All colorings are strong - but some colorings are stronger than the others

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In this talk, I'll touch upon four research projects I've been recently involved in. In each of these projects, a slightly different interpretation of the concept of "strong coloring" is taken.

Due to time restrictions, I won't be able to give all the details, but I will always provide pointers to where these details may be found.

What this talk isn't about

With strong colorings one can:

- characterize cardinal invariants;
- characterize large cardinals and prove variations of Kunen's inconsistency result;
- construct interesting partial orders, topological spaces, uncountable groups and model-theoretic theories.

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The biggest open problem in the study of strong colorings is:

- Can the successor of a singular be Jónsson?

There's a large body of work by Shelah on the subject, and the state of the art is due to Eisworth (2012).
We won't have the time to go through any of them.

Bibliography

The first project we'll discuss is joint with Chris Lambie-Hanson:

[34] **Knaster and friends I: Closed colorings and precalibers**, *Algebra Universalis*, 79(4), Art. 90, 39 pp., 2018.

[35] **Knaster and friends II: The C-sequence number**, *J. Math. Logic*, 21(1):2150002, 54pp, 2021.

[36] **Knaster and friends III: Subadditive colorings**, *in preparation*.

* To access, e.g., paper [34], visit `http://p.assafrinot.com/34`

Bibliography, cont.

The second project is joint with Jing Zhang:

[44] **Transformations of the transfinite plane**, *Forum Math. Sigma*, 25 pp., accepted January 2021.

[45] **Strongest Transformations**, *in preparation*.

Bibliography, cont.

The third project is joint with Tanmay Inamdar:

[46] **Relative club guessing**, *in preparation*.

[47] **Was Ulam right?**, *in preparation*.

Here, we obtain strong colorings, use them to prove club guessing theorems and then use them to get strong colorings at higher cardinals.

Bibliography, cont.

The fourth is joint with Menachem Kojman and Juris Steprāns:

[49] **Advances on strong colorings over partitions**, 21 pp., submitted.

[50] **Sierpinski's onto mapping and partitions**, 15 pp., submitted.

Positive partition relations and chain conditions

In 1930, Ramsey proved that for every coloring $c : [\omega]^2 \to 2$, there exists $A \in [\omega]^\omega$ which is c-homogeneous, i.e., $c \restriction [A]^2$ is constant. This is denoted by $\omega \to (\omega)^2_2$.

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κ is weakly compact iff it is uncountable and $\kappa \to (\kappa)^2_2$ holds.

Evidently, if $\kappa \to (\kappa)^2_2$ then any κ-cc poset is κ-Knaster, hence the square of any κ-cc poset (equivalently, the product of any two κ-cc posets) is again κ-cc.

Recall: a poset $\mathbb{P}$ is κ-Knaster iff any $A \in [\mathbb{P}]^\kappa$ has $B \in [A]^\kappa$ consisting of pairwise compatible conditions.

From homogeneous to bounded and vice versa

Recall: κ is weakly compact iff it is uncountable and $\kappa \to (\kappa)^2_2$.

Theorem (with Inamdar [47])

An infinite cardinal κ is weakly compact iff for every coloring $c : [\kappa]^2 \to \omega$, there exists $A \in [\kappa]^\kappa$ such that for every $\epsilon < \kappa$, $\{c(\epsilon, \alpha) \mid \alpha \in A \setminus (\epsilon + 1)\}$ is finite.

Negative partition relations on ω_1

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The proof is by a diagonalization. Let us sketch it:

- Using CH, enumerate $[\omega_1]^\omega$ as $\langle A_\epsilon \mid \epsilon < \omega_1 \rangle$ with $A_\epsilon \subseteq \epsilon$.

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- Using CH, enumerate $[\omega_1]^\omega$ as $\langle A_\epsilon \mid \epsilon < \omega_1 \rangle$ with $A_\epsilon \subseteq \epsilon$.
- For every infinite $\beta < \omega_1$, $\{A_\epsilon \mid \epsilon < \beta\}$ is a countable family of infinite countable sets, so it admits a *disjoint refinement*, i.e., a pairwise disjoint family $\{A^\beta_\epsilon \mid \epsilon < \beta\}$ with $A^\beta_\epsilon \in [A_\epsilon]^\omega$ for all $\epsilon < \beta$.

