PARTITION RELATIONS FOR TREES II: ENTANGLED LINEAR ORDERS

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Abstract. We find new sufficient conditions for the existence of entangled linear orders, among others, improving a theorem of Shelah and Shafir. The constructions go through an anti-Ramsey colouring for trees and they provide a fine control on the extent of entanglement. As an application, from a strongly Luzin set, we obtain for every positive integer n, an $(\aleph_1, n)$-entangled linear order that is not $(\aleph_1, n+1)$-entangled. The result is more general and it answers a question of Carroy, Levine and Notaro.

1. Introduction

For two cardinals κ, ν, a linear order $(L, <_L)$ is said to be $(\kappa, {<}\nu)$-*entangled* iff for every $\sigma < \nu$, for every κ-sized family $X \subseteq {}^{\sigma}L$ consisting of pairwise disjoint injective σ-tuples, for every map $\tau : \sigma \to 2$, there are $x \neq y$ in X such that for every $i < \sigma$,[1]

$$x(i) <_L y(i) \text{ iff } \tau(i) = 1.$$

These orders were introduced by Abraham and Shelah in [AS81] who proved that Martin's axiom $\mathsf{MA}_{\aleph_1}$ implies that there are no $(\aleph_1, {<}\aleph_0)$-entangled linear orders of size $\aleph_1$, but $\mathsf{MA}_{\aleph_1}$ is compatible with the existence of an $(\aleph_1, n)$-entangled linear orders of size $\aleph_1$ for any prescribed integer n.[2] Later, in [Tod89, §8], Todorčević proved that the Open Graph Axiom implies that there are no $(\aleph_1, 2)$-entangled sets of reals of size $\aleph_1$, hence the Proper Forcing Axiom implies that there are no $(\aleph_1, 3)$-entangled linear orders of size $\aleph_1$. There are ZFC results as well, most notably, the existence of a $(2^{\aleph_0}, {<}\aleph_0)$-entangled linear order of size $2^{\aleph_0}$ due to Todorčević [Tod85] and independently due to Bonnet and Shelah [BS85], and the introduction to [Cha26] offers a rather comprehensive survey concerning entangled orders of size $\leq 2^{\aleph_0}$. Some applications of entangled orders are listed in the introduction to [CLN26]; an additional application is [FR17, Theorem 1].

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[1]Some studies favor the (strictly stronger) *oriented* variation in which the given family X is well-ordered by some relation $\prec$, and the requirement "$x \neq y$" is replaced by "$x \prec y$". The two definitions dovetail for finite values of ν, and coincide for infinite ν's.

[2]By (κ, μ)-*entangled order* we mean a $(\kappa, {<}\nu)$-entangled order for ν the first (finite or infinite) cardinal bigger than μ.

In this paper, we shall obtain entangled linear orders from instances of the partition relations for trees introduced by the authors in [IR25], as follows.

Theorem A. *Consider a structure* $\mathbf{T} = (T, \subsetneq, <_{\mathrm{lex}}, h)$*, where* $T \subseteq {}^{\leq\lambda}\lambda$ *is a streamlined tree of size* κ*,* $h : T \to \kappa$ *is a* $(<\kappa)$*-to-one map, and* $|T \cap {}^{<\lambda}\lambda| = \lambda < \mathrm{cf}(\kappa)$*.*

(1) *If* $\mathbf{T} \overset{\wedge}{\not\to} [\kappa]^{<\nu}_{\lambda}$ *holds with* $\lambda = \lambda^{<\nu}$*, then there exists a* $(\kappa, <\nu)$*-entangled order of size* κ*;*
(2) *If* $\mathbf{T} \overset{\Delta}{\not\to} [\kappa]^{n}_{\lambda}$ *holds for a positive integer* n*, and* T *is uniformly homogeneous, then there exists a* (κ, n)*-entangled order that is not* $(\kappa, n+1)$*-entangled.*

In [SS00, Theorem 2.14], Shelah and Shafir got a $(\lambda^+, <\lambda)$-entangled linear order of size λ^+ for every cardinal $\lambda = \lambda^{<\lambda}$ that satisfies both $\lambda > \beth_\omega$ and $2^\lambda = \lambda^+$.[3] Here, we improve their result (via Theorem A), as follows:

Theorem B. *Suppose that* $\lambda = \lambda^{<\lambda}$ *is an infinite cardinal such that one of the following holds:*

(1) $\lambda > \beth_\omega$*;*
(2) $2^\lambda = \lambda^+$*;*
(3) $\lambda = \mu^+$ *for a cardinal* $\mu = \mu^{\aleph_0}$*.*

Then $\mathbf{T} \overset{\Delta}{\not\to} [\kappa]^{<\lambda,1}_{\lambda}$ *holds for some structure* $\mathbf{T} = (T, \subsetneq, <_{\mathrm{lex}}, h)$*, where* $T \subseteq {}^{\leq\lambda}\lambda$ *is a uniformly homogeneous streamlined tree of size a regular cardinal* $\kappa \in (\lambda, 2^\lambda]$*, and* $h : T \to \kappa$ *is a* $(<\kappa)$*-to-one map. In particular, there is a* $(\kappa, <\lambda)$*-entangled linear order of size* κ*.*

We also get an instance of our partition relation from a strongly Luzin set, as follows.

Theorem C. *If there is a* κ*-strongly Luzin subset of* ${}^\omega\omega$*, then* $\mathbf{T} \overset{\Delta}{\not\to} [\kappa]^{<\omega,1}_{\omega}$ *holds for some structure* $\mathbf{T} = (T, \subsetneq, <_{\mathrm{lex}}, h)$*, where* $T \subseteq {}^{\leq\omega}\omega$ *is a uniformly homogeneous streamlined tree of size* κ*, and* $h : T \to \kappa$ *is a ctbl-to-one map.*

By Theorem A of [CLN26], the continuum hypothesis yields for every positive integer n an $(\aleph_1, n)$-entangled not $(\aleph_1, n+1)$-entangled linear order of size $\aleph_1$. More generally, Theorem 3.1 of the same paper states that if $\mathrm{cov}(\mathcal{M}) = 2^{\aleph_0}$, then for every positive integer n there exists a $(\mathrm{cov}(\mathcal{M}), n)$-entangled not $(\mathrm{cov}(\mathcal{M}), n+1)$-entangled linear order of size κ. Question 3.4 of [CLN26] asks whether the hypothesis $\mathrm{cov}(\mathcal{M}) = 2^{\aleph_0}$ is necessary. Using Theorems A and C here, we obtain a negative answer, as follows.

Corollary D. *Suppose that* $\mathrm{cov}(\mathcal{M}) = \mathrm{cof}(\mathcal{M})$*. For every cardinal* $\kappa \leq \mathrm{cov}(\mathcal{M})$ *with* $\mathrm{cf}(\kappa) = \mathrm{cf}(\mathrm{cov}(\mathcal{M}))$*, for every positive integer* n*, there exists a* (κ, n)*-entangled linear order of size* κ *that is not* $(\kappa, n+1)$*-entangled.*

[3] For infinite cardinals κ, λ, a linear order is $(\kappa, <\lambda)$-entangled in the sense here iff it is (κ, λ)-entangled in the sense of [SS00, Definition 1.1(a)].

1.1. **Organisation of this paper.** In Section 2, we give the basic definitions required, and in particular, we define the three partition relations on trees which are the subject of this paper (Definition 2.7).

In Section 3, we prove some results separating the partition relations that we consider.

In Section 4, we give an application of these partition relations to the construction of entangled and weakly entangled linear orders. We also give an application of these partition relations to entangled Souslin lines, strengthening some results from the literature. Theorem A is proved in this section, and we also prove a 'boosting result' for the partition relations: going from a weaker partition relation to a stronger one from some extra hypotheses.

In Section 5, we give an application of the weakest partition relation to study of productivity of chain conditions.

In Section 6, we obtain the strongest partition relation from the existence of strongly Luzin sets, going towards a proof of Theorem C and Corollary D.

In Section 7, we obtain the strongest partition relation from a weak guessing principle, which forms the bulk of the proof of Theorem B.

In Section 8, we complete the proof of Theorem B and the proof of Corollary D.

1.2. **Notation and conventions.** Throughout this paper, and $\kappa, \theta, \lambda, \chi, \mu, \nu$ denote arbitrary cardinals. E^κ_θ stands for $\{\alpha < \kappa \mid \mathrm{cf}(\alpha) = \theta\}$, and $\mathrm{acc}(\kappa)$ stands for $\{\alpha < \kappa \mid \sup(\alpha) = \alpha > 0\}$. We denote by H_κ the collection of all sets of hereditary cardinality less than κ. We denote the identity function on whatever relevant domain by id.

2. Trees and linearly-ordered sets

2.1. **Linear orders.** Two functions f, g from the ordinals to the universe are *incomparable* iff $\Delta(f, g) < \min\{\mathrm{dom}(f), \mathrm{dom}(g)\}$, where

$$\Delta(f, g) := \min\{\mathrm{dom}(f), \mathrm{dom}(g), \delta \mid \delta \in \mathrm{dom}(f) \cap \mathrm{dom}(g) \ \&\ f(\delta) \neq g(\delta)\}.$$

Definition 2.1 (Default lexicographic ordering)**.** For a subset $T \subseteq {}^{<\kappa}H_\kappa$, fix $\prec$ such that $(H_\kappa, \prec)$ is a well-order extending $(\kappa, \in)$, and then define a linear ordering $<_{\mathrm{lex}}$ of T, as follows:

- ▶ If f and g are comparable, then $f <_{\mathrm{lex}} g$ iff $g \subseteq f$;
- ▶ Otherwise, let $\delta := \Delta(f, g)$, and then $f <_{\mathrm{lex}} g$ iff $f(\delta) \prec g(\delta)$.

2.2. **Trees.** A *tree* is a partially ordered set $\mathbf{T} = (T, <_T)$ such that, for every $x \in T$, the set $x_\downarrow := \{y \in T \mid y <_T x\}$ is well-ordered by $<_T$; its order-type is denoted by $\mathrm{ht}(x)$. For any ordinal α, the α^{th}-level of the tree is the collection $T_\alpha := \{x \in T \mid \mathrm{ht}(x) = \alpha\}$. For a set of ordinals A, we write $T \restriction A$ for $\bigcup_{\alpha \in A} T_\alpha$. The *height* of the tree, denoted $\mathrm{ht}(\mathbf{T})$, is the first ordinal α for which $T_\alpha = \emptyset$. The *normal height* of the tree, denoted $\hat{\mathrm{ht}}(\mathbf{T})$, is the first ordinal α for which $T_{\alpha+1} = \emptyset$. As an example differentiating between the height and the normal height of a tree, the reader may consider the tree generated (via initial segments) by some $S \subseteq {}^{\omega}\omega$. A tree is ς-*splitting* if each of its nodes admit at least ς many immediate successors. A tree is χ-*complete* iff every chain of size less than χ is bounded. A tree is *Hausdorff* iff for any two of its nodes x, y such that $\mathrm{ht}(x) = \mathrm{ht}(y)$ is a limit ordinal, if $x_{\downarrow} = y_{\downarrow}$, then $x = y$. Given two distinct nodes x, y of a Hausdorff tree $\mathbf{T}$, we let $x \wedge y$ denote the highest node z of $\mathbf{T}$ to satisfy $z \leq_T x$ and $z \leq_T y$. In particular, x and y are comparable iff $x \wedge y \in \{x, y\}$.

A κ-*tree* is a tree $\mathbf{T}$ of height κ all of whose levels are of size less than κ. A κ-*Aronszajn tree* is a κ-tree with no chains of size κ. A κ-*Souslin tree* is a κ-Aronszajn tree with no antichains of size κ. A general reference for our terminology on Souslin (and free) trees is [BR17b].

Remark 2.2. A lexicographic ordering of a tree $(T, <_T)$ is any linear ordering $<_L$ of T such that for every $t \in T$, the cone $\{s \in T \mid t \leq_T s\}$ is a convex subset of $(T, <_L)$. If $(T, <_T)$ is a κ-Aronszajn tree then the outcome $(T, <_L)$ is a κ-*Aronszajn line*, i.e., a κ-sized linear order that contains no κ-sized linear order of density less than κ, no copy of $(\kappa, \in)$ nor a copy of $(\kappa, \ni)$.

Definition 2.3. A *streamlined tree* is a collection T of functions from ordinals to the universe that is closed under taking initial segments.

Remark 2.4. We identify a streamlined tree T with the Hausdorff tree $(T, \subsetneq)$; in this case, $\mathrm{ht}(t) = \mathrm{dom}(t)$ for every $t \in T$.

Definition 2.5. A streamlined tree T is *uniformly homogeneous* iff for all $t_0, t_1 \in T$ with $\mathrm{dom}(t_0) < \mathrm{dom}(t_1)$, it is the case that $t_0 * t_1 \in T$, where $t_0 * t_1$ is the unique map with domain $\mathrm{dom}(t_1)$ to satisfy $(t_0 * t_1)(\beta) = t_0(\beta)$ for all $\beta \in \mathrm{dom}(t_0)$, and $(t_0 * t_1)(\beta) = t_1(\beta)$ for all other β's.

