MORE NOTIONS OF FORCING ADD SQUARE

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Abstract. Foreman and Magidor proved that CH yields a countably-closed $\aleph_2$-cc notion of forcing $\mathbb{P}$ for adding $\square_{\omega_1}$. Here we prove that $\mathbb{P}$ may consistently be taken to be an $\aleph_2$-Souslin tree. More generally, it is shown that $\square_\lambda$ may be added by a λ^+-Souslin tree, providing the first analog of the Foreman–Magidor forcing at the level of successors of singulars. Our construction is uniform and applies to inaccessible cardinals as well.

1. Introduction

The square principle $\square_\lambda$ was introduced by Jensen [Jen72] in his study of the fine structure of Gödel's constructible universe. He also devised a countably closed $(\lambda+1)$-strategically closed notion of forcing for adding $\square_\lambda$. Baumgartner introduced a weakening $\square^B_\lambda$ of $\square_\lambda$ that may be added by a $({<}\lambda)$-directed closed notion of forcing, hence is compatible with λ being supercompact (unlike $\square_\lambda$). At the level of a regular $\lambda > \aleph_1$, Baumgartner's square is just one club-shooting away from Jensen's square, and at the level of $\lambda = \aleph_1$, the two principles are logically equivalent. More notions of forcing for adding $\square_{\aleph_1}$ were introduced by Dolinar–Dzamonja [DD13], Krueger [Kru14], and Neeman [Nee17]. A third square principle is Todorčević's $\square(\kappa)$ [Tod87], which follows from $\square_\lambda$ whenever $\kappa = \lambda^+$.

It is an unpublished result of Foreman and Magidor that for an infinite regular cardinal $\lambda = \lambda^{<\lambda}$ there is a λ-closed λ^+-cc notion of forcing that adds Baumgartner's square $\square^B_\lambda$. Alternative posets for introducing $\square^B_\lambda$ are given in [Vel82, §1.3] and [LR19a, Definition 3.12]. In this paper, we show that squares can be added by a poset as good as a Souslin tree. To exemplify:

Theorem A. *For every uncountable cardinal $\lambda = \lambda^{<\lambda}$, if $\diamondsuit(E^{\lambda^+}_\lambda)$ holds,*[1] *then:*

- *there is a countably complete λ^+-Souslin tree that forces $\square_\lambda$;*
- *there is a λ-complete λ^+-Souslin tree that forces $\square^B_\lambda$.*

Our tree constructions make use of the proxy principles, and this has well-known advantages [BRY25] that are apparent here as well. For starters, it provides the first consistent way to add $\square_\lambda$ by a λ^+-cc notion of forcing for λ a singular cardinal (indeed, by forcing with λ^+-Souslin tree). To exemplify one exotic scenario:

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[1] Here, E^κ_λ stands for $\{\alpha < \kappa \mid \mathrm{cf}(\alpha) = \lambda\}$.

Theorem B. *Suppose that λ is a measurable cardinal and $2^\lambda = \lambda^+$. In the forcing extension by Prikry forcing there is a λ^+-Souslin tree that forces $\square_\lambda$.*

Remark 1. The preceding model is a two-step iteration of λ^+-cc notions of forcing. Note that by [CFM01, Theorem 11.1], it is possible that $\square_\lambda$ fails in the intermediate model.

Second, it provides analogous results for inaccessibles. To exemplify:

Theorem C. *Suppose that $\diamondsuit$ holds over a nonreflecting stationary subset of a strongly inaccessible κ. Then there is a κ-Souslin tree that forces $\square(\kappa)$.*

A third advantage, demonstrated here for concreteness at the level of $\aleph_2$ is that it is possible to get an $\aleph_2$-Souslin tree adding a square without assuming diamond on the critical cofinality (as assumed in Theorem A).

Theorem D. *If $2^{2^{\aleph_0}} = \aleph_2$ and there exists a nonreflecting stationary subset of $E^{\aleph_2}_{\aleph_0}$, then:*

(1) *there is an $\aleph_2$-Souslin tree that forces $\square_{\omega_1}$;*
(2) *there is a countably complete $\aleph_2$-Souslin tree that forces $\square(\omega_2)$.*

Remark 2. We refer the reader to Corollary 4.8 for consequences beyond squares, and to Section 4 in general for a whole gallery of applications.

We conclude the paper with a complementary result.

Theorem E. *Assuming the consistency of large cardinals, the conjunction of the following two bullet points is compatible with κ being a successor of a regular uncountable, a successor of a singular, or a strongly inaccessible.*

- *There is a κ-Souslin tree;*
- *Every κ-Souslin tree forces that $\square(\kappa)$ fails.*

1.1. **Organization of this paper.** In Section 2, we provide preliminaries on square and proxy principles.

In Section 3, we prove the main technical result of this paper: constructing a κ-Souslin tree $\mathbf{T}$ from an instance $\mathrm{P}_\xi(\kappa, \kappa, \ldots)$ of the proxy principle such that forcing with $\mathbf{T}$ introduces a narrow instance $\mathrm{P}_\xi(\kappa, 2, \ldots)$.

In Section 4, we derive various corollaries, including Theorems A—D.

In Section 5, we prove Theorem E.

2. Square and proxy principles

A $\mathcal{C}$*-sequence* over a regular uncountable cardinal κ is a sequence $\vec{\mathcal{C}} = \langle \mathcal{C}_\alpha \mid \alpha < \kappa\rangle$ such that for every $\alpha < \kappa$, $\mathcal{C}_\alpha$ is nonempty collection of closed subsets C of α with $\sup(C) = \sup(\alpha)$. It is said to be ξ*-bounded* provided that $\mathrm{otp}(C) \leq \xi$ for every $C \in \bigcup_{\alpha<\kappa} \mathcal{C}_\alpha$. It is said to be *unthreadable* provided that for every club D in κ, there is an $\alpha \in \mathrm{acc}(D)$ such that $D \cap \alpha \notin \mathcal{C}_\alpha$.[2] It is said to be $\mathcal{R}$*-coherent* (for any given binary relation $\mathcal{R}$) provided that for all $C \in \bigcup_{\alpha<\kappa} \mathcal{C}_\alpha$ and $\beta \in \mathrm{acc}(C)$,[3] there is a $\bar{C} \in \mathcal{C}_\beta$ with $\bar{C} \mathrel{\mathcal{R}} C$. The

[2] Clearly, if $\vec{\mathcal{C}}$ is ξ-bounded for some $\xi < \kappa$, then $\vec{\mathcal{C}}$ is unthreadable.

[3] For a set of ordinals x, $\mathrm{acc}(X)$ stand for all $\alpha \in x$ such that $\sup(x \cap \alpha) = \alpha > 0$.

main examples of $\mathcal{R}$ here are the end-extension relation $\sqsubseteq$ and its variant $\sqsubseteq_\chi$ that focuses on clubs of high order-type; they are defined as follows:

- $\bar{C} \sqsubseteq C$ iff there is an ordinal β such that $\bar{C} = C \cap \beta$;
- $\bar{C} \sqsubseteq_\chi C$ iff either $\bar{C} \sqsubseteq C$ or ($\mathrm{otp}(C) < \chi$ and $\mathrm{nacc}(C)$ consists of successor ordinals).

Definition 2.1.

