PARTITION RELATIONS FOR TREES I: INCOMPARABLE TREES

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Abstract. Todorčević proved that Martin's axiom $\mathsf{MA}_{\aleph_1}$ implies that every two coherent $\aleph_1$-Aronszajn trees are comparable. Here, from cardinal arithmetic assumptions, we obtain the failure of the analogous statement for higher trees. In particular, for every $n<\omega$, assuming $2^{\aleph_n}=\aleph_{n+1}$, there is a family of $\aleph_{n+2}$-many pairwise incomparable $\aleph_{n+1}$-coherent $\aleph_{n+1}$-complete $\aleph_{n+2}$-Aronszajn trees. The proof uses an anti-Ramsey colourings for trees recently introduced by the authors.

1. Introduction

Given two trees $\mathbf{S}=(S,<_S)$ and $\mathbf{T}=(T,<_T)$, a map $g:S\rightarrow T$ is a *weak embedding from* $\mathbf{S}$ *to* $\mathbf{T}$ iff for all $s,s'\in S$, $s<_S s'$ implies $g(s)<_T g(s')$.[1] A map g from a subset of S to T is *level-preserving* iff for all $s\in\operatorname{dom}(g)$, $\operatorname{ht}_{\mathbf{S}}(s)=\operatorname{ht}_{\mathbf{T}}(g(s))$. A map g from a subset of S to T is *Lipschitz* iff it is level-preserving and for all $s,s'\in\operatorname{dom}(g)$, $\operatorname{ht}_{\mathbf{S}}(s\wedge_{\mathbf{S}}s')\leq\operatorname{ht}_{\mathbf{T}}(g(s)\wedge_{\mathbf{T}}g(s'))$.

Two trees are *comparable* iff there exists a weak embedding from one of the trees to the other one. This note is motivated by the following theorem of Todorčević (see [Tod07a, Theorem 5.4] or [Tod07b, Theorem 4.3.10]).

Theorem (Todorčević)**.** *Martin's axiom* $\mathsf{MA}_{\aleph_1}$ *implies that every two coherent* $\aleph_1$*-Aronszajn trees are* comparable*.*

Our goal here is to obtain the failure of higher analogs of Todorčević's theorem from hypotheses that are conceivably compatible with potential higher forcing axioms.

Theorem A. *For every* $n<\omega$*, if* $2^{\aleph_n}=\aleph_{n+1}$*, then there is a family of* $\aleph_{n+2}$*-many pairwise incomparable* $\aleph_{n+1}$*-coherent* $\aleph_{n+1}$*-complete* $\aleph_{n+2}$*-Aronszajn trees. Furthermore, no two trees from the family admit a Lipschitz map from an* $\aleph_{n+2}$*-sized subset of one to the other.*

The point is that the existence of a weak embedding between trees gives rise to one that is also level-preserving, and that level-preserving weak embeddings are Lipschitz.

Date: Preprint as of April 30, 2026. For updates, visit http://p.assafrinot.com/73.

[1] Unlike usual embeddings that are required to be injective, weak embeddings may be constant on antichains.

Theorem A follows from the following more general result, since any successor cardinal (in fact, any regular uncountable cardinal that is not greatly Mahlo) admits a *nontrivial C-sequence*.

Theorem B. *Suppose that $\lambda = \lambda^{<\lambda}$ is an uncountable cardinal, and any of the following:*

- $\square(\lambda^+, {<}\lambda)$ *holds and* $\theta = \lambda^+$*;*
- λ *admits a nontrivial C-sequence and* $\theta = \lambda$*.*

Then there is a family of 2^θ many streamlined λ-coherent λ-complete λ^+-Aronszajn trees such that no two of them admit a Lipschitz map from a λ^+-sized subset of one to the other.

The proofs go through obtaining instances of the negative partition relations for trees introduced in [IR25], and reproduced in Definition 2.9 below.

2. Preliminaries

2.1. **Conventions.** Throughout this paper, κ stands for a regular uncountable cardinal, and $\theta, \lambda, \chi, \mu, \nu$ denote arbitrary cardinals. E^κ_θ stands for $\{\alpha < \kappa \mid \mathrm{cf}(\alpha) = \theta\}$, $\mathrm{acc}(\kappa)$ stands for $\{\alpha < \kappa \mid \sup(\alpha) = \alpha > 0\}$, and H_κ stands for the collection of all sets of hereditary cardinality less than κ. Two functions f, g from the ordinals to the universe are said to be *comparable* iff $f \subseteq g$ or $g \subseteq f$; we also let

$$\Delta(f, g) := \min\{\mathrm{dom}(f), \mathrm{dom}(g), \delta \mid \delta \in \mathrm{dom}(f) \cap \mathrm{dom}(g) \ \&\ f(\delta) \neq g(\delta)\},$$

so that f and g are incomparable iff $\Delta(f, g) < \min\{\mathrm{dom}(f), \mathrm{dom}(g)\}$.

2.2. **Trees.** For preliminaries on trees, see the introduction of [BR17]. Given two distinct nodes x, y of a Hausdorff tree $\mathbf{T} = (T, <_T)$, we let $x \wedge y$ denote the highest node z of $\mathbf{T}$ to satisfy $z \leq_T x$ and $z \leq_T y$. In particular, $x \wedge y \in \{x, y\}$ iff x and y are comparable nodes of $\mathbf{T}$.

Definition 2.1. A *streamlined tree* is a collection T of functions from ordinals to the universe that is closed under taking initial segments.

Convention 2.2. We identify a streamlined tree T with the Hausdorff tree $(T, \subsetneq)$; in this case, $\mathrm{ht}(t) = \mathrm{dom}(t)$ for every $t \in T$.

Note that in the context of streamlined trees, a partial function $g : S \to T$ is Lipschitz iff it is dom-preserving and for all $s, s' \in \mathrm{dom}(g)$, $\Delta(s, s') \leq \Delta(g(s), g(s'))$.

Definition 2.3 (Default lexicographic ordering)**.** For a subset $T \subseteq {}^{<\kappa}H_\kappa$, fix $\prec$ such that $(H_\kappa, \prec)$ is a well-order extending $(\kappa, \in)$, and then define a linear ordering $<_{\mathrm{lex}}$ of T, as follows:

- ▶ If f and g are comparable, then $f <_{\mathrm{lex}} g$ iff $g \subseteq f$;
- ▶ Otherwise, let $\delta := \Delta(f, g)$, and then $f <_{\mathrm{lex}} g$ iff $f(\delta) \prec g(\delta)$.