2.3. **Partition relation for trees.**

Definition 2.6 (Rapid families)**.** For a family S of sequences, and a map $h : \bigcup_{x \in S} \mathrm{Im}(x) \rightarrow \mathrm{Ord}$, we say that S is h-*rapid* iff for all $x \neq y$ in S, either $\sup(h[\mathrm{Im}(x)]) < \min(h[\mathrm{Im}(y)])$ or $\sup(h[\mathrm{Im}(y)]) < \min(h[\mathrm{Im}(x)])$.

The next definition was introduced in [IR25] where sufficient conditions for instances of it to hold were obtained, and applications were presented.

Definition 2.7. For a structure $\mathbf{T} = (T, <, \lhd, h)$ such that $(T, <)$ is a Hausdorff tree, $(T, \lhd)$ is a partially ordered set, and $h : T \rightarrow \mathrm{Ord}$ is a map, we define three partition relations, as follows.

(I) *weak* $\mathbf{T} \overset{\wedge}{\nrightarrow} [\kappa]^{<\nu, <\mu}_{\theta}$ asserts the existence of a colouring $c : T \rightarrow \theta$ satisfying that for every nonzero $\sigma < \nu$, for every h-rapid $S \subseteq {}^{\sigma}T$ of size κ, for every $\tau < \theta$, there are $x \neq y$ in S such that the following three hold:
  - (a) for every $i < \sigma$, $c(x(i) \wedge y(i)) = \tau$;
  - (b) for every $i < \sigma$, if $x(i) \lhd y(i)$ then $h(x(i)) < h(y(i))$;
  - (c) $|\{h(x(i) \wedge y(i)) \mid i < \sigma\}| < \mu$.

(II) $\mathbf{T} \overset{\wedge}{\nrightarrow} [\kappa]^{<\nu,<\mu}_{\theta}$ asserts the existence of a colouring $c : T \rightarrow \theta$ witnessing weak $\mathbf{T} \overset{\wedge}{\nrightarrow} [\kappa]^{<\nu,<\mu}_{\theta}$ in addition to satisfying the following. For every $T' \in [T]^{\kappa}$, for every nonzero $\sigma < \nu$, there exists an h-rapid $S' \subseteq {}^{\sigma}T'$ of size κ such that for every $S \subseteq S'$ of size κ, and every $\tau \in {}^{\sigma}\theta$, there are $x \neq y$ in S such that the following three hold:
 (a) for every $i < \sigma$, $c(x(i) \wedge y(i)) = \tau(i)$;
 (b) same as Clause (b) above;
 (c) same as Clause (c) above.

(III) $\mathbf{T} \overset{\Delta}{\nrightarrow} [\kappa]^{<\nu,<\mu}_{\theta}$ asserts the existence of a colouring $d : \mathrm{Im}(h) \rightarrow \theta$ satisfying following. For every nonzero $\sigma < \nu$, for every h-rapid $S \subseteq {}^{\sigma}T$ of size κ, for every $\tau < \theta$, there are $x \neq y$ in S such that the following three hold:
 (a) for every $i < \sigma$, $d(h(x(i) \wedge y(i))) = \tau$;
 (b) same as Clause (b) above;
 (c) same as Clause (c) above.

Remark 2.8. The partition relation of Clause (III) implies that of Clause (I), and is superior in the context of uniformly homogeneous or coherent trees. As seen in [IR26a, Lemma 2.11(2)], in case that $\nu \leq \omega$, $\theta = \kappa$ and $(T, <)$ is a κ-tree and h is $(<\kappa)$-to-one, the partition relation of Clause (I) is no weaker than that of Clause (II). An additional result in this spirit will be obtained as part of Lemma 4.3 below. Another one may be extracted from the proof of [IR25, Lemma 8.7], and it reads as follows: for every structure $\mathbf{T} = (T, <, \lhd, h)$ such that $(T, <)$ is special λ^+-Aronszajn tree, if weak $\mathbf{T} \overset{\wedge}{\nrightarrow} [\kappa]^{<\nu,1}_{\theta}$ holds for $\theta = \lambda$, then it also holds for $\theta = \lambda^+$.

Convention 2.9. We may replace the quadruple $\mathbf{T}$ by a triple $(T, <, \lhd)$, and then we mean $\mathbf{T} := (T, <, \lhd, \mathrm{ht})$.[4] We may replace $\mathbf{T}$ by a streamlined tree T, and then we mean $\mathbf{T} := (T, \subsetneq, <_{\mathrm{lex}}, \mathrm{dom})$. We may replace the superscript $<\nu$ by ν, and then we mean that σ is equal to ν. We may replace the superscript $<\mu$ by μ, in which case Clause (c) becomes "$|\{h(x(i) \wedge y(i)) \mid i < \sigma\}| \leq \mu$". We may omit μ, in which case requirement (c) is waived.

Theorem 2.10. *Suppose that $\lambda = \lambda^{<\lambda}$ is an uncountable cardinal admitting a λ-Aronszajn tree. Then there is a uniformly homogeneous streamlined λ^+-Aronszajn tree T such that $T \overset{\Delta}{\nrightarrow} [\lambda^+]^{<\lambda,1}_{\theta}$ holds for every regular $\theta < \lambda$.*

Proof. The proof is identical to that of [IR26a, Proposition 3.3], except that instead of invoking [IR26b, Corollary 2.7], we invoke [IR26b, Lemma 2.6(2)] with $(\mu, \nu, \kappa) := (\lambda, \lambda, \lambda^+)$. For this, it remains to verify for every regular $\theta < \lambda$, $\mathsf{onto}(\mathcal{I}^{\lambda^+}_{\lambda}, \theta)$ and $\mathsf{onto}(\mathcal{J}^{\lambda}_{\lambda}, \theta)$ both hold. The former is covered by [IR26b, Corollary 6.22(iii)]. The latter follows from [IR26b, Theorem 3.1] together with [IR24, Theorem 4.1(1)]. □

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[4] As explained in the beginning of Section 4 below, h cannot be dropped in general.

3. The role of dimension

In this short section, we give consistent examples of κ-trees T for which $T \overset{\Delta}{\not\to} [\kappa]^{<\nu,1}_{\kappa}$ holds and weak $T \overset{\wedge}{\not\to} [\kappa]^{\nu}_{1}$ fails.

Lemma 3.1. *If there is a κ-Souslin tree, then there is a streamlined κ-Souslin tree T such that $T \overset{\Delta}{\not\to} [\kappa]^{1}_{\kappa}$ holds, but weak $T \overset{\wedge}{\not\to} [\kappa]^{2}_{1}$ fails.*

Proof. Assuming there is a κ-Souslin tree, we may fix one such T that is *binary*, i.e., a streamlined 2-splitting subset of ${}^{<\kappa}2$ (see [BR17b, Lemma 2.4 and §7]). As T is a streamlined κ-Souslin tree, $T \overset{\Delta}{\not\to} [\kappa]^{1}_{\kappa}$ holds by [IR26a, Proposition 3.1], For each $t \in T$, let $\tilde{t} : \mathrm{dom}(t) \to 2$ be the unique function to satisfy that for every $\alpha < \mathrm{dom}(t)$:

$$\tilde{t}(\alpha) = 1 - t(\alpha).$$

Now, let $T' := \{t, \tilde{t} \mid t \in T\}$.

Claim 3.1.1. *T' is a streamlined κ-Souslin tree.*

Proof. This follows from the pigeonhole principle, as T' is the union of the Souslin tree T and the (isomorphic) Souslin tree $\{\tilde{t} \mid t \in T\}$. $\square$

We shall verify that $T' \overset{\wedge}{\not\to} [\kappa]^{2}_{1}$ fails by introducing a ht-rapid subfamily $X \subseteq {}^{2}T$ such that for all $x \neq y$ in X, Clause (b) of Definition 2.7(I) fails. For each $\alpha < \kappa$, pick $t_\alpha \in T_\alpha$ and let $x_\alpha := (t_\alpha{}^\frown\langle 0\rangle, \tilde{t}_\alpha{}^\frown\langle 0\rangle)$. Then $X := \{x_\alpha \mid \alpha < \kappa\}$ is ht-rapid. Now, given $x \neq y$ in X, consider $\alpha, \beta < \kappa$ such that $x = x_\alpha$ and $y = x_\beta$. Without loss of generality, $\alpha < \beta$. Put $\delta := \Delta(x_\alpha(0), x_\beta(0))$.

- If $\delta = \alpha{+}1$, then $x_\beta(0) <_{\mathrm{lex}} x_\alpha(0)$. It also follows that $\Delta(\tilde{t}_\alpha{}^\frown\langle 0\rangle, \tilde{t}_\beta) = \alpha$ with $(\tilde{t}_\alpha{}^\frown\langle 0\rangle)(\alpha) = 0 < 1 = \tilde{t}_\beta(\alpha)$, so that $x_\alpha(1) <_{\mathrm{lex}} x_\beta(1)$.
- If $\delta = \alpha$, then $t_\alpha{}^\frown\langle 1\rangle \subseteq t_\beta$, so $\Delta(t_\alpha{}^\frown\langle 0\rangle, t_\beta) = \alpha$ with $(t_\alpha{}^\frown\langle 0\rangle)(\alpha) = 0 < 1 = t_\beta(\alpha)$, so that $x_\alpha(0) <_{\mathrm{lex}} x_\beta(0)$. It also follows that $\tilde{t}_\alpha{}^\frown\langle 0\rangle \subseteq \tilde{t}_\beta$, and hence $x_\beta(1) <_{\mathrm{lex}} x_\alpha(1)$.
- If $\delta < \alpha$, then $\Delta(x_\alpha(0), x_\beta(0)) = \delta$ and $x_\alpha(0)(\delta) < x_\beta(0)(\delta)$ iff $x_\alpha(1)(\delta) > x_\beta(1)(\delta)$. So $x_\alpha(0) <_{\mathrm{lex}} x_\beta(0)$ iff $x_\beta(1) <_{\mathrm{lex}} x_\alpha(1)$.

This completes the proof. It follows from [IR26a, Proposition 3.1] that the tree T' is not 3-free. For the readers benefit, we give a direct proof of this conclusion, as follows. For every nonzero $\alpha < \kappa$, let

$$a_\alpha := \begin{cases} (t_\alpha{}^\frown\langle 0\rangle, \tilde{t}_\alpha{}^\frown\langle 0\rangle) & \text{if } t_\alpha(0) = 0; \\ (\tilde{t}_\alpha{}^\frown\langle 0\rangle, t_\alpha{}^\frown\langle 0\rangle) & \text{if } \tilde{t}_\alpha(0) = 0. \end{cases}$$

It is clear that $A := \{a_\alpha \mid \alpha < \kappa\}$ is a subset of $(T')^2$ consisting of nodes above $(\langle 0\rangle, \langle 1\rangle)$, and that A is an antichain. $\square$

Remark 3.2. Here is a sample corollary of the proof of the above lemma. Suppose that $2^{\aleph_0} = \aleph_1$ and let $\kappa := 2^{\aleph_1}$. Let $h : {}^{\leq\omega_1}\omega_1 \to \kappa$ be any $({<}\kappa)$-to-one map. Then weak $\mathbf{T} \overset{\wedge}{\not\to} [\kappa]^{2}_{1}$ fails, for $\mathbf{T} := ({}^{\leq\omega_1}\omega_1, \subsetneq, <_{\mathrm{lex}}, h)$. More generally, it tells us that when considering a tree $\mathbf{T}$ consisting of a weak

Kurepa tree adjoined by some of its cofinal branches, the choice of these branches needs to be an educated one in order for this negative partition relation to hold. A successful example may be found in Theorem 7.2 below.

Proposition 3.3. *Suppose that:*

- $(T,<)$ *is a* κ*-tree for* κ *a regular uncountable cardinal;*
- $\nu\in[2,\kappa)$ *is a cardinal;*
- *weak* $(T,<,\emptyset,h)\overset{\wedge}{\nrightarrow}[\kappa]^{<\nu}_2$ *holds, where* $h:T\rightarrow\kappa$ *is* $({<}\kappa)$*-to-one.*

Then $(T,<)$ *has no ascent path of nonzero width less than* ν*.*

Proof. By a pigeonhole argument, as in the proof of [IR26a, Lemma 2.11(1)]. $\square$

Lemma 3.4. *Suppose that:*

- T *is a streamlined* κ*-tree for* κ *a regular uncountable cardinal;*
- $\nu\in[2,\kappa)$ *is a cardinal such that* $\lambda^{<\nu}<\kappa$ *for every* $\lambda<\kappa$*;*
- *weak* $T\overset{\wedge}{\nrightarrow}[\kappa]^{<\nu}_1$ *holds.*

Then $(T,\subsetneq)$ *has no ascending path of nonzero width less than* ν*.*

Proof. Suppose not. This means that we may fix some nonzero $\sigma<\nu$ and a sequence $\langle f_\alpha\mid\alpha<\kappa\rangle$ such that for every $\alpha<\kappa$, f_α is a map from σ to T_α, and for all $\alpha<\beta<\kappa$, there are $i,j<\sigma$ such that $f_\alpha(i)\subsetneq f_\beta(j)$. Fix a regular cardinal $\chi\in(\sigma,\nu]$. For every $\alpha\in E^\kappa_\chi$, pick $\epsilon_\alpha<\alpha$ such that for all $i,j<\sigma$, if $f_\alpha(i)\neq f_\alpha(j)$, then $f_\alpha(i)\restriction\epsilon_\alpha\neq f_\alpha(j)\restriction\epsilon_\alpha$. Since $|\lambda^\sigma|<\kappa$ for every $\lambda<\kappa$, we may find a stationary $U\subseteq E^\kappa_\chi$, an $\epsilon<\kappa$ and a map $z\in{}^\sigma T_\epsilon$ such that for every $\alpha\in U$, $\langle f_\alpha(i)\restriction\epsilon_\alpha\mid i<\sigma\rangle=z$. Next, as $S:=\{f_\alpha\mid\alpha\in U\}$ is an ht-rapid family, we may fix $x\neq y$ in S such that the following two hold:

(a) for every $i<\sigma$, $c(x(i)\wedge y(i))=0$;
(b) for every $i<\sigma$, if $x(i)<_{\text{lex}}y(i)$ then $\operatorname{ht}(x(i))<\operatorname{ht}(y(i))$.