- A *C-sequence over κ* is one $\langle C_\alpha \mid \alpha < \kappa\rangle$ for which $\langle \{C_\alpha\} \mid \alpha < \kappa\rangle$ is a $\mathcal{C}$-sequence over κ;
- Jensen's $\square_\lambda$ asserts the existence of a λ-bounded $\sqsubseteq$-coherent C-sequence over λ^+;
- Baumgartner's $\square^B_\lambda$ asserts the existence of a λ-bounded $\sqsubseteq_\lambda$-coherent C-sequence over λ^+;
- Todorčević's $\square(\kappa)$ asserts the existence of an unthreadable $\sqsubseteq$-coherent C-sequence over κ.
- Todorčević's $\square(\kappa, {<}\mu)$ asserts the existence of an unthreadable $\sqsubseteq$-coherent $\mathcal{C}$-sequence $\langle \mathcal{C}_\alpha \mid \alpha < \kappa\rangle$ such that $|\mathcal{C}_\alpha| < \mu$ for every $\alpha < \kappa$.

Remark 2.2. We mentioned in the introduction that $\square_{\aleph_1}$ and $\square^B_{\aleph_1}$ are logically equivalent. Note that, more generally, the existence of a λ-bounded $\sqsubseteq_{\aleph_1}$-coherent C-sequence over λ^+ implies that $\square_\lambda$ holds. Indeed, given such a sequence $\langle C_\alpha \mid \alpha < \lambda^+\rangle$, for every $\alpha \in E^{\lambda^+}_\omega$, if there exists a $\beta \in E^{\lambda^+}_{>\omega}$ such that $\alpha \in \mathrm{acc}(C_\beta)$, then replace C_α by $C_\beta \cap \alpha$, and otherwise, replace C_α by a cofinal subset of α of order-type ω. The outcome of this systematic modification at ordinals of countable cofinality is a $\square_\lambda$-sequence.

We now present a special case of the parametrized proxy principle $\mathrm{P}^-(\ldots)$.

Definition 2.3 (special case of [BR17, Definition 1.5])**.** Suppose:

- κ is a regular uncountable cardinal;
- $\xi, \sigma, \chi \leq \kappa$ are ordinals;
- $\mu, \theta \leq \kappa$ are cardinals;
- $\mathcal{S}$ is a nonempty family of stationary subsets of κ.

$\mathrm{P}^-_\xi(\kappa, \mu, \sqsubseteq_\chi, \theta, \mathcal{S}, \mu, \sigma)$ asserts the existence of a ξ-bounded $\sqsubseteq_\chi$-coherent $\mathcal{C}$-sequence $\vec{\mathcal{C}} = \langle \mathcal{C}_\alpha \mid \alpha < \kappa\rangle$ such that the two hold:

(1) for every $\alpha < \kappa$, $|\mathcal{C}_\alpha| < \mu$;
(2) for every sequence $\langle B_i \mid i < \theta\rangle$ of cofinal subsets of κ, for every $S \in \mathcal{S}$, there are stationarily many $\alpha \in S$ such that, for every $i < \min\{\alpha, \theta\}$, for every $C \in \mathcal{C}_\alpha$, $\sup\{\gamma \in C \mid \mathrm{succ}_\sigma(C \setminus \gamma) \subseteq B_i\} = \alpha$.

Remark 2.4. $\mathrm{succ}_\sigma(C) := \{\beta \in \mathrm{nacc}(C) \mid 0 < \mathrm{otp}(C \cap \beta) \leq \sigma\}$ is the set of the first σ many successor elements of C, should they exist. In particular, for every $\gamma \in C$ such that $\sup(\mathrm{otp}(C \setminus \gamma)) \geq \sigma$, $\mathrm{succ}_\sigma(C \setminus \gamma)$ is nothing but the next σ many successor elements of C above γ. In the special case $\sigma = 1$, the ending of requirement (2) above is equivalent to asserting that $\sup(\mathrm{nacc}(C) \cap B_i) = \alpha$.

Remark 2.5. For a discussion on the motivation of the proxy principles and a comparison between them and the square and diamond principles, we refer the reader to [BRY25, §2].

Definition 2.6. $\mathrm{P}_\xi(\kappa,\mu,\sqsubseteq_\chi,\theta,\mathcal{S},\mu,\sigma)$ asserts that $\mathrm{P}^-_\xi(\kappa,\mu,\sqsubseteq_\chi,\theta,\mathcal{S},\mu,\sigma)$ and $\diamondsuit(\kappa)$ both hold.

Convention 2.7. We may omit ξ in which case we mean that ξ is equal to κ.

Fact 2.8 ([Tod87])**.** *$\square(\kappa,{<}\kappa)$ entails the existence of a κ-Aronszajn tree.*

Fact 2.9 ([BR19b, Proposition 2.2])**.** *$\mathrm{P}(\kappa,\kappa,\sqsubseteq,1,\{\kappa\},\kappa,1)$ entails the existence of a κ-Souslin tree.*

3. The example

3.1. **Setup.** Hereafter, κ denotes an arbitrary regular uncountable cardinal, and H_κ stands for the collection of all sets of hereditary carnality less than κ. We write E^κ_χ for the set $\{\alpha\in E^\kappa \mid \mathrm{cf}(\alpha)=\chi\}$. $E^\kappa_{<\chi}$ and $E^\kappa_{\geq\chi}$ are defined analogously.

Definition 3.1. A κ-tree $\mathbf{T}=(T,<_T)$ is said to be *streamlined* iff all of the following hold:

- $<_T$ is $\subsetneq$;
- T is a subset of ${}^{<\kappa}H_\kappa$;[4]
- for all $t\in T$ and $\alpha<\mathrm{dom}(t)$, $t\restriction\alpha\in T$.

In order to seal antichains in streamlined κ-trees, we shall make use of the following verbose version of $\diamondsuit(\kappa)$.

Fact 3.2 ([BR17, Lemma 2.2])**.** *$\diamondsuit(\kappa)$ is equivalent to the existence of a sequence $\langle\Omega_\beta\mid\beta<\kappa\rangle$ satisfying that for all $\Omega\subseteq H_\kappa$ and $p\in H_{\kappa^+}$, there exists an elementary submodel $\mathcal{M}\prec H_{\kappa^+}$ containing p, such that $\mathcal{M}\cap\kappa\in\kappa$ and $\mathcal{M}\cap\Omega=\Omega_{\mathcal{M}\cap\kappa}$.*

Hereafter, fix a well-ordering $\lhd_\kappa$ of H_κ, and let $\langle\Omega_\beta\mid\beta<\kappa\rangle$ be as in Fact 3.2.

Definition 3.3 (canonical extension action)**.** For every $T\in H_\kappa$, denote $\beta(T):=0$ unless there is a $\beta<\kappa$ such that $T\subseteq{}^{\leq\beta}H_\kappa$ and $T\nsubseteq{}^{<\beta}H_\kappa$, in which case, we do the following:

- let $\beta(T)$ denote the unique β such that $T\subseteq{}^{\leq\beta}H_\kappa$ and $T\nsubseteq{}^{<\beta}H_\kappa$.
- let $T_{\beta(T)}:=\{t\in T\mid\mathrm{dom}(t)=\beta(T)\}$.
- for all $s\in T$ and $C\subseteq\beta(T)$, if $\sup(C)<\mathrm{dom}(s)$, consider the sets:
  - $Q_0:=\{t\in T_{\beta(T)}\mid s\subseteq t\}$, and
  - $Q_1:=\{t\in T_{\beta(T)}\mid\exists r\in\Omega_\beta(s\cup r\subseteq t)\}$.