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The proof is by a diagonalization. Let us sketch it:

- Using CH, enumerate $[\omega_1]^\omega$ as $\langle A_\epsilon \mid \epsilon < \omega_1 \rangle$ with $A_\epsilon \subseteq \epsilon$.
- For every infinite $\beta < \omega_1$, $\{A_\epsilon \mid \epsilon < \beta\}$ is a countable family of infinite countable sets, so it admits a *disjoint refinement*, i.e., a pairwise disjoint family $\{A^\beta_\epsilon \mid \epsilon < \beta\}$ with $A^\beta_\epsilon \in [A_\epsilon]^\omega$ for all $\epsilon < \beta$.
- Pick any $c : [\omega_1]^2 \to \omega_1$ such that for every infinite $\beta < \omega_1$, $c[A^\beta_\epsilon \times \{\beta\}] = \beta$ for every $\epsilon < \beta$.

Negative partition relations on ω_1, cont.

The above coloring c in fact demonstrates $\omega_1 \nrightarrow [\omega \circledast \omega_1]^2_{\omega_1}$, i.e., for every $A \in [\omega_1]^\omega$ and $B \in [\omega_1]^{\omega_1}$, $c[A \times B] = \omega_1$.

Negative partition relations on ω_1, cont.

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Theorem (Erdős, Hajnal and Milner, 1966)

CH *entails* $\omega_1 \nrightarrow \left[{}^{\omega \circledast \omega_1}\!/_{1 \circledast \omega_1}\right]^2_{\omega_1}$, *i.e., a coloring* $c : [\omega_1]^2 \to \omega_1$ *such that for all* $A \in [\omega_1]^\omega$ *and* $B \in [\omega_1]^{\omega_1}$, *there is* $\alpha \in A$ *such that* $c[\{\alpha\} \times B] = \omega_1$.

Negative partition relations on ω_1, cont.

Jensen (1972) got $\omega_1 \not\to [\omega_1]^2_\omega$ from a Souslin tree.

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Negative partition relations on ω_1, cont.

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Erdős and Hajnal (1978) got $\omega_1 \not\to [\omega_1]^2_\omega$ from a Luzin set.

Can the two be improved to yield $\omega_1 \not\to [\omega_1]^2_{\omega_1}$?

Pumping-up, free of charge

Let $c : [\omega_1]^2 \to \omega$ be a witness to $\omega_1 \not\to [\omega_1]^2_\omega$.
We shall pump it up to get a witness to $\omega_1 \not\to [\omega_1]^2_{\omega_1}$.

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For every $\beta < \omega_1$, fix a bijection $e_\beta : \omega \leftrightarrow \omega + \beta$.
Define $c^+ : [\omega_1]^2 \to \omega_1$ via $c^+(\alpha, \beta) := e_\beta(c(\alpha, \beta))$.

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Given an uncountable $B \subseteq \omega_1$ and a prescribed color $\tau < \omega_1$,
find $n < \omega$ such that $B' := \{\beta \in B \mid e_\beta(n) = \tau\}$ is uncountable.

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find $n < \omega$ such that $B' := \{\beta \in B \mid e_\beta(n) = \tau\}$ is uncountable.
Find $(\alpha, \beta) \in [B']^2$ such that $c(\alpha, \beta) = n$. Then $c^+(\alpha, \beta) = \tau$.

Negative partition relations on ω_1, cont.

There are additional ways to get the consistency of strong colorings at ω_1, but the holy grail is getting them in ZFC. The primary tool for this purpose is Todorcevic's method of walks on ordinals.

Theorem (Todorcevic, 1987)

$\omega_1 \not\to [\omega_1]^2_{\omega_1}$.

Theorem (Moore, 2006)

$\omega_1 \not\to [\omega_1 \circledast \omega_1]^2_{\omega_1}$.

Theorem (Peng-Wu, 2018)

$\mathrm{Pr}_1(\omega_1, \omega_1, \omega_1, n)$ *for every positive integer* n.

The principle $\mathrm{Pr}_1(\ldots)$ will be defined momentarily.

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So $\{(\{\alpha\}, \{\alpha\}) \mid \alpha < \omega_1\}$ is an uncountable antichain in $\mathbb{P}_0 \times \mathbb{P}_1$.