Fix $\alpha,\beta<\kappa$ such that $x=f_\alpha$ and $y=f_\beta$. Without loss of generality, $\alpha<\beta$. Fix $i,j<\sigma$ such that $f_\alpha(i)\subsetneq f_\beta(j)$. In particular, $z(i)=f_\alpha(i)\restriction\epsilon=f_\beta(j)\restriction\epsilon=z(j)$. So $f_\beta(i)\restriction\epsilon_\beta=f_\beta(j)\restriction\epsilon_\beta$ and hence $f_\beta(i)=f_\beta(j)$. Altogether, $f_\alpha(i)\subsetneq f_\beta(j)=f_\beta(i)$, and hence $f_\beta(i)<_{\text{lex}}f_\alpha(i)$, contradicting Clause (b). $\square$

Corollary 3.5. *If* $\mathsf{V}=\mathsf{L}$*, then for every regular uncountable cardinal* κ*, for every infinite* $\nu\leq\operatorname{cf}(\sup(\operatorname{Reg}(\kappa)))$*, there exists a streamlined* κ*-tree such that* $T\overset{\Delta}{\nrightarrow}[\kappa]^{<\nu,1}_\kappa$ *holds and weak* $T\overset{\wedge}{\nrightarrow}[\kappa]^\nu_1$ *fails.*

Proof. Appeal to [BR17b, Theorem 1.8] with $(\theta,\eta,X):=(\nu,\nu,{}^{<\kappa}1)$. $\square$

Corollary 3.6. *Assuming the consistency of a weakly compact cardinal, it is consistent that all of the following hold:*

1. *there exists an* $\aleph_2$*-Aronszajn tree;*
2. *weak* $T\overset{\wedge}{\nrightarrow}[\aleph_2]^{\aleph_0}_1$ *fails for every streamlined* $\aleph_2$*-Aronszajn tree* T*;*

(3) *weak* $(T,<,\emptyset,h) \overset{\Diamond}{\not\to} [\aleph_2]^{\aleph_0}_2$ *fails for every* $\aleph_2$*-Aronszajn tree* $(T,<)$ *and every* $\aleph_1$*-to-one map* $h : T \to \aleph_2$.

Proof. By [LHL18, Theorem 1.3], Proposition 3.3 and Lemma 3.4. $\square$

4. Entangled linear orders

In [Tod85, p. 298], Todorčević remarks that a $(\kappa,3)$-entangled order of size κ has density smaller than κ, and Shelah–Shafir show [SS00, Lemma 2.1] that an oriented $(\kappa,2)$-entangled order (in the sense of Footnote 1) of size κ has density smaller than κ. It thus follows from Remark 2.2 that all κ-sized subsets of the lexicographic ordering of a κ-Aronszajn tree are neither $(\kappa,3)$-entangled linear orders nor oriented $(\kappa,2)$-entangled linear orders. Bearing in mind these limitations, in this section we shall obtain:

(i) A $(\kappa,2)$-entangled κ-Souslin line from a lexicographic ordering of a 3-free κ-Souslin tree;
(ii) A $(\kappa,{<}\nu)$-entangled linear order from a lexicographic ordering of a tree of size κ and whose height and width are smaller than κ;
(iii) A (κ,n)-entangled linear order that is not $(\kappa,n+1)$-entangled for a prescribed positive integer n;
(iv) A weakly $(\kappa,{<}\nu)$-entangled linear order (see Definition 4.5 below) from a lexicographic ordering of a κ-Aronszajn tree.

Clauses (ii)–(iv) will make use of the partition relations of Definition 2.7. The first two will moreover demonstrate the utility of the fourth parameter h in this definition. Note that in Clauses (ii) and (iv), once ν is infinite, $(\kappa,{<}\nu)$-entangled orders are moreover oriented $(\kappa,{<}\nu)$-entangled.

4.1. Entangled Souslin line.

In [Kru20], Krueger gave a forcing construction of an $(\aleph_1,2)$-entangled $\aleph_1$-Souslin line. Recently, in [CLN26] Carroy, Levine, and Notaro constructed such a line from $\Diamond$. In this short subsection, we obtain such a line from any 3-free $\aleph_1$-Souslin tree. More generally, for every regular uncountable cardinal κ, the existence of a 3-free κ-tree implies the existence of a $(\kappa,2)$-entangled κ-Souslin line.[5] To simplify the proof of the upcoming Theorem 4.2, we commence with the following lemma.

Lemma 4.1. *For every infinite cardinal* κ*, there is a linear ordering* $\prec$ *of* κ *such that for every* $\alpha<\kappa$*, there is an* $n<\omega$ *with* $n \prec \alpha \prec n+1$.

Proof. The ordering is illustrated in Figure 1 below. Formally, given $i \neq j$ in κ, we let $i \prec j$ iff any of the following occurs:

- i and j are infinite and $i<j$;
- i and j are odd finite and $i<j$;

[5] A sufficient condition for the existence of a free κ-Souslin tree is the proxy principle $\mathrm{P}(\kappa,\kappa,\sqsubseteq,\kappa)$ [BR21, Theorem 6.27]. A fine control on the extent of freeness is available assuming stronger proxy principles (see [BR17b, Theorem 1.8] for an infinite extent and [BRY26, Theorem 4.10] for a finite extent). At the level of $\kappa=\aleph_1$, the strongest proxy principle already follows from $\Diamond$ [BR17a, Theorem 3.6(1)].

- i and j are even finite and $j < i$;
- i is even and finite, whereas j is not;
- j is odd and finite, whereas i is not.

The verification is left to the reader. □

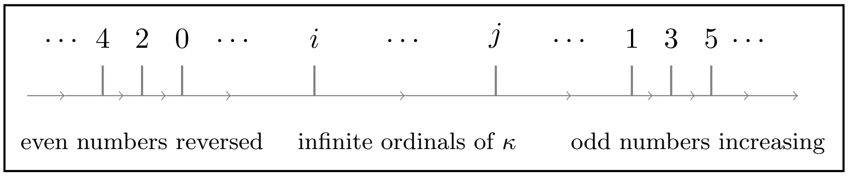

FIGURE 1. The linear order $(\kappa, \prec)$ defined in Lemma 4.1.

Theorem 4.2. *Let κ denote a regular uncountable cardinal. Suppose there is a 3-free κ-Souslin tree. Then there is a $(\kappa, 2)$-entangled κ-Souslin line.*

Proof. As seen in the proof of [RS23, Proposition 2.16], the restriction of any κ-Souslin tree to a sparse enough club of levels is infinitely-splitting, so we may let $\mathbf{T}$ be an infinitely-splitting 3-free normal κ-Souslin tree. By a straightforward refinement of the procedure of [BR21, Lemma 2.5(2)], we may also assume that $\mathbf{T} = (T, \subsetneq)$ where T is a downward-closed subfamily of ${}^{<\kappa}\kappa$ satisfying that for every $t \in T$, $\{\iota < \kappa \mid t^\frown\langle\iota\rangle \in T\}$ is an (infinite) ordinal. Let $\prec$ be a linear ordering of κ as in Lemma 4.1. Define a lexicographic ordering $\lhd$ of T as follows:

- for all $t \subsetneq s$ in T, let $s \lhd t$;
- for all incomparable s and t in T, consider $\delta := \Delta(s, t)$ and let $s \lhd t$ iff $s(\delta) \prec t(\delta)$.

By a standard fact (see [Tod84, Lemma 6.2]), we may fix an $L \in [T]^\kappa$ such that $\mathbf{L} := (L, \lhd)$ is a κ-Souslin line. To see that $\mathbf{L}$ is $(\kappa, 2)$-entangled, let $X \subseteq {}^2L$ be a family of κ-many pairwise disjoint injective 2-tuples. We have two tasks:

(i) find $x \neq y$ in X such that $x(0) \lhd y(0)$ and $x(1) \lhd y(1)$, and
(ii) find $x \neq y$ in X such that $x(0) \lhd y(0)$ and $y(1) \lhd x(1)$.

Case 1: Suppose that there are κ-many $x \in X$ such that $x(0)$ and $x(1)$ are comparable. In this case, by possibly flipping X, we may assume that $x(0) \subsetneq x(1)$ for κ many $x \in X$. By the pigeonhole principle and by the choice of $\prec$, we may moreover assume the existence of an $n < \omega$ for which the set X' of all $x \in X$ such that:

- $x(0) \subsetneq x(1)$, and
- $n \prec x(1)(\mathrm{dom}(x(0))) \prec n + 1$,

has size κ. Now, back to our two tasks:

(i) Since $A := \{x(0)^\frown\langle n\rangle \mid x \in X'\}$ is a κ-sized subset of T, and T is a κ-Souslin tree, we may find a pair $b \subsetneq a$ of elements of A. Pick $x, y \in X'$ such that $a = x(0)^\frown\langle n\rangle$ and $b = y(0)^\frown\langle n\rangle$, so that

$$y(0)^\frown\langle n\rangle \subseteq x(0) \subsetneq x(1).$$

In particular, $x(0) \lhd y(0)$. Since $y(0)^\frown\langle\iota\rangle \subseteq y(1)$ for some ι with $n \prec \iota \prec n+1$, we get that $y(1) \wedge x(1) = y(0)$, and taking $\delta := \operatorname{ht}(y(0))$, we have $x(1)(\delta) = n \prec y(1)(\delta)$, so that $x(1) \lhd y(1)$.

(ii) Since $A := \{x(0)^\frown\langle n+1\rangle \mid x \in X'\}$ is a κ-sized subset of T, we may pick $x, y \in X'$ such that

$$y(0)^\frown\langle n+1\rangle \subseteq x(0) \subsetneq x(1).$$

In particular, $x(0) \lhd y(0)$. Since $y(0)^\frown\langle\iota\rangle \subseteq y(1)$ for some ι with $n \prec \iota \prec n+1$, we get that $y(1) \wedge x(1) = y(0)$, and taking $\delta := \operatorname{ht}(y(0))$, we have $y(1)(\delta) \prec n+1 = x(1)(\delta)$, so that $y(1) \lhd x(1)$.

<u>Case 2:</u> Suppose that $x(0)$ is incomparable with $x(1)$ for every $x \in X$. Consider $D := \{\Delta(x(0), x(1)) \mid x \in X\}$. There are two subcases to consider:
<u>Case 2.1:</u> Suppose that D is cofinal in κ. By possibly flipping and shrinking X, we may assume that $x(0) \lhd x(1)$ for every $x \in X$. Fix an injective sequence $\langle x_\alpha \mid \alpha < \kappa\rangle$ of elements of X and satisfying that for every $\beta < \kappa$,

$$\sup\{\operatorname{dom}(x_\alpha(i)) \mid \alpha < \beta, i < 2\} < \Delta(x_\beta(0), x_\beta(1)).$$

Back to our two tasks:

(i) Since T is Souslin, we may find a pair $\alpha < \beta$ of ordinals in κ such that $x_\alpha(0) \subsetneq x_\beta(0)$. In particular, $x_\beta(0) \lhd x_\alpha(0)$ and $x_\alpha(0) \subsetneq x_\beta(0) \wedge x_\beta(1) \subsetneq x_\beta(1)$. So $x_\beta(1) \wedge x_\alpha(1) = x_\alpha(0) \wedge x_\alpha(1)$ and then $x_\alpha(0) \lhd x_\alpha(1)$ implies $x_\beta(1) \lhd x_\alpha(1)$.

(ii) Since T is Souslin, we may find a pair $\alpha < \beta$ of ordinals in κ such that $x_\alpha(1) \subsetneq x_\beta(1)$. In particular, $x_\beta(1) \lhd x_\alpha(1)$ and $x_\alpha(1) \subsetneq x_\beta(0) \wedge x_\beta(1) \subsetneq x_\beta(0)$. So $x_\alpha(0) \wedge x_\beta(0) = x_\alpha(0) \wedge x_\alpha(1)$ and then $x_\alpha(0) \lhd x_\alpha(1)$ implies $x_\alpha(0) \lhd x_\beta(0)$.

