcal{M} \prec H_{\omega_2}$ such that:*

- *$p \in \mathcal{M}$;*
- *$\beta := \mathcal{M} \cap \omega_1$ is an ordinal in R_i;*
- *$\mathcal{M} \cap \Omega = \Omega_\beta$.*

We now arrive at the main result of this section, namely, Theorem A.

Theorem 2.21. *Suppose that $\diamondsuit^*(\omega_1)$ holds. Then there is an $\mathbb{R}$-embeddable almost-Souslin $\aleph_1$-tree T such that X_T is perfectly normal, but $(X_T)^2$ is not cmc.*

Proof. As CH holds, let $\phi : \omega_1 \leftrightarrow H_{\omega_1}$ be any bijection, and let $\lhd_{\omega_1}$ be the induced well-ordering of H_{ω_1}. As $\diamondsuit(\omega_1)$ holds, let $\langle R_i \mid i < \omega_1 \rangle$ and $\langle \Omega_\beta \mid \beta < \omega_1 \rangle$ be given by Fact 2.20. Let $\pi : \omega_1 \to \omega_1$ be the unique function satisfying $\alpha \in R_{\pi(\alpha)}$ for all $\alpha < \omega_1$. Then, set $\psi := \phi \circ \pi$. Finally, as $\diamondsuit^*(\omega_1)$ holds, let $\langle C_\alpha \mid \alpha < \omega_1 \rangle$ be given by Fact 2.19. We may assume that $\min(C_\alpha) = 1$ for every $\alpha \in \Lambda$.

We shall construct an $\aleph_1$-tree $T \subseteq \mathcal{T}$ along with a subset $E \subseteq \omega_1$ and a $\subseteq^2$-antichain $\langle a_\epsilon \mid \epsilon \in E\rangle \in \prod_{\epsilon\in E}(T_\epsilon)^2$.

This $\subseteq^2$-antichain will induce a tree-coarsening $\trianglelefteq$ of $\subseteq^2$ defined by letting $(x_0, x_1) \trianglelefteq (y_0, y_1)$ iff:

(ℵ) $(x_0, x_1) \subseteq^2 (y_0, y_1)$, and
(ℶ) for every $j < 2$, if $a_\epsilon \not\subseteq^2 (x_j, x_{1-j})$ for every $\epsilon \in E \cap (\mathrm{dom}(x_0) + 1)$, then $a_\epsilon \not\subseteq^2 (y_j, y_{1-j})$ for every $\epsilon \in E \cap (\mathrm{dom}(y_0) + 1)$.

The construction of the tree T is by recursion on $\alpha < \omega_1$, where at stage α, we determine a countable set T_α (the α^{th} level of T), decide whether α is in E, and if it is, determine also an element $a_\alpha \in (T_\alpha)^2$.

For all $\beta < \alpha < \omega_1$, we will ensure that T_β and T_α are coordinated.

For any ordinal $\alpha < \omega_1$ such that $\langle T_\beta \mid \beta < \alpha\rangle$ has already been determined and for every $B \subseteq \alpha$, we shall write $T\restriction B := \bigcup_{\beta\in B} T_\beta$, and we observe that the restriction of $\trianglelefteq$ to $(T\restriction\alpha)^2$ is determined by the initial segment $\langle a_\epsilon \mid \epsilon \in E \cap \alpha\rangle$ of the eventual $\subseteq^2$-antichain. We also observe the following:

Claim 2.21.1. *For all $\alpha < \omega_1$ and $(x_0, x_1), (y_0, y_1)$ in $(T\restriction\alpha)^2$, if the interval $(\mathrm{dom}(x_0), \alpha)$ is disjoint from E, then*

$$(x_0, x_1) \trianglelefteq (y_0, y_1) \iff (x_0, x_1) \subseteq^2 (y_0, y_1). \qquad \square$$

The preparations are over, and we now turn to the construction. We start by setting $T_0 := \{\emptyset\}$ and deciding that $0 \notin E$.

Next, given $\alpha < \omega_1$ such that T_α has already been defined, we set

$$\begin{aligned} T_{\alpha+1} &:= \{x \in {}^{\alpha+1}\mathbb{Q} \mid (x \restriction \alpha) \in T_\alpha\} \cap \mathcal{T} \\ &= \{t^\frown\langle q\rangle \mid t \in T_\alpha,\ q \in \mathbb{Q},\ q > c(t)\}, \end{aligned}$$

and decide not to include $\alpha + 1$ in E. As T_α is a countable set, so is $T_{\alpha+1}$.

Claim 2.21.2. *For every $\beta < \alpha + 1$, T_β and $T_{\alpha+1}$ are coordinated.*

Proof. As $\alpha+1 \notin E$, Claim 2.21.1 implies that for all $(x_0, x_1) \in T_\alpha \times T_\alpha$ and $(y_0, y_1) \in T_{\alpha+1} \times T_{\alpha+1}$, $(x_0, x_1) \trianglelefteq (y_0, y_1)$ iff $(x_0, x_1) \subseteq^2 (y_0, y_1)$. Thus, by Lemma 2.18 and the induction hypothesis, it suffices to prove that T_α and $T_{\alpha+1}$ are coordinated. To this end, let $q \in \bar{\mathbb{Q}}$ be given, and we shall go over the three clauses of Definition 2.16:

(0) Given a finite $W \subseteq T_{\alpha+1}$, set

$$r := \min\{q, c(w) - c(w \restriction \alpha) \mid w \in W\},$$

and then fix a system $\langle q_t \mid t \in T_\alpha\rangle$ of rational numbers such that $c(t) < q_t < c(t) + r$ for every $t \in T_\alpha$. Define a map $e : T_\alpha \to T_{\alpha+1}$ via $e(t) := t^\frown\langle q_t\rangle$. As $\alpha + 1 \notin E$ and $r \leq q$, e is a q-elevator. For every $w \in W$, $c(e(w \restriction \alpha)) = q_{w\restriction\alpha} < c(w \restriction \alpha) + r \leq c(w)$, so that $e(w \restriction \alpha) \neq w$. Thus, it is also the case that $\mathrm{Im}(e) \cap W = \emptyset$.

(1) Let $x \in T_\alpha$, $y \in U(x, q) \cap T_{\alpha+1}$ and a finite set $W \subseteq T_{\alpha+1} \setminus \{y\}$ be given; we need to find a q-elevator $e : T_\alpha \to T_{\alpha+1} \setminus W$ such that

$e(x) = y$. Obtain r and $\langle q_t \mid t \in T_\alpha\rangle$ as in Clause (0), and then define a map $e : T_\alpha \to T_{\alpha+1}$ via

$$e(t) := \begin{cases} y, & \text{if } t = x; \\ t^\frown\langle q_t\rangle, & \text{otherwise.} \end{cases}$$

As $\alpha + 1 \notin E$, e is a q-elevator. As $y \notin W$, it is also the case that $\mathrm{Im}(e) \cap W = \emptyset$, just as in Clause (0).

(2) Let $\{x_0, x_1\} \in [T_\alpha]^2$, $(y_0, y_1) \in (U(x_0, q) \cap T_{\alpha+1}) \times (U(x_1, q) \cap T_{\alpha+1})$ such that $(x_0, x_1) \trianglelefteq (y_0, y_1)$, and a finite set $W \subseteq T_{\alpha+1} \setminus \{y_0, y_1\}$ be given; we need to find a q-elevator $e : T_\alpha \to T_{\alpha+1} \setminus W$ such that $e(x_0) = y_0$ and $e(x_1) = y_1$. Obtain r and $\langle q_t \mid t \in T_\alpha\rangle$ as in Clause (0), and then define a map $e : T_\alpha \to T_{\alpha+1}$ via

$$e(t) := \begin{cases} y_0, & \text{if } t = x_0; \\ y_1, & \text{if } t = x_1; \\ t^\frown\langle q_t\rangle, & \text{otherwise.} \end{cases}$$

Then, e is a q-elevator as sought. □

Now, fix a given $\alpha \in \Lambda$ such that $\langle T_\beta \mid \beta < \alpha\rangle$ and $\langle a_\epsilon \mid \epsilon \in E \cap \alpha\rangle$ have already been successfully defined. In particular, for all $\gamma < \beta < \alpha$, T_γ and T_β are coordinated.

Consider the collection $\mathcal{B}^\alpha := \{t \in {}^\alpha\mathbb{Q} \mid \forall\beta < \alpha\,(t \restriction \beta \in T_\beta)\}$ of all cofinal branches through $T \restriction \alpha$. For each $x \in T \restriction C_\alpha$ we shall carefully identify some element $\mathbf{b}^\alpha_x \in \mathcal{B}^\alpha \cap \mathcal{T}$ with $x \subseteq \mathbf{b}^\alpha_x$, and we shall then define the α^{th} level of the tree to be

$$T_\alpha := \{\mathbf{b}^\alpha_x \mid x \in T \restriction C_\alpha\}.$$

To this end, we denote by $\langle \beta_n \mid n < \omega\rangle$ the increasing enumeration of C_α, and we plan to construct, recursively, a sequence $\langle (e_n, q_n) \mid n < \omega\rangle$, where, for every $n < \omega$, $e_n : T_{\beta_n} \to T_{\beta_{n+1}}$ is a q_{n+1}-elevator.[10] For each $x \in T \restriction C_\alpha$, those e_n's will determine $\mathbf{b}^\alpha_x$ as the limit $\bigcup \mathrm{Im}(b^\alpha_x)$ of the unique $\subseteq$-increasing sequence b^α_x satisfying the following three properties:

(i) $\mathrm{dom}(b^\alpha_x) = \{\beta_n \mid n < \omega\} \setminus \mathrm{dom}(x)$;
(ii) $b^\alpha_x(\mathrm{dom}(x)) = x$;
(iii) for every $n < \omega$ with $\beta_n \geq \mathrm{dom}(x)$, $b^\alpha_x(\beta_{n+1}) = e_n(b^\alpha_x(\beta_n))$.

In particular, for the unique $k < \omega$ such that $\mathrm{dom}(x) = \beta_k$,[11] it would be the case that $b^\alpha_x(\beta_n) \in T_{\beta_n}$ for every $n \in \omega \setminus k$.

As for the q_n's, we set $q_0 := 1$, and announce at the outset that for every $n < \omega$, there will be three possible cases; in the first two, we shall let $q_{n+1} := \frac{q_n}{2}$, and in the third, we shall let $q_{n+1} := \frac{q_n}{8}$. Consequently, for every $n < \omega$, if we fall into the first two cases, then $(\sum_{m=n+1}^\infty q_m) \leq q_n$, and otherwise, $(\sum_{m=n+1}^\infty q_m) \leq \frac{q_n}{4}$. In particular, $\lim_{n\to\infty} q_n = 0$.

[10] Strictly speaking, the notation should have been β^α_n, e^α_n and q^α_n, but we suppress the superscript for brevity as we will always be working in the context of a fixed value of α.

[11] Recall that $\mathrm{dom}(x)$ is the height of x in the tree, and that $x \in T \restriction C_\alpha$.

We now turn to the actual construction. Suppose $n<\omega$ is such that the sequence $\langle (e_k,q_{k+1})\mid k<n\rangle$ has already been defined. In particular, Clauses (i)–(iii) have already determined $b^\alpha_x(\beta_n)$ for every $x\in T\restriction(C_\alpha\cap\beta_{n+1})=\bigcup_{k\le n}T_{\beta_k}$. As announced, the definition of (e_n,q_{n+1}) is divided into three cases. They read as follows:

Case I: Suppose that all of the following hold:

- $\Omega_{\beta_{n+1}}$ is a subset of $T\restriction\beta_{n+1}$,
- $w:=\psi(\beta_{n+1})$ is an element of $T\restriction(C_\alpha\cap\beta_{n+1})$, so that, in particular, $b^\alpha_w(\beta_n)$ is an element of T_{β_n}, and
- the set $Q^\alpha_{n+1}:=\{t\in U(b^\alpha_w(\beta_n),\frac{q_n}{2})\cap T_{\beta_{n+1}}\mid \exists a\in\Omega_{\beta_{n+1}}\,(a\subseteq t)\}$ is nonempty.

In this case, set $q_{n+1}:=\frac{q_n}{2}$, and choose some q_{n+1}-elevator $e_n:T_{\beta_n}\rightarrow T_{\beta_{n+1}}$ such that $e_n(b^\alpha_w(\beta_n))=\min(Q^\alpha_{n+1},\lhd_{\omega_1})$, which must exist by Clause (1) of coordination of T_{β_n} and $T_{\beta_{n+1}}$.