Remark 2.4. If $T \subseteq {}^{<\kappa}H_\kappa$ is a streamlined κ-Aronszajn tree then $(T, <_{\text{lex}})$ is a κ-*Aronszajn line*, i.e., a κ-sized linear order that contains no κ-sized linear order of density less than κ, no copy of $(\kappa, \in)$ nor a copy of $(\kappa, \ni)$.

Definition 2.5. A streamlined tree T is λ-*coherent* iff $|\{\beta \in \operatorname{dom}(t_0) \cap \operatorname{dom}(t_1) \mid t_0(\beta) \neq t_1(\beta)\}| < \lambda$ for all $t_0, t_1 \in T$.

Fact 2.6 (essentially Specker, [Spe49])**.** *For every cardinal $\lambda = \lambda^{<\lambda}$, there exists a streamlined λ-coherent, λ-complete λ^+-Aronszajn tree $T \subseteq {}^{<\lambda^+}\lambda$.*

Fact 2.7 (König, [Kön03, Theorem 3.9])**.** *If $\square(\kappa)$ holds, then there exists a streamlined coherent (that is, $\aleph_0$-coherent) κ-Aronszajn tree $T \subseteq {}^{<\kappa}2$.*

2.3. Partition relation for trees.

Definition 2.8 (Rapid families)**.** For a family S of sequences, and a map $h : \bigcup_{x \in S} \operatorname{Im}(x) \to \operatorname{Ord}$, we say that S is h-*rapid* iff for all $x \neq y$ in S, either $\sup(h[\operatorname{Im}(x)]) < \min(h[\operatorname{Im}(y)])$ or $\sup(h[\operatorname{Im}(y)]) < \min(h[\operatorname{Im}(x)])$.

The next definition was introduced in [IR25], where sufficient conditions for instances of it to hold were obtained, and applications were presented.

Definition 2.9. For a structure $\mathbf{T} = (T, <, \lhd, h)$ such that $(T, <)$ is a Hausdorff tree, $(T, \lhd)$ is a partially ordered set, and $h : T \to \operatorname{Ord}$ is a map, we define three partition relations, as follows.

- (I) *weak* $\mathbf{T} \overset{\wedge}{\nrightarrow} [\kappa]^{<\nu,<\mu}_\theta$ asserts the existence of a colouring $c : T \to \theta$ satisfying that for every nonzero $\sigma < \nu$, for every h-rapid $S \subseteq {}^\sigma T$ of size κ, for every $\tau < \theta$, there are $x \neq y$ in S such that the following three hold:
  - (a) for every $i < \sigma$, $c(x(i) \wedge y(i)) = \tau$;
  - (b) for every $i < \sigma$, if $x(i) \lhd y(i)$ then $h(x(i)) < h(y(i))$;
  - (c) $|\{h(x(i) \wedge y(i)) \mid i < \sigma\}| < \mu$.
- (II) $\mathbf{T} \overset{\wedge}{\nrightarrow} [\kappa]^{<\nu,<\mu}_\theta$ asserts the existence of a colouring $c : T \to \theta$ witnessing weak $\mathbf{T} \overset{\wedge}{\nrightarrow} [\kappa]^{<\nu,<\mu}_\theta$ in addition to satisfying the following. For every $T' \in [T]^\kappa$, for every nonzero $\sigma < \nu$, there exists an h-rapid $S' \subseteq {}^\sigma T'$ of size κ such that for every $S \subseteq S'$ of size κ, and every $\tau \in {}^\sigma\theta$, there are $x \neq y$ in S such that the following three hold:
  - (a) for every $i < \sigma$, $c(x(i) \wedge y(i)) = \tau(i)$;
  - (b) same as Clause (b) above;
  - (c) same as Clause (c) above.
- (III) $\mathbf{T} \overset{\Delta}{\nrightarrow} [\kappa]^{<\nu,<\mu}_\theta$ asserts the existence of a colouring $d : \operatorname{Im}(h) \to \theta$ satisfying following. For every nonzero $\sigma < \nu$, for every h-rapid $S \subseteq {}^\sigma T$ of size κ, for every $\tau < \theta$, there are $x \neq y$ in S such that the following three hold:
  - (a) for every $i < \sigma$, $d(h(x(i) \wedge y(i))) = \tau$;
  - (b) same as Clause (b) above;
  - (c) same as Clause (c) above.

Convention 2.10. We may replace the quadruple $\mathbf{T}$ by a triple $(T,<,\lhd)$, and then we mean $\mathbf{T}:=(T,<,\lhd,\mathrm{ht})$. We may replace $\mathbf{T}$ by a streamlined tree T, and then we mean $\mathbf{T}:=(T,\subsetneq,<_{\mathrm{lex}},\mathrm{dom})$. We may replace the superscript $<\nu$ by ν, and then we mean that σ is equal to ν. We may replace the superscript $<\mu$ by μ, in which case Clause (c) becomes "$|\{h(x(i)\wedge y(i))\mid i<\sigma\}|\leq\mu$". We may omit μ, in which case requirement (c) is waived.

The utility of the parameter μ in the partition relations of Definition 2.9 will be demonstrated in Section 4 below. It is clear that the partition relation of Clause (III) implies that of Clause (I). We now give a sufficient condition for Clause (I) to be no weaker than Clause (II).

Lemma 2.11. *Suppose that weak* $\mathbf{T}\overset{\wedge}{\nrightarrow}[\kappa]^{<\nu,<\mu}_{\theta}$ *holds, where:*

- κ *is a regular uncountable cardinal;*
- $\nu\in[2,\kappa)$ *is a cardinal such that* $\lambda^{<\nu}<\kappa$ *for every* $\lambda<\kappa$*;*
- $\mathbf{T}=(T,<,\lhd,h)$ *and* h *is* $(<\kappa)$*-to-one.*

All of the following hold:

1. *If* $\theta>1$*, then* $(T,<)$ *has no chains of size* κ*;*
2. *If* $\theta=\kappa$ *and* $(T,<)$ *is a* κ*-tree, then* $\mathbf{T}\overset{\wedge}{\nrightarrow}[\kappa]^{<\nu,<\mu}_{\theta}$ *holds.*

Proof. (1) Assuming $\theta>1$, by monotonicity, we may fix a colouring $c:T\to 2$ witnessing weak $\mathbf{T}\overset{\wedge}{\nrightarrow}[\kappa]^1_2$. Now, if $(T,<)$ were to have a chain B of size κ, then we could find some $C\in[B]^\kappa$ on which C is constant with value, say, j. As h is $(<\kappa)$-to-one, we may find an $S\subseteq C$ of size κ on which h is injective. So, S is h-rapid, and then, by the choice of c, there are $x\neq y$ in S such that $c(x\wedge y)=1-j$. However, $x\wedge y\in\{x,y\}\subseteq C$ and hence $c(x\wedge y)=j$. This is a contradiction.