<u>Case 2.2:</u> Suppose that D is not cofinal in κ. By possibly flipping and shrinking X, we may assume the existence of $\delta < \kappa$ and $z \in T_{\delta+1} \times T_{\delta+1}$ such that for all $x \in X$ and $i < 2$, $x(i) \restriction (\Delta(x(0), x(1)) + 1) = z(i)$. As T is 3-free, $T^* := z(0)^\uparrow \otimes z(1)^\uparrow$ is a κ-Souslin tree with respect to the order $\subseteq^2$. Fix an injective sequence $\langle x_\alpha \mid \alpha < \kappa\rangle$ of elements of X and satisfying that for every $\beta < \kappa$,

$$\sup\{\beta, \operatorname{dom}(x_\alpha(i)) \mid \alpha < \beta, i < 2\} < \min\{\operatorname{dom}(x_\beta(i)) \mid i < 2\}.$$

Back to our two tasks:

(i) As T is normal, for each $\alpha < \kappa$, let us pick $x'_\alpha \in T \otimes T$ such that $x_\alpha(0) \subseteq x'_\alpha(0)$, $x_\alpha(1) \subseteq x'_\alpha(1)$ and

$$\operatorname{dom}(x'_\alpha(0)) = \operatorname{dom}(x'_\alpha(1)) = \max\{\operatorname{dom}(x_\alpha(i)) \mid i < 2\}.$$

As $\{(x'_\alpha(0), x'_\alpha(1)) \mid \alpha < \kappa\}$ is a κ-sized subset of T^*, we may find a pair $\alpha < \beta$ of ordinals in κ such that $x'_\alpha \subsetneq^2 x'_\beta$. In particular, $x_\alpha(0) \subsetneq x_\beta(0)$ and $x_\alpha(1) \subsetneq x_\beta(1)$. Consequently, $x_\beta(0) \lhd x_\alpha(0)$ and $x_\beta(1) \lhd x_\alpha(1)$.

(ii) As $P := \{(x_\alpha(0) \restriction \alpha, x_\alpha(1) \restriction \alpha)) \mid \alpha < \kappa\}$ is a κ-sized subset of T^*, a standard fact (see [RS23, Lemma 2.13]) provides a $(t_0, t_1) \in T^*$ that has κ many extensions in T^* such that any extension of (t_0, t_1) in T^* is extended by some element of P. As T is infinitely-splitting, for each $i < 2$, we may fix a pair $t_{i0} \lhd t_{i1}$ of immediate successors of t_i. Finally, find a pair $p \in P$ that extends (t_{00}, t_{11}) and a pair $q \in P$ that extends (t_{01}, t_{10}). Find $\alpha \neq \beta$ in κ such that $p(i) \subsetneq x_\alpha(i)$ and $q(i) \subsetneq x_\beta(i)$ for every $i < 2$. Then $x_\alpha(0) \lhd x_\beta(0)$ and $x_\beta(1) \lhd x_\alpha(1)$.

This completes the proof. $\square$

4.2. **Entangled linear orders.** The next result establishes Clause (1) of Theorem A.

Lemma 4.3. *Suppose that:*

- $T \subseteq {}^{\leq\lambda}\lambda$ *is a streamlined tree of size* κ*;*
- $|T \cap {}^{<\lambda}\lambda| = \lambda = \lambda^{<\nu} < \mathrm{cf}(\kappa)$*;*
- $h : T \to \kappa$ *is a* $(<\kappa)$*-to-one map;*
- *weak* $(T, \subsetneq, <_{\mathrm{lex}}, h) \overset{\wedge}{\not\to} [\kappa]^{<\nu,<\mu}_\lambda$ *holds.*

Then:

(0) *weak* $(T, \subsetneq, \emptyset, h) \overset{\wedge}{\not\to} [\kappa]^1_{\lambda^+}$ *fails;*

(1) $(T, \subsetneq, <_{\mathrm{lex}}, h) \overset{\wedge}{\not\to} [\kappa]^{<\nu,<\mu}_\lambda$ *holds;*

(2) *some lexicographic ordering of* T *makes it a* $(\kappa, <\nu)$*-entangled order.*

Proof. We first dispose of Clause (0): for all $s \neq t$ in T, it is the case that $s \wedge t$ belongs to $T \cap {}^{<\lambda}\lambda$, and hence weak $(T, \subsetneq, \emptyset, h) \overset{\wedge}{\not\to} [\kappa]^1_\theta$ fails for every cardinal $\theta > |T \cap {}^{<\lambda}\lambda|$, in particular, for $\theta := \lambda^+$.

Next, let $c : T \to \lambda$ be a witness for weak $(T, \subsetneq, <_{\mathrm{lex}}, h) \overset{\wedge}{\not\to} [\kappa]^{<\nu,<\mu}_\lambda$. Using $\lambda^{<\nu} = \lambda$, fix a bijection

$$\pi : \lambda \leftrightarrow \bigcup\{{}^z\lambda \mid z \subseteq T \cap {}^{<\lambda}\lambda \ \&\ |z| < \nu\}.$$

Derive a new colouring $\hat{c} : T \to \lambda$, as follows. Given $s \in T$, consider $g := \pi(c(s))$, and then, if there exists a unique $t \in \mathrm{dom}(g)$ such that $t \subseteq s$, then let $\hat{c}(s) := g(t)$ for this t. Otherwise, let $\hat{c}(s) := 0$.

Claim 4.3.1. *For every nonzero* $\sigma < \nu$*, for every* h*-rapid* $S \subseteq {}^\sigma T$ *of size* κ*, and every* $\tau \in {}^\sigma\lambda$*, there are* $x \neq y$ *in* S *such that the following three hold:*

(a) *for every* $i < \sigma$*,* $\hat{c}(x(i) \wedge y(i)) = \tau(i)$*;*

(b) *for every* $i < \sigma$*, if* $x(i) <_{\mathrm{lex}} y(i)$ *then* $h(x(i)) < h(y(i))$*;*

(c) $|\{h(x(i) \wedge y(i)) \mid i < \sigma\}| < \mu$*.*

Proof. Let σ, S and τ be as above. As $|T \cap {}^{<\lambda}\lambda| < \kappa$, and by possibly passing to a κ-sized subset of S, we may assume that $S \subseteq {}^\sigma(T_\lambda)$. It follows that for every $x \in S$, we may fix a large enough $\beta_x < \lambda$ such that $i \mapsto x(i) \restriction \beta_x$ is injective over σ. Fix $\beta < \lambda$ and $z \in {}^\sigma(T_\beta)$ for which the following set has size κ:

$$S' := \{x \in S \mid \langle x(i) \restriction \beta_x \mid i < \sigma\rangle = z\}.$$

Define a map $g : \mathrm{Im}(z) \to \lambda$ via $g(z(i)) := \tau(i)$.

As c witnesses weak $(T, \subsetneq, <_{\mathrm{lex}}, h) \xrightarrow{\wedge}\!\!\!\!\!\to [\kappa]^{<\nu,<\mu}_\lambda$, we may now find $x \neq y$ in S' such that all of the following hold:

- for every $i < \sigma$, $c(x(i) \wedge y(i)) = \pi^{-1}(g)$;
- for every $i < \sigma$, if $x(i) <_{\mathrm{lex}} y(i)$ then $h(x(i)) < h(y(i))$;
- $|\{h(x(i) \wedge y(i)) \mid i < \sigma\}| < \mu$.

By the definition of S', for every $i < \sigma$,

$$\{j < \sigma \mid z(j) \subseteq x(j)\} = \{i\} = \{j < \sigma \mid z(j) \subseteq y(j)\},$$

and hence $\hat{c}(x(i) \wedge y(i)) = g(z(i)) = \tau(i)$, as sought. □

(1) By the claim, $\hat{c}$ witnesses (a strong form of) $(T, \subsetneq, <_{\mathrm{lex}}, h) \xrightarrow{\wedge}\!\!\!\!\!\to [\kappa]^{<\nu,<\mu}_\lambda$.
(2) Define a lexicographic ordering $\lhd$ of T as follows:

- for all $t \subsetneq s$ in T, let $s \lhd t$;
- for all incomparable s and t in T, consider $\delta := \Delta(s, t)$:
  - if $\hat{c}(s \wedge t) = 1$, then $s \lhd t$ iff $s(\delta) < t(\delta)$;
  - if $\hat{c}(s \wedge t) \neq 1$, then $s \lhd t$ iff $s(\delta) > t(\delta)$.

To see that $(T, \lhd)$ is $(\kappa, {<}\nu)$-entangled, fix a nonzero $\sigma < \nu$, a κ-sized $S \subseteq {}^\sigma T$ consisting of pairwise disjoint injective σ-tuples, and some type $\tau : \sigma \to 2$. As $\sigma < \mathrm{cf}(\kappa)$ and h is $({<}\kappa)$-to-one, by possibly shrinking S, we may assume it is h-rapid. Then, as before, we may also assume that $S \subseteq {}^\sigma(T_\lambda)$. Using Claim 4.3.1, pick $x \neq y$ in S such that the following two hold:

(a) for every $i < \sigma$, $\hat{c}(x(i) \wedge y(i)) = \tau(i)$;
(b) for every $i < \sigma$, if $x(i) <_{\mathrm{lex}} y(i)$ then $h(x(i)) < h(y(i))$.

As S is h-rapid, we may assume without loss of generality that $\sup\{h(x(i)) \mid i < \sigma\} < \min\{h(y(i)) \mid i < \sigma\}$. Consider an arbitrary $i < \sigma$. By Clause (b), $x(i) <_{\mathrm{lex}} y(i)$. As $\mathrm{dom}(x(i)) = \lambda = \mathrm{dom}(y(i))$, Definition 2.1 implies that $x(i)(\delta) < y(i)(\delta)$ for $\delta := \Delta(x(i), y(i))$. Then by definition of $\lhd$, $x(i) \lhd y(i)$ iff $\hat{c}(x(i) \wedge y(i)) = 1$. So, by Clause (a), $x(i) \lhd y(i)$ iff $\tau(i) = 1$, as sought. □

The next result establishes Clause (2) of Theorem A.

Lemma 4.4. *Suppose that:*

- $T \subseteq {}^{\leq\lambda}\lambda$ *is a uniformly homogeneous streamlined tree of size* κ*;*
- $|T \cap {}^{<\lambda}\lambda| = \lambda < \mathrm{cf}(\kappa)$*;*
- $h : T \to \kappa$ *is a* $({<}\kappa)$*-to-one map;*
- $(T, \subsetneq, <_{\mathrm{lex}}, h) \xrightarrow{\Delta}\!\!\!\!\!\to [\kappa]^n_\lambda$ *holds for a given positive integer* n*.*

Then there is a (κ, n)*-entangled not* $(\kappa, n+1)$*-entangled order of size* κ*.*

Proof. As $|T\cap {}^{<\lambda}\lambda|<\kappa$, we may let $\langle f_\alpha\mid \alpha<\kappa\rangle$ be an injective sequence of elements of T_λ such that $|\{\iota<\lambda\mid f_\alpha(\iota)\neq f_\beta(\iota)\}|=\lambda$ for all $\alpha<\beta<\kappa$. Fix a large enough $\epsilon<\lambda$ such that $\{f_m\restriction\epsilon\mid m<n+1\}$ has size $n+1$. Let Z be the collection of all $z\in[T]^{\le n}$ such that:

- $(z,\subseteq)$ is an antichain;
- $\{\mathrm{dom}(t)\mid t\in z\}\subseteq[\epsilon,\lambda)$;
- for some $m<n+1$, $f_m\restriction\epsilon\in z$.

As $|T\cap{}^{<\lambda}\lambda|=\lambda$, we may fix a bijection $\pi:\lambda\leftrightarrow Z$. Let $d:\mathrm{Im}(h)\to\lambda$ be a witness for $(T,\subsetneq,<_{\mathrm{lex}},h)\xrightarrow{\Delta}\!\!\!\!\!\not\,\,\,\,[\kappa]^n_\lambda$. Derive a colouring $c:T\to 2$, as follows. Given $s\in T$, consider $z:=\pi(d(h(s)))$, and then, if there exists a $t\in z$ such that $t\subseteq s$, then let $c(s):=1$. Otherwise, let $c(s):=0$. Lexicographically order T_λ, as follows. Given $s\neq t$ in T_λ, set $\delta:=\Delta(s,t)$, and:

- if $c(s\wedge t)=1$, then $s\lhd t$ iff $s(\delta)<t(\delta)$;
- if $c(s\wedge t)=0$, then $s\lhd t$ iff $s(\delta)>t(\delta)$.

We start by showing that $(T_\lambda,\lhd)$ is (κ,n)-entangled.

Claim 4.4.1. *For every κ-sized $S\subseteq{}^n(T_\lambda)$ consisting of pairwise disjoint injective n-tuples, for every $\tau:n\to 2$, there are $x\neq y$ in S such that for every $i<2$, $x(i)\lhd y(i)$ iff $\tau(i)=1$.*

Proof. Let S and τ be as above. By possibly shrinking S, we may assume it is h-rapid. By symmetry, we may assume that $|\tau^{-1}\{0\}|\ge|\tau^{-1}\{1\}|$, so that $I:=\{i<n\mid\tau(i)=1\}$ has size less than n. For every $x\in S$, fix a large enough $\beta_x\in[\epsilon,\lambda)$ such that $i\mapsto x(i)\restriction\beta_x$ is injective over n. Fix $\beta<\lambda$ and $v\in{}^n(T_\beta)$ for which the following set has size κ:

$$S':=\{x\in S\mid\langle x(i)\restriction\beta_x\mid i<n\rangle=v\}.$$

As $\{f_m\restriction\epsilon\mid m<n+1\}$ is an antichain of size greater than n, we may fix an $m<n+1$ such that $f_m\restriction\epsilon$ is not extended by $v(i)$ for every $i<n$. It follows that the following set is in Z:

$$z:=\{v(i)\mid i\in I\}\cup\{f_m\restriction\epsilon\}.$$

As d witnesses $(T,\subsetneq,<_{\mathrm{lex}},h)\xrightarrow{\Delta}\!\!\!\!\!\not\,\,\,\,[\kappa]^n_\lambda$, we may now find $x\neq y$ in S' such that all the following two hold:

(a) for every $i<n$, $d(h(x(i)\wedge y(i)))=\pi^{-1}(z)$;
(b) for every $i<n$, if $x(i)<_{\mathrm{lex}}y(i)$ then $h(x(i))<h(y(i))$.