[4]Together with the previous bullet point this means that $x<_T y$ iff x is an initial segment of y.

If Q_1 is nonempty, then let $\mathsf{extend}(s,T,C) := \min(Q_1, \lhd_\kappa)$. If Q_1 is empty but Q_0 is nonempty, then let $\mathsf{extend}(s,T,C) := \min(Q_0, \lhd_\kappa)$. Otherwise, let $\mathsf{extend}(s,T,C) := s$.

- for all $s \in T$ and $C \subseteq \beta(T)$, if $\sup(C) = \mathrm{dom}(s)$, consider the sets:
  - $Q_0 := \{t \in T_{\beta(T)} \mid (s^\frown\langle C\rangle) \subseteq t\}$, and
  - $Q_1 := \{t \in T_{\beta(T)} \mid \exists r \in \Omega_\beta((s^\frown\langle C\rangle) \cup r \subseteq t)\}$.

  If Q_1 is nonempty, then let $\mathsf{extend}(s,T,C) := \min(Q_1, \lhd_\kappa)$. If Q_1 is empty but Q_0 is nonempty, then let $\mathsf{extend}(s,T,C) := \min(Q_0, \lhd_\kappa)$. Otherwise, let $\mathsf{extend}(s,T,C) := \mathsf{extend}(s,T,\emptyset)$.

3.2. **The construction.**

Theorem 3.4. *Suppose:*

- *$\chi < \kappa$ is a pair of infinite regular cardinals such that $\nu^{<\chi} < \kappa$ for every $\nu < \kappa$;*
- *$\mathcal{S}$ is a nonempty collection of stationary subsets of $E^\kappa_{\geq\chi}$;*
- *ξ, σ are nonzero ordinals $\leq \kappa$;*
- *$\mathrm{P}_\xi(\kappa, \mu, \sqsubseteq_\chi, \theta, \mathcal{S}, \mu, \sigma)$ holds with $\mu = \kappa$.*

Then there exists a streamlined χ-complete κ-Souslin tree $\mathbf{T}$ such that in the forcing extension by $\mathbf{T}$, $\mathrm{P}_\xi(\kappa, \mu, \sqsubseteq_\chi, \theta, \mathcal{S}, \mu, \sigma)$ holds with $\mu = 2$.

Proof. Let $\vec{\mathcal{C}} = \langle \mathcal{C}_\alpha \mid \alpha < \kappa\rangle$ be a $\mathrm{P}^-_\xi(\kappa, \kappa, \sqsubseteq_\chi, \theta, \mathcal{S}, \kappa, \sigma)$-sequence. As $\mathcal{S} \subseteq \mathcal{P}(E^\kappa_{\geq\chi})$ and as $\nu^{<\chi} < \kappa$ for every $\nu < \kappa$, we may assume that for every $\alpha \in \mathrm{acc}(\kappa) \cap E^\kappa_{<\chi}$, $\mathcal{C}_\alpha$ contains a club C in α of order-type $\mathrm{cf}(\alpha)$ such that $\mathrm{nacc}(C)$ consists of successor ordinals. We may also assume that $\mathcal{C}_{\alpha+1} = \{\{\alpha\}, \{0, \alpha\}\}$ for every $\alpha < \kappa$. Lastly, we may also assume that $C \setminus \bar\alpha \in \mathcal{C}_\alpha$ for all $\bar\alpha < \alpha < \kappa$ and $C \in \mathcal{C}_\alpha$.

For every $\alpha < \kappa$, let $\mathcal{C}^1_\alpha := \{C \in \mathcal{C}_\alpha \mid \forall\gamma \in \mathrm{acc}(C)\,[C \cap \gamma \in \mathcal{C}_\gamma]\}$. Note that for every $C \in \mathcal{C}^1_\alpha$ and $\beta \in \mathrm{acc}(C)$, we have $C \cap \beta \in \mathcal{C}^1_\beta$. Also note that $\mathcal{C}^1_\alpha = \mathcal{C}_\alpha$ for every $\alpha \in E^\kappa_{\geq\chi}$.

Next, write $\mathcal{C} := \bigcup_{\alpha<\kappa} \mathcal{C}_\alpha$. Let $\mathcal{T}$ be the collection of all functions t such that:

- $t \in \prod_{\beta<\alpha} \mathcal{C}_\beta$ for some $\alpha < \kappa$;
- for every $\beta < \alpha$, for every $\gamma \in \mathrm{acc}(t(\beta))$, $t(\gamma) \sqsubseteq_\chi t(\beta)$.

We shall construct a sequence $\langle T_\alpha \mid \alpha < \kappa\rangle$ such that, for every $\alpha < \kappa$, T_α will constitute the α^{th}-level of our ultimate tree T. We make the following promises:

(1) for every $\alpha < \kappa$, T_α will be a nonempty subset of $\mathcal{T} \cap {}^\alpha\mathcal{C}$ of size $< \kappa$;
(2) for all $\beta < \alpha < \kappa$ and $t \in T_\alpha$, $t \restriction \beta \in T_\beta$;
(3) for all $\alpha < \kappa$ and $t \in T_\alpha$, there will be a $C \in \mathcal{C}_\alpha$ with $t^\frown\langle C\rangle \in T_{\alpha+1}$;
(4) for every $\alpha \in \mathrm{acc}(\kappa) \cap E^\kappa_{<\chi}$, $T_\alpha = \{t \in {}^\alpha\mathcal{C} \mid \forall\beta < \alpha\,(t \restriction \beta \in T_\beta)\}$;[5]
(5) for every $\alpha \in E^\kappa_{\geq\chi}$, $T_\alpha = \{\mathbf{b}^C_x \mid C \in \mathcal{C}_\alpha, x \in T_{\min(C)}\}$, where:

[5] Put differently, the tree we construct is χ-complete.

(6) for all $\alpha \in \mathrm{acc}(\kappa)$, $C \in \mathcal{C}^1_\alpha$ and $x \in T_{\min(C)}$, $\mathbf{b}^C_x$ will be some distinguished element of T_α satisfying the following three requirements:
  (i) $\mathbf{b}^C_x \restriction \min(C) = x$;
  (ii) for every pair $\beta^- < \beta$ of consecutive ordinals in C,
$$\mathbf{b}^C_x \restriction \beta = \mathsf{extend}(\mathbf{b}^C_x \restriction \beta^-, T \restriction (\beta+1), C \cap \beta^-);$$
  (iii) for every $\gamma \in \mathrm{acc}(C)$, $\mathbf{b}^C_x(\gamma) = C \cap \gamma$.