(2) Suppose that $(T,<)$ is a κ-tree and that $c:T\to\kappa$ is a colouring witnessing weak $\mathbf{T}\overset{\wedge}{\nrightarrow}[\kappa]^{<\nu,<\mu}_{\kappa}$. By Clause (1), $(T,<)$ is a κ-Aronszajn tree. We shall need the following direct consequence of this fact.

Claim 2.11.1. *For every* $T'\in[T]^\kappa$*, for every* $\sigma<\nu$*, there exists an* $\alpha<\kappa$ *for which* $Z_\alpha:=\{z\in T_\alpha\mid|\{t\in T'\mid z<t\}|=\kappa\}$ *satisfies* $|Z_\alpha|\geq|\sigma|$*.*

Proof. Suppose not, and let T' and σ be a counterexample. For every $z\in T$ and $\alpha<\mathrm{ht}(z)$, let $z\restriction\alpha$ denote the unique $y\in T_\alpha$ to satisfy $y<z$. By transitivity, for all $\alpha<\beta<\kappa$ and $z\in Z_\beta$, we have $z\restriction\alpha\in Z_\alpha$.

Fix a regular cardinal $\chi\in[\sigma,\nu]$. As $(T,<)$ is a Hausdorff tree, for each $\alpha\in E^\kappa_\chi$, we may fix a large enough $\delta_\alpha<\alpha$ such that $z\mapsto z\restriction\delta_\alpha$ is injective over Z_α. As $(T,<)$ is a κ-tree and $\lambda^{<\nu}<\kappa$ for every $\lambda<\kappa$, we may fix some $\delta<\kappa$ and a subset $Y\in[T_\delta]^{<\sigma}$ for which the following set is stationary:

$$A:=\{\alpha\in E^\kappa_\chi\mid\{z\restriction\delta_\alpha\mid z\in Z_\alpha\}=Y\}.$$

Fix $y\in Y$. Then for every $\alpha\in A$, there is a unique $z_\alpha\in Z_\alpha$ such that $y<z_\alpha$. Let $\alpha<\beta$ be a pair of ordinals from A. As $z_\beta\in Z_\beta$, we have that $z_\beta\restriction\alpha\in Z_\alpha$ and it extends y, so that $z_\beta\restriction\alpha=z_\alpha$ necessarily. So $\{z_\alpha\mid\alpha\in A\}$ is a κ-sized chain in T. This is a contradiction. □

Fix a bijection $\pi : \kappa \leftrightarrow \bigcup\{{}^{Z}\kappa \mid Z \in [T]^{<\nu}, Z \neq \emptyset\}$, and define a colouring $c' : T \to \kappa$ as follows. Given $t \in T$, first denote $\pi(c(t))$ by $g : Z \to \kappa$. Then, if there exists a unique $z \in Z$ such that $z < t$, then let $c'(t) := g(z)$ for this z. Otherwise, let $c'(t) := \min(\mathrm{Im}(g))$. We verify that c' witnesses $\mathbf{T} \overset{\wedge}{\not\to} [\kappa]^{<\nu,<\mu}_{\kappa}$.

(I) Given $\sigma < \nu$, an h-rapid $S \subseteq {}^{\sigma}T$ of size κ, and a prescribed colour $\tau < \kappa$, fix an arbitrary $z \in T$, and consider the constant function $g : \{z\} \to \{\tau\}$. Next, by the choice of c, fix $x \neq y$ in S such that the following three hold:
  - for every $i < \sigma$, $c(x(i) \wedge y(i)) = \pi^{-1}(g)$;
  - for every $i < \sigma$, if $x(i) \lhd y(i)$ then $h(x(i)) < h(y(i))$;
  - $|\{h(x(i) \wedge y(i)) \mid i < \sigma\}| < \mu$.

  It is clear that for every $i < \sigma$, $c'(x(i) \wedge y(i)) = \tau$.

(II) Given $\sigma < \nu$ and $T' \in [T]^{\kappa}$, since T is a κ-tree, by appealing to Claim 2.11.1, we find some $\alpha < \kappa$ and an injective $z \in {}^{\sigma}(T_{\alpha})$ such that $|\{t \in T' \mid z(i) \subseteq t\}| = \kappa$ for every $i < \sigma$. As h is $(<\kappa)$-to-one, it follows that we may find an h-rapid $S' \subseteq {}^{n}T'$ of size κ satisfying that for every $x \in S$, it is the case that $z(i) < x(i)$ for every $i < \sigma$. Finally, given $S \subseteq S'$ of size κ, and $\tau \in {}^{\sigma}\kappa$, define $g : \{z(i) \mid i < \sigma\} \to \kappa$ via $g(z(i)) := \tau(i)$. By the choice of c, fix $x \neq y$ in S such that the following three hold:
  - for every $i < \sigma$, $c(x(i) \wedge y(i)) = \pi^{-1}(g)$;
  - for every $i < \sigma$, if $x(i) \lhd y(i)$ then $h(x(i)) < h(y(i))$;
  - $|\{h(x(i) \wedge y(i)) \mid i < \sigma\}| < \mu$.

  It is clear that for every $i < \sigma$, $c'(x(i) \wedge y(i)) = \tau(i)$. □

3. Instances of the partition relation

We start with a fact announced without a proof in [IR25]:

Proposition 3.1. *Suppose that T is a streamlined ν-free κ-Souslin tree. Then $T \overset{\Delta}{\not\to} [\kappa]^{<\nu,1}_{\kappa}$ holds.*

Proof. First, to avoid superficial complications, and as there are less than κ many nodes in T with fewer than κ many extensions in T, we shall identify T with a *prolific* tree, i.e., a tree satisfying that for every $\alpha < \kappa$, for every $t \in T_{\alpha}$, for every $\tau < \alpha$, $t^{\frown}\langle\tau\rangle$ is in T (see [RS23, Proposition 2.16]).