As S' is h-rapid, we may assume without loss of generality that $\sup\{h(x(i))\mid i<\sigma\}<\min\{h(y(i))\mid i<\sigma\}$. By the definition of S', for every $i<n$,

$$\{j<n\mid v(j)\subseteq x(j)\}=\{i\}=\{j<n\mid v(j)\subseteq y(j)\}.$$

In addition, for every $i<n$, $f_m\restriction\epsilon$ is not extended by $v(i)$, let alone by $x(i)\wedge y(i)$. Altogether, and recalling the definition of z, for every $i<n$, $c(x(i)\wedge y(i))=1$ iff $i\in I$.

Let $i < n$. By Clause (b), $x(i) <_{\text{lex}} y(i)$. Definition 2.1 then implies that $x(i)(\delta) < y(i)(\delta)$ for $\delta := \Delta(x(i), y(i))$. Then by definition of $\lhd$, $x(i) \lhd y(i)$ iff $c(x(i) \wedge y(i)) = 1$ iff $i \in I$ iff $\tau(i) = 1$, as sought. $\square$

We now turn to show that $(T, \lhd)$ is not $(\kappa, n+1)$-entangled. For every $\alpha < \kappa$, let

$$x_\alpha := \langle (f_m \restriction \epsilon) * f_\alpha \mid m < n+1 \rangle.$$

Then $S := \{x_\alpha \mid \alpha < \kappa\}$ is a pairwise disjoint subfamily of ${}^n(T_\lambda)$ of size κ.

Claim 4.4.2. *The constant type $\tau : n+1 \to \{1\}$ is not realized over S.*

Proof. Suppose not, and pick $\alpha \neq \beta$ such that $x_\alpha(i) \lhd x_\beta(i)$ for every $i < n+1$. Set $\delta := \Delta(x_\alpha(0), x_\beta(0))$, $z := \pi(h(\delta))$, $\xi := x_\alpha(0)(\delta)$, and $\zeta := x_\beta(0)(\delta)$. Recalling the definition of x_α and x_β, we infer that for every $i < n+1$:

(1) $\Delta(x_\alpha(i), x_\beta(i)) = \delta \geq \epsilon$;
(2) $\pi(h(d(x_\alpha(i) \wedge x_\beta(i)))) = z$;
(3) $x_\alpha(i)(\delta) = \xi$ and $x_\beta(i)(\delta) = \zeta$.

Now, there are two options to consider:

▶ If $\xi < \zeta$, then the definition of $\lhd$ implies that $c(x_\alpha(i) \wedge x_\beta(i)) = 1$ for every $i < n+1$. By Clause (2) together with the definition of c, for every $i < n+1$, we may fix a $t_i \in z$ that is extended by $x_\alpha(i) \wedge x_\beta(i)$. As $|z| \leq n$, let us fix $i < j < n+1$ such that $t_i = t_j$. As $\{\mathrm{dom}(t) \mid t \in z\} \subseteq [\epsilon, \lambda)$, it follows that $(x_\alpha(i) \wedge x_\beta(i)) \restriction \epsilon = (x_\alpha(j) \wedge x_\beta(j)) \restriction \epsilon$, contradicting the fact that the former is $f_i \restriction \epsilon$ and the latter is $f_j \restriction \epsilon$.

▶ Otherwise, the definition of $\lhd$ implies that $c(x_\alpha(i) \wedge x_\beta(i)) = 0$ for every $i < n+1$. However, $z \in Z$, so there is an $m < n+1$ such that $f_m \restriction \epsilon \in z$, in which case $c(x_\alpha(m) \wedge x_\beta(m)) = 1$. This is a contradiction. $\square$

This completes the proof. $\square$

4.3. **Weakly entangled linear orders.** The next definition is a natural generalisation of the non-oriented version of [Kru20, Definition 3.7].

Definition 4.5. For two cardinals κ, ν, a linear order $\mathbf{L} = (L, <_L)$ is said to be *weakly $(\kappa, {<}\nu)$-entangled* iff given

- a map $\tau : \sigma \to 2$ with $\sigma < \nu$,
- a sequence $\langle I_i \mid i < \sigma \rangle$ of pairwise disjoint open intervals of $\mathbf{L}$, and
- a κ-sized family $X \subseteq {}^\sigma L$ consisting of pairwise disjoint injective σ-tuples satisfying $x(i) \in I_i$ for all $x \in X$ and $i < \sigma$,

there are $x \neq y$ in X such that for every $i < \sigma$,

$$x(i) <_L y(i) \text{ iff } \tau(i) = 1.$$

The next lemma is proved in a similar way to that of Lemma 4.3:

Lemma 4.6. *Suppose that:*

- *κ is a regular uncountable cardinal;*

- $\nu \in [2, \kappa)$ *is a cardinal such that* $\lambda^{<\nu} < \kappa$ *for every* $\lambda < \kappa$*;*
- $T \subseteq {}^{<\kappa}\kappa$ *is a streamlined* κ*-tree such that weak* $T \overset{\wedge}{\not\to} [\kappa]^{<\nu}_{\kappa}$ *holds.*

Then some lexicographic ordering of T *makes it a weakly* $(\kappa, {<}\nu)$*-entangled order.* □

Corollary 4.7. *If there is* ν*-free* κ*-Souslin tree, then there is a weakly* $(\kappa, {<}\nu)$*-entangled* κ*-Souslin line.*

Proof. By [IR26a, Proposition 3.1] and Lemma 4.6. □

5. Chain condition

In [Tod18, Cor 7], Todorčević proved that CH gives rise to two countably closed $\aleph_2$-cc posets whose product is not $\aleph_2$-cc. The proof generalises to show that if $\lambda = \lambda^{<\lambda}$ is an uncountable cardinal smaller than the first measurable cardinal, then there are two λ-closed λ^+-cc posets whose product is not λ^+-cc. Here we present yet another easy variation of that proof that factors through Definition 2.7. A sample corollary reads as follows.

Corollary 5.1. *If* λ *is a strongly inaccessible cardinal that is not weakly compact, then for every pair* $\vartheta < \theta$ *of nonzero cardinals smaller than* λ *there is a* λ*-complete poset* $\mathbb{P}$ *such that* $\mathbb{P}^{\vartheta}$ *has the* λ^+*-cc, but* $\mathbb{P}^{\theta}$ *does not.*

Proof. Appeal to Lemma 5.2 with $(\kappa, \nu) := (\lambda^+, \lambda)$ and the λ^+-Aronszajn tree T provided by Theorem 2.10. □

On first reading of the next lemma, we recommend focusing on the special case where $(\nu, \vartheta, \theta) := (\omega, n, n+1)$ for some positive integer n.

Lemma 5.2. *Suppose that:*

1. T *is a streamlined tree of regular size* κ*;*
2. ν *is an infinite cardinal and* $\lambda^{<\nu} < \kappa$ *for every* $\lambda < \kappa$*;*
3. $\vartheta < \theta$ *are nonzero cardinals smaller than* κ *with* $\vartheta < \mathrm{cf}(\nu)$*;*
4. $h : T \to \kappa$ *is a* $({<}\kappa)$*-to-one map;*
5. *weak* $(T, \subsetneq, <_{\mathrm{lex}}, h) \overset{\wedge}{\not\to} [\kappa]^{<\nu}_{\theta}$ *holds.*

Then there is a poset $\mathbb{P}$ *such that all of the following hold:*

- (i) $\mathbb{P}$ *has size* κ*;*
- (ii) $\mathbb{P}$ *is* $({<}\nu)$*-compact, in particular it is* ν*-closed;*
- (iii) $\mathbb{P}^{\vartheta}$ *has the* κ*-cc;*
- (iv) $\mathbb{P}^{\theta}$ *does not have the* κ*-cc.*

Proof. We simply combine Todorčević's proof [Tod18] with that of [LHR18, Lemma 3.3]. Let $c : T \to \theta$ witness that weak $(T, \subsetneq, <_{\mathrm{lex}}, h) \overset{\wedge}{\not\to} [\kappa]^{<\nu}_{\theta}$ holds. Define $\mathbb{P} = (P, \leq)$, as follows:

- P is the collection of all pairs (τ, a) such that $\tau < \theta$, $a \in [T]^{<\nu}$, and for all $s \neq t$ in a, $c(s \wedge t) \neq \tau$;
- $(\tau', a') \leq (\tau, a)$ iff $\tau' = \tau$ and $a' \supseteq a$.

It is clear that $\mathbb{P}$ has size κ and that any centered subset of $\mathbb{P}$ of size less then ν is bounded, that is, $\mathbb{P}$ is $(<\nu)$-compact.

Claim 5.2.1. *$\mathbb{P}^\vartheta$ has the κ-cc.*

Proof. Let $Q \subseteq P^\vartheta$ of size κ be given. For each $q \in Q$, write q as $\langle(\tau_j^q, a_j^q) \mid j < \vartheta\rangle$. By possibly shrinking Q, we may assume the existence of a sequence $\langle\tau_j \mid j < \vartheta\rangle$ such that, every $j < \vartheta$:

- for every $q \in Q$, $\tau_j^q = \tau_j$;
- $\{a_j^q \mid q \in Q\}$ forms a Δ-system with some root r_j;
- $\{\bigcup_{j<\vartheta}(a_j^q \setminus r_j) \mid q \in Q\}$ is a pairwise disjoint family of size κ consisting of sets of some fixed size $\mu < \nu$.

Denote $\lambda := \hat{\mathrm{ht}}(T)$, and consider the following two options:

Case 1: If $\lambda = \kappa$, then hypothesis (2) implies that $|T_\alpha| < \kappa$ for every $\alpha < \kappa$. Fix a regular cardinal $\chi \in (\mu, \nu]$. For each $\alpha \in E^\kappa_\chi$, pick $q_\alpha \in Q$ such that $\mathrm{ht}(t) \geq \alpha$ for every $t \in \bigcup_{j<\vartheta}(a_j^{q_\alpha} \setminus r_j)$, and then, for every $j < \vartheta$, let $x_j^\alpha : \sigma_j^\alpha \to T_\alpha$ be some injective enumeration of $\{t \restriction \alpha \mid t \in a_j^{q_\alpha} \setminus r_j\}$.

Case 2: If $\lambda < \kappa$, then hypothesis (2) implies that $|\bigcup_{\alpha<\lambda} T_\alpha| < \kappa$, so in this case by possibly shrinking Q further, we may assume that $(a_j^q \setminus r_j) \subseteq T_\lambda$ for all $q \in Q$ and $j < \vartheta$. Set $\chi := \omega$. For each $\alpha \in E^\kappa_\chi$, pick $q_\alpha \in Q$ such that $h(t) \geq \alpha$ for every $t \in \bigcup_{j<\vartheta}(a_j^{q_\alpha} \setminus r_j)$, and then, for every $j < \vartheta$, fix a bijection $x_j^\alpha : \sigma_j^\alpha \leftrightarrow a_j^{q_\alpha} \setminus r_j$.

Our next step uses hypotheses (2) and (4) to fix a stationary $U \subseteq E^\kappa_\chi$, $\epsilon < \lambda$, and a sequence $\langle z_j \mid j < \vartheta\rangle$ such that for every $j < \vartheta$:

(1) z_j is an injection from some ordinal σ_j to T_ϵ;
(2) for every $\alpha \in U$, $\langle x_j^\alpha(i) \restriction \epsilon \mid i < \sigma_j^\alpha\rangle = z_j$;
(3) for every pair $\alpha < \beta$ of ordinals from U,

$$\sup\{h(x_j^\alpha(i)) \mid i < \sigma_j\} < \min\{h(x_j^\beta(i)) \mid i < \sigma_j\}.$$

Finally, let $\tau^* := \min(\theta \setminus \{\tau_j \mid j < \vartheta\})$, and then pick $\alpha \neq \beta$ in U such that the following two hold:

(a) for all $j < \vartheta$ and $i < \sigma_j$, $c(x_j^\alpha(i) \wedge x_j^\beta(i)) = \tau^*$;
(b) for all $j < \vartheta$ and $i < \sigma_j$, if $x_j^\alpha(i) <_{\mathrm{lex}} x_j^\beta(i)$ then $h(x_j^\alpha(i)) < h(x_j^\beta(i))$.