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So $\{(\{\alpha\}, \{\alpha\}) \mid \alpha < \omega_1\}$ is an uncountable antichain in $\mathbb{P}_0 \times \mathbb{P}_1$.

What about the chain condition of each of the factors?

Negative partition relations and chain conditions, cont.

Lemma

For $i < 2$, $\mathbb{P}_i = \{x \in [\omega_1]^{<\omega} \mid c\text{“}[x]^2 \subseteq \{i\}\}$ has the ccc provided that for all $n < \omega$ and uncountable pairwise disjoint family $\mathcal{A} \subseteq [\omega_1]^n$, there is a pair $a < b$ in $\mathcal{A}$ such that $c[a \times b] = \{i\}$.

Negative partition relations and chain conditions, cont.

Definition (Shelah, 1988)

$\mathrm{Pr}_1(\kappa, \kappa, \theta, \chi)$ asserts the existence of a coloring $c : [\kappa]^2 \to \theta$ such that for every $\sigma < \chi$, every pairwise disjoint family $\mathcal{A} \subseteq [\kappa]^\sigma$ with $|\mathcal{A}| = \kappa$, and every $\tau < \theta$, there is a pair $a < b$ in $\mathcal{A}$ such that $c[a \times b] = \{\tau\}$.

Galvin's theorem may be broken into two parts:

- CH $\implies$ $\mathrm{Pr}_1(\omega_1, \omega_1, 2, \omega)$;
- $\mathrm{Pr}_1(\omega_1, \omega_1, 2, \omega)$ $\implies$ there are ccc posets $\mathbb{P}_0, \mathbb{P}_1$ for which $\mathbb{P}_0 \times \mathbb{P}_1$ is not ccc.

Illustration of $\mathrm{Pr}_1(\kappa, \kappa, \theta, \chi)$

The key here is the shift in focus: from increasing the number of colors (3^{rd} parameter) to increasing the dimension (4^{th} parameter). For instance, $\chi = 3$ already takes care of rectangles.

Replacing non-homogeneous by unbounded, again

Definition (Shelah, 1988)

$\mathsf{Pr}_1(\kappa,\kappa,\theta,\chi)$: $\exists$coloring $c:[\kappa]^2\to\theta$ s.t. for every $\sigma<\chi$, every pairwise disjoint family $\mathcal{A}\subseteq[\kappa]^\sigma$ with $|\mathcal{A}|=\kappa$, and every $\tau<\theta$, there is a pair $a<b$ in $\mathcal{A}$ such that $c[a\times b]=\{\tau\}$.

Definition (with Lambie-Hanson [34])

$\mathsf{U}(\kappa,\kappa,\theta,\chi)$: $\exists$coloring $c:[\kappa]^2\to\theta$ s.t. for every $\sigma<\chi$, every pairwise disjoint family $\mathcal{A}\subseteq[\kappa]^\sigma$ with $|\mathcal{A}|=\kappa$, and every $\tau<\theta$, there is $\mathcal{B}\in[\mathcal{A}]^\kappa$ such that for every pair $a<b$ in $\mathcal{B}$

$$\min(c[a\times b])>\tau.$$

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Note that unlike $\mathrm{Pr}_1(\ldots)$, there's no reason for $\mathrm{U}(\ldots)$ to be monotone in its 3^{rd} parameter (number of colors),

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Note that unlike $\mathrm{Pr}_1(\ldots)$, there's no reason for $\mathrm{U}(\ldots)$ to be monotone in its 3^{rd} parameter (number of colors), yet some monotonicity results may be found in [35].
Note also that unlike $\mathrm{Pr}_1(\kappa,\kappa,\ldots)$ that asks for a pair in $\mathcal{A}$, $\mathrm{U}(\kappa,\kappa,\ldots)$ asks for $\mathcal{B}\in[\mathcal{A}]^\kappa$ such that all pairs from $\mathcal{B}$ will do.

Negative partition relations and chain conditions, cont.

For infinite regular cardinals $\theta \leq \chi < \kappa$ such that $\forall \lambda < \kappa(\lambda^{<\chi})$:

(Galvin, 1980) $\mathrm{Pr}_1(\kappa, \kappa, 2, \chi)$ yields two χ-closed posets $\mathbb{P}_0, \mathbb{P}_1$ such that:

- $\mathbb{P}_0$ and $\mathbb{P}_1$ have the κ-cc;
- $\mathbb{P}_0 \times \mathbb{P}_1$ doesn't have the κ-cc.