Fix a surjection $d : \kappa \to \kappa$ that satisfies $|\{\delta < \kappa \mid d(\delta) = \tau\}| = \kappa$ for every $\tau < \kappa$. To see that this works, fix a nonzero $\sigma < \nu$, a ht-rapid $S \subseteq {}^{\sigma}T$ of size κ, and a prescribed colour $\tau < \kappa$. Fix a regular cardinal $\lambda \in [\sigma, \nu]$. For each $\alpha \in E^{\kappa}_{\lambda}$, we do the following:

- fix $\langle x^{\alpha}_i \mid i < \sigma\rangle$ in S such that $\mathrm{ht}(x^{\alpha}_i) > \alpha$ for every $i < \sigma$;
- let $I_{\alpha} \subseteq \sigma$ be a maximal set on which $i \mapsto x^{\alpha}_i \restriction \alpha$ is injective;
- let $\epsilon_{\alpha} < \alpha$ be large enough such that $i \mapsto x^{\alpha}_i \restriction \epsilon_{\alpha}$ is injective over I_{α}.

Since T is ν-free, it follows from [BR17, Lemma 7.7(1)] that $\mu^{<\nu}<\kappa$ for every $\mu<\kappa$. So since T is a κ-tree we may fix a stationary $U\subseteq E^\kappa_\lambda$, $\epsilon<\kappa$, $I\subseteq\sigma$ and $\langle x_i\mid i\in I\rangle\in {}^I(T_\epsilon)$ such that:

(1) for every $\alpha\in U$, $\langle x^\alpha_i\restriction\epsilon_\alpha\mid i\in I_\alpha\rangle=\langle x_i\mid i\in I\rangle$;
(2) for all $\alpha,\beta\in U$, $i\in I$ and $j<\sigma$, $x^\alpha_i\restriction\alpha\subseteq x^\alpha_j$ iff $x^\beta_i\restriction\alpha\subseteq x^\beta_j$;
(3) for every pair $\alpha<\beta$ of ordinals from U, $\sup\{\mathrm{ht}(x^\alpha_i)\mid i<\sigma\}<\beta$.

As $|I|\le\sigma<\nu$ and T is ν-free, $T^*:=\bigotimes_{i\in I}x_i^\uparrow$ is a κ-Souslin tree (recall [BR17, Definition 1.3]). As $S':=\{\langle x^\alpha_i\restriction\alpha\mid i\in I\rangle\mid\alpha\in U\}$ is a κ-sized subset of T^*, a standard fact (see [RS23, Lemma 2.14]) yields a node $\langle t_i\mid i\in I\rangle$ in T^* that has κ many extensions in T^* and such that every extension of it in T^* is extended by some node from S'. By possibly passing to an extension of $\langle t_i\mid i\in I\rangle$ in T^*, we may assume that $\delta:=\mathrm{dom}(t_0)$ satisfies $d(\delta)=\tau$. Fix an $\alpha\in U$ such that $x^\alpha_i\restriction\alpha$ extends $t_i{}^\frown\langle0\rangle$ for every $i\in I$, and then fix a $\beta\in U\setminus\alpha$ such that $x^\beta_i\restriction\beta$ extends $t_i{}^\frown\langle1\rangle$ for every $i\in I$. For every $j<\sigma$, by Clause (2), there exists some $i\in I$ such that $d(\mathrm{ht}(x^\alpha_j\wedge x^\beta_j))=d(\mathrm{ht}(x^\alpha_i\wedge x^\beta_i))=d(\delta)=\tau$, and

$$x^\alpha_j(\delta)=(t_i{}^\frown\langle0\rangle)(\delta)=0<1=(t_i{}^\frown\langle1\rangle)(\delta)=x^\beta_j(\delta),$$

so that $x^\alpha_j<_{\mathrm{lex}}x^\beta_j$, and by Clause (3), $\mathrm{ht}(x^\alpha_j)<\mathrm{ht}(x^\beta_j)$. □

Proposition 3.2. *Suppose:*

- $\nu\le\lambda<\kappa$ *are infinite regular cardinals such that* $\mu^{<\nu}<\kappa$ *for every* $\mu<\kappa$*;*
- T *is a streamlined* λ*-coherent* κ*-Souslin tree.*

Then $T\overset{\Delta}{\not\to}[\kappa]^{<\nu,1}_\kappa$ *holds.*

Proof. By a standard fact (see [BR17, Lemma 2.4]), the restriction of T to some club is normal and splitting. For notational simplicity, and in order to avoid relativizing all of our calculations to such a club, we hereafter assume that every node of T admits κ many successors and at least two immediate successors. Fix a surjection $d:\kappa\to\kappa$ that satisfies $|\{\delta<\kappa\mid d(\delta)=\tau\}|=\kappa$ for every $\tau<\kappa$. To see that this works, fix a nonzero $\sigma<\nu$, a ht-rapid $S\subseteq{}^\sigma T$ of size κ, and a prescribed colour $\tau<\kappa$. As S is ht-rapid, for every $\alpha<\kappa$, we may fix $\langle x^\alpha_i\mid i<\sigma\rangle$ in S such that $\mathrm{ht}(x^\alpha_i)>\alpha$ for every $i<\sigma$. As T is λ-coherent, we may fix a stationary $U\subseteq E^\kappa_\lambda$ and some $\epsilon<\kappa$ such that the following hold:

(1) for every $\alpha\in U$, for every $\beta\in[\epsilon,\alpha)$, the map $i\mapsto x^\alpha_i(\beta)$ is constant over σ.

As T is a κ-tree and using the fact that $\mu^{<\nu}<\kappa$ for every $\mu<\kappa$, we may then find a stationary $U'\subseteq U$ such that the following hold as well:

(2) the map $\alpha\mapsto\langle x^\alpha_i\restriction\epsilon\mid i<\sigma\rangle$ is constant over U';
(3) for every pair $\alpha<\beta$ of ordinals from U', $\sup\{\mathrm{ht}(x^\alpha_i)\mid i<\sigma\}<\beta$.