Subclaim 5.2.1.1. *Let $j < \vartheta$. Then $x_j^\alpha(i)$ and $x_j^\beta(i)$ are incomparable for every $i < \sigma_j$.*

Proof. Let $i < \sigma_j$. If $\lambda = \kappa$, then we get from Clauses (b) and (3) that $x_j^\alpha(i) <_{\mathrm{lex}} x_j^\beta(i)$ iff $h(x_j^\alpha(i)) < h(x_j^\beta(i))$ iff $\alpha < \beta$, and hence $x_j^\alpha(i)$ and $x_j^\beta(i)$ are incomparable by Definition 2.1.

If $\lambda < \kappa$, then $\mathrm{ht}(x_j^\alpha(i)) = \lambda = \mathrm{ht}(x_j^\alpha(i))$, and we are done. □

Subclaim 5.2.1.2. *Let $j < \vartheta$ and $\{s,t\} \in [a_j^{q_\alpha} \cup a_j^{q_\beta}]^2$. Then $c(s \wedge t) \neq \tau_j$.*

Proof. To avoid trivialities, assume that $\{s,t\} \nsubseteq [a_j^{q_\alpha}]^2 \cup [a_j^{q_\beta}]^2$. Say $(s,t) \in (a_j^{q_\alpha} \setminus r_j) \times (a_j^{q_\beta} \setminus r_j)$. Pick $i, i' < \sigma_j$ such that $x_j^\alpha(i) \subseteq s$ and $x_j^\beta(i') \subseteq t$.

- ▶ If $i = i'$, then since $x_j^\alpha(i)$ and $x_j^\beta(i)$ are incomparable, $c(s \wedge t) = c(x_j^\alpha(i) \wedge x_j^\beta(i)) = \tau^*$ by Clause (a). In particular, $c(s \wedge t) \neq \tau_j$.
- ▶ If $i \neq i'$, then pick $s' \in (a_j^{q_\beta} \setminus r_j)$ such that $x_j^\beta(i) \subseteq s'$. By Clause (2). $z_j(i) \subseteq x_j^\alpha(i) \subseteq s$, $z_j(i) \subseteq x_j^\beta(i) \subseteq s'$ and $z_j(i') \subseteq x_j^\beta(i') \subseteq t$. Recalling that z_j is injective with image contained in T_ϵ, it follows that $s \wedge t = s' \wedge t$. But $s', t \in a_j^{q_\beta}$, so that $c(s \wedge t) = c(s' \wedge t)$ and the latter is nonequal to τ_j. □

Altogether, $\langle (\tau_j, a_j^{q_\alpha} \cup a_j^{q_\beta}) \mid j < \vartheta \rangle$ is a legitimate condition witnessing that q_α and q_β are compatible. □

Finally, $\mathbb{P}^\theta$ does not have the κ-cc, since

$$\{\langle (j, \{t\}) \mid j < \theta \rangle \mid t \in T\}$$

is readily seen to be a κ-sized antichain in $\mathbb{P}^\theta$. □

6. Strongly Luzin sets

In this section, we shall obtain instances of the partition relations of Definition 2.7 from a strongly Luzin set.

Definition 6.1. For an uncountable cardinal κ, a subset L of a polish space X is *κ-strongly Luzin* iff it has size κ, and for every positive integer n, every pairwise disjoint subfamily S of ${}^nL$ of size κ consisting of injective tuples, and every meager subset M of ${}^nX$, we have $|M \cap S| < \kappa$.

In [Tod89, Proposition 6.0], it is shown that Martin's Axiom (possibly with the continuum hypothesis) implies the existence of a $\mathfrak{c}$-strongly Luzin subset of $\mathbb{R}$. A standard variation reads as follows. For completeness, we also give a detailed proof.

Proposition 6.2. *Suppose* $\mathrm{cov}(\mathcal{M}) = \mathrm{cof}(\mathcal{M}) = \mu$. *For every cardinal $\kappa \leq \mu$ with $\mathrm{cf}(\kappa) = \mathrm{cf}(\mu)$, every Polish space X admits a κ-strongly Luzin subset.*

Proof. Fix a Polish space X. We break the proof into four steps.

Step 1: Using $\mathrm{cof}(\mathcal{M}) = \mu$, let $\langle (F_\beta, n_\beta) \mid \beta < \mu \rangle$ be a sequence of pairs such that:

- for every $\beta < \mu$, F_β is a meager subset of ${}^{n_\beta}X$, and
- for every $n < \omega$ and every meager $M \subseteq {}^nX$, there is a $\beta < \mu$ such that $n_\beta = n$ and $M \subseteq F_\beta$.

Step 2: Given any subset $R \subseteq X$, for every positive integer n, and every subset $Z \subseteq {}^nX$, we shall define a corresponding collection $\mathbb{M}(Z,R)$ of no more than $|R| + \aleph_0$ meager subsets of X. The definition is by recursion over all positive $n < \omega$. For $n = 1$, i.e., $Z \subseteq X$, we let $\mathbb{M}(Z,R) := \{Z\}$ in case that Z is meager, and $\mathbb{M}(Z,R) := \emptyset$ otherwise. Next, suppose that n is a positive integer and for every positive $d \leq n$ and every $Z \subseteq {}^dX$, the corresponding collection $\mathbb{M}(Z,R)$ has already been successfully defined. Now, given a subset Z of ${}^{n+1}X$, let $\mathbb{M}(Z,R)$ be the union of the following two sets:

(i) $\bigcup_{j\leq n}\bigcup_{\bar{r}\in {}^jR}\mathbb{M}(\{v \in {}^{n+1-j}X \mid \bar{r}^\frown v \in Z\}, R)$, and
(ii) $\bigcup_{j<n}\bigcup_{\bar{r}\in {}^jR}\mathbb{M}(\{r \in X \mid \{v \in {}^{n-j}X \mid \bar{r}^\frown\langle r\rangle^\frown v \in Z\} \text{ is non-meager}\}, R)$.

Step 3: Recursively construct a sequence $\langle r_\gamma \mid \gamma < \mu\rangle$ of elements of X, as follows. Given $\gamma < \mu$, let

(iii) $R_\gamma := \{r_\beta \mid \beta < \gamma\}$, and
(iv) $M_\gamma := \bigcup_{\beta\leq\gamma}\mathbb{M}(F_\beta, R_\gamma)$.

As R_γ is a subset of X of size less than $\mathrm{cov}(\mathcal{M})$, and M_γ consists of fewer than $\mathrm{cov}(\mathcal{M})$ many meager subsets of X (since $\mathrm{cov}(\mathcal{M})$ may be singular, we use here that each $\mathbb{M}(F_\beta, R_\gamma)$ consists of no more than $|R_\gamma| + \aleph_0$ meager subsets of X), we may pick r_γ such that:

(v) $r_\gamma \in X \setminus (R_\gamma \cup \bigcup M_\gamma)$.

Step 4: Given a cardinal $\kappa \leq \mu$ with $\mathrm{cf}(\kappa) = \mathrm{cf}(\mu)$, fix a cofinal subset Γ of μ of size κ, and we shall verify that $L := \{r_\gamma \mid \gamma \in \Gamma\}$ is a κ-strongly Luzin subset of X.

Claim 6.2.1. *Suppose that n is a positive integer, S is a pairwise disjoint subfamily of ${}^nL$ of size κ consisting of injective tuples, and M is a meager subset of ${}^nX$. Then $|M \cap S| < \kappa$.*

Proof. Suppose not. By shrinking, we may assume that $S \subseteq M$. For each $s \in S$, fix an injection $\pi_s : n \to \Gamma$ such that $s(i) = r_{\pi_s(i)}$ for every $i < n$. By possibly shrinking S further and applying a permutation to both S and M, we may assume that π_s is increasing for every $s \in S$.[6]

Fix a $\beta < \mu$ such that $n_\beta = n$ and $M \subseteq F_\beta$. Then pick an $s \in S$ such that $\mathrm{Im}(\pi_s)\cap\beta = \emptyset$. We shall reach a contradiction by showing that $s \notin M$. For notational simplicity, we hereafter write $\langle\gamma_i \mid i < n\rangle$ for π_s.

If $n = 1$, then $\mathbb{M}(F_\beta, R_{\gamma_0}) = \{F_\beta\} \subseteq M_{\gamma_0}$, so that $r_{\gamma_0} \in X \setminus F_\beta \subseteq X \setminus M$, and we are done. Next, suppose that $n = \bar{n} + 1$ for a positive integer $\bar{n}$. Set $F_{\beta,0} := F_\beta$, and for every $i < \bar{n}$, let

$$F_{\beta,i+1} := \{v \in {}^{\bar{n}-i}X \mid \langle r_{\gamma_i}\rangle^\frown v \in F_{\beta,i}\}.$$

The following should be clear.

[6] Permutation is a homeomorphism, so preserves nowhere dense sets, hence also meager sets.

Subclaim 6.2.1.1. *$F_{\beta,i} = \{v \in {}^{n-i}X \mid \langle r_{\gamma_0}, \ldots, r_{\gamma_{i-1}}\rangle^\frown v \in F_\beta\}$ for every $i \in [1, n)$.* □

Subclaim 6.2.1.2. *$F_{\beta,i} \in M_{\gamma_i}$ for every $i < n$.*

Proof. By induction on $i < n$. For $i = 0$, we just need to know that $F_\beta \in M_{\gamma_0}$ which we have already pointed out. Next, given $i < \bar{n}$ for which $F_{\beta,i}$ is in M_{γ_i}, it follows in particular that $F_{\beta,i}$ is meager, so by the Kuratowski–Ulam theorem,

$$Z_i := \{r \in X \mid \{v \in {}^{\bar{n}-i}X \mid \langle r\rangle^\frown v \in F_{\beta,i}\} \text{ is non-meager}\}$$

is meager, and hence $\mathbb{M}(Z_i, R_{\gamma_i}) = \{Z_i\}$. Meanwhile, by Subclaim 6.2.1.1,

$$Z_i = \{r \in X \mid \{v \in {}^{\bar{n}-i}X \mid \langle r_{\gamma_0}, \ldots, r_{\gamma_{i-1}}, r\rangle^\frown v \in F_\beta\} \text{ is non-meager}\}.$$

So the instance $\bar{r} := \langle r_{\gamma_0}, \ldots, r_{\gamma_{i-1}}\rangle$ of Clause (ii) in the definition of $\mathbb{M}(F_\beta, R_{\gamma_i})$ ensures that $Z_i \in \mathbb{M}(F_\beta, R_{\gamma_i})$. By Clause (iv), then, $Z_i \in M_{\gamma_i}$. By Clause (v), then, $r_{\gamma_i} \in X \setminus Z_i$ which implies that the following set is meager:

$$\{v \in {}^{\bar{n}-i}X \mid \langle r_{\gamma_i}\rangle^\frown v \in F_{\beta,i}\}.$$

That is, $F_{\beta,i+1}$ is meager and hence $\mathbb{M}(F_{\beta,i+1}, R_{\gamma_{i+1}}) = \{F_{\beta,i+1}\}$. By Subclaim 6.2.1.1 and the instance $\bar{r} := \langle r_{\gamma_0}, \ldots, r_{\gamma_i}\rangle$ of Clause (i) in the definition of $\mathbb{M}(F_\beta, R_{\gamma_{i+1}})$, $\mathbb{M}(F_{\beta,i+1}, R_{\gamma_{i+1}}) \subseteq \mathbb{M}(F_\beta, R_{\gamma_{i+1}})$. So by Clause (iv), $F_{\beta,i+1} \in M_{\gamma_{i+1}}$, as sought. □

As $F_{\beta,\bar{n}}$ is in $M_{\gamma_{\bar{n}}}$, Clause (v) ensures that $r_{\gamma_{\bar{n}}} \notin F_{\beta,\bar{n}}$, so that $\langle r_{\gamma_0}, \ldots, r_{\gamma_{\bar{n}}}\rangle \in {}^nX \setminus F_\beta \subseteq {}^nX \setminus M$, and we are done. □

This completes the proof. □

Remark 6.3. Any κ-strongly Luzin subset X of $\mathbb{R}$ is $(\kappa, {<}\omega)$-entangled. In addition $X \cup \mathbb{Q}$ is a $(\kappa, {<}\omega)$-entangled set of reals that is κ-concentrated on $\mathbb{Q}$ — an object obtained from CH (with $\kappa = \aleph_1$) in [SW26, Theorem 2.9].

We are now ready to prove Theorem C.