(with Lambie-Hanson [34]) $\mathrm{U}(\kappa, \kappa, \theta, \chi)$ yields a sequence of χ-closed posets $\langle \mathbb{P}_i \mid i < \theta \rangle$ such that:

- $\prod_{i<\tau} \mathbb{P}_i$ is κ-Knaster for all $\tau < \theta$;
- $\prod_{i<\theta} \mathbb{P}_i$ does not have the κ-cc.

(with Lambie-Hanson [36]) A closed subadditive witness to $\mathrm{U}(\kappa, \kappa, \theta, \chi)$ yields χ-closed posets $\langle \mathbb{P}_i \mid i < \theta \rangle$ s.t.:

- $\prod_{i<\tau} \mathbb{P}_i$ is κ-stationarily layered for all $\tau < \theta$;
- $\prod_{i<\theta} \mathbb{P}_i$ does not have the κ-cc.

Galvin's theorem, generalizations and restrictions

(Galvin, 1980. Todorcevic 1988) If $\mathfrak{c} = \aleph_1$ (resp. $\mathfrak{b} = \aleph_1$), then $\mathsf{Pr}_1(\omega_1, \omega_1, \omega_1, \omega)$ holds (diagonalization. oscillation).

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(with Kojman and Steprāns [50]) $\mathrm{non}(\mathcal{M}) = \aleph_1$ is equivalent to $\mathsf{Pr}_1(\omega_1, {}^{\omega}\circledast\omega_1\!/\!1\circledast\omega_1, \omega_1, \omega)$. This improves results of Sierpiński (1934), Erdős-Hajnal-Milner (1966), Todorcevic (1987), Miller (2014) and Guzmán (2017)

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(Shelah, 1990–1997) $\mathsf{Pr}_1(\lambda^+, \lambda^+, \lambda^+, \omega)$ holds for every $\lambda \geq \aleph_1$ (walks for λ regular, oscillation for λ singular).

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(with Kojman and Steprāns [50]) $\mathrm{non}(\mathcal{M}) = \aleph_1$ is equivalent to $\mathsf{Pr}_1(\omega_1, {}^{\omega}\circledast\omega_1\!/\!1_{\circledast\omega_1}, \omega_1, \omega)$. This improves results of Sierpiński (1934), Erdős-Hajnal-Milner (1966), Todorcevic (1987), Miller (2014) and Guzmán (2017)

(Shelah, 1990–1997) $\mathsf{Pr}_1(\lambda^+, \lambda^+, \lambda^+, \omega)$ holds for every $\lambda \geq \aleph_1$ (walks for λ regular, oscillation for λ singular).

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(with Lambie-Hanson [34]) For any λ, $\mathrm{U}(\lambda^+, \lambda^+, \mathrm{cf}(\lambda), \lambda)$ holds.

(with Lambie-Hanson [34]) If λ is regular, then $\mathrm{U}(\lambda^+, \lambda^+, \theta, \lambda)$ holds for any infinite cardinal $\theta \leq \lambda$ (walks).
If λ is singular and $2^\lambda = \lambda^+$, then $\mathrm{U}(\lambda^+, \lambda^+, \theta, \mathrm{cf}(\lambda))$ holds for any inf. cardinal $\theta \leq \lambda$ (diamond+walks).

More: If λ is a singular limit of strongly compacts, then various instances of $\mathrm{U}(\lambda^+, \ldots, \mathrm{cf}(\lambda)^+)$ and $\mathrm{Pr}_1(\lambda^+, \ldots, \mathrm{cf}(\lambda)^+)$ must fail.

At the level of inaccessibles

Let κ denote a weakly inaccessible, and $\chi < \kappa$ infinite & regular.

([15] and [18]) If $E^{\kappa}_{\geq\chi}$ admits a nonreflecting stationary set or if $\square(\kappa)$ holds, then $\mathrm{Pr}_1(\kappa,\kappa,\kappa,\chi)$ (walks + $\mathrm{P}\ell_6$).