As $A := \{x_0^\alpha \restriction \alpha \mid \alpha \in U'\}$ is a κ-sized subset of the κ-Souslin tree T, another standard fact (see [RS23, Lemma 2.14]), yields a node $t \in T$ such that every extension of t in T is extended by some node from A. By possibly extending t, we may assume that $\delta := \mathrm{ht}(t)$ is bigger than ϵ and satisfying $d(\delta) = \tau$. Let $t_0 <_{\mathrm{lex}} t_1$ be some pair of immediate successors of t. Now fix $\alpha \in U'$ such that $x_0^\alpha \restriction \alpha$ extends t_0, and then fix $\beta \in U'$ above α such that $x_0^\beta \restriction \beta$ extends t_1. So $\Delta(x_\alpha^0, x_\beta^0) = \Delta(t_0, t_1) = \delta < \alpha$, $x_0^\alpha(\delta) = t_0(\delta)$ and $x_0^\beta(\delta) = t_1(\delta)$. By Clause (2), $\Delta(x_i^\alpha, x_i^\beta) \geq \epsilon$ for every $i < \sigma$. By Clause (1), then, for every $i < \sigma$, $\Delta(x_i^\alpha, x_i^\beta) = \delta$, $x_i^\alpha(\delta) = t_0(\delta)$ and $x_i^\beta(\delta) = t_1(\delta)$, so that $d(\mathrm{ht}(x_i^\alpha \wedge x_i^\beta)) = d(\delta) = \tau$, $x_i^\alpha <_{\mathrm{lex}} x_i^\beta$ and (by Clause (3)) $\mathrm{ht}(x_i^\alpha) < \mathrm{ht}(x_i^\beta)$. □

In Todorčević's note [Tod18] that was written before the introduction of Definition 2.9, it is proved that CH implies the existence of some streamlined $\aleph_2$-Aronszajn tree T for which Clauses (a) and (c) of $T \overset{\Delta}{\not\to} [\aleph_2]^{\aleph_0,1}_{\aleph_1}$ hold. The note also contains a discussion on how this may be generalized to higher cardinals. Here we present two additional generalizations that make use of recent results from [IR26].

Proposition 3.3. *Suppose that $\lambda = \lambda^{<\lambda}$ is an uncountable cardinal admitting a nontrivial C-sequence. Then any streamlined λ^+-Aronszajn tree T as in Fact 2.6 satisfies $T \overset{\Delta}{\not\to} [\lambda^+]^{<\lambda,1}_{\lambda}$.*

Proof. Denote $\kappa := \lambda^+$, so that $T \subseteq {}^{<\kappa}\lambda$. For every nonzero $\sigma < \lambda$, for every rapid family $S \subseteq {}^{\sigma}T$, let $D(S)$ be the collection of all $\delta < \kappa$ for which there are $x \neq y$ in S such that the following two hold:

(a) for every $i < \sigma$, $\Delta(x(i), y(i)) = \delta$;
(b) for every $i < \sigma$, $x(i) <_{\mathrm{lex}} y(i)$ iff $\mathrm{dom}(x(i)) < \mathrm{dom}(y(i))$.

Now let $\mathcal{F}$ be the collection of all sets $E \subseteq \kappa$ for which there are nonzero $\sigma < \lambda$ and a rapid $S \subseteq {}^{\sigma}T$ such that $D(S) \subseteq E$. Trivially, $\mathcal{F}$ is closed under taking supersets. We claim it is a λ-complete filter. To verify, suppose that we are given $\nu < \lambda$, and for every $j < \nu$, we are given a nonzero $\sigma_j < \lambda$ and a rapid $S_j \subseteq {}^{\sigma_j}T$ of size κ. For all $j < \nu$ and $\alpha < \kappa$, fix $\langle x_{j,i}^\alpha \mid i < \sigma_j\rangle$ in S_j such that $\mathrm{ht}(x_{j,i}^\alpha) > \alpha$ for every $i < \sigma_j$. Fix an ordinal $\sigma < \lambda$ and some bijection $\pi : \sigma \leftrightarrow \bigcup_{j<\nu}\{j\} \times \sigma_j$. For every $\alpha < \kappa$, define $\langle x_i^\alpha \mid i < \sigma\rangle$ via $x_i^\alpha := x_{\pi(i)}^\alpha$. As T is a κ-tree, we may find some club $C \subseteq \kappa$ for which $S := \{\langle x_i^\alpha \mid i < \sigma\rangle \mid \alpha \in C\}$ is a rapid subfamily of ${}^{\sigma}T$. Clearly, $D(S) \subseteq \bigcap_{j<\nu} D(S_j)$.

Claim 3.3.1. *$\mathcal{F}$ is a uniform nonempty filter.*

Proof. Fix a nonzero $\sigma < \lambda$, a rapid $S \subseteq {}^{\sigma}T$ of size κ, and an $\varepsilon < \kappa$, and we shall show that $D(S) \setminus \varepsilon \neq \emptyset$. As S is rapid, for every $\alpha < \kappa$, we may fix $\langle x_i^\alpha \mid i < \sigma\rangle$ in S such that $\mathrm{ht}(x_i^\alpha) > \alpha$ for every $i < \sigma$. As in the proof of Proposition 3.2, since T is a λ-coherent λ^+-tree and $\lambda^{<\lambda} = \lambda$, we may fix a stationary $U' \subseteq E^\kappa_\lambda$ and some $\epsilon \in [\varepsilon, \kappa)$ such that all of the following hold:

(1) for every $\alpha \in U'$, for every $\beta \in [\epsilon, \alpha)$, the map $i \mapsto x_i^\alpha(\beta)$ is constant over σ;
(2) the map $\alpha \mapsto \langle x_i^\alpha \restriction \epsilon \mid i < \sigma \rangle$ is constant over U';
(3) for every pair $\alpha < \beta$ of ordinals from U', $\sup\{\mathrm{dom}(x_i^\alpha) \mid i < \sigma\} < \beta$.

By Remark 2.4, $(T, <_{\mathrm{lex}})$ contains no copies of $(\kappa, \ni)$, so we may pick a pair $\alpha < \beta$ of ordinals in U' such that $x_0^\alpha \restriction \alpha <_{\mathrm{lex}} x_0^\beta \restriction \beta$. Recalling Definition 2.3, this means that $\delta := \Delta(x_0^\alpha \restriction \alpha, x_0^\beta \restriction \beta)$ is smaller than α, and $x_0^\alpha(\delta) < x_0^\beta(\delta)$. By Clause (2), $\delta \geq \epsilon$ and, more generally, $\Delta(x_i^\alpha, x_i^\beta) \geq \epsilon$ for every $i < \sigma$. By Clause (1), then, for every $i < \sigma$, $\Delta(x_i^\alpha, x_i^\beta) = \delta$ and $x_i^\alpha(\delta) = x_0^\alpha(\delta) < x_0^\beta(\delta) = x_i^\beta(\delta)$, so that $x_i^\alpha <_{\mathrm{lex}} x_i^\beta$. □