Claim 3.4.1. *Suppose that $\zeta \leq \kappa$ is such that Promises* (3)–(5) *and* (6)(i) *hold for every $\alpha < \zeta$. Then $T \restriction \zeta := \bigcup_{\alpha<\zeta} T_\zeta$ is a* normal *tree, that is, for all $\beta < \alpha < \zeta$ and $s \in T_\beta$, there is a $t \in T_\alpha$ such that $s \subsetneq t$.*

Proof. Let $\beta < \zeta$ and $s \in T_\beta$. We prove by induction on $\alpha \in [\beta, \zeta)$ the existence of $t \in T_\alpha$ such that $s \subsetneq t$. The base case is trivial, the successor case follows from Promise (3), and the case $\alpha \in \mathrm{acc}(\zeta \setminus \beta) \cap E^\kappa_{<\chi}$ then follows from Promise (5). Finally, given $\alpha \in \mathrm{acc}(\zeta \setminus \beta) \cap E^\kappa_{\geq\chi}$, pick $C \in \mathcal{C}_\alpha = \mathcal{C}^1_\alpha$. As C is a club in α, we may find some $\bar\alpha \in C$ above β. By the induction hypothesis, we may find $x \in T_{\bar\alpha}$ such that $s \subsetneq x$. As $\bar C := C \setminus \bar\alpha$ belongs to $\mathcal{C}_\alpha$ and $x \in T_{\min(\bar C)}$, we get from Promise (6)(i) that $s \subsetneq x \subseteq \mathbf{b}^{\bar C}_x \in T_\alpha$. □

We are now ready for the recursive construction. We start by letting $T_0 := \{\emptyset\}$, noting that T_0 is indeed a subset of $\mathcal{T}$. Next, for every $\alpha < \kappa$ such that T_α has already been successfully constructed, we let
$$T_{\alpha+1} := \{t^\frown\langle C\rangle \mid t \in T_\alpha, C \in \mathcal{C}_\alpha\} \cap \mathcal{T}.$$

Claim 3.4.2. *Promises* (1)–(3) *are maintained.*

Proof. If $\alpha = 0$, then $T_{\alpha+1} = \{t\}$ for the unique $t \in \mathcal{T}$ to satisfy $\mathrm{dom}(t) = 1$, namely, $t : \{0\} \to \mathcal{C}_0$. If $\alpha = \beta + 1$ is a successor ordinal, then $T_{\alpha+1}$ is equal to $\{t^\frown\langle\{\beta\}\rangle, t^\frown\langle\{0,\beta\}\rangle \mid t \in T_\alpha\}$ which again takes care of Promises (1)–(3).

If $\alpha \in \mathrm{acc}(\kappa) \cap E^\kappa_{<\chi}$, then given $t \in T_\alpha$, we may fix $C \in \mathcal{C}_\alpha$ of order-type less than χ such that $\mathrm{nacc}(C)$ consists of successor ordinals. We claim that $t^\frown\langle C\rangle$ is in $\mathcal{T}$, hence in $T_{\alpha+1}$. This requires that $t(\gamma) \sqsubseteq_\chi C$ for every $\gamma \in \mathrm{acc}(C)$, which holds trivially by our choice of C.

If $\alpha \in \mathrm{acc}(\kappa) \cap E^\kappa_{\geq\chi}$, then given $t \in T_\alpha$, by Promise (5), we may fix $C \in \mathcal{C}_\alpha = \mathcal{C}^1_\alpha$ and $x \in T_{\min(C)}$ such that $t = \mathbf{b}^C_x$, and we claim that $t^\frown\langle C\rangle$ is in $\mathcal{T}$, hence in $T_{\alpha+1}$. This requires that $t(\gamma) \sqsubseteq_\chi C$ for every $\gamma \in \mathrm{acc}(C)$, which is guaranteed by Promise (6)(iii). □

Next, suppose that $\alpha \in \mathrm{acc}(\kappa)$ is such that $\langle T_\beta \mid \beta < \alpha\rangle$ has already been successfully defined. Let $C \in \mathcal{C}^1_\alpha$ and $x \in T_{\min(C)}$. We shall obtain $\mathbf{b}^C_x$ as the limit $\bigcup \mathrm{Im}(b^C_x)$ of a $\subsetneq$-increasing sequence $b^C_x \in \prod_{\beta \in C} T_\beta$ canonically obtained by recursion, as follows:

- $b^C_x(\min(C)) := x$.
- for every pair $\beta^- < \beta$ of consecutive elements of C such that $b^C_x(\beta^-)$ has already been defined, let
$$b^C_x(\beta) := \mathsf{extend}(b^C_x(\beta^-), T \restriction (\beta+1), C \cap \beta^-).$$

- for every $\beta \in \mathrm{acc}(C)$ such that $b_x^C \restriction \beta$ has already been defined, let $b_x^C(\beta) := \bigcup \mathrm{Im}(b_x^C \restriction \beta)$.

We verify that the construction indeed produced a $\subsetneq$-increasing sequence of nodes in $T \restriction \alpha$. For every pair $\beta^- < \beta$ of consecutive elements of C, since $T \restriction (\beta+1)$ is a normal tree thus far, $\mathsf{extend}(b_x^C(\beta^-), T \restriction (\beta+1), \ldots)$ provides an extension of $b_x^C(\beta^-)$ belonging to T_β, so indeed $b_x^C(\beta^-) \subsetneq b_x^C(\beta) \in T_\beta$. For every $\beta \in \mathrm{acc}(C)$, Clauses (i) and (ii) of Promise (6) yield that $b_x^C(\gamma) = \mathbf{b}_x^{C\cap\beta} \restriction \gamma$ for every $\gamma \in C \cap \beta$, and hence $b_x^C(\beta) = \mathbf{b}_x^{C\cap\beta}$ which was placed in T_β in an earlier stage of the recursion, since $C \cap \beta \in \mathcal{C}_\beta^1$.

Next, if $\mathrm{cf}(\alpha) < \chi$, then define T_α according to Promise (4). It is trivial to verify that $T_\alpha \subseteq \mathcal{T}$. If $\mathrm{cf}(\alpha) \geq \chi$. then define T_α according to Promise (5).

Claim 3.4.3. *Promises* (1)–(6) *are maintained.*

Proof. Promise (1) is maintained in case $\mathrm{cf}(\alpha) < \chi$, since $|T \restriction \alpha| < \kappa$ and then the cardinal arithmetic hypothesis implies $|T_\alpha| < \kappa$. It is also maintained in case $\mathrm{cf}(\alpha) \geq \chi$ because in this case $|T_\alpha| \leq |T \restriction \alpha| \cdot |\mathcal{C}_\alpha| < \kappa$. The verification of Promise (2) is immediate. Promise (3) does not come into play. So, we are left with verifying Clause (iii) of Promise (6). Let $C \in \mathcal{C}_\alpha^1$ and $x \in T_{\min(C)}$, and we shall prove that $\mathbf{b}_x^C(\gamma) = C \cap \gamma$ for every $\gamma \in \mathrm{acc}(C)$. By induction. Specifically, given $\beta \in \mathrm{acc}(C)$ such that $\mathbf{b}_x^C(\gamma) = C \cap \gamma$ for every $\gamma \in \mathrm{acc}(C \cap \beta)$, we argue as follows. Let $\beta^+ := \min(C \setminus \beta+1)$, so that $\mathbf{b}_x^C(\beta) = b_x^C(\beta^+)(\beta)$. It is the case that

$$b_x^C(\beta^+) = \mathsf{extend}(b_x^C(\beta), T \restriction (\beta^+ + 1), C \cap \beta).$$

We have that $\sup(C \cap \beta) = \beta = \mathrm{dom}(b_x^C(\beta))$, so by the definition of $\mathsf{extend}(\ldots)$, it suffices to show that $t := b_x^C(\beta)^\frown\langle C \cap \beta\rangle$ belongs to $T_{\beta+1}$. As $b_x^C(\beta) \in T_\beta$, the definition of $T_{\beta+1}$ implies that we need to show that t belongs to $\mathcal{T}$, i.e., that for every $\gamma \in \mathrm{acc}(t(\beta))$, $t(\gamma) \sqsubseteq_\chi t(\beta)$. But $t(\beta) = C \cap \beta$ and $t(\gamma) = C \cap \gamma$ for every $\gamma \in \mathrm{acc}(C \cap \beta)$, so we are good. $\square$

Having constructed all levels of the tree, we let $T := \bigcup_{\alpha<\kappa} T_\alpha$, so that $\mathbf{T} := (T, \subsetneq)$ is our streamlined κ-tree.