Case II: Suppose that all of the following hold:

- $\Omega_{\beta_{n+1}}$ is a function from $(T\restriction\beta_{n+1})^2$ to ω,
- $k:=\psi(\beta_{n+1})$ is an element of ω, and
- the set P^α_{n+1} of all pairs $(t_1,t_2)\in(T_{\beta_{n+1}})^2$ satisfying all of the following is nonempty:
  - $(t_1,t_2)\in U(b^\alpha_{\langle 1\rangle}(\beta_n),\frac{q_n}{2})\times U(b^\alpha_{\langle 2\rangle}(\beta_n),\frac{q_n}{2})$,[12]
  - $(b^\alpha_{\langle 1\rangle}(\beta_n),b^\alpha_{\langle 2\rangle}(\beta_n))\trianglelefteq(t_1,t_2)$, and
  - there is $\tau\in[\beta_n,\beta_{n+1})$ such that $\Omega_{\beta_{n+1}}(t_1\restriction\tau,t_2\restriction\tau)=k$.

In this case, set $(t_1,t_2):=\min(P^\alpha_{n+1},\lhd_{\omega_1})$, $q_{n+1}:=\frac{q_n}{2}$, and choose some q_{n+1}-elevator $e_n:T_{\beta_n}\rightarrow T_{\beta_{n+1}}$ such that $e_n(b^\alpha_{\langle j\rangle}(\beta_n))=t_j$ for $j\in\{1,2\}$, which must exist by Clause (2) of coordination of T_{β_n} and $T_{\beta_{n+1}}$.

Case III: Otherwise. In this case, set $q_{n+1}:=\frac{q_n}{8}$ and choose any q_{n+1}-elevator $e_n:T_{\beta_n}\rightarrow T_{\beta_{n+1}}$, which must exist by Clause (0) of coordination of T_{β_n} and $T_{\beta_{n+1}}$.

We record the following crucial features that follow from the above construction together with Definition 2.13, Remark 2.14, and Proposition 2.9(4):

Claim 2.21.3. *For all $n<m<\omega$ and $x,x'\in T\restriction(C_\alpha\cap\beta_{n+1})$, the following hold:*

- $\mathbf{b}^\alpha_x\restriction\beta_n=b^\alpha_x(\beta_n)$;
- $(b^\alpha_x(\beta_n),b^\alpha_{x'}(\beta_n))\trianglelefteq(b^\alpha_x(\beta_m),b^\alpha_{x'}(\beta_m))$;
- *if e_n was defined according to Case III, then $\mathbf{b}^\alpha_x\in U(b^\alpha_x(\beta_n),\frac{q_n}{4})$;*[13] *otherwise, $\mathbf{b}^\alpha_x\in U(b^\alpha_x(\beta_n),q_n)$. So, in either case, $\mathbf{b}^\alpha_x\in\mathcal{T}$.* □

Having completed the recursive construction of $\langle (e_n,q_n)\mid n<\omega\rangle$, for each $x\in T\restriction C_\alpha$, the corresponding ascending sequence b^α_x and its limit $\mathbf{b}^\alpha_x:=$

[12]Recall that $\beta_0=\min(C_\alpha)=1$, so that $\langle 1\rangle$ and $\langle 2\rangle$ are in $T\restriction(C_\alpha\cap\beta_{n+1})$ and therefore, in particular, both $b^\alpha_{\langle 1\rangle}(\beta_n)$ and $b^\alpha_{\langle 2\rangle}(\beta_n)$ are in T_{β_n}.

[13]The importance of $\frac{q_n}{4}$ will become clear in the proof of Subclaim 2.21.8.1(2).

$\bigcup \mathrm{Im}(b^\alpha_x)$ have been completely determined, so we now set, as promised,

$$T_\alpha := \{\mathbf{b}^\alpha_x \mid x \in T \restriction C_\alpha\},$$

which is a subset of $\mathcal{T}$ by the preceding claim. As $T \restriction \alpha$ is a countable set, so is T_α.

Claim 2.21.4. *For every $y \in T_\alpha$, there are co-finitely many $m < \omega$ such that $y = \mathbf{b}^\alpha_{y\restriction\beta_m}$.*

Proof. Given $y \in T_\alpha$, by the definition of T_α there is some $x \in T \restriction C_\alpha$ such that $y = \mathbf{b}^\alpha_x$. Choose any such x, and let $k < \omega$ be such that $\mathrm{dom}(x) = \beta_k$. Consider any given $m \in [k, \omega)$; we will show that $\mathbf{b}^\alpha_{y\restriction\beta_m} = y$.[14] Observe that $\mathbf{b}^\alpha_{y\restriction\beta_m} = \bigcup\{b^\alpha_{y\restriction\beta_m}(\beta_n) \mid m \leq n < \omega\}$ and, since $k \leq m$ and the sequence b^α_x is $\subseteq$-increasing, $y = \mathbf{b}^\alpha_x = \bigcup\{b^\alpha_x(\beta_n) \mid k \leq n < \omega\} = \bigcup\{b^\alpha_x(\beta_n) \mid m \leq n < \omega\}$. Thus, it suffices to show that $b^\alpha_{y\restriction\beta_m}(\beta_n) = b^\alpha_x(\beta_n)$ for all $n \in [m, \omega)$. We prove this by induction:

- For $n = m$, Clause (ii) (see page 10) gives
$$b^\alpha_{y\restriction\beta_m}(\beta_m) = y \restriction \beta_m = \mathbf{b}^\alpha_x \restriction \beta_m = b^\alpha_x(\beta_m).$$
- For every $n \in [m, \omega)$ such that $b^\alpha_{y\restriction\beta_m}(\beta_n) = b^\alpha_x(\beta_n)$, as $\mathrm{dom}(x) = \beta_k \leq \beta_m = \mathrm{dom}(y \restriction \beta_m) \leq \beta_n$, Clause (iii) gives
$$b^\alpha_{y\restriction\beta_m}(\beta_{n+1}) = e_n(b^\alpha_{y\restriction\beta_m}(\beta_n)) = e_n(b^\alpha_x(\beta_n)) = b^\alpha_x(\beta_{n+1}),$$
completing the induction and thereby proving the claim. □

At this point, we need to decide whether to include α in E, and if so, then also to determine the identity of a_α.

If Ω_α happens to be a function from $(T \restriction \alpha)^2$ to ω, then consider the set

$$K_\alpha := \{k < \omega \mid \sup\{\tau < \alpha \mid \Omega_\alpha(\mathbf{b}^\alpha_{\langle 1\rangle} \restriction \tau, \mathbf{b}^\alpha_{\langle 2\rangle} \restriction \tau) = k\} = \alpha\},$$

and in case $K_\alpha \neq \emptyset$, we include α in E and set $a_\alpha := (\mathbf{b}^\alpha_{\langle 1\rangle}, \mathbf{b}^\alpha_{\langle 2\rangle})$. Otherwise, we do not include α in E.

Claim 2.21.5. *$\langle a_\epsilon \mid \epsilon \in E \cap (\alpha + 1)\rangle$ is a $\subseteq^2$-antichain.*

Proof. By the induction hypothesis, $\langle a_\epsilon \mid \epsilon \in E \cap \alpha\rangle$ is a $\subseteq^2$-antichain. Thus, it suffices to consider the case $\alpha \in E$ and prove that $a_\epsilon \nsubseteq^2 a_\alpha$ for every $\epsilon \in E \cap \alpha$. To this end, let $\epsilon \in E \cap \alpha$ be given. Choose a large enough $m < \omega$ such that $\beta_m > \epsilon$. Appealing to Claim 2.21.3 with $n := 0$, $x := \langle 1\rangle$ and $x' := \langle 2\rangle$, we obtain $(\langle 1\rangle, \langle 2\rangle) \trianglelefteq (b^\alpha_{\langle 1\rangle}(\beta_m), b^\alpha_{\langle 2\rangle}(\beta_m))$. By Clause ($\sqsupset$), from $E \cap 2 = \emptyset$, we obtain $a_\epsilon \nsubseteq^2 (b^\alpha_{\langle 1\rangle}(\beta_m), b^\alpha_{\langle 2\rangle}(\beta_m))$. In particular, $a_\epsilon \nsubseteq^2 (\mathbf{b}^\alpha_{\langle 1\rangle}, \mathbf{b}^\alpha_{\langle 2\rangle})$. But the latter is equal to a_α, so we are done. □

Claim 2.21.6. *For every $\beta < \alpha$, T_β and T_α are coordinated.*

[14] The same proof will show that $(y = \mathbf{b}^\alpha_{y\restriction\beta_k}) \implies (y = \mathbf{b}^\alpha_{y\restriction\beta_{k+1}})$ for every $k < \omega$, but we shall not need that.

Proof. Before we start, for every $m<\omega$, we define a map $f_m : T_{\beta_m} \to T_\alpha$ via $f_m(x) := \mathbf{b}^\alpha_x$. While we cannot guarantee that f_m is a q_m-elevator, we can nevertheless prove the following subclaim.

Subclaim 2.21.6.1. *Suppose that $\bar{e} : T_\beta \to T_{\beta_m}$ is a p-elevator with $p \in \mathbb{Q}$, $m<\omega$ and $\beta<\beta_m$. If $\alpha \notin E$ or if $\{\mathbf{b}^\alpha_{\langle 1\rangle} \restriction \beta_m, \mathbf{b}^\alpha_{\langle 2\rangle} \restriction \beta_m\} \nsubseteq \mathrm{Im}(\bar{e})$, then $f_m \circ \bar{e}$ is a $(p+q_m)$-elevator from T_β to T_α.*

Proof. It follows from Claim 2.21.3 that $f_m(x) \in U(x, q_m)$ for every $x \in T_{\beta_m}$. Furthermore, by Claim 2.21.3 and Clause (❏), for every $(x_0,x_1) \in (T_{\beta_m})^2$, if $a_\epsilon \not\sqsubseteq^2 (x_0,x_1)$ for every $\epsilon \in E \cap (\beta_m+1)$, then $a_\epsilon \not\sqsubseteq^2 (f_m(x_0), f_m(x_1))$ for every $\epsilon \in E\cap\alpha$. So if, in addition, $\alpha \notin E$, then f_m is a q_m-elevator, and then $f_m \circ \bar{e}$ is a $(p+q_m)$-elevator by Remark 2.14.

On the other hand, if $\alpha \in E$, then the fact that f_m is *not* a q_m-elevator is only because of the pair $(x_0,x_1) := (\mathbf{b}^\alpha_{\langle 1\rangle}\restriction\beta_m, \mathbf{b}^\alpha_{\langle 2\rangle}\restriction\beta_m)$ in $(T_{\beta_m})^2$, which satisfies $(f_m(x_0), f_m(x_1)) = (\mathbf{b}^\alpha_{\langle 1\rangle}, \mathbf{b}^\alpha_{\langle 2\rangle}) = a_\alpha$, so that $(x_0,x_1) \ntrianglelefteq (f_m(x_0), f_m(x_1))$ for this pair. Thus, we still have that both $f_m \restriction (T_{\beta_m} \setminus \{\mathbf{b}^\alpha_{\langle 1\rangle} \restriction \beta_m\})$ and $f_m \restriction (T_{\beta_m} \setminus \{\mathbf{b}^\alpha_{\langle 2\rangle} \restriction \beta_m\})$ are q_m-elevators. So, for every $i \in \{1,2\}$ such that $\mathbf{b}^\alpha_{\langle i\rangle} \restriction \beta_m \notin \mathrm{Im}(\bar{e})$, it is the case that

$$f_m \circ \bar{e} = (f_m \restriction (T_{\beta_m} \setminus \{\mathbf{b}^\alpha_{\langle i\rangle} \restriction \beta_m\})) \circ \bar{e}$$

is the composition of a q_m-elevator and a p-elevator. Again, we are done by Remark 2.14. □

Let $\beta<\alpha$ and $q \in \bar{\mathbb{Q}}$ be given. We go over the clauses of Definition 2.16:

(0) Given a finite $W \subseteq T_\alpha$, first, by possibly enlarging it, we ensure $\mathbf{b}^\alpha_{\langle 1\rangle} \in W$. We then find a large enough $m<\omega$ such that:
   - $q_m < q$; and
   - $\beta_m > \beta$.