Altogether, $\mathcal{F}$ is a λ-complete uniform nonempty filter over λ^+. If it is λ^+-complete, then by Ulam's theorem, it is not weakly λ^+-saturated. If it is λ^+-incomplete, then since λ admits a nontrivial C-sequence, [IR26, Corollary 2.7] implies that $\mathcal{F}$ is not weakly λ-saturated. So, in both cases, we may fix a map $d : \kappa \to \lambda$ such that $d[E] = \lambda$ for every $E \in \mathcal{F}$. Then d witnesses that $T \xrightarrow{\Delta}\!\!\!\!\!\!/\;\; [\kappa]^{<\lambda,1}_\lambda$ holds, and we are done. □

Proposition 3.4. *Suppose that:*

- *$\nu \leq \lambda < \kappa$ are regular uncountable cardinals such that $\mu^{<\nu} < \kappa$ for every $\mu < \kappa$;*
- *T is a streamlined λ-coherent κ-Aronszajn tree;*
- *$\square(\kappa, {<}\nu)$ holds.*

Then $T \xrightarrow{\Delta}\!\!\!\!\!\!/\;\; [\kappa]^{<\nu,1}_\kappa$ holds.

Proof. A proof similar to that of Proposition 3.3 yields that the collection $\mathcal{F}$ of all sets $E \subseteq \kappa$ for which there are nonzero $\sigma < \nu$ and a rapid $S \subseteq {}^\sigma T$ such that $D(S) \subseteq E$ is a ν-complete uniform nonempty filter over κ. Since $\square(\kappa, {<}\nu)$ holds, [IR26, Theorem D] implies that $\mathcal{F}$ is not weakly κ-saturated. As in the proof of Proposition 3.3, it follows that $T \xrightarrow{\Delta}\!\!\!\!\!\!/\;\; [\kappa]^{<\nu,1}_\kappa$ holds. □

Remark 3.5. By [IR25, Theorem E], for every regular $\kappa \geq \aleph_2$, if $\square(\kappa)$ holds then there exists a streamlined κ-Aronszajn tree T such that $T \xrightarrow{\wedge}\!\!\!\!\!\!/\;\; [\kappa]^{n,1}_\kappa$ holds for every $n < \omega$. By the preceding proposition and Fact 2.7, assuming additionally $\mu^{\aleph_0} < \kappa$ for every $\mu < \kappa$, moreover $T \xrightarrow{\Delta}\!\!\!\!\!\!/\;\; [\kappa]^{\aleph_0,1}_\kappa$ holds.

4. A family of incomparable trees

The next lemma is inspired by the proof of [Tod07a, Proposition 8.4].

Lemma 4.1. *Suppose that T is a χ-complete streamlined κ-tree and weak $T \xrightarrow{\wedge}\!\!\!\!\!\!/\;\; [\kappa]^{2,1}_\theta$ holds, with $\theta \geq 2$. Then there is a sequence $\langle T^A \mid A \in \mathcal{P}(\theta)\rangle$ of χ-complete streamlined κ-Aronszajn trees such that for all $A, B \in \mathcal{P}(\theta)$:*

(1) *$(T \restriction \mathrm{acc}(\kappa), \subseteq)$ and $(T^A \restriction \mathrm{acc}(\kappa), \subseteq)$ are order-isomorphic, there is a weak embedding from T to T^A, and if T is normal, then so is T^A;*

(2) *if* $A \subseteq B$*, then there is a level-preserving weak embedding from* T^A *to* T^B*;*
(3) *if* $A \nsubseteq B$*, then there is no Lipschitz map from a* κ*-sized subset of* T^A *to* T^B*;*
(4) *if* $A \neq \emptyset$*, then there is no Lipschitz map from a* κ*-sized subset of* T^A *to* T*.*

If $T \overset{\Delta}{\nrightarrow} [\kappa]^{2,1}_{\theta}$ *holds, then* (1)–(4) *can be achieved in addition to the following:*

(5) *For every* $S \subseteq \mathrm{acc}(\kappa)$ *such that* T *is* S*-coherent,*[2] *all trees in* $\langle T^A \mid A \in \mathcal{P}(\theta)\rangle$ *are* S*-coherent.*

Proof. By possibly applying an injection from $\{t(\alpha) \mid t \in T, \alpha \in \mathrm{dom}(t)\}$ to κ, we may assume that $T \subseteq {}^{<\kappa}\kappa$. Define a map $\pi : {}^{<\kappa}\kappa \rightarrow {}^{<\kappa}\kappa$, as follows. Given $\alpha < \kappa$, $i < 2$ and $s : (2 \cdot \alpha) + i \rightarrow \kappa$, let $\pi(s)$ be the unique function from α to κ to satisfy

$$(\pi(s))(\beta) = s((2 \cdot \beta) + 1), \quad (\beta < \alpha).$$

Fix a witness c for weak $T \overset{\wedge}{\nrightarrow} [\kappa]^{2,1}_{\theta}$. For each $A \subseteq \theta$, we shall define a map $\pitchfork_A : T \rightarrow {}^{<\kappa}\kappa$ that will satisfy $\pi(\pitchfork_A(t)) = t$ for every $t \in T$. Indeed, given $\alpha < \kappa$ and $t \in T_\alpha$, let $\pitchfork_A(t)$ be the unique function from $(2 \cdot \alpha)$ to κ to satisfy that for all $\beta < \alpha$ and $i < 2$:

$$(\pitchfork_A(t))(2 \cdot \beta + i) = \begin{cases} 0, & \text{if } i = 0 \text{ and } c(t \restriction \beta) \in A; \\ t(\beta), & \text{otherwise.} \end{cases}$$

Consider $T^A := \{s \restriction \epsilon \mid s \in \mathrm{Im}(\pitchfork_A), \epsilon \leq \mathrm{dom}(s)\}$, and note that:

(i) $\pi[T^A] \subseteq T$;
(ii) for all $t, t' \in T$ with $\mathrm{dom}(t) < \mathrm{dom}(t')$:
  - $t \subseteq t'$ iff $\pitchfork_A(t) \subseteq \pitchfork_A(t')$;
  - $\pi(\pitchfork_A(t)) = t$.
(iii) for all $s, s' \in T^A$ with $\mathrm{dom}(s) < \mathrm{dom}(s')$:
  - $s \subseteq s'$ iff $\pi(s) \subseteq \pi(s')$;
  - if $\mathrm{dom}(s) = 2 \cdot \alpha$ for some α, then $\pitchfork_A(\pi(s)) = s$;
  - if $\mathrm{dom}(s) = 2 \cdot \alpha + 1$ for some α, then $\pitchfork_A(\pi(s)) = s \restriction (2 \cdot \alpha)$.