Claim 3.4.4. $\mathbf{T}$ *admits no antichains of size* κ.

Proof. Suppose not, and let $A \subseteq T$ be a maximal antichain of size κ. By [BR19b, Claim 2.2.2], the following set is stationary in κ:

$$B := \{\beta \in \mathrm{acc}(\kappa) \mid A \cap (T \restriction \beta) = \Omega_\beta \text{ is a maximal antichain in } T \restriction \beta\}.$$

Using the hitting property of the proxy principle, fix an $\alpha \in E^\kappa_{\geq\chi}$ such that $\sup(\mathrm{nacc}(C) \cap B) = \alpha$ for every $C \in \mathcal{C}_\alpha$. We shall prove that $A \subseteq T \restriction \alpha$. To this end consider any $z \in T \restriction (\kappa \setminus \alpha)$. Set $y := z \restriction \alpha$, so that $y \in T_\alpha$ and $y \subseteq z$. Recalling Promise (5), pick $C \in \mathcal{C}_\alpha$ and $x \in T_{\min(C)}$ such that $y = \mathbf{b}_x^C$. Fix $\beta \in \mathrm{nacc}(C) \cap B$ with $\mathrm{dom}(x) < \beta < \alpha$. Denote $\beta^- := \sup(C \cap \beta)$. Then $\beta^- < \beta$ is a pair of consecutive elements of C, so by Promise (6)(ii),

$$\mathbf{b}_x^C \restriction \beta = \mathsf{extend}(\mathbf{b}_x^C \restriction \beta^-, T \restriction (\beta+1), C \cap \beta^-).$$

Since $\beta \in B$, β is a limit ordinal and $\Omega_\beta = A \cap (T \restriction \beta)$ is a maximal antichain in $T \restriction \beta$. It follows that for every $s' \in T_{\beta^-} \cup T_{\beta^-+1}$ there is an $r \in \Omega_\beta$ such that $s' \cup r \in T \restriction \beta$, and by the normality of $T \restriction (\beta+1)$, there is then a $t \in T_\beta$ with $s' \cup r \subseteq t$. This verifies that the definition of $\mathsf{extend}(\mathbf{b}^C_x \restriction \beta^-, T \restriction (\beta+1), C \cap \beta^-)$ will end up being $\min(Q_1, \lhd_\kappa)$,[6] so that $\mathbf{b}^C_x \restriction \beta$ extends some $r \in \Omega_\beta = A \cap (T \restriction \beta)$. Since r is an element of the antichain A, and $r \subseteq b^C_x(\beta) \subsetneq \mathbf{b}^C_x = y \subseteq z$, we infer that $z \notin A$. □

Claim 3.4.5. $\mathbf{T}$ *admits no chains of size* κ.

Proof. Suppose not, and pick $b : \kappa \to \mathcal{C}$ such that $b \restriction \alpha \in T_\alpha$ for every $\alpha < \kappa$. For every infinite $\alpha < \kappa$, as $\mathcal{C}_{\alpha+1} = \{\{\alpha\}, \{0, \alpha\}\}$, the definition of $T_{\alpha+2}$ implies that

$$t_\alpha := b \restriction (\alpha+1)^\frown \langle b(\alpha+1) \mathbin{\triangle} \{0\}\rangle$$

is an element of $T_{\alpha+2} \setminus \{b \restriction (\alpha+2)\}$. So $\{t_\alpha \mid \omega \le \alpha < \kappa\}$ is a κ-sized antichain, contradicting Claim 3.4.4. □

So, $\mathbf{T}$ is a κ-Souslin tree. Next, work in $V[G]$, where G is $\mathbf{T}$-generic over V. Set $b := \bigcup G$, so that $b : \kappa \to \mathcal{C}$ is a map such that $b \restriction \alpha \in T_\alpha$ for every $\alpha < \kappa$. As $T_\alpha \subseteq \mathcal{T} \cap {}^\alpha\mathcal{C}$ for every $\alpha < \kappa$, b is a ξ-bounded $\sqsubseteq_\chi$-coherent C-sequence over κ, which we hereafter denote by $\vec{C} = \langle C_\alpha \mid \alpha < \kappa\rangle$. The next claim is unnecessary, but its proof is short and may be of interest to those readers whose primary interest is in classical square principles.

Claim 3.4.6. $\vec{C}$ *is unthreadable.*

Proof. Given a club $D \subseteq \kappa$, use the κ-cc of $\mathbf{T}$ to find a ground model club $D' \subseteq D$. Working in the ground model, invoke the hitting property of the proxy principle with respect to $B := \operatorname{acc}(D')$, and find some $\alpha \in E^\kappa_{\ge\chi}$ such that $\sup(\operatorname{nacc}(C) \cap B) = \alpha$ for every $C \in \mathcal{C}_\alpha$. Then $\sup(\operatorname{nacc}(b(\alpha)) \cap \operatorname{acc}(D')) = \alpha$, in particular $\sup(\operatorname{nacc}(b(\alpha)) \cap \operatorname{acc}(D)) = \alpha$, and hence $\alpha \in \operatorname{acc}(D)$ with $b(\alpha) \neq D \cap \alpha$. □

Our next task is verifying that $\mathrm{P}^-_\xi(\kappa, 2, \sqsubseteq_\chi, \theta, \mathcal{S}, 2, \sigma)$ holds in $V[G]$.

Back in V, as $\diamondsuit(\kappa)$ holds and $\mathbf{T}$ is a κ-sized κ-cc forcing, the beginning of the proof of [BR19b, Theorem 3.4] provides a sequence $\langle Z_\beta \mid \beta < \kappa\rangle \in V$ satisfying that for every $B \in \mathcal{P}^{V[G]}(\kappa)$, there exists some $X_B \in \mathcal{P}^V(\kappa)$ such that:

(1) $V \models X_B$ is stationary;
(2) $V[G] \models X_B \subseteq \{\beta < \kappa \mid Z_\beta = B \cap \beta\}$.

In particular, $\diamondsuit(\kappa)$ holds in $V[G]$. Working in $V[G]$, for every $\alpha \in \operatorname{acc}(\kappa)$, let $C^\bullet_\alpha := \operatorname{Im}(g_\alpha)$, where $g_\alpha : C_\alpha \to \alpha$ is defined by stipulating:

$$g_\alpha(\beta) := \begin{cases} \beta, & \text{if } \beta \in \operatorname{acc}(C_\alpha);\\ \min(Z_\beta \cup \{\beta\}), & \text{if } \beta = \min(C_\alpha);\\ \min\big((Z_\beta \cup \{\beta\}) \setminus (\sup(C_\alpha \cap \beta)+1)\big), & \text{otherwise.}\end{cases}$$

[6] Recall Definition 3.3.