Theorem 6.4. *Suppose that there exists a κ-strongly Luzin subset of ${}^\omega\omega$. Then there exist a uniformly homogeneous streamlined tree $T \subseteq {}^{\leq\omega}\omega$ of size κ and a surjection $h : T \to \kappa$ such that:*

1. *$h \restriction T_\omega$ is injective;*
2. *for every $n < \omega$, $h^{-1}\{n\} = {}^n\omega$;*
3. *$(T, \subsetneq, <_{\mathrm{lex}}, h) \overset{\Delta}{\nrightarrow} [\kappa]^{<\omega,1}_\omega$ holds.*

Proof. Let L be a κ-strongly Luzin subset of ${}^\omega\omega$. Consider

$$T := \{(q * r) \restriction m \mid q \in {}^{<\omega}\omega, r \in L, m \leq \omega\},$$

which is a uniformly homogeneous streamlined subtree of ${}^{\leq\omega}\omega$ of size κ. Let $h : T \to \kappa$ be any surjection satisfying Clauses (1) and (2). Let $d : \omega \to \omega$ be any surjection such that the preimage of any singleton is infinite. To see that d witnesses $(T, \subsetneq, <_{\mathrm{lex}}, h) \overset{\Delta}{\nrightarrow} [\kappa]^{<\omega,1}_\omega$, let n be a positive integer and let an h-rapid $S \subseteq {}^nT$ of size κ be given. We start with some reductions:

- By possibly shrinking S we may assume the existence of $m < \omega$, $q \in {}^m\omega$ and a pairwise disjoint family $S' \subseteq {}^nL$ of size κ such that for every $s \in S'$, we have that $\langle q(i) * s(i) \mid i < n\rangle$ is in S.
- By possibly shrinking S', we may assume that for all $s \neq s'$ in S', $\langle s(i)\restriction m \mid i < n\rangle = \langle s'(i)\restriction m \mid i < n\rangle$. Consequently, for every $i < n$, $\Delta(q(i) * s(i), q(i) * s'(i)) = \Delta(s(i), s'(i))$.
- By possibly shrinking S' further, we may assume that for all $s \neq s'$ in S' for all $i, j < n$, $s(i) = s(j)$ iff $s'(i) = s'(j)$. Thus, by possibly passing to a smaller n, we may assume that all elements of S' are injective.

Thus, hereafter, we simply assume that $S \subseteq {}^nL$ is a pairwise disjoint family of size κ, consisting of injective tuples.

Claim 6.4.1. *S is somewhere dense. Moreover, there is an $m < \omega$ and $z \in {}^n({}^m\omega)$ such that for every $\langle y_i \mid i < n\rangle \in {}^n({}^{<\omega}\omega)$ and every $\epsilon < \kappa$, there is an $s \in S$ with $\min\{h(s(i)) \mid i < n\} > \epsilon$ and such that $z(i)^\frown y(i) \subseteq s(i)$ for every $i < n$.*

Proof. Suppose not. For each $z \in \bigcup_{m<\omega} {}^n({}^m\omega)$, pick $\langle y_i^z \mid i < n\rangle$ and $\epsilon^z < \kappa$ such that

- $\langle y_i^z \mid i < n\rangle \in {}^n({}^{<\omega}\omega)$, and
- for every $s \in S$, for some $i < n$, either $h(s(i)) \leq \epsilon^z$ or $z(i)^\frown y_i^z \nsubseteq s(i)$.

For every $z \in \bigcup_{m<\omega} {}^n({}^m\omega)$,

$$U_z := \{s \in {}^n({}^\omega\omega) \mid \forall i < n\, (z(i)^\frown y_i^z \subseteq s(i))\},$$

is open, and $U := \bigcup\{U_z \mid z \in \bigcup_{m<\omega} {}^n({}^m\omega)\}$ is dense-open, so its complement M is meager, which implies that $|M \cap S| < \kappa$. Let $\epsilon := \sup\{\epsilon^z \mid z \in \bigcup_{m<\omega} {}^n({}^m\omega)$. Pick $s \in S \setminus M$ such that $\min\{h(s(i)) \mid i < n\} > \epsilon$. Then $s \in S \cap U$, and we may find a $z \in \bigcup_{m<\omega} {}^n({}^m\omega)$ such that $s \in S \cap U_z$. It follows that $z(i)^\frown y_i^z \subseteq s(i)$ for every $i < n$, and also $\min\{h(s(i)) \mid i < n\} > \epsilon \geq \epsilon^z$, contradicting the choice of $\langle y_i^z \mid i < n\rangle$. □

Let $z \in {}^n({}^m\omega)$ be given by the claim. Given a prescribed colour $\tau < \omega$, we may use the claim to extend z to ensure that $d(m) = \tau$. Now, pick $x \in S$ such that, for every $i < n$, $z(i)^\frown\langle 0\rangle \subseteq x(i)$, and then pick $y \in S$ such that, for every $i < n$, $h(x(i)) < h(y(i))$ and $z(i)^\frown\langle 1\rangle \subseteq y(i)$. Then:

(1) for every $i < n$, $d(h(x(i) \wedge y(i))) = d(m) = \tau$;
(2) for every $i < n$, $x(i) <_{\mathrm{lex}} y(i)$ and $h(x(i)) < h(y(i))$;
(3) $\{h(x(i) \wedge y(i)) \mid i < n\} = \{m\}$.

So we are done. □

Proposition 6.2 and Theorem 6.4 have to do with Polish spaces, but their proofs generalise to address the product space ${}^\lambda\lambda$ for any cardinal λ satisfying $\lambda = \lambda^{<\lambda}$. A sample corollary of these generalisations reads as follows.

Corollary 6.5. *Suppose that $\lambda = \lambda^{<\lambda}$ is an infinite cardinal satisfying $2^\lambda = \lambda^+$. Then there exist a uniformly homogeneous streamlined tree $T \subseteq {}^{\leq\lambda}\lambda$ of size λ^+ and a surjection $h : T \to \lambda^+$ such that:*

(1) *$h \restriction T_\lambda$ is injective;*
(2) *for every $\alpha < \lambda$, $h^{-1}\{\alpha\} = {}^\alpha\lambda$;*
(3) *$(T, \subsetneq, <_{\text{lex}}, h) \overset{\Delta}{\nrightarrow} [\lambda^+]^{<\lambda,1}_\lambda$ holds.* □

7. More instances of the partition relation

In this section, we shall obtain instances of the partition relations of Definition 2.7 using the following weakening of Jensen's diamond.

Definition 7.1 (Shelah, [She83, §2])**.** For a stationary subset S of a regular uncountable cardinal λ, $\mathrm{D}\ell^*(S)$ asserts the existence of a sequence $\langle \mathcal{P}_\delta \mid \delta \in S\rangle$ such that:

- for every $\delta \in S$, $\mathcal{P}_\delta \subseteq \mathcal{P}(\delta)$ with $|\mathcal{P}_\delta| < \lambda$;
- for every $A \subseteq \lambda$, the set $\{\delta \in S \mid A \cap \delta \notin \mathcal{P}_\delta\}$ is nonstationary.

Theorem 7.2. *Suppose that λ is a regular uncountable cardinal, $S \subseteq \lambda$ is stationary and $\mathrm{D}\ell^*(S)$ holds. Then there exist a uniformly homogeneous streamlined tree $T \subseteq {}^{\leq\lambda}\lambda$ of size a regular cardinal $\kappa \in (\lambda, 2^\lambda]$ and a surjection $h : T \to \kappa$ such that:*

(1) *$h \restriction T_\lambda$ is injective;*
(2) *for every $\alpha < \lambda$, $h^{-1}\{\alpha\} = {}^\alpha\lambda$;*
(3) *$(T, \subsetneq, <_{\text{lex}}, h) \overset{\Delta}{\nrightarrow} [\kappa]^{<\lambda,1}_\lambda$ holds.*

Proof. Clearly, $\mathrm{D}\ell^*(S')$ holds for every stationary $S' \subseteq S$. In addition, by [IR23, Corollary 3.11], any stationary subset of a regular uncountable cardinal can be refined to a stationary subset that carries an *amenable C-sequence*. That is, by possibly shrinking S, we may assume the existence of a sequence $\vec{C} = \langle C_\delta \mid \delta \in S\rangle$ such that the following two hold:

- for every $\delta \in S$, $\delta \in \mathrm{acc}(\lambda)$ and C_δ is a club in δ;
- for every club $D \subseteq \lambda$, $\{\delta \in S \mid D \cap \delta \subseteq C_\delta\}$ is nonstationary in λ.

As $\mathrm{D}\ell^*(S)$ holds, it is the case that $\lambda^{<\lambda} = \lambda$, so we may let $\langle \varrho^\iota \mid \iota < \lambda\rangle$ be an injective enumeration of ${}^{<\lambda}\lambda$. Let κ denote the least size of an unbounded family in the quasi-ordered set $({}^\lambda\lambda, <_S)$, where $f <_S g$ iff $\{\delta \in S \mid g(\delta) \leq f(\delta)\}$ is nonstationary. It easily follows that $\kappa = \mathrm{cf}(\kappa) \in (\lambda, 2^\lambda]$ and we may fix a strictly increasing unbounded sequence $\langle f_\xi \mid \xi < \kappa\rangle$ in $({}^\lambda\lambda, <_S)$. For every $(\iota, \xi) \in \lambda \times \kappa$, let $f^\iota_\xi := \varrho^\iota * f_\xi$, and consider the streamlined tree $T := \{f^\iota_\xi \restriction \alpha \mid \iota < \lambda, \xi < \kappa, \alpha \leq \lambda\}$ which is clearly uniformly homogeneous. Let $h : T \to \kappa$ be any surjection satisfying Clauses (1) and (2). Note that for every nonzero $\sigma < \lambda$, for every h-rapid $H \subseteq {}^\sigma T$ of size κ, there exist a map $\varpi : \sigma \to \lambda$ and an id-rapid $X \subseteq {}^\sigma\kappa$ of size κ such that:

- for every $x \in X$, $\langle f^{\varpi(i)}_{x(i)} \mid i \in \mathrm{dom}(x)\rangle \in H$;

- for all $x \neq y$ in X, $\sup(x) < \min(y)$ iff $\sup\{h(f^{\varpi(i)}_{x(i)}) \mid i \in \mathrm{dom}(x)\} < \min\{h(f^{\varpi(i)}_{y(i)}) \mid i \in \mathrm{dom}(y)\}$.

The last observation motivates the next definition. For every map $\varpi : \sigma \to \lambda$ with a nonzero $\sigma < \lambda$, for every id-rapid $X \subseteq {}^{\sigma}\kappa$ of size κ, let $D^{\varpi}(X)$ be the collection of all $\delta \in S$ for which there are $x \neq y$ in X such that for every $i < \sigma$:

(a) $\Delta(f^{\varpi(i)}_{x(i)}, f^{\varpi(i)}_{y(i)}) = \delta$;
(b) if $f^{\varpi(i)}_{x(i)} <_{\mathrm{lex}} f^{\varpi(i)}_{y(i)}$ then $x(i) < y(i)$.

Let $\mathcal{F}$ be the collection of all $E \subseteq S$ for which there are a map $\varpi : \sigma \to \lambda$ with a nonzero $\sigma < \lambda$ and an id-rapid $X \subseteq {}^{\sigma}\kappa$ of size κ such that $D^{\varpi}(X) \subseteq E$. It is clear that $\mathcal{F}$ is a filter over S. It is not hard to see it is λ-complete, but for our purpose, what matters is the following restricted form of normality.

Claim 7.2.1. *For every club $C \subseteq \kappa$, for every regressive map $\varphi : S \cap C \to \lambda$, there exists a $\beta < \lambda$ such that $\varphi^{-1}\{\beta\} \in \mathcal{F}^+$.*

Proof. Suppose not, and let $\varphi : S \cap C \to \lambda$ be a counterexample. For every $\beta < \kappa$, since $S_\beta := \{\delta \in S \mid \varphi(\delta) = \beta\}$ is not in $\mathcal{F}^+$, we may pick a map $\varpi_\beta : \sigma_\beta \to \lambda$ with a nonzero $\sigma_\beta < \lambda$ and an id-rapid $X_\beta \subseteq {}^{\sigma_\beta}\kappa$ of size κ such that $S_\beta \cap D^{\varpi_\beta}(X_\beta) = \emptyset$. For every $\gamma < \lambda$, denote $\Sigma_\gamma := \sum_{\beta<\gamma} \sigma_\beta$. Note that $\{\delta < \lambda \mid \Sigma_\delta = \delta\}$ is a club in λ. Let $\varpi : \lambda \to \lambda$ be the unique map to satisfy that for every $\beta < \lambda$, $\varpi(\Sigma_\beta + i) = \varpi_\beta(i)$ for every $i < \sigma_\beta$. Recursively construct a map $\psi : \lambda \to {}^{<\lambda}\kappa$ such that:

- $\mathrm{Im}(\psi)$ is a pairwise disjoint family of sequences;
- for every $\gamma < \lambda$, $\mathrm{dom}(\psi(\gamma)) = \Sigma_\gamma$;
- for all $\beta < \gamma < \lambda$, $\langle \psi(\gamma)(\Sigma_\beta + i) \mid i < \sigma_\beta \rangle$ is in X_β.

Subclaim 7.2.1.1. *There are $\delta \in S \cap C$ and $t \in {}^{\delta}({}^{\delta}\lambda)$ such that for all $(\alpha, \xi) \in \lambda \times \kappa$, there is an $x \in \mathrm{Im}(\psi)$ such that:*

- *$\min(x) > \xi$,*
- *$\langle f^{\varpi(i)}_{x(i)} \restriction \delta \mid i < \Sigma_\delta \rangle = t$, and*
- *$\min\{f^{\varpi(i)}_{x(i)}(\delta) \mid i \in \mathrm{dom}(x)\} > \alpha$.*

Proof. Suppose not. For every $\delta \in S \cap C$, for every $t \in {}^{\delta}({}^{\delta}\lambda)$, pick $(\alpha_t, \xi_t) \in \lambda \times \kappa$ such that for every $x \in \mathrm{Im}(\psi)$ with $\min(x) > \xi_t$, if $\langle f^{\varpi(i)}_{x(i)} \restriction \delta \mid i < \Sigma_\delta \rangle = t$, then

$$(\sharp) \qquad \alpha_t \geq \min\{f^{\varpi(i)}_{x(i)}(\delta) \mid i \in \mathrm{dom}(x)\}.$$

As $\lambda^{<\lambda} = \lambda$, the ordinal $\xi^* := \sup\{\xi_t \mid t \in \bigcup_{\delta \in S \cap C} {}^{\delta}({}^{\delta}\lambda)\}$ is smaller than κ. As $\mathrm{D}\ell^*(S)$ holds, a standard argument (as in [BR17a, Lemma 2.2]) yields a sequence $\langle \mathcal{G}_\delta \mid \delta \in S \rangle$ such that:

- for every $\delta \in S$, $|\mathcal{G}_\delta| < \lambda$;

- for every sequence $\langle g_i \mid i<\lambda\rangle$ of functions from λ to λ, there are club many $\delta\in S$ such that $\langle g_i\restriction\delta \mid i<\delta\rangle\in\mathcal{G}_\delta$.