Consequently, $(T \restriction \mathrm{acc}(\kappa), \subseteq)$ and $(T^A \restriction \mathrm{acc}(\kappa), \subseteq)$ are order-isomorphic, and if T is normal then so is T^A. By Lemma 2.11, T is a κ-Aronszajn tree. So, by the said existence of isomorphism, so is T^A. It also follows from Clauses (ii) and (iii) that T^A is χ-complete, and that $\pitchfork_A$ is a weak embedding from T to T^A.

Claim 4.1.1. *Let* $A \subseteq B \subseteq \theta$*. Then there is a level-preserving weak embedding from* T^A *to* T^B*.*

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[2]This is a refinement of Definition 2.5: T is *S-coherent* iff for all $\alpha \in S$ and $t_0, t_1 \in T_\alpha$, there is an $\epsilon < \alpha$ such that $t_0(\beta) = t_1(\beta)$ for every $\beta \in [\epsilon, \alpha)$.

Proof. Define $g : T^A \to {}^{<\kappa}\kappa$ as follows. Given $s \in T^A$, set $\alpha := \mathrm{dom}(s)$, and define $g(s) : \alpha \to \kappa$ by letting for all $\beta < \kappa$ and $i < 2$ with $2 \cdot \beta + i < \alpha$:

$$g(s)(2\cdot\beta+i) = \begin{cases} 0, & \text{if } i = 0 \text{ and } c(\pi(s) \restriction \beta) \in B; \\ s(2\cdot\beta+i), & \text{otherwise.} \end{cases}$$

By Clause (iii), for every pair $s \subsetneq s'$ of nodes in T^A, we have $g(s) \subsetneq g(s')$. Thus, we are left with showing that $\mathrm{Im}(g) \subseteq T^B$. To this end, let $s \in T^A$. Recalling the definition of T^A, let us pick $t \in T$ such that $s \subseteq \pitchfork_A(t)$. As $g(s) \subseteq g(\pitchfork_A(t))$ and $\pitchfork_B(t) \in T^B$, it thus suffices to prove that $g(\pitchfork_A(t)) = \pitchfork_B(t)$.

Let $\beta < \kappa$ and $i < 2$ be such that $2 \cdot \beta + i < \mathrm{dom}(t)$. By expanding the definition of the map g and then the definition of the map $\pitchfork_A$, and recalling Clause (ii), we see that

$$\begin{aligned} g(\pitchfork_A(t))(2\cdot\beta+i) &= \begin{cases} 0, & \text{if } i = 0 \text{ and } c(t \restriction \beta) \in B; \\ (\pitchfork_A(t))(2\cdot\beta+i), & \text{otherwise.} \end{cases} \\ &= \begin{cases} 0, & \text{if } i = 0 \text{ and } c(t \restriction \beta) \in B; \\ 0, & \text{if } i = 0 \text{ and } c(t \restriction \beta) \in A \setminus B; \\ t(\beta), & \text{otherwise.} \end{cases} \end{aligned}$$

However, $A \subseteq B$, which means that the middle case does not exist. So, there are only the top and bottom cases and they coincide with the definition of $(\pitchfork_B(t))(2 \cdot \beta + i)$, so we are done. □

Claim 4.1.2. *Let $A, B \in \mathcal{P}(\theta)$ with $A \nsubseteq B$. Then there is no Lipschitz map from a κ-sized subset of T^A to T^B. Furthermore, for every $X \subseteq T^A$ of size κ and every $({<}\kappa)$-to-one map $g : X \to T^B$, there are $s \neq s'$ in X such that $\Delta(g(s), g(s')) < \Delta(s, s')$.*

Proof. Fix $\tau \in A \setminus B$. Since T^A and T^B are κ-trees, given a map $g : X \to T^B$ as above, we may fix a κ-sized dom-rapid subfamily S of $\{(\pi(s), \pi(g(s))) \mid s \in X\}$. As c witnesses weak $T \overset{\wedge}{\nrightarrow} [\kappa]^{2,1}_{\theta}$, we may pick $(x, y) \neq (x', y')$ in S and $\delta < \kappa$ such that:

- $c(x \wedge x') = \tau = c(y \wedge y')$;
- if $x <_{\mathrm{lex}} x'$, then $\mathrm{dom}(x) < \mathrm{dom}(x')$;
- if $y <_{\mathrm{lex}} y'$, then $\mathrm{dom}(y) < \mathrm{dom}(y')$;
- $\mathrm{dom}(x \wedge x') = \delta = \mathrm{dom}(y \wedge y')$.

Without loss of generality, $\mathrm{dom}(x) < \mathrm{dom}(x')$. So, since S is dom-rapid, also $\mathrm{dom}(y) < \mathrm{dom}(y')$. It thus follows that $x <_{\mathrm{lex}} x'$ and $y <_{\mathrm{lex}} y'$. Recalling Definition 2.3, this means that x is incomparable with x', and y is incomparable with y'. Altogether, $\delta \in \mathrm{dom}(x) \cap \mathrm{dom}(x') \cap \mathrm{dom}(y) \cap \mathrm{dom}(y')$.

Fix $s, s' \in X$ such that $(x, y) = (\pi(s), \pi(g(s)))$ and $(x', y') = (\pi(s'), \pi(g(s')))$. By Clause (iii), $\pitchfork_A(x) \subseteq s$, $\pitchfork_A(x') \subseteq s'$, $\pitchfork_B(y) \subseteq g(s)$ and $\pitchfork_B(y') \subseteq g(s')$. We now analyze $s \wedge s'$ and $g(s) \wedge g(s')$:

- ▶ As $x \restriction \delta = x' \restriction \delta$, we have $\pitchfork_A(x) \restriction (2\cdot\delta) = \pitchfork_A(x') \restriction (2\cdot\delta)$. As $c(x \wedge x') = \tau \in A$, we have $\pitchfork_A(x)(2\cdot\delta) = 0 = \pitchfork_A(x')(2\cdot\delta)$. Finally, $\pitchfork_A(x)(2\cdot\delta+1) = x(\delta)$ and $\pitchfork_A(x')(2\cdot\delta+1) = x'(\delta)$. Thus, altogether, $\mathrm{dom}(s \wedge s') = \mathrm{dom}(\pitchfork_A(x) \wedge \pitchfork_A(x')) = 2\cdot\delta+1$.
- ▶ As $y \restriction \delta = y' \restriction \delta$, we have $\pitchfork_B(y) \restriction (2\cdot\delta) = \pitchfork_B(y') \restriction (2\cdot\delta)$. As $c(y \wedge y') = \tau \notin B$, we have $\pitchfork_B(y)(2\cdot\delta) = y(\delta)$ and $\pitchfork_B(y')(2\cdot\delta) = y'(\delta)$. Altogether, $\mathrm{dom}(g(s) \wedge g(s')) = \mathrm{dom}(\pitchfork_B(y) \wedge \pitchfork_B(y')) = 2\cdot\delta$.