For completeness, we also set $C^\bullet_0 := \emptyset$ and $C^\bullet_{\alpha+1} := \{\alpha\}$ for every $\alpha < \kappa$. By [BR19a, Lemma 2.8], $\vec{C}^\bullet := \langle C^\bullet_\alpha \mid \alpha < \kappa\rangle$ is yet another ξ-bounded $\sqsubseteq_\chi$-coherent C-sequence over κ. To verify it witnesses $\mathrm{P}^-_\xi(\kappa, 2, \sqsubseteq_\chi, \theta, \mathcal{S}, 2, \sigma)$, let $\langle B_i \mid i < \theta\rangle$ be a given sequence of cofinal subsets of κ, and let $S \in \mathcal{S}$; we need to find stationarily many $\alpha \in S$ such that

$$\sup\{\gamma \in C^\bullet_\alpha \mid \mathrm{succ}_\sigma(C^\bullet_\alpha \setminus \gamma) \subseteq B_i\} = \alpha.$$

For each $i < \theta$, fix $X_i \in \mathcal{P}^V(\kappa)$ such that:

(1) $V \models X_i$ is stationary;
(2) $V[G] \models X_i \subseteq \{\beta < \kappa \mid Z_\beta = B_i \cap \beta\}$.

As any club in $V[G]$ covers a club from V, we may shrink the stationary set X_i to also ensure the following:

(3) $V[G] \models X_i \subseteq \{\beta < \kappa \mid \sup(B_i \cap \beta) = \beta\}$.

Working in V, since $S \in \mathcal{S}$, there is a stationary subset $\bar{S}$ of S such that for every $\alpha \in \bar{S}$, for every $C \in \mathcal{C}_\alpha$, for every $i < \min\{\alpha, \theta\}$,

$$\sup\{\gamma \in C \mid \mathrm{succ}_\sigma(C \setminus \gamma) \subseteq X_i\} = \alpha.$$

In particular, for every $i < \min\{\alpha, \theta\}$, the following set is cofinal in α:

$$\Gamma_i := \{\gamma \in C_\alpha \mid \mathrm{succ}_\sigma(C_\alpha \setminus \gamma) \subseteq X_i\}.$$

Work in $V[G]$. As it is a κ-cc forcing extension of V, $\bar{S}$ remains stationary. In addition, a moment's reflection makes it clear that for every $i < \theta$, for every $\beta \in \mathrm{nacc}(C_\alpha) \cap X_i$, it is the case that $g_\alpha(\beta) \in \mathrm{nacc}(C^\bullet_\alpha) \cap B_i$.[7] It follows that for every $i < \min\{\alpha, \theta\}$, for every $\gamma \in \Gamma_i$, it is the case that $g_\alpha(\gamma)$ is an element of $C^\bullet_\alpha$ satisfying that $\mathrm{succ}_\sigma(C^\bullet_\alpha \setminus \gamma) \subseteq B_i$. So we are done. □

4. Corollaries

We now present a gallery of corollaries to Theorem 3.4.

Corollary 4.1. *Suppose κ is a regular uncountable cardinal, and $\mathrm{P}(\kappa, \kappa, \sqsubseteq, 1, \{\kappa\}, \kappa, 1)$ holds. Then there is a κ-Souslin tree that forces $\square(\kappa)$.*

Proof. By Theorem 3.4, there is a streamlined κ-Souslin tree $\mathbf{T}$ such that in the forcing extension by $\mathbf{T}$, $\mathrm{P}(\kappa, 2, \sqsubseteq, 1, \{\kappa\}, 2, 1)$ holds. By [BR17, Lemma 3.2], then, $\square(\kappa)$ holds in the extension. □

The next corollary establishes Theorem C.

Corollary 4.2. *Suppose $\diamondsuit$ holds over a nonreflecting stationary subset of a strongly inaccessible κ. Then there is a κ-Souslin tree that forces $\square(\kappa)$.*

Proof. By [BR21, Theorem 4.26], in particular, $\mathrm{P}(\kappa, \kappa, \sqsubseteq, 1, \{\kappa\}, \kappa, 1)$ holds. Now appeal to Corollary 4.1. □

[7] See the proof of [BR19b, Claim 3.4.5] for details.

Corollary 4.3. *Suppose λ is an infinite cardinal, and* $\mathrm{P}_\lambda(\lambda^+,\lambda^+,\sqsubseteq,1,\{\lambda^+\},\lambda^+,1)$ *holds. Then there is a λ^+-Souslin tree that forces $\square_\lambda$.*

Proof. By Theorem 3.4, there is a streamlined λ^+-Souslin tree $\mathbf{T}$ such that in the forcing extension by $\mathbf{T}$, $\mathrm{P}_\lambda(\lambda^+,2,\sqsubseteq,1,\{\lambda^+\},2,1)$ holds. In particular, in the extension, there is a λ-bounded $\sqsubseteq$-coherent C-sequence over λ^+, i.e., $\square_\lambda$ holds. □

The next corollary establishes Theorem B.

Corollary 4.4. *Suppose λ is a measurable cardinal satisfying $2^\lambda=\lambda^+$. In the forcing extension by Prikry forcing there is a λ^+-Souslin tree that forces $\square_\lambda$.*

Proof. By [BR19b, Main Theorem], the hypotheses imply that in the forcing extension by Prikry forcing, $\mathrm{P}(\lambda^+,\lambda^+,\sqsubseteq,1,\{\lambda^+\},\lambda^+,1)$ holds. Now appeal to Corollary 4.3. □

The next two corollaries establish together Theorem A.

Corollary 4.5. *Suppose $\lambda=\lambda^{<\lambda}$ is an infinite cardinal and $\diamondsuit(E^{\lambda^+}_\lambda)$ holds. Then there is a λ-complete λ^+-Souslin tree $\mathbf{T}$ that forces $\square^B_\lambda$. In particular:*

(1) *If $\lambda=\aleph_1$, then $\mathbf{T}$ forces $\square_\lambda$;*
(2) *If $\lambda>\aleph_1$ is a successor cardinal, then $\mathbf{T}$ also forces the existence of a λ-complete λ^+-super-Souslin tree.*

Proof. By [BR17, Theorem 5.6], the hypotheses imply that $\mathrm{P}_\lambda(\lambda^+,2,\mathcal{R},\lambda^+,\{E^{\lambda^+}_\lambda\},2,\sigma)$ holds for $\mathcal{R}:={}_\lambda\sqsubseteq$ and every $\sigma<\lambda$, in particular, $\sigma:=1$. As $\lambda^{<\lambda}=\lambda$, it follows that $\mathrm{P}_\lambda(\lambda^+,\lambda^+,\mathcal{R},\lambda^+,\{E^{\lambda^+}_\lambda\},2,1)$ holds for $\mathcal{R}:=\sqsubseteq$, in particular, for $\mathcal{R}:=\sqsubseteq_\lambda$. By Theorem 3.4, then, there exists a streamlined λ-complete λ^+-Souslin tree $\mathbf{T}$ such that in the forcing extension by $\mathbf{T}$, $\mathrm{P}_\lambda(\lambda^+,2,\sqsubseteq_\lambda,\lambda^+,\{E^{\lambda^+}_\lambda\},2,1)$ holds. In particular, in the extension, there is a λ-bounded $\sqsubseteq_\lambda$-coherent C-sequence over λ^+, i.e., $\square^B_\lambda$ holds.