   Set $p := q - q_m$ and $\bar{W} := \{w \restriction \beta_m \mid w \in W\}$. As T_β and T_{β_m} are coordinated, we may fix a p-elevator $\bar{e} : T_\beta \to T_{\beta_m}$ such that $\mathrm{Im}(\bar{e}) \cap \bar{W} = \emptyset$. In particular, $\mathbf{b}^\alpha_{\langle 1\rangle} \restriction \beta_m \notin \mathrm{Im}(\bar{e})$. By the subclaim, then, $e := f_m \circ \bar{e}$ is a q-elevator from T_β to T_α. In addition, $\mathrm{Im}(e) \cap W = \emptyset$, so we are done.

(1) Let $x \in T_\beta$, $y \in U(x,q) \cap T_\alpha$ and a finite set $W \subseteq T_\alpha \setminus \{y\}$ be given; we need to find a q-elevator $e : T_\beta \to T_\alpha \setminus W$ such that $e(x)=y$. First, choose some $i \in \{1,2\}$ such that $\mathbf{b}^\alpha_{\langle i\rangle} \neq y$. By possibly enlarging W, we may assume that $\mathbf{b}^\alpha_{\langle i\rangle} \in W$. Choose a large enough $p \in \mathbb{Q} \cap (0,q)$ such that $y \in U(x,p)$. Recalling Claim 2.21.4, we then find a large enough $m<\omega$ such that:
   - $q_m < (q-p)$;
   - $\beta_m > \beta$; and
   - $\mathbf{b}^\alpha_{w\restriction\beta_m} = w$ for every $w \in W \cup \{y\}$.

   Set $\bar{W} := \{w \restriction \beta_m \mid w \in W\}$ and $\bar{y} := y \restriction \beta_m$, so that $\bar{y} \in U(x,p) \cap T_{\beta_m}$ and $\bar{W} \subseteq T_{\beta_m} \setminus \{\bar{y}\}$. As T_β and T_{β_m} are coordinated, fix a p-elevator $\bar{e} : T_\beta \to T_{\beta_m} \setminus \bar{W}$ such that $\bar{e}(x) = \bar{y}$. In particular, $\mathbf{b}^\alpha_{\langle i\rangle} \restriction \beta_m \notin \operatorname{Im}(\bar{e})$. By the subclaim, then, $e := f_m \circ \bar{e}$ is a q-elevator from T_β to T_α. In addition, $\operatorname{Im}(e) \cap W = \emptyset$ and $e(x) = y$, so we are done.

(2) Let $\{x_0, x_1\} \in [T_\beta]^2$, $(y_0, y_1) \in (U(x_0, q) \cap T_\alpha) \times (U(x_1, q) \cap T_\alpha)$ such that $(x_0, x_1) \trianglelefteq (y_0, y_1)$, and a finite set $W \subseteq T_\alpha \setminus \{y_0, y_1\}$ be given; we need to find a q-elevator $e : T_\beta \to T_\alpha \setminus W$ such that $e(x_0) = y_0$ and $e(x_1) = y_1$. Choose a large enough $p \in \mathbb{Q} \cap (0, q)$ such that $y_0 \in U(x_0, p)$ and $y_1 \in U(x_1, p)$. Then, again recalling Claim 2.21.4, find a large enough $m < \omega$ such that:
- $q_m < (q - p)$;
- $\beta_m > \beta$; and
- $\mathbf{b}^\alpha_{w\restriction\beta_m} = w$ for every $w \in W \cup \{y_0, y_1\}$.

We now consider three cases:

(2.1) If $\alpha \notin E$, then set $\bar{W} := \{w \restriction \beta_m \mid w \in W\}$, $\bar{y}_0 := y_0 \restriction \beta_m$ and $\bar{y}_1 := y_1 \restriction \beta_m$. Clearly, $(\bar{y}_0, \bar{y}_1) \in (U(x_0, p) \cap T_{\beta_m}) \times (U(x_1, p) \cap T_{\beta_m})$, $(x_0, x_1) \trianglelefteq (\bar{y}_0, \bar{y}_1)$ and $\bar{W} \subseteq T_{\beta_m} \setminus \{\bar{y}_0, \bar{y}_1\}$. As T_β and T_{β_m} are coordinated, we may fix a p-elevator $\bar{e} : T_\beta \to T_{\beta_m} \setminus \bar{W}$ such that $\bar{e}(x_j) = \bar{y}_j$ for every $j < 2$. Recalling that $\alpha \notin E$, by the subclaim, $e := f_m \circ \bar{e}$ is a q-elevator from T_β to T_α. In addition, $\operatorname{Im}(e) \cap W = \emptyset$ and $e(x_j) = y_j$ for every $j < 2$, so we are done.

(2.2) If $\alpha \in E$ but $\{y_0, y_1\} \neq \{\mathbf{b}^\alpha_{\langle 1\rangle}, \mathbf{b}^\alpha_{\langle 2\rangle}\}$, then choose some $i \in \{1, 2\}$ such that $\mathbf{b}^\alpha_{\langle i\rangle} \notin \{y_0, y_1\}$. By possibly enlarging W, we may assume that $\mathbf{b}^\alpha_{\langle i\rangle} \in W$. Set $\bar{W} := \{w\restriction\beta_m \mid w \in W\}$, $\bar{y}_0 := y_0\restriction\beta_m$ and $\bar{y}_1 := y_1 \restriction \beta_m$. As in the previous case, fix a p-elevator $\bar{e} : T_\beta \to T_{\beta_m} \setminus \bar{W}$ such that $\bar{e}(x_j) = \bar{y}_j$ for every $j < 2$. In particular, $\mathbf{b}^\alpha_{\langle i\rangle} \restriction \beta_m \notin \operatorname{Im}(\bar{e})$. By the subclaim, then, $e := f_m \circ \bar{e}$ is a q-elevator from T_β to T_α. In addition, $\operatorname{Im}(e) \cap W = \emptyset$ and $e(x_j) = y_j$ for every $j < 2$, so we are done.

(2.3) If $\alpha \in E$ and for some $j < 2$,

$$(y_j, y_{1-j}) = (\mathbf{b}^\alpha_{\langle 1\rangle}, \mathbf{b}^\alpha_{\langle 2\rangle}) = a_\alpha,$$

then from $(x_0, x_1) \trianglelefteq (y_0, y_1)$, Clause ($\sqsupset$) implies that there exists an $\epsilon \in E \cap (\beta + 1)$ such that

$$\begin{aligned} a_\epsilon &\subseteq^2 (x_j, x_{1-j}) \\ &\subseteq^2 (y_j, y_{1-j}) = a_\alpha, \end{aligned}$$

contradicting Claim 2.21.5. So this case does not exist. □

Finally, let $T := \bigcup_{\alpha<\omega_1} T_\alpha$. This completes the construction of our $\aleph_1$-tree. As T is a subset of $\mathcal{T}$, it is $\mathbb{R}$-embeddable.

Claim 2.21.7. *T is almost-Souslin.*

Proof. Let A be a given antichain in T, and we shall show that $H := \{\mathrm{dom}(a) \mid a \in A\}$ is nonstationary. For every $i < \omega_1$, let B_i denote the set of all $\beta \in R_i$ such that there exists a countable elementary submodel $\mathcal{M} \prec H_{\omega_2}$ containing $\{A, T\}$, satisfying $\mathcal{M} \cap \omega_1 = \beta$ and $\mathcal{M} \cap A = \Omega_\beta$.

Subclaim 2.21.7.1. *Let $i < \omega_1$.*

1. *B_i is stationary in ω_1;*
2. *there is a club $D_i \subseteq \omega_1$ such that* $\sup(C_\alpha \cap B_i) = \alpha$ *for every $\alpha \in D_i$.*

Proof. (1) Let C be an arbitrary club in ω_1, and we shall prove that $B_i \cap C \neq \emptyset$. Set $p := \{C, A, T\}$ and $\Omega := A$. As $\langle \Omega_\beta \mid \beta < \omega_1\rangle$ and $\langle R_i \mid i < \omega_1\rangle$ were given by Fact 2.20, we may find a countable elementary submodel $\mathcal{M} \prec H_{\omega_2}$ containing p such that $\beta := \mathcal{M} \cap \omega_1$ is in R_i and $\mathcal{M} \cap \Omega = \Omega_\beta$. As the club C belongs to $\mathcal{M}$, $\beta \in C$, and hence $\mathcal{M}$ witnesses that $\beta \in B_i \cap C$.

(2) By Clause (1) and the fact that $\langle C_\alpha \mid \alpha < \omega_1\rangle$ was given by Fact 2.19. □

Let $\langle D_i \mid i < \omega_1\rangle$ be given by the subclaim. So, the following set is a club in ω_1:

$$D := \{\alpha \in \Lambda \mid T \restriction \alpha \subseteq \phi[\alpha] \ \&\ \forall i < \alpha\, (\alpha \in D_i)\}.$$

We claim that D is disjoint from H. Suppose not; pick $\alpha \in D \cap H$, and then pick $a^* \in A \cap T_\alpha$. Recalling the definition of T_α, we may pick an $x \in T \restriction C_\alpha$ such that $a^* = \mathbf{b}^\alpha_x$. As $\alpha \in D$, $T \restriction \alpha \subseteq \phi[\alpha]$, so we may fix an $i < \alpha$ such that $\phi(i) = x$. As $\alpha \in D$ and $i < \alpha$, $\sup(C_\alpha \cap B_i) = \alpha$, so letting $\langle \beta_n \mid n < \omega\rangle$ denote the increasing enumeration of C_α, we may fix a large enough $n < \omega$ such that $\beta_{n+1} \in B_i \setminus (\mathrm{dom}(x) + 1)$. In particular, $\beta_{n+1} \in R_i$, so that

$$\psi(\beta_{n+1}) = \phi(\pi(\beta_{n+1})) = \phi(i) = x.$$

As $\beta_{n+1} \in B_i$, we may now fix a countable elementary submodel $\mathcal{M} \prec H_{\omega_2}$ containing $\{A, T\}$ satisfying $\mathcal{M} \cap \omega_1 = \beta_{n+1}$ and $\mathcal{M} \cap A = \Omega_{\beta_{n+1}}$. By elementarity of $\mathcal{M}$, all of the following hold:

- $\mathcal{M} \cap T = T \restriction \beta_{n+1}$;
- $\mathcal{M} \cap A = A \cap (T \restriction \beta_{n+1})$;
- for all $y \in T \restriction \beta_{n+1}$ and $q \in \mathbb{Q}$, if $U(y, q) \cap A$ is nonempty, then so is $U(y, q) \cap A \cap (T \restriction \beta_{n+1})$.

Consider $y := b^\alpha_x(\beta_n)$. As $\Omega_{\beta_{n+1}}$ is a subset of $T \restriction \beta_{n+1}$, and $\psi(\beta_{n+1})$ is equal to x which is an element of $T \restriction (C_\alpha \cap \beta_{n+1})$, we have that $\mathbf{b}^\alpha_x \restriction \beta_{n+1} = e_n(y)$, where e_n is defined either according to Case I or according to Case III. In each case we will derive a contradiction:

► If e_n is defined according to Case I, then $\mathbf{b}^\alpha_x \restriction \beta_{n+1}$ belongs to Q^α_{n+1} which means that it extends some element a of $\Omega_{\beta_{n+1}} = A \cap (T \restriction \beta_{n+1})$, and then $a \subseteq \mathbf{b}^\alpha_x \restriction \beta_{n+1} \subseteq \mathbf{b}^\alpha_x = a^*$, contradicting the fact that a and a^* are two distinct elements of the antichain A.

► If e_n is defined according to Case III, then Claim 2.21.3 implies that $a^* = \mathbf{b}^\alpha_x \in U(y, \frac{q_n}{4})$. In particular, $U(y, \frac{q_n}{2}) \cap A \neq \emptyset$. By elementarity as above, it follows that $U(y, \frac{q_n}{2}) \cap A \cap (T \restriction \beta_{n+1})$ — equivalently, $U(b^\alpha_x(\beta_n), \frac{q_n}{2}) \cap \Omega_{\beta_{n+1}}$ — is nonempty. Choose $a \in U(b^\alpha_x(\beta_n), \frac{q_n}{2}) \cap \Omega_{\beta_{n+1}}$. Choose a large enough $p \in \mathbb{Q} \cap (0, \frac{q_n}{2})$ such that $a \in U(b^\alpha_x(\beta_n), p)$, and set $r := \frac{q_n}{2} - p$. As $T_{\mathrm{dom}(a)}$ and $T_{\beta_{n+1}}$ are coordinated, fix an r-elevator $e : T_{\mathrm{dom}(a)} \to T_{\beta_{n+1}}$, and set $t := e(a)$. Then $t \in U(b^\alpha_x(\beta_n), \frac{q_n}{2}) \cap T_{\beta_{n+1}}$ is an extension of a. In particular, t witnesses that Q^α_{n+1} is nonempty, contradicting the fact that we are in Case III. □

Since T is $\mathbb{R}$-embeddable and almost-Souslin, [Han83, Theorem 3] implies that X_T is perfect. As a topological space is perfectly normal iff it is perfect and normal, our next task is proving that X_T is normal. Recalling Fact 2.3(2), we now turn to prove the following.