Altogether, $\Delta(g(s), g(s')) < \Delta(s, s')$, as sought. □

Claim 4.1.3. *Let $A \in \mathcal{P}(\theta) \setminus \{\emptyset\}$. Then there is no Lipschitz map from a κ-sized subset of T^A to T. Furthermore, for every $X \subseteq T^A$ of size κ and every $({<}\kappa)$-to-one map $g : X \to T$, there are $s \neq s'$ in X such that $\Delta(g(s), g(s')) < \Delta(s, s')$.*

Proof. This is very similar to the proof of Claim 4.1.2. Fix $\tau \in A$. Given a map $g : X \to T$ as above, fix a κ-sized dom-rapid subfamily S of $\{(\pi(s), g(s)) \mid s \in X\}$. Pick $(x, y) \neq (x', y')$ in S and $\delta < \kappa$ such that:

- $c(x \wedge x') = \tau = c(y \wedge y')$;
- $\mathrm{dom}(x \wedge x') = \delta = \mathrm{dom}(y \wedge y')$;
- $\delta \in \mathrm{dom}(x) \cap \mathrm{dom}(x') \cap \mathrm{dom}(y) \cap \mathrm{dom}(y')$.

Fix $s, s' \in X$ such that $(x, y) = (\pi(s), g(s))$ and $(x', y') = (\pi(s'), g(s'))$. So $\Delta(g(s), g(s')) = \delta$, and since $\tau \in A$, we have $\Delta(s, s') = 2\cdot\delta+1$. □

Suppose now that $T \overset{\Delta}{\not\to} [\kappa]^{2,1}_\theta$ holds, as witnessed by a colouring $d : \kappa \to \theta$. In particular, the map $t \mapsto d(\mathrm{dom}(t))$ witnesses weak $T \overset{\wedge}{\not\to} [\kappa]^{2,1}_\theta$, so we may assume that our initial c is derived from d in this very way.

Let S denote the collection of all $\alpha \in \mathrm{acc}(\kappa)$ such that all $t_0, t_1 \in T_\alpha$ agree on a tail, i.e., there is an $\epsilon < \alpha$ such that $t_0(\beta) = t_1(\beta)$ for every $\beta \in [\epsilon, \alpha)$.

Claim 4.1.4. *Let $A \subseteq \theta$. Then T^A is S-coherent.*

Proof. Let $\alpha \in S$. Let s_0, s_1 be two elements of the α^{th}-level of T^A. It follows that we may fix t_0, t_1 from the α^{th}-level of T such that $s_j = \pitchfork_A(t_j)$ for each $j < 2$. Fix a large enough $\epsilon < \alpha$ such that $t_0(\beta) = t_1(\beta)$ for every

$\beta \in [\epsilon, \alpha)$. Then, for every $\beta \in [\epsilon, \alpha)$, for every $i < 2$,

$$\begin{aligned} s_0(2\cdot\beta+i) &= (\psi_A(t_0))(2\cdot\beta+i) \\ &= \begin{cases} 0, & \text{if } i=0 \text{ and } c(t_0 \restriction \beta) \in A; \\ t_0(\beta), & \text{otherwise.} \end{cases} \\ &= \begin{cases} 0, & \text{if } i=0 \text{ and } d(\beta) \in A; \\ t_0(\beta), & \text{otherwise.} \end{cases} \\ &= \begin{cases} 0, & \text{if } i=0 \text{ and } d(\beta) \in A; \\ t_1(\beta), & \text{otherwise.} \end{cases} \\ &= \begin{cases} 0, & \text{if } i=0 \text{ and } c(t_1 \restriction \beta) \in A; \\ t_0(\beta), & \text{otherwise.} \end{cases} \\ &= (\psi_A(t_1))(2\cdot\beta+i) = s_1(2\cdot\beta+i), \end{aligned}$$

so that s_0 and s_1 agree on a tail. □

This completes the proof □

We now arrive at Theorem B.

Corollary 4.2. *Suppose that $\lambda = \lambda^{<\lambda}$ is an uncountable cardinal, and any of the following:*

- *$\square(\lambda^+, {<}\lambda)$ holds and $\theta = \lambda^+$;*
- *λ admits a nontrivial C-sequence and $\theta = \lambda$.*

Then there is a family of 2^θ many streamlined λ-coherent λ-complete λ^+-Aronszajn trees such that no two of them admit a Lipschitz map from a λ^+-sized subset of one to the other.

Proof. Using Fact 2.6, let T be a streamlined λ-coherent λ-complete λ^+-Aronszajn tree T. By Propositions 3.3 and 3.4, $T \overset{\Delta}{\nrightarrow} [\lambda^+]^{<\lambda,1}_\theta$ holds. By Lemma 4.1, then, there is a sequence $\langle T^A \mid A \in \mathcal{P}(\theta)\rangle$ of λ-coherent, λ-complete streamlined λ^+-Aronszajn trees such that for all $A, B \in \mathcal{P}(\theta)$, if $A \nsubseteq B$, then there is no Lipschitz map from a λ^+-sized subset of T^A to T^B. Let $\mathcal{A}$ be an antichain in $\mathcal{P}(\theta)$ of size 2^θ. Then $\langle T^A \mid A \in \mathcal{A}\rangle$ is as sought. □

By feeding Lemma 4.1 with [IR25, Theorem A], we get another conclusion:

Corollary 4.3. *Suppose that κ is the successor of a regular uncountable cardinal. Then the following are equivalent:*

- *There is a special κ-Aronszajn tree;*
- *There is a family of 2^κ many streamlined special κ-Aronszajn trees such that no two of them admit a Lipschitz map from a κ-sized subset of one to the other.* □

Acknowledgments

A part of the first author's work on this paper was made when they were a postdoctoral researcher at the Ben Gurion University of the Negev in Beer-sheva, Israel, supported by the Israel Science Foundation (grant agreement 665/20). The second author is partially supported by the Israel Science Foundation (grant agreement 203/22).

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