Finally, Clause (1) follows from the fact that $\square_{\aleph_1}$ and $\square^B_{\aleph_1}$ are logically equivalent. Clause (2) follows from Corollary 1.7 of [LR19a] together with the sentence prior to Theorem B'' of the same paper. □

Corollary 4.6. *Suppose $\lambda=\lambda^{<\lambda}$ is an uncountable cardinal and $\diamondsuit(E^{\lambda^+}_\lambda)$ holds. Then there is a countably complete λ^+-Souslin tree $\mathbf{T}$ that forces $\square_\lambda$.*

Proof. As explained in the proof of Corollary 4.5, $\mathrm{P}_\lambda(\lambda^+,\lambda^+,\sqsubseteq,\lambda^+,\{E^{\lambda^+}_\lambda\},2,1)$ holds. In particular, $\mathrm{P}_\lambda(\lambda^+,\lambda^+,\sqsubseteq_{\aleph_1},\lambda^+,\{E^{\lambda^+}_\lambda\},2,1)$ holds. By Theorem 3.4, then, there exists a countably complete λ^+-Souslin tree $\mathbf{T}$ such that in the forcing extension by $\mathbf{T}$, $\mathrm{P}_\lambda(\lambda^+,2,\sqsubseteq_{\aleph_1},\lambda^+,\{E^{\lambda^+}_\lambda\},2,1)$ holds. In particular, in the extension, there is a λ-bounded $\sqsubseteq_{\aleph_1}$-coherent C-sequence over λ^+. By Remark 2.2, then, $\square_\lambda$ holds. □

The next corollary establishes Theorem D.

Corollary 4.7. *Suppose $\lambda = 2^{<\lambda} < 2^\lambda = \lambda^+$ is uncountable cardinal, and there exists a stationary subset of $E^{\lambda^+}_{\neq\mathrm{cf}(\lambda)}$ that does not reflect.*

1. *If λ is regular or if $\square^*_\lambda$ holds, then there is a λ^+-Souslin tree that forces $\square_\lambda$;*
2. *If λ is regular, then there is a λ-complete λ^+-Souslin tree that forces $\square(\lambda^+)$.*

Proof. (1) If λ is a regular cardinal then the cardinal arithmetic hypothesis implies that $\square^*_\lambda$ holds. Thus, by [BR19c, Corollary 3.15], either of the hypothesis of Clause (1) imply that $\mathrm{P}_\lambda(\lambda^+, \lambda^+, \sqsubseteq, {<}\lambda, \{E^{\lambda^+}_\eta\}, 2, {<}\lambda)$ holds for some infinite regular cardinal $\eta < \lambda$. In particular, $\mathrm{P}_\lambda(\lambda^+, \lambda^+, \sqsubseteq, 1, \{\lambda^+\}, \lambda^+, 1)$ holds. By Theorem 3.4, then, there exists a streamlined λ-complete λ^+-Souslin tree $\mathbf{T}$ such that in the forcing extension by $\mathbf{T}$, $\mathrm{P}_\lambda(\lambda^+, 1, \sqsubseteq, 2, \{\lambda^+\}, 2, 1)$ holds. In particular, in the extension, there is a λ-bounded $\sqsubseteq$-coherent C-sequence over λ^+, i.e., $\square_\lambda$ holds.

(2) By [BR19c, Theorem A], the hypotheses imply that $\mathrm{P}(\lambda^+, \lambda^+, \mathcal{R}, 1, \{E^{\lambda^+}_\lambda\}, 2, 1)$ holds for $\mathcal{R} := \sqsubseteq^*$. By [BR21, Corollary 4.36], it moreover holds for $\mathcal{R} := \sqsubseteq$. By Theorem 3.4, then, there exists a streamlined λ-complete λ^+-Souslin tree $\mathbf{T}$ such that in the forcing extension by $\mathbf{T}$, $\mathrm{P}(\lambda^+, 1, \sqsubseteq_\lambda, 2, \{E^{\lambda^+}_\lambda\}, 2, 1)$ holds. By [BR17, Lemma 3.2], then, $\square(\lambda^+)$ holds in the extension. $\square$

To put the next corollary in context, recall that Laver proved (see [LS81]) that in the generic extension after Lévy-collapsing a weakly compact cardinal to $\aleph_2$, there are no $\aleph_2$-Aronszajn trees with an ω-ascent path, and that Todorčević proved [Tod81, Theorem 4.4] that in the same generic extension, no $\aleph_2$-Aronszajn tree is coherent.

Corollary 4.8. *Suppose $2^{2^{\aleph_0}} = \aleph_2$ and $E^{\aleph_2}_{\aleph_0}$ admits a nonreflecting stationary subset. Then there is a countably complete $\aleph_2$-Souslin tree $\mathbf{T}$ such that:*

1. *$\mathbf{T}$ forces the existence of a uniformly coherent $\aleph_2$-Souslin tree;*
2. *$\mathbf{T}$ forces the existence of an $\aleph_2$-Souslin tree with an ω-ascent path.*

Proof. As seen in the proof of Corollary 4.7(2), there exists a streamlined countably complete $\aleph_2$-Souslin tree $\mathbf{T}$ such that in the forcing extension by $\mathbf{T}$, $\mathrm{P}(\aleph_2, 2, \sqsubseteq_{\aleph_1}, 1, \{E^{\aleph_2}_{\aleph_1}\}, 2, 1)$ holds. Equivalently, in this case, $\mathrm{P}(\aleph_2, 2, \sqsubseteq, 1, \{E^{\aleph_2}_{\aleph_1}\}, 2, 1)$ holds. By [LR19b, Theorem 3.11(3)], moreover, $\mathrm{P}(\aleph_2, 2, \sqsubseteq, \aleph_2, \{E^{\aleph_2}_{\aleph_1}\}, 2, 1)$ holds.

(1) By [BR21, Theorem 6.35], $\mathrm{P}(\aleph_2, 2, \sqsubseteq, \aleph_2, \{\aleph_2\}, 2, 1)$ is sufficient for the construction of a uniformly coherent $\aleph_2$-Souslin tree in the extension.