Claim 2.21.8. *T has property γ.*

Proof. Let $A \subseteq T$ be a given antichain, and we shall find a club $D \subseteq \omega_1$ and two disjoint open sets U and V such that $A \subseteq U$ and $T \restriction D \subseteq V$. Consider the club D from the proof of Claim 2.21.7, which we already know is disjoint from $H := \{\mathrm{dom}(a) \mid a \in A\}$, that is, $A \cap (T \restriction D) = \emptyset$.

As A is an antichain, it is closed discrete, so by Fact 2.3(1), we may fix a pairwise disjoint family $\langle O_a \mid a \in A\rangle$ such that, for every $a \in A$, O_a is an open neighborhood of a. For every $a \in A$, we define an open subset $U_a \subseteq O_a$ as follows:

- if $\mathrm{dom}(a) \notin \Lambda$, then let $U_a := \{a\}$;
- otherwise, pick $a' \subsetneq a$ such that $(a', a] \subseteq O_a$, and then, using $\mathrm{dom}(a) \in \Lambda \setminus D$ and the definition of $c(a)$, pick some $a'' \in T$ such that
  - $a' \subsetneq a'' \subsetneq a$,
  - $c(a'') \geq \frac{c(a')+c(a)}{2}$, and
  - $\mathrm{dom}(a'') > \sup(D \cap \mathrm{dom}(a))$,

  and finally let $U_a := (a'', a]$.

Altogether, $U := \bigcup_{a \in A} U_a$ is an open neighborhood of A. Our next task is defining, for every $z \in T \restriction D$, an open neighborhood V_z of z which is disjoint from U. This way, $V := \bigcup_{z \in T \restriction D} V_z$ together with U will be as sought.

To this end, let $z \in T \restriction D$ be given. Write $\alpha := \mathrm{dom}(z)$ and let $\langle \beta_n \mid n < \omega\rangle$ denote the increasing enumeration of C_α. Pick $x \in T \restriction C_\alpha$ such that $z = \mathbf{b}^\alpha_x$. As in the proof of Claim 2.21.7, fix a large enough $n < \omega$ such that all of the following hold:

- $\beta_{n+1} > \mathrm{dom}(x)$;
- $\psi(\beta_{n+1}) = x$;
- there exists a countable elementary submodel $\mathcal{M} \prec H_{\omega_2}$ containing $\{A, T\}$ satisfying $\mathcal{M} \cap \omega_1 = \beta_{n+1}$ and $\mathcal{M} \cap A = \Omega_{\beta_{n+1}}$.

Recall that $z \restriction \beta_{n+1} = e_n(b^\alpha_x(\beta_n))$, where e_n is defined either according to Case I or according to Case III, and $z \restriction \beta_n = b^\alpha_x(\beta_n)$. Before we can define V_z, we need to observe the following.

Subclaim 2.21.8.1. *Suppose that e_n is defined according to Case III. For every $a \in A$ such that $U_a \cap (z \restriction \beta_{n+1}, z] \neq \emptyset$, both of the following hold:*

(1) *a and z are incomparable;*
(2) $\mathrm{dom}(a) \in \Lambda$ *and* $a' \subsetneq z \restriction \beta_n$.

Proof. By Claim 2.21.3, we know that $z \in U(z \restriction \beta_n, \frac{q_n}{4})$. Fix a given $a \in A$ as in the hypothesis.

(1) Suppose that a and z are comparable, and we will derive a contradiction in each of the following three cases:

- If $a \subsetneq z \restriction \beta_{n+1}$, then $\mathrm{dom}(a) < \beta_{n+1}$, so that $U_a \cap (z \restriction \beta_{n+1}, z] = \emptyset$.
- If $z \restriction \beta_{n+1} \subseteq a \subseteq z$, then $a \in U(z \restriction \beta_n, \frac{q_n}{4}) \cap A$, so that the reflection argument at the end of the proof of Claim 2.21.7 implies that we must have been in Case I.
- If $z \subsetneq a$, then $\mathrm{dom}(z) \in D \cap \mathrm{dom}(a)$, so that either $U_a = \{a\}$, or $U_a = (a'', a]$ with
$$\mathrm{dom}(a'') > \sup(D \cap \mathrm{dom}(a)) \geq \mathrm{dom}(z),$$
in either case yielding $U_a \cap (z \restriction \beta_{n+1}, z] = \emptyset$.

(2) From $U_a \cap (z \restriction \beta_{n+1}, z] \neq \emptyset$, we infer that $\mathrm{dom}(a) \in \Lambda$ and, in addition to a', $z \restriction \beta_n$ is below a. So either $a' \subsetneq z \restriction \beta_n$ or $z \restriction \beta_n \subseteq a'$.

Towards a contradiction, suppose that $z \restriction \beta_n \subseteq a'$. If $a \in U(z \restriction \beta_n, \frac{q_n}{2})$, then again by the argument of the last paragraph of the proof of Claim 2.21.7, we must have been in Case I. Thus, in fact, $c(a) \geq c(z \restriction \beta_n) + \frac{q_n}{2}$. Together with $a' \supseteq z \restriction \beta_n$, this yields
$$c(a'') \geq \frac{c(a') + c(a)}{2} \geq \frac{c(z \restriction \beta_n) + c(z \restriction \beta_n) + \frac{q_n}{2}}{2} = c(z \restriction \beta_n) + \frac{q_n}{4}.$$

Recalling that $z \in U(z \restriction \beta_n, \frac{q_n}{4})$, we get
$$c(a'') \geq c(z \restriction \beta_n) + \frac{q_n}{4} > c(z),$$
which implies that the intervals $U_a = (a'', a]$ and $(\emptyset, z]$ are disjoint, contradicting the hypothesis. □

Finally, the definition of V_z is divided as follows, where in all cases V_z will be $(z \restriction \beta_m, z]$ for some $m \in [n+1, \omega)$:

► If e_n is defined according to Case I, then let $V_z := (z \restriction \beta_{n+1}, z]$.

► If e_n is defined according to Case III, then we consider two subcases:

►► If $(z \restriction \beta_{n+1}, z] \cap U = \emptyset$, then again let $V_z := (z \restriction \beta_{n+1}, z]$.

►► Otherwise, pick $a \in A$ such that $(z \restriction \beta_{n+1}, z] \cap U_a \neq \emptyset$. By Subclaim 2.21.8.1(1), a and z are incomparable, so we may fix a large enough $m \in [n+1, \omega)$ such that $(z \restriction \beta_m, z] \cap U_a = \emptyset$, and we let $V_z := (z \restriction \beta_m, z]$.

Subclaim 2.21.8.2. *$V_z \cap U = \emptyset$.*

Proof. Suppose not, and pick $b \in A$ such that $V_z \cap U_b \neq \emptyset$. Let us come back to the above division, deriving a contradiction in each case:

▶ If e_n is defined according to Case I, then $z\restriction\beta_{n+1} = e_n(b^\alpha_x(\beta_n))$ belongs to Q^α_{n+1}, which means that it extends some element a of $\Omega_{\beta_{n+1}} = A\cap(T\restriction\beta_{n+1})$. But since $(z\restriction\beta_{n+1}, z]\cap(b'', b] \neq \emptyset$, we get that $z\restriction\beta_{n+1} \subsetneq b$, so that a and b are two distinct comparable elements of the antichain A. This is a contradiction.

▶ If e_n is defined according to Case III, then since $(z \restriction \beta_{n+1}, z] \cap U \neq \emptyset$, it was the case that we picked an $a \in A$ such that $(z \restriction \beta_{n+1}, z] \cap U_a \neq \emptyset$, and let $V_z := (z \restriction \beta_m, z]$ for some $m \in [n + 1, \omega)$ such that $(z \restriction \beta_m, z] \cap U_a = \emptyset$. In particular, $V_z \cap U_a = \emptyset$, so $a \neq b$. By Subclaim 2.21.8.1(2), $a' \subsetneq z \restriction \beta_n$ and $b' \subsetneq z \restriction \beta_n$, so $z \restriction \beta_n \in (a', a] \cap (b', b] \subseteq O_a \cap O_b$, contradicting the fact that O_a and O_b are disjoint. □

This completes the proof. □

Next, to help us verify that $(X_T)^2$ is not cmc, we define an auxiliary map $g : E \to \omega$ via $g(\alpha) := \min(K_\alpha)$.

Claim 2.21.9. *For every function $f : T^2 \to \omega$ there are stationarily many $\alpha \in E$ such that*

$$\sup\{\tau < \alpha \mid f(\mathbf{b}^\alpha_{\langle 1\rangle} \restriction \tau, \mathbf{b}^\alpha_{\langle 2\rangle} \restriction \tau) = g(\alpha)\} = \alpha.$$

Proof. Given $f : T^2 \to \omega$, for every $k < \omega$, let B_k be the set of all $\beta \in R_{\phi^{-1}(k)}$ such that there exists a countable elementary submodel $\mathcal{M} \prec H_{\omega_2}$ such that all of the following hold:

- $\{T, f\} \in \mathcal{M}$;
- $\beta = \mathcal{M} \cap \omega_1$, and
- $\Omega_\beta = f \restriction (T \restriction \beta)^2$.

A proof similar to that of Subclaim 2.21.7.1 establishes that

$$A := \{\alpha \in B_0 \mid \forall k < \omega\, [\sup(C_\alpha \cap B_k) = \alpha]\}$$

is a stationary subset of B_0.

Subclaim 2.21.9.1. *$A \subseteq E$.*

Proof. Suppose not, and let $\alpha \in A \setminus E$. Let $\langle \beta_n \mid n < \omega\rangle$ denote the increasing enumeration of C_α. By Claim 2.21.3, for every $n < \omega$,

$$(b^\alpha_{\langle 1\rangle}(\beta_n), b^\alpha_{\langle 2\rangle}(\beta_n)) \trianglelefteq (b^\alpha_{\langle 1\rangle}(\beta_{n+1}), b^\alpha_{\langle 2\rangle}(\beta_{n+1})).$$

So, since $\alpha \notin E$, we obtain furthermore that, for every $n < \omega$,

$$(b^\alpha_{\langle 1\rangle}(\beta_n), b^\alpha_{\langle 2\rangle}(\beta_n)) \trianglelefteq (\mathbf{b}^\alpha_{\langle 1\rangle}, \mathbf{b}^\alpha_{\langle 2\rangle}).$$

As $\alpha \in A \subseteq B_0$, the following equation holds

$$K_\alpha = \{k < \omega \mid \sup\{\tau < \alpha \mid f(\mathbf{b}^\alpha_{\langle 1\rangle} \restriction \tau, \mathbf{b}^\alpha_{\langle 2\rangle} \restriction \tau) = k\} = \alpha\}.$$

We shall show that $k := f(\mathbf{b}^\alpha_{\langle 1\rangle}, \mathbf{b}^\alpha_{\langle 2\rangle})$ belongs to K_α, in particular, $K_\alpha \neq \emptyset$, contradicting the fact that $\alpha \notin E$.

From $\alpha \in A$, we infer that the set $N := \{n < \omega \mid \beta_{n+1} \in B_k\}$ is infinite. Thus, to prove that $k \in K_\alpha$, it suffices to prove that for every $n \in N$, there exists some $\tau \in [\beta_n, \beta_{n+1})$ such that $f(\mathbf{b}^\alpha_{\langle 1\rangle} \restriction \tau, \mathbf{b}^\alpha_{\langle 2\rangle} \restriction \tau) = k$.

To this end, let $n \in N$. As $\Omega_{\beta_{n+1}} = f \restriction (T \restriction \beta_{n+1})^2$ and $\psi(\beta_{n+1}) = k$, we have that $b^\alpha_{\langle j \rangle}(\beta_{n+1}) = e_n(b^\alpha_{\langle j \rangle}(\beta_n))$ for $j \in \{1, 2\}$, where e_n is defined either according to Case II or according to Case III.