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(with Zhang [45]) If $\square(\kappa)+\diamondsuit^*(\kappa)$ holds, then $\mathrm{Pr}_1(\kappa,\kappa,\kappa,\kappa)$ (diagonalization+walks+diamond result from [47]).

If $\diamondsuit(S)$ holds for $S\subseteq\mathrm{Reg}(\kappa)$ stat. nonreflecting, then $\mathrm{Pr}_1(\kappa,\kappa,\kappa,\kappa)$ (walks along a proxy from [23]).

(with Lambie-Hanson [34]) If $\square(\kappa)$ holds, then $\mathrm{U}(\kappa,\kappa,\theta,\kappa)$ holds for any infinite cardinal $\theta\leq\kappa$ (walks).

(with Lambie-Hanson [34]) If $E^\kappa_{\geq\chi}$ admits a stationary set that does not reflect at regulars, then $\mathrm{U}(\kappa,\kappa,\theta,\chi)$ holds for any infinite cardinal $\theta\leq\kappa$ (walks).

(with Lambie-Hanson [35]) It is consistent that $\mathrm{Pr}_1(\kappa,\kappa,\kappa,\omega)$ holds but $\mathrm{U}(\kappa,\kappa,\theta,\omega)$ fails for any $\theta<\kappa$ (forcing over a model with a weakly compact).

Partition relations and chain conditions at ω_2

(Shelah, 1997) $\mathrm{Pr}_1(\omega_2, \omega_2, 2, \omega)$ holds, hence there is an ω_2-cc poset whose square is not ω_2-cc.

(Todorcevic, 2017) Assuming CH, a weak form of $\mathrm{Pr}_1(\omega_2, \omega_2, 2, \omega_1)$ holds, sufficient to get an ω_2-cc σ-closed poset whose square is not ω_2-cc.

(with Lambie-Hanson [34]) $\mathrm{U}(\omega_2, \omega_2, \omega, \omega_1)$ holds, so there is an ω_2-Knaster (assuming CH, also σ-closed) poset whose ω^{th} power is not ω_2-cc.

Question

Is there any positive partition relation consistent at the level of ω_2? (without lifting the axiom of choice, that is)

Strong colorings over partitions

Hereafter, κ denotes a regular uncountable cardinal, and $\nu, \mu, \theta, \lambda$ are (possibly finite) cardinals $\leq \kappa$.

Recall

For a coloring $c : [\kappa]^2 \to \theta$, a subset $A \subseteq \kappa$ is c-homogeneous iff there is $\tau < \theta$ such that $c(\alpha, \beta) = \tau$ for all $(\alpha, \beta) \in [A]^2$.

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For a "partition" $p : [\kappa]^2 \to \mu$ and a "coloring" $c : [\kappa]^2 \to \theta$,
a subset $A \subseteq \kappa$ is (p, c)-homogeneous iff there is a map $\tau : \mu \to \theta$ such that $c(\alpha, \beta) = \tau(p(\alpha, \beta))$ for all $(\alpha, \beta) \in [A]^2$.

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p-weak compactness

Given a partition $p : [\kappa]^2 \to \mu$, $\kappa \to_p (\kappa)^2_2$ asserts that every coloring $c : [\kappa]^2 \to 2$ admits a (p, c)-homogeneous set of size κ.

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Note that every set is (c, c)-homogeneous and that
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More generally

Given a partition $p : [\kappa]^2 \to \mu$, $\kappa \to_p (\kappa)^2_\theta$ asserts that every coloring $c : [\kappa]^2 \to \theta$ admits a (p, c)-homogeneous set of size κ.

ω_2 may consistently be p-weakly compact

Theorem (with Kojman and Steprāns [49])

Consistently, there exists $p : [\omega_2]^2 \to \omega_1$ *for which* $\omega_2 \to_p (\omega_2)^2_{\omega_1}$.