Next, as $|\mathcal{G}_\delta|<\lambda$ for every $\delta\in S$, we may define a function $\chi: S\rightarrow\lambda$ via

$$\chi(\delta):=\sup\{\alpha_t \mid t\in\mathcal{G}_\delta\}.$$

As $\langle f_\xi \mid \xi<\kappa\rangle$ is unbounded in $({}^\lambda\lambda,<_S)$, we may fix an ordinal $\zeta<\kappa$ such that $S_{\chi,\zeta}:=\{\delta\in S \mid \chi(\delta)\le f_\zeta(\delta)\}$ is stationary. As $\mathrm{Im}(\psi)$ consists of κ-many pairwise disjoint sequences, we may fix an $x\in\mathrm{Im}(\psi)$ with $\min(x)>\max\{\xi^*,\zeta\}$. For each $i\in\mathrm{dom}(x)$, fix a club $D_i\subseteq\lambda$ such that $f_\zeta(\delta)<f_{x(i)}(\delta)$ for every $\delta\in S\cap D_i$. Let us also fix a club $D\subseteq C$ such that $\langle f^{\varpi(i)}_{x(i)}\restriction\delta \mid i<\delta\rangle$ is in $\mathcal{G}_\delta$, and $\Sigma_\delta=\delta$, for every $\delta\in S\cap D$. Now pick $\delta\in S_{\chi,\zeta}\cap D\cap\bigcap_{i\in\mathrm{dom}(x)}D_i$ above $\sup\{\mathrm{dom}(\varrho^{\varpi(i)}) \mid i\in\mathrm{dom}(x)\}$. Then $\delta\in S\cap C$ and the following two hold:

(i) $t:=\langle f^{\varpi(i)}_{x(i)}\restriction\delta \mid i<\Sigma_\delta\rangle$ is in $\mathcal{G}_\delta$;

(ii) $\chi(\delta)\le f_\zeta(\delta)<f_{x(i)}(\delta)=f^{\varpi(i)}_{x(i)}(\delta)$ for every $i\in\mathrm{dom}(x)$.

By Clause (i), $\alpha_t\le\chi(\delta)$. By Clause (ii), $\chi(\delta)<\min\{f^{\varpi(i)}_{x(i)}(\delta) \mid i\in\mathrm{dom}(x)\}$. This is a contradiction to the choice of α_t in ($\sharp$). □

Let δ and t be given by the subclaim. Put $\beta:=\varphi(\delta)$. Pick $x\in\mathrm{Im}(\psi)$ as in the conclusion of the claim for $\alpha:=\sup\{f^{\varpi(i)}_{\bar{x}(i)}(\delta) \mid i\in\mathrm{dom}(\bar{x}),\bar{x}\in\psi[\beta+1]\}$, and $\xi:=0$. Then pick $y\in\mathrm{Im}(\psi)$ as in the conclusion of the claim for $\alpha':=\sup\{f^{\varpi(i)}_{x(i)}(\delta) \mid i\in\mathrm{dom}(x)\}$ and $\xi':=\sup(x)$. In particular, x and y are elements of $\psi[\lambda\setminus(\beta+1)]$ with $\sup(x)<\min(y)$. Recalling the features of ψ, it is the case that $\bar{x}:=\langle x(\Sigma_\beta+i) \mid i<\sigma_\beta\rangle$ and $\bar{y}:=\langle y(\Sigma_\beta+i) \mid i<\sigma_\beta\rangle$ are both in X_β. However, for every $i<\Sigma_\delta$, it is the case that $\Delta(f^{\varpi(i)}_{x(i)},f^{\varpi(i)}_{y(i)})=\delta$ and $f^{\varpi(i)}_{x(i)}(\delta)\le\alpha'<f^{\varpi(i)}_{y(i)}(\delta)$. So $\bar{x}$ and $\bar{y}$ witness together that $\delta\in S_\beta\cap D^{\varpi_\beta}(X_\beta)$. This is a contradiction. □

Claim 7.2.2. *There is an $\eta<\lambda$ such that the following set has size λ:*

$$B_\eta:=\{\beta<\lambda \mid \{\delta\in S\setminus(\eta+1) \mid \min(C_\delta\setminus\eta)=\beta\}\in\mathcal{F}^+\}.$$

Proof. Suppose not, and pick $g:\lambda\rightarrow\lambda$ such that $B_\eta\subseteq g(\eta)$ for every $\eta<\lambda$.

Subclaim 7.2.2.1. $S':=\{\delta\in S \mid \forall\eta<\delta\,[\min(C_\delta\setminus\eta)<g(\eta)]\}$ *is stationary.*

Proof. Suppose not. Fix a bijection $\pi:\lambda\times\lambda\leftrightarrow\lambda$. Fix a club $C\subseteq\{\delta<\lambda \mid \pi[\delta\times\delta]=\delta\}$ disjoint from S', and define a regressive map $\varphi: S\cap C\rightarrow\lambda$, as follows. Given $\delta\in S\cap C$, pick the least $\eta<\delta$ such that $\min(C_\delta\setminus\eta)\ge g(\eta)$ and then let

$$\varphi(\delta):=\pi(\eta,\min(C_\delta\setminus\eta)).$$

Using Claim 7.2.1, fix $(\eta,\beta)\in\lambda\times\lambda$ such that $\varphi^{-1}\{\pi(\eta,\beta)\}\in\mathcal{F}^+$. This means that $\{\delta\in S\cap C\setminus(\eta+1) \mid \min(C_\delta\setminus\eta)=\beta\ge g(\eta)\}\in\mathcal{F}^+$, that is, $\beta\in B_\eta\setminus g(\eta)$, contradicting our choice of g. □

Consider the club $D := \{\alpha < \lambda \mid g[\alpha] \subseteq \alpha\}$. By the amenability of $\vec{C}$, we may now pick a $\delta \in S'$ such that $D \cap \delta \nsubseteq C_\delta$. Pick $\alpha \in D \cap \delta \setminus C_\delta$. As $\delta \in S'$, for every $\eta < \delta$, $\min(C_\delta \setminus \eta) < g(\eta)$. As $\alpha \in D$, for every $\eta < \alpha$, $\min(C_\delta \setminus \eta) < g(\eta) < \alpha$. Altogether, for every $\eta < \alpha$, $\eta \leq \min(C_\delta \setminus \eta) < \alpha$. So α is an accumulation point of the club C_δ, contradicting the fact that $\alpha \notin C_\delta$. □

Let η be given by the claim so that $|B_\eta| = \lambda$. Define a map $c : S \to \lambda$ via

$$c(\delta) := \begin{cases} \operatorname{otp}(B_\eta \cap \min(C_\delta \setminus \eta)), & \text{if } \delta > \eta; \\ 0, & \text{otherwise.} \end{cases}$$

Then $c[E] = \lambda$ for every $E \in \mathcal{F}$. Let $d : \kappa \to \lambda$ be any map extending c. Then d witnesses that $(T, \subsetneq, <_{\text{lex}}, h) \overset{\Delta}{\not\to} [\kappa]^{<\lambda,1}_\lambda$ holds, and we are done. □

Corollary 7.3. *Suppose that λ is a strongly inaccessible that is not greatly Mahlo. Then the conclusion of Theorem 7.2 holds with $\kappa := \mathfrak{b}_\lambda$.*

Proof. As λ is strongly inaccessible, $\mathrm{D}\ell^*(\lambda)$ holds. By [IR23, Lemma 3.4 and Corollary 3.10], any regular uncountable cardinal that is not greatly Mahlo carries a (strongly) amenable C-sequence. Altogether, we can run the proof of Theorem 7.2 using $S := \lambda$, obtaining the desired tree T and map $h : T \to \kappa$ for κ being the least size of an unbounded family in the quasi-ordered set $({}^\lambda\lambda, \lhd)$, where $f \lhd g$ iff $\{\delta < \lambda \mid g(\delta) \leq f(\delta)\}$ is nonstationary. By [CS95, Theorem 6], κ coincides with $\mathfrak{b}_\lambda$. □

Remark 7.4. Using [IR26b, Corollary 3.5], we get that for every strongly inaccessible cardinal λ that is not weakly compact, for every cardinal $\theta < \lambda$, $(T, \subsetneq, <_{\text{lex}}, h) \overset{\Delta}{\not\to} [\mathfrak{b}_\lambda]^{<\lambda,1}_\theta$ holds for a tree T and a map h as in Theorem 7.2.

8. Applications

We now arrive at Theorem B.

Corollary 8.1. *Suppose that $\lambda = \lambda^{<\lambda}$ is an infinite cardinal such that one of the following holds:*

(1) $\lambda > \beth_\omega$;
(2) $2^\lambda = \lambda^+$;
(3) $\lambda = \mu^+$ *for a cardinal* $\mu = \mu^{\aleph_0}$.

Then $\mathbf{T} \overset{\Delta}{\not\to} [\kappa]^{<\lambda,1}_\lambda$ holds for some structure $\mathbf{T} = (T, \subsetneq, <_{\text{lex}}, h)$, where $T \subseteq {}^{\leq\lambda}\lambda$ is a uniformly homogeneous streamlined tree of size a regular cardinal $\kappa \in (\lambda, 2^\lambda]$, and $h : T \to \kappa$ is a $({<}\kappa)$-to-one map. In particular, there is a $(\kappa, {<}\lambda)$-entangled linear order of size κ.

Proof. The "In particular" part follows from Lemma 4.3.

(1) Assuming $\lambda > \beth_\omega$, by the main result of [She00], we may fix some regular $\theta < \beth_\omega$ such that $\operatorname{cf}([\mu]^\theta, \supseteq) < \lambda$ for all $\mu \in [\theta, \lambda)$. Then a standard

application of $\lambda^{<\lambda} = \lambda$ (see, e.g., the proof of [FR18, Lemma 2.3]) yields that $\mathrm{D}\ell^*(E^\lambda_\theta)$ holds. Now appeal to Theorem 7.2.

(2) By Corollary 6.5.

(3) By [Gre76, Lemma 2.1], for every cardinal $\lambda = \mu^+ = 2^\mu > \mu^{\aleph_0}$, $\diamondsuit^*(E^\lambda_\omega)$ holds. In particular, $\mathrm{D}\ell^*(E^\lambda_\omega)$ holds. Now appeal to Theorem 7.2. □

We now arrive at Theorem D.

Corollary 8.2. *Suppose that* $\mathrm{cov}(\mathcal{M}) = \mathrm{cof}(\mathcal{M})$. *For every cardinal* $\kappa \leq \mathrm{cov}(\mathcal{M})$ *with* $\mathrm{cf}(\kappa) = \mathrm{cf}(\mathrm{cov}(\mathcal{M}))$, *there are:*

- *A* $(\kappa, {<}\omega)$*-entangled linear order of size* κ*;*
- *For every positive integer* n*, a* (κ, n)*-entangled linear order of size* κ *that is not* $(\kappa, n+1)$*-entangled.*

Proof. By [Mil82], $\mathrm{cov}(\mathcal{M})$ always has uncountable cofinality, so $\mathrm{cf}(\kappa)$ is uncountable as well. By Proposition 6.2, under these hypotheses there is a κ-strongly Luzin subset of ${}^\omega\omega$. By Theorem 6.4, this implies that there is a uniformly homogeneous streamlined tree $T \subseteq {}^{\leq\omega}\omega$ of size κ and a surjection $h : T \to \kappa$ such that:

(1) $h \restriction T_\omega$ is injective;
(2) for every $n < \omega$, $h^{-1}\{n\} = {}^n\omega$;
(3) $(T, \subsetneq, <_{\mathrm{lex}}, h) \overset{\Delta}{\not\to} [\kappa]^{<\omega,1}_\omega$ holds.

In particular, the function h is ${<}\kappa$-to-one. Now, by Lemma 4.3, there is a $(\kappa, {<}\omega)$-entangled linear order of size κ. Also, by Lemma 4.4, for every positive integer n, a (κ, n)-entangled linear order of size κ that is not $(\kappa, n+1)$-entangled. □

We conclude the paper with one last corollary.

Corollary 8.3. *For every regular uncountable cardinal* λ*, the following are equivalent:*

(1) *There is a special* λ^+*-Aronszajn line;*
(2) *For every positive integer* n*, there is a special* λ^+*-Aronszajn line that is weakly* (λ^+, n)*-entangled.*

Proof. By Lemma 4.6 and [IR25, Theorem A]. □

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