▶ If e_n is defined according to Case II, then $(b^\alpha_{\langle 1 \rangle}(\beta_{n+1}), b^\alpha_{\langle 2 \rangle}(\beta_{n+1}))$ belongs to P^α_{n+1} which means that there indeed exists some $\tau \in [\beta_n, \beta_{n+1})$ such that $f(\mathbf{b}^\alpha_{\langle 1 \rangle} \restriction \tau, \mathbf{b}^\alpha_{\langle 2 \rangle} \restriction \tau) = k$.

▶ If e_n is defined according to Case III, then Claim 2.21.3 implies that $\mathbf{b}^\alpha_{\langle j \rangle} \in U(b^\alpha_{\langle j \rangle}(\beta_n), \frac{q_n}{4}) \subseteq U(b^\alpha_{\langle j \rangle}(\beta_n), \frac{q_n}{2})$ for $j \in \{1, 2\}$. Pick a countable $\mathcal{M} \prec H_{\omega_2}$ that includes $\{T, f\}$ such that $\mathcal{M} \cap \omega_1 = \beta_{n+1}$ and $\Omega_{\beta_{n+1}} = f \restriction (T \restriction \beta_{n+1})^2$. So, $\mathcal{M}$ reflects the above properties of the pair $(\mathbf{b}^\alpha_{\langle 1 \rangle}, \mathbf{b}^\alpha_{\langle 2 \rangle})$, meaning that there exists a pair $(r_1, r_2) \in (T \restriction \beta_{n+1})^2$ such that

- $r_1 \in U(b^\alpha_{\langle 1 \rangle}(\beta_n), \frac{q_n}{2})$,
- $r_2 \in U(b^\alpha_{\langle 2 \rangle}(\beta_n), \frac{q_n}{2})$,
- $(b^\alpha_{\langle 1 \rangle}(\beta_n), b^\alpha_{\langle 2 \rangle}(\beta_n)) \trianglelefteq (r_1, r_2)$, and
- $f(r_1, r_2) = k$.

As $T_{\mathrm{dom}(r_1)}$ and $T_{\beta_{n+1}}$ are coordinated, it follows that P^α_{n+1} is nonempty, contradicting the fact that we are in Case III. □

Consider any given $\alpha \in A$, so that $\Omega_\alpha = f \restriction (T \restriction \alpha)^2$. By Subclaim 2.21.9.1, $\alpha \in E$, which means that $K_\alpha \neq \emptyset$. So, by the definition of g,

$$\sup\{\tau < \alpha \mid f(\mathbf{b}^\alpha_{\langle 1 \rangle} \restriction \tau, \mathbf{b}^\alpha_{\langle 2 \rangle} \restriction \tau) = g(\alpha)\} = \alpha,$$

as sought. □

Claim 2.21.10. $(X_T)^2$ *is not cmc.*

Proof. By [Git74, Theorem 2.2], a space is cmc iff for every $\subseteq$-decreasing sequence $\langle D_n \mid n < \omega \rangle$ of closed sets that vanishes (i.e., $\bigcap_{n<\omega} D_n = \emptyset$), there exists a $\subseteq$-decreasing vanishing sequence $\langle U_n \mid n < \omega \rangle$ of open sets such that $U_n \supseteq D_n$ for every $n < \omega$.

As $\{a_\epsilon \mid \epsilon \in E\}$ is a $\subseteq^2$-antichain in T^2, Proposition 2.7 implies that for every $n < \omega$, $D_n := \{a_\epsilon \mid \epsilon \in E \ \&\ g(\epsilon) \geq n\}$ is a closed (discrete) subset of $(X_T)^2$. Clearly, $\langle D_n \mid n < \omega \rangle$ is decreasing and vanishing. Towards a contradiction, suppose that $\langle U_n \mid n < \omega \rangle$ is a vanishing decreasing sequence of open subsets of $(X_T)^2$ such that $U_n \supseteq D_n$ for every $n < \omega$. Define a function $f : T^2 \to \omega$ by letting $f(x_0, x_1)$ be the least $n < \omega$ such that $(x_0, x_1) \notin U_n$. Now using Claim 2.21.9 pick $\alpha \in E$ such that

$$\sup\{\tau < \alpha \mid f(\mathbf{b}^\alpha_{\langle 1 \rangle} \restriction \tau, \mathbf{b}^\alpha_{\langle 2 \rangle} \restriction \tau) \leq g(\alpha)\} = \alpha.$$

Consider $n := g(\alpha)$, so that $(\mathbf{b}^\alpha_{\langle 1 \rangle}, \mathbf{b}^\alpha_{\langle 2 \rangle}) = a_\alpha \in D_n \subseteq U_n$. Since U_n is open in $(X_T)^2$, we may fix $y_1 \subsetneq \mathbf{b}^\alpha_{\langle 1 \rangle}$ and $y_2 \subsetneq \mathbf{b}^\alpha_{\langle 2 \rangle}$ such that $(y_1, \mathbf{b}^\alpha_{\langle 1 \rangle}] \times (y_2, \mathbf{b}^\alpha_{\langle 2 \rangle}] \subseteq U_n$. Fix a large enough $\tau < \alpha$ such that $\max\{\mathrm{dom}(y_1), \mathrm{dom}(y_2)\} < \tau$ and $f(\mathbf{b}^\alpha_{\langle 1 \rangle} \restriction \tau, \mathbf{b}^\alpha_{\langle 2 \rangle} \restriction \tau) \leq n$.

It follows that $(\mathbf{b}^\alpha_{\langle 1\rangle}\restriction\tau,\mathbf{b}^\alpha_{\langle 2\rangle}\restriction\tau)\in(y_1,\mathbf{b}^\alpha_{\langle 1\rangle}]\times(y_2,\mathbf{b}^\alpha_{\langle 2\rangle}]\subseteq T^2\cap U_n$, which must mean that $f(\mathbf{b}^\alpha_{\langle 1\rangle}\restriction\tau,\mathbf{b}^\alpha_{\langle 2\rangle}\restriction\tau)>n$. This is a contradiction. □

This completes the proof. □

Remark 2.22.
(1) By Claim 2.21.8 and [Har83, Theorem 2.1], X_T is hereditarily collectionwise normal.
(2) In [Bes94], Bešlagić constructed from $\diamondsuit(\omega_1)$ a perfectly normal space X such that X^2 is not cmc but is normal. In contrast, our tree-based example witnesses the non-productivity of normality. This is because the proof of Claim 2.21.10 moreover shows that the closed subspace T^2 of $(X_T)^2$ is not cmc. So, by [Fle80, Corollary 4.2], the closed subspace T^2 cannot be normal, let alone $(X_T)^2$.
(3) An inspection of the preceding construction makes it clear that for the sake of getting an $\mathbb{R}$-embeddable $\aleph_1$-tree T such that T^2 is not cmc, $\diamondsuit(\omega_1)$ suffices. As $(T^2,\subseteq^2)$ would be an $\mathbb{R}$-embeddable $\aleph_1$-tree in this case, this shows that $\diamondsuit(\omega_1)$ yields an $\mathbb{R}$-embeddable $\aleph_1$-tree that is not cmc, a result announced by Hanazawa on [Han83, p. 61].[15]
(4) A minor tweak of the preceding construction will secure the following anti-uniformization feature, a strong form of Claim 2.21.9. There exist an ω-bounded ladder system $\langle A_\alpha\mid\alpha<\omega_1\rangle$ and a function $g:T^2\to\omega$ such that for every function $f:T^2\to\omega$ there are stationarily many $\alpha\in E$ such that for every $(z_0,z_1)\in T_\alpha\times T_\alpha$,
$$\sup\{\tau\in A_\alpha\mid f(z_0\restriction\tau,z_1\restriction\tau)\le g(z_0,z_1)\}=\alpha.$$
Such an almost-Souslin tree can be viewed as an instance of diamond whose square is an instance of uniformization.
(5) The preceding construction can also be tweaked to obtain, for every positive integer k, an $\aleph_1$-tree $T\subseteq{}^{<\omega_1}\omega$ consisting of a finite-to-one maps (in particular, T is $\mathbb{R}$-embeddable) such that $(T^k,\subseteq^k)$ is almost-Souslin and T^{k+1} is not cmc.
(6) It is an open problem whether every $\mathbb{R}$-embeddable cmc $\aleph_1$-tree is perfect. An affirmative answer holds assuming MA_{ω_1} [Fre84, Corollary 23B/41J] or $\mathsf{V}=\mathsf{L}$ or PMEA [Nyi93, p. 420], but the general case is open. A purported counterexample from $\diamondsuit^*(\omega_1)$ was constructed in [Han83, Theorem 4], but it is flawed in view of the fact that $\mathsf{V}=\mathsf{L}$ implies $\diamondsuit^*(\omega_1)$. The error was acknowledged in [Han84, p. 20].

2.4. **Additional set-theoretic comments.** The topologist who has had enough set theory can stop reading here. For the interested reader, we mention that the ladder system in the conclusion of Fact 2.19 is a particular instance of the proxy principles $\mathrm{P}^-(\dots)$ from [BRY25], namely, $\mathrm{P}^-_\omega(\omega_1,2,\sqsubseteq,1,\mathrm{NS}^+_{\omega_1},2,1)$. The existence of such a ladder system does not require CH, let

[15]The first consistent example of an $\mathbb{R}$-embeddable $\aleph_1$-tree that is not special was given by Baumgartner in his dissertation [Bau70].

alone $\diamondsuit^*(\omega_1)$, as it is for instance introduced by the forcing to add a single Cohen real (see [Rin15, Claim 2.3.3]).

Even the conjunction $\mathrm{P}^-_\omega(\omega_1, 2, \sqsubseteq, 1, \mathrm{NS}^+_{\omega_1}, 2, 1) \wedge \diamondsuit(\omega_1)$, which is denoted by $\mathrm{P}_\omega(\omega_1, 2, \sqsubseteq, 1, \mathrm{NS}^+_{\omega_1}, 2, 1)$ according to [BR17, Definition 1.6], is strictly weaker than $\diamondsuit^*(\omega_1)$, as can be seen from the proof of Corollary 2.24 below. Meanwhile, since the application of $\diamondsuit^*(\omega_1)$ in the proof of Theorem 2.21 is limited to Facts 2.19 and 2.20, we arrive at the following conclusion.

Theorem 2.23. *$\mathrm{P}_\omega(\omega_1, 2, \sqsubseteq, 1, \mathrm{NS}^+_{\omega_1}, 2, 1)$ implies the existence of an $\mathbb{R}$-embeddable almost-Souslin $\aleph_1$-tree T such that X_T is perfectly normal, but $(X_T)^2$ is not cmc.* □

Corollary 2.24. *Suppose that $\diamondsuit(\omega_1)$ holds. If $\mathbb{P}$ is the notion of forcing to add a single Cohen real or any other ccc notion of forcing of size at most $\aleph_1$ that is not ${}^\omega\omega$-bounding, then in $V^{\mathbb{P}}$ there is an $\mathbb{R}$-embeddable almost-Souslin $\aleph_1$-tree T such that X_T is perfectly normal, but $(X_T)^2$ is not cmc.*

Proof. Given $\mathbb{P}$ as in the hypothesis, since we have assumed $\diamondsuit(\omega_1)$ in V, by [BR19c, Proposition 3.1(1) and Theorem 3.4], both $\mathrm{P}^*(E^{\omega_1}_\omega, \omega)$ and $\diamondsuit(\omega_1)$ hold in $V^{\mathbb{P}}$.[16] By Remark (iv) after [BR19c, Definition 3.3], $\mathrm{P}^*(E^{\omega_1}_\omega, \omega)$ stands for $\mathrm{P}^-_\omega(\omega_1, \infty, \sqsubseteq, 1, \mathrm{NS}^+_{\omega_1}, 2, {<}\infty)$, which is easily seen to imply $\mathrm{P}^-_\omega(\omega_1, 2, \sqsubseteq, 1, \mathrm{NS}^+_{\omega_1}, 2, 1)$. Altogether, $\mathrm{P}_\omega(\omega_1, 2, \sqsubseteq, 1, \mathrm{NS}^+_{\omega_1}, 2, 1)$ holds in $V^{\mathbb{P}}$, and we can appeal to Theorem 2.23.[17] □

An argument of Todorčević [Tod05, p. 44] building on [Roi79, Theorem 1.3] shows that an extra hypothesis concerning the ground model is indeed necessary here. For instance, it shows that in the forcing extension by adding a single Cohen or random real over a model of MA_{ω_1}, for every $\mathbb{R}$-embeddable $\aleph_1$-tree T, the space $(X_T)^2$ is cmc (in fact, it is a Q-space).