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Theorem (with Kojman and Steprāns [49])

Consistently, there exists $p : [\omega_2]^2 \to \omega_1$ *for which* $\omega_2 \to_p (\omega_2)^2_{\omega_1}$. *In fact, assuming* $\lambda^{<\lambda} = \lambda$ *and Generalized Martin's Axiom for* λ^+, *there exists a partition* $p : [\lambda^+]^2 \to \lambda$ *such that every coloring* $c : [\lambda^+]^2 \to \lambda$ *may be covered by* λ *many* (p, c)*-homogeneous sets.*

Recall: GMA_{λ^+} asserts that for any poset $\mathbb{Q}$ of size $< 2^\lambda$, if

(a) $\mathbb{Q}$ is well-met;

(b) $\mathbb{Q}$ is $<\lambda$-closed with greatest lower bounds;

(c) $\mathbb{Q}$ satisfy the λ^+-stationary-cc,

then for every sequence of λ^+ many dense sets in $\mathbb{Q}$ there is a pseudo-generic filter over $\mathbb{Q}$ that meets them all.

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In fact, assuming $\lambda^{<\lambda} = \lambda$ *and Generalized Martin's Axiom for* λ^+,
there exists a partition $p : [\lambda^+]^2 \to \lambda$ *such that every coloring*
$c : [\lambda^+]^2 \to \lambda$ *may be covered by* λ *many* (p, c)*-homogeneous sets*
$\langle A_i \mid i < \lambda \rangle$ *such that each* A_i *is large:* $|A_i| = \lambda^+$ *and* $p``[A_i]^2 = \lambda$.

ω_2 may consistently be p-weakly compact, cont.

Theorem (with Kojman and Steprāns [49])

Assuming $\lambda^{<\lambda} = \lambda$ and Generalized Martin's Axiom for λ^+, there is a partition $p : [\lambda^+]^2 \to \lambda$ such that every coloring $c : [\lambda^+]^2 \to \lambda$ may be covered by λ many "large" (p, c)-homogeneous sets.

About the proof

Given a partition $p : [\lambda^+]^2 \to \lambda$ and a coloring $c : [\lambda^+]^2 \to \lambda$, consider the poset $\mathbb{Q}(p, c)$ consisting of functions $f : a \to \lambda$ with $a \in [\lambda^+]^{<\lambda}$ such that $f^{-1}\{i\}$ is (p, c)-homogeneous for all $i < \lambda$.

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The only tricky part is finding a suitable p for which $\mathbb{Q}(p, c)$ has a very strong chain condition (λ^+-stationary-cc) for any coloring c.

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This prompts the study of partitions...

A study of partitions

For a partition $p : [\kappa]^2 \to \mu$ and a cardinal λ:

(1) p has injective fibers iff for $\alpha < \alpha' < \beta$, $p(\alpha, \beta) \neq p(\alpha', \beta)$.

(2) p has λ-almost-disjoint fibers iff for all $\beta < \beta' < \kappa$:

$$|\{p(\alpha, \beta) \mid \alpha < \beta\} \cap \{p(\alpha, \beta') \mid \alpha < \beta\}| < \lambda.$$

(3) p has λ-coherent fibers iff for all $\beta < \beta' < \kappa$:

$$|\{\alpha < \beta \mid p(\alpha, \beta) \neq p(\alpha, \beta')\}| < \lambda.$$

(4) p has λ-Cohen fibers iff for every injection $g : a \to \mu$ with $a \in [\kappa]^{<\lambda}$, there are cofinally many $\beta < \kappa$ such that $g(\alpha) = p(\alpha, \beta)$ for all $\alpha \in a$.

Clauses (1) and (2) will secure the strong condition. Clause (4) will secure that each of the (p, c)-homogeneous sets be "large".

Partitions makes a difference

We mentioned that Luzin sets and Souslin trees give rise to very strong colorings. It turns out they cannot overcome partitions!

Theorem (with Kojman and Steprāns [49])

It is consistent that the following hold simultaneously:

- *There exist a Luzin set and a Souslin tree;*
- *There exists a partition* $p : [\omega_1]^2 \to \omega$ *such that all colorings* $c : [\omega_1]^2 \to \omega$ *are "p-special" (i.e., for any* c, ω_1 *may be covered by countably many* (p, c)*-homogeneous sets).*

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The proof is by a finite support iteration of the explicit (non well-met) version of $\mathbb{Q}(p, c)$ for a well-chosen ground model p, using bookkeeping to take care of all possible c's.

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The proof is by a finite support iteration of the explicit (non well-met) version of $\mathbb{Q}(p, c)$ for a well-chosen ground model p, using bookkeeping to take care of all possible c's.
Each $\mathbb{Q}(p, c)$ has the Knaster property, so any ground model Souslin tree will survive. The heart of the matter is to preserve Luzin sets.