(2) By [BR21, Corollary 6.12], $\square(\aleph_2)$ together with $2^{2^{\aleph_0}} = \aleph_2$ suffices for the construction of $\aleph_2$-Souslin tree with an ω-ascent path. $\square$

Corollary 4.9. *Suppose $\lambda = \lambda^{<\lambda}$ is an uncountable cardinal and $2^\lambda = \lambda^+$. Then $\mathrm{Add}(\lambda,1)$ introduces a λ-complete λ^+-Souslin tree that forces the existence of a full λ^+-Souslin tree.*

Proof. Recall that $V^{\mathrm{Add}(\lambda,1)} \models \diamondsuit(\lambda)$. Meanwhile, by [BR21, Theorem 5.7], in $V^{\mathrm{Add}(\lambda,1)}$, $\mathrm{P}_\lambda(\lambda^+, 2, \mathcal{R}, \lambda^+, \{E^{\lambda^+}_\lambda\}, \sigma, 1)$ holds for $\mathcal{R} := {}_\lambda\sqsubseteq$ and every $\sigma < \lambda$, in particular, $\sigma := 1$. As $\lambda^{<\lambda} = \lambda$, it follows that (in $V^{\mathrm{Add}(\lambda,1)}$) $\mathrm{P}_\lambda(\lambda^+, \lambda^+, \mathcal{R}, \lambda^+, \{E^{\lambda^+}_\lambda\}, 2, 1)$ holds for $\mathcal{R} := \sqsubseteq$, in particular for $\mathcal{R} := \sqsubseteq_\lambda$. By Theorem 3.4, then, in the same model there exists a streamlined λ-complete λ^+-Souslin tree $\mathbf{T}$ such that in the forcing extension by $\mathbf{T}$, $\mathrm{P}_\lambda(\lambda^+, 2, \sqsubseteq_\lambda, \lambda^+, \{E^{\lambda^+}_\lambda\}, 2, 1)$ holds. As $\mathbf{T}$ is λ^+-distributive, $\diamondsuit(\lambda)$ remains to hold. As $\mathbf{T}$ has size λ^+, $2^\lambda = \lambda^+$ remains to hold. Now, appeal to [RYY24, Theorem 5.1]. $\square$

5. A complementary result

In this section, we establish Theorem E. Throughout this section, κ stands for a regular uncountable cardinal.

Definition 5.1. κ is said to have the *tree property* iff there are no κ-Aronszajn trees.

Let us now recall an argument of Kunen from [Kun78, §3]. Denote by $\mathbb{S}_\kappa$ the notion of forcing consisting of the collection of all normal streamlined homogeneous subtrees of ${}^{<\kappa}2$ that are either empty or of a successor height; the order on $\mathbb{S}_\kappa$ is defined by taking end extensions.

Fact 5.2 ([Kun78, §3])**.** *For any $\mathbb{S}_\kappa$-generic filter G, $T(G) := \bigcup G$ is a normal streamlined homogeneous κ-Souslin tree in $V[G]$.*

For brevity, let us denote $T(G)$ by T, and its canonical name by $\dot{T}$.

Fact 5.3 ([Kun78, §3])**.** *If $\kappa = \kappa^{<\kappa}$, then $\mathbb{S}_\kappa * \dot{T}$ is forcing equivalent to $\mathrm{Add}(\kappa,1)$.*

We now apply Kunen's idea as follows.

Lemma 5.4. *Suppose $\kappa = \kappa^{<\kappa}$, the tree property holds at κ and is moreover indestructible under $\mathrm{Add}(\kappa,1)$. Then for any $\mathbb{S}_\kappa$-generic G, the forcing extension $V[G]$ satisfies both of the following:*

1. *There exists a κ-Souslin tree;*
2. *Every κ-Souslin tree forces that $\square(\kappa,{<}\mu)$ fails for every $\mu < \kappa$.*

Proof. Clause (1) is taken care of by Fact 5.2, so we turn to address Clause (2). To this end, let T' be an arbitrary κ-Souslin tree in $V[G]$, and let b' be a T'-generic branch over $V[G]$. Recall that we denote by T the κ-tree added by G.

Claim 5.4.1. *There is a T-generic branch b over $V[G]$ such that $V[G][b'] \subseteq V[G][b]$.*

Proof. Suppose not. Fix $p \in T'$ that forces for any T'-generic branch b' over $V[G]$ with $p \subseteq b'$, there is no T-generic branch b over $V[G]$ such that $V[G][b'] \nsubseteq V[G][b]$. Now let b be a T-generic branch over $V[G]$. Then $V[G][b]$ is a forcing extension of $V[G]$ by $\mathrm{Add}(\kappa, 1)$ by Fact 5.3. Thus, by our assumption, the tree property at κ holds in $V[G][b]$. Therefore, it follows that T' has a cofinal branch, say b', in $V[G][b]$ with $p \subseteq b'$. Since T' is κ-Souslin, it follows that b' is generic over $V[G]$. Thus, $V[G][b'] \subseteq V[G][b]$, contradicting the assumption that p forces that $V[G][b'] \nsubseteq V[G][b]$. $\square$

In $V[G][b']$, let $\vec{\mathcal{C}} = \langle \mathcal{C}_\alpha \mid \alpha < \kappa\rangle$ be a $\sqsubseteq$-coherent $\mathcal{C}$-sequence over κ such that $(\sup_{\alpha<\kappa} |\mathcal{C}_\alpha|)^+ < \kappa$; we need to prove that it admits a thread.

Let b be given by the claim, so that $V[G][b'] \subseteq V[G][b]$. Recall that by Fact 5.3, the tree property holds at κ in $V[G][b]$, so Fact 2.8 implies that $\square(\kappa, {<}\kappa)$ fails, in particular, we may fix in $V[G][b]$ a club D threading $\vec{\mathcal{C}}$. As $V[G][b]$ is a κ-cc forcing extension of $V[G]$, we may fix a subclub D' of D lying in $V[G]$. Working back in its extension $V[G][b']$, for every $\alpha \in \mathrm{acc}(D')$, it is the case that $D' \cap \alpha \subseteq D \cap \alpha \in \mathcal{C}_\alpha$. By [HLH17, Lemma 2.4], then, $\vec{\mathcal{C}}$ admits a thread. $\square$

We are now ready to derive a strong form of Theorem E.

Corollary 5.5. *Assuming the consistency of large cardinals, the conjunction of the following two bullet points is compatible with κ being a successor of a regular uncountable, a successor of a singular, or a strongly inaccessible.*

- *There is a κ-Souslin tree;*
- *Every κ-Souslin tree forces that $\square(\kappa, {<}\mu)$ fails for every $\mu < \kappa$.*

Proof. **Inaccessible:** If there is a weakly compact cardinal κ, then by Kunen's original argument [Kun78, §3], there is a forcing extension in which κ is a strongly inaccessible having the tree property and it is moreover indestructible under $\mathrm{Add}(\kappa, 1)$. Since κ is inaccessible, we have $\kappa^{<\kappa} = \kappa$ in this model, so we are in conditions to appeal to Lemma 5.4.

Successor of regular uncountable: As proved in [LHL18, Lemma 4.14], starting from a weakly compact cardinal and an infinite cardinal $\mu = \mu^{<\mu}$, one obtains a model in which the tree property holds at $\kappa := \mu^{++}$ and it is indestructible under $\mathrm{Add}(\kappa, 1)$. Moreover, $\kappa^{<\kappa} = \kappa$ holds in this model.

Successor of singular: By [MS96, Theorem 3.1], for every increasing sequence $\langle \lambda_n \mid n < \omega\rangle$ of supercompact cardinals, the tree property holds at $\kappa := (\sup_{n<\omega}(\lambda_n))^+$. By a suitable forcing preparation, it is possible to ensure that λ_n be a supercompact that is indestructible under $\mathrm{Add}(\kappa, 1)$ for each $n < \omega$. Consequently, the tree property holds at κ and is indestructible under $\mathrm{Add}(\kappa, 1)$. In addition, $\kappa^{<\kappa} = \kappa$ since the Singular Cardinal Hypothesis holds at $\lambda := \sup_{n<\omega}(\lambda_n)$. $\square$

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