3. Clause (1) of Theorem B

Throughout this section, κ stands for a regular uncountable cardinal, $\mathrm{Reg}(\kappa)$ denotes the collection of all infinite regular cardinals below κ, and for $\sigma \in \mathrm{Reg}(\kappa)$, we let $E^\kappa_\sigma := \{\alpha < \kappa \mid \mathrm{cf}(\alpha) = \sigma\}$. For a set of ordinals A we write $\mathrm{acc}(A) := \{\alpha \in A \mid \sup(A \cap \alpha) = \alpha > 0\}$ and $\mathrm{nacc}(A) := A \setminus \mathrm{acc}(A)$.

Definition 3.1.

(1) A *C-sequence over κ* is a sequence $\vec{C} = \langle C_\alpha \mid \alpha < \kappa \rangle$ such that, for every $\alpha < \kappa$, C_α is a closed subset of α with $\sup(C_\alpha) = \sup(\alpha)$;

(2) For two sets of ordinals C and D, we write $C \sqsubseteq D$ iff $C = D \cap \varepsilon$ for some ordinal ε;

(3) A C-sequence $\langle C_\alpha \mid \alpha < \kappa \rangle$ is *coherent* iff $C_\beta \sqsubseteq C_\alpha$ for all $\alpha < \kappa$ and all $\beta \in \mathrm{acc}(C_\alpha)$;

[16] See also the third section of [BRY25, Table 3.2].

[17] To demonstrate that the hypothesis of Theorem 2.23 is consistently weaker than that of Theorem 2.21, note that if $\diamondsuit^*(\omega_1)$ failed in V, then it remains failing in $V^{\mathbb{P}}$. Indeed, by [Kun80, Lemma VII.5.5 and Exercise VII.H1], a ccc notion of forcing cannot force $\diamondsuit^*(\omega_1)$.

(4) A C-sequence $\langle C_\alpha \mid \alpha < \kappa\rangle$ is *weakly coherent* iff $|\{C_\alpha \cap \epsilon \mid \alpha < \kappa\}| < \kappa$ for every $\epsilon < \kappa$.

Definition 3.2 ([IR25, §4])**.** Let $\vec{C} = \langle C_\alpha \mid \alpha < \kappa\rangle$ be a C-sequence.

- The set of *lower-regressive levels* of $\vec{C}$ is the following:
$$R(\vec{C}) := \{\alpha \in \mathrm{acc}(\kappa) \mid \forall \beta \in (\alpha, \kappa)\,(\mathrm{otp}(C_\beta \cap \alpha) < \alpha)\}.$$
- The set of *avoiding levels* of $\vec{C}$ is the following:
$$A(\vec{C}) := \{\alpha \in \mathrm{acc}(\kappa) \mid \forall \beta \in (\alpha, \kappa)\,(\sup(C_\beta \cap \alpha) < \alpha)\}.$$

As explained in [BRY25], constructions of κ-trees are typically guided by some 'good' C-sequences such as those whose existence is asserted by *proxy principles*. The general definition of these principles may be found as [BRY25, Definition 2.5, §2.4 and §2.6], but for our purpose, the following special case suffices.[18]

Definition 3.3. $\mathrm{P}_<(\kappa, 2, \sqsubseteq, \kappa)$ asserts $\diamondsuit(\kappa)$ as well as the existence of a C-sequence $\vec{C} = \langle C_\alpha \mid \alpha < \kappa\rangle$ such that all of the following hold:

(1) $R(\vec{C})$ covers a club;
(2) for all $\alpha < \kappa$ and all $\beta \in \mathrm{acc}(C_\alpha)$, $C_\beta = C_\alpha \cap \beta$;[19]
(3) for every sequence $\langle B_i \mid i < \kappa\rangle$ of cofinal subsets of κ, there are stationarily many $\alpha < \kappa$ such that, for every $i < \alpha$,
$$\sup(\mathrm{nacc}(C_\alpha) \cap B_i) = \alpha.$$

Remark 3.4. To compare, the proxy principle $\mathrm{P}_\xi(\kappa, 2, \sqsubseteq, \kappa)$ is the outcome of replacing Clause (1) by the uniform requirement that $\mathrm{otp}(C_\alpha) \leq \xi$ for all $\alpha < \kappa$. For a successor cardinal κ, having $\mathrm{P}_\xi(\kappa, 2, \sqsubseteq, \kappa)$ with $\xi < \kappa$ is consistent [BR17, Theorem 3.6] and quite useful [Sha24, Main Theorem], but what do we do at inaccessibles? A natural generalization would require that $\mathrm{otp}(C_\alpha) < \alpha$ for club many $\alpha < \kappa$. However, for an inaccessible cardinal κ, every $\vec{C}$ satisfying (2) and (3) must satisfy that $\{\alpha \in E^\kappa_\omega \mid \mathrm{otp}(C_\alpha) = \alpha\}$ is stationary in κ. Thus, Clause (1) turns out to be the right way to go.

We now introduce a poset $\mathbb{P}$ that forces $\mathrm{P}_<(\kappa, 2, \sqsubseteq, \kappa)$. In the next section we shall demonstrate the utility of this axiom.

Definition 3.5. Let $\mathbb{P}$ be the forcing notion in which each condition is either $\emptyset$ or a pair $p = (\langle C^p_\alpha \mid \alpha \leq \gamma^p\rangle, D^p)$ such that:

(1) $\gamma^p \in \mathrm{acc}(\kappa)$;
(2) D^p is a closed set of limit ordinals with $\max(D^p) = \gamma^p$;
(3) for all $\alpha \leq \gamma^p$, C^p_α is a closed subset of α with $\sup(C^p_\alpha) = \sup(\alpha)$;
(4) for all $\alpha < \beta \leq \gamma^p$:
    (a) if $\alpha \in \mathrm{acc}(C^p_\beta)$, then $C^p_\beta \cap \alpha = C^p_\alpha$;
    (b) if $\alpha \in D^p$, then $\mathrm{otp}(C^p_\beta \cap \alpha) < \alpha$.

[18] Compare the upcoming definition with the conclusion of Fact 2.19.

[19] That is, $\vec{C}$ is coherent.

We order $\mathbb{P}$ by assigning $\emptyset$ as the maximal element $\mathbb{1}_{\mathbb{P}}$, and otherwise requiring end-extension on both coordinates.

Before we analyze the features of the poset $\mathbb{P}$, we recall the notion of strategic closure.

Definition 3.6. For a notion of forcing $\mathbb{P}$ and an ordinal σ, $\Game_\sigma(\mathbb{P})$ denotes the following two-player game of perfect information:

Two players, named **I** and **II**, take turns to play conditions from $\mathbb{P}$ for σ many moves, with **I** playing at odd stages and **II** at even stages (including all limit stages). **II** must play $\mathbb{1}_{\mathbb{P}}$ at move zero. Let p_i be the condition played at move i; the player who plays p_i loses immediately unless $p_i \leq p_j$ for all $j < i$. If neither player loses at any stage $i < \sigma$, then **II** wins.

$\mathbb{P}$ is said to be *σ-strategically-closed* iff **II** has a winning strategy for $\Game_\sigma(\mathbb{P})$. It is said to be *$<\sigma$-strategically-closed* iff it is τ-strategically-closed for all $\tau < \sigma$.

Lemma 3.7. (1) *for every nonempty $p \in \mathbb{P}$, for every $\beta < \kappa$, there exists a $q \leq p$ with $\gamma^q > \beta$ and $\operatorname{otp}(C^q_{\gamma^q}) = \omega$.*

(2) *for every $\mathbb{P}$-name $\dot{B}$ for a cofinal subset of κ, for every $\sigma \in \operatorname{acc}(\kappa)$, for every nonempty $p \in \mathbb{P}$ such that $\operatorname{otp}(C^p_{\gamma^p}) < \gamma^p$, there exists a $q \leq p$ such that all of the following hold:*
   - $C^p_{\gamma^p} \sqsubseteq C^q_{\gamma^q}$;
   - $\operatorname{otp}(C^q_{\gamma^q}) = \operatorname{otp}(C^p_{\gamma^p}) + \omega \cdot \sigma < \gamma^q$;
   - *q forces that $\{C^q_{\gamma^q}(\operatorname{otp}(C^p_{\gamma^p}) + \omega \cdot \iota + 1) \mid \iota < \sigma\}$ is a subset of $\dot{B}$.*

(3) *$\mathbb{P}$ is $<\kappa$-strategically-closed.*

(4) *$\mathbb{P}$ is countably closed.*

(5) *$\mathbb{P}$ has size $\kappa^{<\kappa}$.*

Proof. (1) Given a nonempty $p \in \mathbb{P}$ and $\beta < \kappa$, define $q \leq p$ as follows:
- $\gamma^q := \gamma^p + \beta + \omega$;
- $C^q_\alpha := C^p_\alpha$ for every $\alpha \leq \gamma^p$;
- $C^q_\alpha := \alpha \setminus \gamma^p$ whenever $\gamma^p < \alpha < \gamma^q$;
- $C^q_{\gamma^q} := \gamma^q \setminus (\gamma^p + \beta)$, a tail of γ^q of order-type ω;
- $D^q := D^p \cup \{\gamma^q\}$.

It is clear that q is as sought.

Next, we prove Clauses (2) and (3) simultaneously. So, let $\dot{B}$ be a $\mathbb{P}$-name for a cofinal subset of κ, let $\sigma \in \operatorname{acc}(\kappa)$, and we shall play the game $\Game_{\sigma+1}(\mathbb{P})$, producing a decreasing sequence of conditions $\langle p_i \mid i \leq \sigma\rangle$, where $p_0 = \mathbb{1}_{\mathbb{P}}$ and p_1 is an arbitrary condition (as dictated by the rules of the game), and p_2 is any extension of p_1 satisfying $\operatorname{otp}(C^{p_2}_{\gamma^{p_2}}) < \gamma^{p_2}$ (such an extension exists, by Clause (1)), so p_2 plays the role of p from Clause (2).

While describing the winning strategy for **II** in $\Game_{\sigma+1}(\mathbb{P})$, we shall be producing an auxiliary sequence $\langle \beta_i \mid \text{even } i \in [4, \sigma)\rangle$ of ordinals, and the strategy for **II** will ensure the following features:

(i) $\mathrm{otp}(C^{p_2}_{\gamma^{p_2}}) < \gamma^{p_2}$;
(ii) $\gamma^{p_4} > \gamma^{p_2} + \omega \cdot \sigma$;
(iii) for every even $j \in [4, \sigma]$,

$$\mathrm{acc}(C^{p_j}_{\gamma^{p_j}}) = \mathrm{acc}(C^{p_2}_{\gamma^{p_2}}) \cup \{\gamma^{p_i} \mid \text{even } i \in [2, j)\};$$

(iv) for every even $i \in [2, \sigma)$, the ordinal β_{i+2} will be equal to $\min(C^{p_{i+2}}_{\gamma^{p_{i+2}}} \setminus (\gamma^{p_i} + 1))$ and it will be the case that $p_{i+2} \Vdash \beta_{i+2} \in \dot{B}$.

Note that the combination of Clauses (i)–(iii) implies the following:

(v) for every even $i \in [2, \sigma]$, $\mathrm{otp}(C^{p_i}_{\gamma^{p_i}}) < \gamma^{p_i}$,

because for every $j \in [4, \sigma]$, $\mathrm{otp}(C^{p_j}_{\gamma^{p_j}}) \leq \gamma^{p_2} + \omega \cdot j < \gamma^{p_4} \leq \gamma^{p_j}$. Additionally, the combination of Clauses (iii) and (iv) implies the following:

(vi) for every even $j \in [4, \sigma)$, $\mathrm{otp}(C^{p_j}_{\gamma^{p_j}} \setminus \beta_j) = \omega$.

We now turn to the description of the strategy for **II**:

- ▶ Start by letting $p_0 := \mathbb{1}_{\mathbb{P}}$.
- ▶ Once p_1 was played, p_2 will be an extension of p_1 satisfying Clause (i);
- ▶ For every even $i \in [2, \sigma)$ such that p_{i+1} has already been defined, we first let q_{i+1} be some extension of p_{i+1} that decides that some ordinal β_{i+2} belongs to $\dot{B} \setminus (\gamma^{p_{i+1}} + \omega \cdot \sigma + 1)$. By Clause (1), we may assume that $\gamma^{q_{i+1}} > \beta_{i+2}$. We then define p_{i+2} as follows:
  - $\gamma^{p_{i+2}} := \gamma^{q_{i+1}} + \omega$;
  - $C^{p_{i+2}}_{\alpha} := C^{q_{i+1}}_{\alpha}$ for every $\alpha \leq \gamma^{q_{i+1}}$;
  - $C^{p_{i+2}}_{\alpha+1} := \{\alpha\}$ for every $\alpha \in \gamma^{p_{i+2}} \setminus \gamma^{q_{i+1}}$;
  - $C^{p_{i+2}}_{\gamma^{p_{i+2}}} := C^{p_i}_{\gamma^{p_i}} \cup \{\gamma^{p_i}, \beta_{i+2}\} \cup \{\gamma^{q_{i+1}} + n \mid n < \omega\}$;
  - $D^{p_{i+2}} := D^{q_{i+1}} \cup \{\gamma^{p_{i+2}}\}$.

  Evidently, $\mathrm{acc}(C^{p_{i+2}}_{\gamma^{p_{i+2}}}) = \mathrm{acc}(C^{p_i}_{\gamma^{p_i}}) \cup \{\gamma^{p_i}\}$, so that requirement (iii) is maintained. Clause (iv) is satisfied by design, and so is Clause (ii) in case we are at stage $i = 2$. To verify that p_{i+2} is a legitimate condition, it suffices to focus on Clause (4)(b) of Definition 3.5, with $\beta := \gamma^{p_{i+2}}$ and some $\alpha \in D^{p_{i+2}} \cap \beta$. That is, to verify that $\mathrm{otp}(C^{p_{i+2}}_{\gamma^{p_{i+2}}} \cap \alpha) < \alpha$ for every $\alpha \in D^{q_{i+1}}$. As $\max(D^{q_{i+1}}) = \gamma^{q_{i+1}}$ and $C^{p_{i+2}}_{\gamma^{p_{i+2}}} \cap \gamma^{q_{i+1}} = C^{p_i}_{\gamma^{p_i}} \cup \{\gamma^{p_i}, \beta_{i+2}\}$, it suffices to verify that $\mathrm{otp}(C^{p_i}_{\gamma^{p_i}} \cap \alpha) < \alpha$ for every $\alpha \in D^{q_{i+1}} \cap (\gamma^{p_i} + 1) = D^{p_i}$. For $\alpha < \gamma^{p_i}$, this follows from the fact that p_i is a legitimate condition. For $\alpha = \gamma^{p_i}$, this follows from Clause (v).
- ▶ Given a $j \in \mathrm{acc}(\sigma + 1)$ such that $\langle p_i \mid i < j\rangle$ has already been determined, define p_j as follows:
  - $\gamma^{p_j} := \sup_{i<j} \gamma^{p_i}$;
  - $C^{p_j}_{\alpha} := C^{p_i}_{\alpha}$ for every $\alpha < \gamma^{p_j}$, using a large enough $i < j$;
  - $C^{p_j}_{\gamma^{p_j}} := \bigcup\{C^{p_i}_{\gamma^{p_i}} \mid \text{even } i \in [2, j)\}$;
  - $D^{p_j} := \bigcup_{i<j} D^{p_i} \cup \{\gamma^{p_j}\}$.

It is clear that requirement (iii) is maintained. To verify that p_j is a legitimate condition, we verify Clause (4)(b) of Definition 3.5 with respect to $\beta := \gamma^{p_j}$ and a given $\alpha \in D^{p_j} \cap \beta$. Find an even $i < j$ such that $\alpha < \gamma^{p_i}$. Then $C^{p_j}_{\gamma^{p_j}} \cap \alpha = C^{p_i}_{\gamma^{p_i}} \cap \alpha$ and the latter has order-type less than α. Altogether, p^j is a legitimate condition extending p^i for all $i < j$.

This completes the presentation of a successful strategy for **II** in the game $\Game_{\sigma+1}(\mathbb{P})$. In addition, $q := p_\sigma$ satisfies that $q \leq p_2$ and all of the following hold:

- $C^{p_2}_{\gamma^{p_2}} \sqsubseteq C^q_{\gamma^q}$;
- $\mathrm{otp}(C^q_{\gamma^q}) = \mathrm{otp}(C^{p_2}_{\gamma^{p_2}}) + \omega \cdot \sigma < \gamma^q$;
- $\{C^q_{\gamma^q}(\mathrm{otp}(C^{p_2}_{\gamma^{p_2}}) + \omega \cdot \iota + 1) \mid \iota < \sigma\}$ is equal to $\{\beta_i \mid \text{even } i \in [4, \sigma)\}$ and is forced by q to be subset of $\dot{B}$.

Finally, Clauses (4) and (5) are standard and are left to the reader. □

Corollary 3.8. *If $\kappa^{<\kappa} = \kappa$, then $\mathbb{P}$ preserves all cardinals.*

Proof. By Clauses (3)–(5) of Lemma 3.7. □

Corollary 3.9. *For every $p \in \mathbb{P}$, for every $\mathbb{P}$-name $\dot{B}$ of a cofinal subset of κ, there exists a $q \leq p$ that forces that the order-type of the intersection of* $\mathrm{nacc}(C^q_{\gamma^q})$ *and $\dot{B}$ is equal to γ^q.*

Proof. Let p and $\dot{B}$ as above be given. Define a sequence $\langle p_n \mid n < \omega \rangle$ by recursion on $n < \omega$, as follows. First, using Lemma 3.7(1), find a $p_0 \leq p$ for which $\mathrm{otp}(C^{p_0}_{\gamma^{p_0}}) < \gamma^{p_0}$. Then, given $n < \omega$ such that p_n has already been defined, appeal to Lemma 3.7(2) to obtain a condition $p_{n+1} \leq p_n$ such that all of the following hold:

(1) $C^{p_n}_{\gamma^{p_n}} \sqsubseteq C^{p_{n+1}}_{\gamma^{p_{n+1}}}$;
(2) $\mathrm{otp}(C^{p_{n+1}}_{\gamma^{p_{n+1}}}) < \gamma^{p_{n+1}}$;
(3) p_{n+1} forces that the intersection of $\mathrm{nacc}(C^{p_{n+1}}_{\gamma^{p_{n+1}}} \setminus \gamma^{p_n})$ and $\dot{B}$ has order-type $\geq \gamma^{p_n}$.

Now, define a condition q as follows:

- $\gamma^q := \sup_{n<\omega} \gamma^{p_n}$;[20]
- $C^q_\alpha := C^{p_n}_\alpha$ for every $\alpha < \gamma^q$, using a large enough $n < \omega$;
- $C^q_{\gamma^q} := \bigcup\{C^{p_n}_{\gamma^{p_n}} \mid n < \omega\}$;
- $D^q := \bigcup_{n<\omega} D^{p_n} \cup \{\gamma^q\}$.

As in the limit case from the proof of the previous Lemma, q is a legitimate condition. By Clauses (2) and (3), q forces that the order-type of the intersection of $\mathrm{nacc}(C^q_{\gamma^q})$ and $\dot{B}$ is equal to γ^q. □

Lemma 3.10. *$\mathbb{P}$ forces that $\kappa^{<\kappa} = \kappa$.*

[20]Note that $\mathrm{cf}(\gamma^q) = \omega$. This is aligned with Remark 3.4.

Proof. Work in $V[H]$ for H a $\mathbb{P}$-generic over V. By Lemma 3.7(1), we may define a sequence $\vec{C} := \langle C_\alpha \mid \alpha < \kappa\rangle$ by letting $C_\alpha := C^p_\alpha$ for some nonempty condition $p \in H$ with $\gamma^p \geq \alpha$. For all $\beta, \gamma < \kappa$, denote

$$X_{\beta,\gamma} := \{\iota < \beta \mid \gamma + \omega \cdot \iota \in C_{\gamma+\omega\cdot\iota+\omega}\}.$$

Claim 3.10.1. *For every $X \in [\kappa]^{<\kappa}$, there are $\beta, \gamma < \kappa$ such that $X = X_{\beta,\gamma}$.*

Proof. By Lemma 3.7(3), V and $V[H]$ have the same bounded subsets of κ. Thus, it suffices to prove that for every condition p in $\mathbb{P}$ and every $X \in [\kappa]^{<\kappa}$ in V, there are a condition $q \leq p$ and ordinals $\beta, \gamma < \kappa$ such that q forces that X coincides with $X_{\beta,\gamma}$. To this end, work in V and let p and X be as above. By possibly extending p, we may assume it is nonempty, say $p = (\langle C^p_\alpha \mid \alpha \leq \gamma^p\rangle, D^p)$. Find a large enough $\beta < \kappa$ such that $X \subseteq \beta$. Define $q = (\langle C^q_\alpha \mid \alpha \leq \gamma^q\rangle, D^q)$ with $\gamma^q := \gamma^p + \omega\cdot\beta$ and $D^q := D^p \cup \{\gamma^q\}$, by letting for every $\alpha \leq \gamma^q$:

$$C^q_\alpha := \begin{cases} C^p_\alpha, & \text{if } \alpha \leq \gamma^p;\\ \{\bar\alpha\}, & \text{if } \alpha > \gamma^p \ \&\ \alpha = \bar\alpha + 1;\\ \operatorname{acc}(\alpha \setminus \gamma^p), & \text{if } \alpha > \gamma^p \ \&\ \alpha \in \operatorname{acc}(\operatorname{acc}(\gamma^q + 1));\\ \{\gamma^p + \omega\cdot\iota + n \mid n < \omega\}, & \text{if } \alpha = \gamma^p + \omega\cdot\iota + \omega \ \&\ \iota \in X;\\ \{\gamma^p + \omega\cdot\iota + n \mid 0 < n < \omega\}, & \text{if } \alpha = \gamma^p + \omega\cdot\iota + \omega \ \&\ \iota \notin X.\end{cases}$$

It is clear that q is a legitimate condition extending p and forcing that X_{β,γ^p} coincides with X. □

It immediately follows that $\kappa^{<\kappa} = \kappa$. □

Corollary 3.11. $\mathrm{P}_<(\kappa, 2, \sqsubseteq, \kappa)$ *holds in the generic extension by* $\mathbb{P}$.

Proof. Work in $V[H]$ for H a $\mathbb{P}$-generic over V. By Lemma 3.7(1), we may define a sequence $\vec{C} := \langle C_\alpha \mid \alpha < \kappa\rangle$ by letting $C_\alpha := C^p_\alpha$ for some nonempty condition $p \in H$ with $\gamma^p \geq \alpha$. Also note that $D := \bigcup\{D^p \mid p \in H\}$ is a club in κ. Clearly, $\vec{C}$ is a coherent C-sequence such that $R(\vec{C})$ covers D. For every $\alpha \in \operatorname{acc}(\kappa)$, let $X_\alpha := \{C_\alpha(\omega\cdot\iota + 1) \mid \iota < \operatorname{otp}(\operatorname{acc}(C_\alpha))\}$.[21]

Let $\sigma \in \operatorname{Reg}(\kappa)$. By Clause (2) of Lemma 3.7, for every cofinal $B \subseteq \kappa$, there exists an $\alpha \in E^\kappa_\sigma$ such that X_α is a cofinal subset of α and $\sup(X_\alpha \setminus B) < \alpha$. Together with $\kappa^{<\kappa} = \kappa$ (that we obtain from Lemma 3.10), this easily implies that $\diamondsuit(E^\kappa_\sigma)$ holds.[22] In particular, we may let $\Phi^\omega : \mathcal{K}(\kappa) \to \mathcal{K}(\kappa)$ be the postprocessing function given by [LHR19, Lemma 3.8]. Define a C-sequence $\vec{C^\bullet} := \langle C^\bullet_\alpha \mid \alpha < \kappa\rangle$ by letting $C^\bullet_0 := \emptyset$, $C^\bullet_{\beta+1} := \{\beta\}$ for every $\beta < \kappa$ and $C^\bullet_\alpha := \Phi^\omega(C_\alpha)$ for every $\alpha \in \operatorname{acc}(\kappa)$. Then $\vec{C^\bullet}$ is a coherent C-sequence such that $R(\vec{C^\bullet}) \supseteq R(\vec{C}) \supseteq D$. Thus, we are left with verifying that $\vec{C^\bullet}$ satisfies Clause (3) of Definition 3.3. To this end, let $\vec{B} = \langle B_i \mid$

[21] Here, $C_\alpha(\xi)$ stands for the unique $\beta \in C_\alpha$ satisfying $\operatorname{otp}(C_\alpha \cap \beta) = \xi$.

[22] See the proof of [RYY26, Claim 6.6.2].

$i < \kappa\rangle$ be a sequence of cofinal subsets of κ, and let E be a club in κ, and we shall find a $\gamma \in E$ such that $\sup(\mathrm{nacc}(C^\bullet_\gamma) \cap B_i) = \gamma$ for every $i < \gamma$.

Let G be the stationary set given by [LHR19, Lemma 3.8] with respect to our sequence $\vec{B}$. As $G \cap E$ is in particular a cofinal subset of κ, by Corollary 3.9, we may fix a $\gamma \in \mathrm{acc}(\kappa)$ such that $\mathrm{otp}(\mathrm{nacc}(C_\gamma) \cap G \cap E) = \gamma$. Clearly, $\gamma \in E$ and $\mathrm{otp}(\mathrm{nacc}(C_\gamma) \cap G) = \gamma$. By Clause (2) of [LHR19, Lemma 3.8], the latter implies that $\sup(\mathrm{nacc}(C^\bullet_\gamma) \cap B_i) = \gamma$ for every $i < \gamma$, as sought. □

Question 3.12. Suppose that $\mathsf{V} = \mathsf{L}$ and κ is an inaccessible that is non-Mahlo. Does $\mathrm{P}_<(\kappa, 2, \sqsubseteq, \kappa)$ hold?

An affirmative answer may possibly follow from the arguments of [BR19b, §4].

4. Clause (2) of Theorem B

Throughout this section, κ stands for a regular uncountable cardinal.

All necessary background on set-theoretic trees may be found in standard textbooks, as well as at [BR17, §2]. In addition, for our purpose, we shall need the following definition.

Definition 4.1 (Todorčević, [Tod87, p. 266])**.** A tree $(T, <_T)$ of height κ is *special* iff there exists a map $g : T \to T$ satisfying the following:

- for every non-minimal $x \in T$, $g(x) <_T x$;
- for every $y \in T$, $g^{-1}\{y\}$ is covered by less than κ many antichains.

Remark 4.2. In case that $\kappa = \lambda^+$ is a successor cardinal, a tree of height κ is special (in the above sense) iff it may be covered by λ many antichains [Tod85, Theorem 14].

Definition 4.3 ([Tod84, p. 273], [ASS87, Convention 4.1(4)])**.** Let $\mathbb{Q}_\kappa$ denote the linear order consisting of all nonempty finite sequences of ordinals in κ, with the ordering $q <_{\mathbb{Q}_\kappa} p$ iff either $p \subsetneq q$ or $q(n) < p(n)$ for the least $n < \omega$ such that $q(n) \neq p(n)$.

The next easy fact is an immediate corollary to a claim from [IR25, §2], but we give the details here for completeness.

Proposition 4.4. *Suppose that $\mathbf{T} = (T, <_T)$ is a tree of height κ and there exists a map $f : T \to \mathbb{Q}_\kappa$ satisfying the following two requirements:*

1. *f is strictly increasing, i.e., for all $x <_T x'$ in T, $f(x) <_{\mathbb{Q}_\kappa} f(x')$;*
2. *the set $\{\alpha < \kappa \mid f[T_\alpha] \subseteq {}^{<\omega}\alpha\}$ covers a club.*

Then $\mathbf{T}$ is special.

Proof. Fix a club $C \subseteq \kappa$ such that $f[T_\alpha] \subseteq {}^{<\omega}\alpha$ for every $\alpha \in C$. Fix a bijection $\pi : {}^{<\omega}\kappa \leftrightarrow \kappa$, and consider the club $D := \{\alpha \in \mathrm{acc}(\kappa) \cap C \mid \pi[{}^{<\omega}\alpha] = \alpha\}$. Clearly, $\pi(f(x)) < \mathrm{ht}(x)$ for every $x \in T$ with $\mathrm{ht}(x) \in D$.

For all $\alpha < \kappa$ and $x \in T$ with $\mathrm{ht}(x) > \alpha$, let $x \restriction \alpha$ denote the unique $y <_T x$ with $\mathrm{ht}(y) = \alpha$. Define a map $g : T \rightarrow T$, as follows:

$$g(x) := \begin{cases} x \restriction (\pi(f(x)) + 1), & \text{if } \mathrm{ht}(x) \in D; \\ x \restriction \sup(D \cap \mathrm{ht}(x)), & \text{otherwise.} \end{cases}$$

It is clear that $g(x) = x$ for every minimal $x \in T$ and that $g(x) <_T x$ for every non-minimal $x \in T$. Now, given $y \in T$, there are two cases to consider:

▶ If $\mathrm{ht}(y)$ is a successor ordinal, then for the unique $p \in {}^{<\omega}\kappa$ such that $\mathrm{ht}(y) = \pi(p) + 1$, for all $x, x' \in g^{-1}\{y\}$, we have $f(x) = p = f(x')$ and hence x, x' are incomparable. That is, $(g^{-1}\{y\}, <_T)$ is an antichain.

▶ Otherwise, $\epsilon := \mathrm{ht}(y)$ is a (possibly zero) limit ordinal, and then $g^{-1}\{y\}$ is a subset of $\{x \in T \mid \mathrm{ht}(x) < \min(D \setminus (\epsilon + 1))\}$, which is a set consisting of fewer than κ many levels of T. In particular, $g^{-1}\{y\}$ is the union of less than κ many antichains of $(T, <_T)$. □

Definition 4.5 ([BR21, Definition 2.3])**.** A set T is a *streamlined tree* iff there exists some cardinal θ such that $T \subseteq {}^{<\theta}H_\theta$ and, for all $t \in T$ and $\beta < \mathrm{dom}(t)$, $t \restriction \beta \in T$.

We identify a streamlined tree T with the abstract set-theoretic tree $(T, \subsetneq)$.

Definition 4.6. For a subset $T \subseteq {}^{<\kappa}H_\kappa$ and a positive integer n, denote $T^n := \{\vec{x} : n \xrightarrow{1-1} T \mid i \mapsto \mathrm{dom}(\vec{x}(i)) \text{ is constant}\}$. The ordering $<_{T^n}$ of T^n is defined as follows:

$$\vec{x} <_{T^n} \vec{y} \iff \bigwedge_{i<n} \vec{x}(i) \subsetneq \vec{y}(i).$$

Note that the sequences in T^n are injective. This requirement was not imposed back in Section 2 (see Definition 2.5) because it is inessential in the context in which the main notion of a square is $(X_T)^2$, and in that section we opted to prefer simplicity over generality.

Definition 4.7 (Derived trees)**.** For a streamlined tree T and a positive integer n, an *n-derived tree of T* is a collection of the form

$$T(w_0, \ldots, w_{n-1}) := \{(x_0, \ldots, x_{n-1}) \in T^n \mid \forall i < n\, (x_i \cup w_i \in T)\},$$

for some node $(w_0, \ldots, w_{n-1}) \in T^n$.

Definition 4.8. A streamlined κ-tree T is $(n+1)$-*free* iff all of its n-derived trees are κ-Souslin.

Remark 4.9. $(T^n, <_{T^n})$ is the union of all n-derived trees of T, and can be thought of as the n^{th} power of T. In particular, if $(T^n, <_{T^n})$ admits a map as in Proposition 4.4, then all n-derived trees of T are special. By focusing on the n-derived trees, rather than the full n^{th} power of T, we avoid the obvious obstacles to the desired trees' being either Souslin or special. By contrasting the n-derived trees of a streamlined tree T with its $(n+1)$-derived trees, the upcoming Theorem 4.10 realizes the optimal tension between freeness and having a special power.

A forcing construction of an n-free $\aleph_1$-Souslin tree all of whose derived trees of dimension n are special was given in [Kru20, Corollary 5.5]. A uniform combinatorial construction of an n-free λ^+-Souslin tree T such that $(T^n, <_{T^n})$ admits a strictly increasing map to $\mathbb{Q}_\lambda$ was given in [BRY25, §6], assuming $\mathrm{P}_\lambda(\lambda^+, 2, \sqsubseteq, \lambda^+)$. By [BR17, Theorem 5.1(2)], in the special case of $\lambda = \aleph_0$, this proxy hypothesis is equivalent to $\diamondsuit(\omega_1)$, hence the said construction verified the statement from [SF10, p. 130] that it "seems quite plausible that one can construct an n-self-specializing tree under $\diamondsuit$".

The construction given in [BRY25, §6] is abstract enough to generalize to limit cardinals κ assuming $\mathrm{P}_\kappa(\kappa, 2, \sqsubseteq, \kappa)$, where the map now goes to $\mathbb{Q}_\kappa$. In view of Proposition 4.4, the challenge remaining is to ensure that Clause (2) of the proposition will hold for this map. This naturally depends on the very C-sequence used in the construction, and this is where Clause (1) of Definition 3.3 demonstrates its utility.

Theorem 4.10. *Suppose that $\mathrm{P}_<(\kappa, 2, \sqsubseteq, \kappa)$ holds, and $\chi \in [2, \omega)$. Then there exists a χ-free, streamlined κ-Souslin tree T and strictly increasing $f : T^\chi \to \mathbb{Q}_\kappa$ such that $f[(T_\alpha)^\chi] \subseteq {}^{<\omega}\alpha$ for club many $\alpha < \kappa$. In addition, T is slim, prolific and club-regressive.*[23]

Proof. Let $\vec{C} = \langle C_\alpha \mid \alpha < \kappa\rangle$ be a C-sequence satisfying Clauses (1)–(3) of Definition 3.3. Derive $\vec{D} = \langle D_\alpha \mid \alpha \in \mathrm{acc}(\kappa)\rangle$ from $\vec{C}$ as in the beginning of the proof of [BRY25, Theorem 6.11] and note that $\vec{D}$ is a coherent C-sequence with $R(\vec{D}) = R(\vec{C})$. Running the exact same abstract proof of [BRY25, Theorem 6.11], modulo replacing λ by κ throughout, we obtain a χ-free, slim, prolific, club-regressive, streamlined κ-Souslin tree T and a strictly increasing map $f : T^\chi \to \mathbb{Q}_\kappa$. Thus, we are left with verifying that $f[(T_\alpha)^\chi] \subseteq {}^{<\omega}\alpha$ for club many $\alpha < \kappa$.

As χ is finite and T is a κ-tree, we may fix a club $E \subseteq \kappa$ such that, for every $\alpha \in E$ and every $\gamma < \alpha$, $f[(T_\gamma)^\chi] \subseteq {}^{<\omega}\alpha$. As $R(\vec{D})$ covers a club, we may shrink E and also assume it is a subset of $R(\vec{D})$ and that all ordinals in E are indecomposable, that is, $\beta + \gamma < \alpha$ for all $\alpha \in E$ and all $\beta, \gamma < \alpha$.

Now, given $\alpha \in E$ and $\vec{w} = \langle w_i \mid i < \chi\rangle$ in $(T_\alpha)^\chi$, we remind the reader that the definition of $f_\alpha(\vec{w})$ (right after [BRY25, Claim 6.11.6]) went as follows. First, for each $i < \chi$, we found an $x_i \in T \restriction D_\alpha$ of minimal height such that $\mathbf{b}^\alpha_{x_i} = w_i$. Then we set $\gamma := \sup\{\mathrm{dom}(x_i) \mid i < \chi\}$, and then we denoted $(\varphi_1 \circ f_\gamma)(\langle w_i \restriction \gamma \mid i < \chi\rangle)$ by $p^\frown\langle\xi\rangle$. Finally, as χ is an integer, it follows that $\gamma < \alpha$. Thus, there are two cases to consider:

- ▶ If $\alpha \in A(\vec{D})$, then $f_\alpha(\vec{w}) = p$ which is an element of ${}^{<\omega}\alpha$, since $\gamma < \alpha$ and $\alpha \in E$.

[23]The definitions of all additional properties may found at [BRY25, §4.2].

▶ Otherwise, $f_\alpha(\vec{w}) = p^\frown\langle\xi+\sigma\rangle$, where $\sigma := \operatorname{otp}(D_\alpha \setminus (\gamma+1))$. As $\alpha \in R(\vec{C}) \setminus A(\vec{C})$ and since $\vec{D}$ is coherent, we infer that $\operatorname{otp}(D_\alpha) < \alpha$. In addition, since $\alpha \in E$ and $\gamma < \alpha$, $p^\frown\langle\xi\rangle$ is in ${}^{<\omega}\alpha$. Altogether, $p^\frown\langle\xi+\sigma\rangle$ is an element of ${}^{<\omega}\alpha$. □

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