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Lemma (the choice of p)

$\mathfrak{d} = \aleph_1$ iff there exists a partition $p : [\omega_1]^2 \to \omega$ with injective, ω-almost-disjoint (and ω-Cohen) fibers satisfying the following: For every function $h : \epsilon \to \omega$ with $\epsilon < \omega_1$ there exists $\gamma < \omega_1$ such that for every $b \in [\omega_1 \setminus \gamma]^{<\aleph_0}$ there exists $\Delta \in [\epsilon]^{<\omega}$ such that:

- $p \restriction ((\epsilon \setminus \Delta) \times b)$ is injective, and
- for all $\alpha \in \epsilon \setminus \Delta$ and $\beta \in b$, $h(\alpha) < p(\alpha, \beta)$.

Positive or negative? A duality

Theorem (Chen-Kojman-Steprāns, 2020)

There consistently exists a partition $p : [\omega_1]^2 \to \omega$ *for which* $\omega_1 \to_p [\omega_1]^2_{\omega,\mathit{finite}}$ *holds. That is, for every coloring* $c : [\omega_1]^2 \to \omega$ *there is* $A \in [\omega_1]^{\omega_1}$ *such that for every* $j < \omega$*:*

$$\{c(\alpha,\beta) \mid (\alpha,\beta) \in [A]^2 \ \&\ p(\alpha,\beta) = j\} \text{ is finite.}$$

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Theorem (with Kojman and Steprāns [49])

Assuming Martin's Axiom, for every partition $p : [\omega_1]^2 \to \omega$ *TFAE:*

1. $\omega_1 \to_p [\omega_1]^2_{\omega,finite}$;
2. *There is* $A \in [\omega_1]^{\omega_1}$ *s.t.* $p \restriction [A]^2$ *witnesses* $\mathsf{U}(\omega_1,\omega_1,\omega,\omega)$.

Pump-up theorems

Recall

$\mathrm{Pr}_1(\kappa,\kappa,\theta,\chi)$: $\exists$coloring $c:[\kappa]^2\to\theta$ s.t. for every $\sigma<\chi$, every pairwise disjoint family $\mathcal{A}\subseteq[\kappa]^\sigma$ with $|\mathcal{A}|=\kappa$, and every $\tau<\theta$, there is a pair $a<b$ in $\mathcal{A}$ such that

$$c(\alpha,\beta)=\tau \text{ for all } (\alpha,\beta)\in a\times b.$$

Definition (Chen-Kojman-Steprāns, 2020)

Let $p:[\kappa]^2\to\mu$ denote an arbitrary partition.

$\mathrm{Pr}_1(\kappa,\kappa,\theta,\chi)_p$: $\exists$coloring $c:[\kappa]^2\to\theta$ s.t. for every $\sigma<\chi$, every pairwise disjoint family $\mathcal{A}\subseteq[\kappa]^\sigma$ with $|\mathcal{A}|=\kappa$, and $\tau:\mu\to\theta$, there is a pair $a<b$ in $\mathcal{A}$ such that

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Pump-up theorems

Fact (Chen-Kojman-Steprāns, 2020)

For any partition $p : [\kappa]^2 \to \mu$*:*

- $\mathrm{Pr}_1(\kappa, \kappa, \theta^\mu, \chi)$ *entails* $\mathrm{Pr}_1(\kappa, \kappa, \theta, \chi)_p$.
- $\mathrm{Pr}_1(\kappa, \kappa, \theta, 2)_p$ *iff there exists a coloring* $c : [\kappa]^2 \to \theta$ *such that for every* $A \in [\kappa]^\kappa$*, there is a cell* $j < \mu$ *such that* $\{c(\alpha, \beta) \mid (\alpha, \beta) \in [A]^2 \ \&\ p(\alpha, \beta) = j\} = \theta$.

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$$c(\alpha, \beta) = \tau(p(\alpha, \beta)) \text{ for all } (\alpha, \beta) \in a \times b.$$

Pump-up theorems, cont.

We have seen how to "stretch" a witness to $\omega_1 \not\to [\omega_1]^2_\omega$ into a witness to $\omega_1 \not\to [\omega_1]^2_{\omega_1}$. But such one-dimensional stretch doesn't respect a two-dimensional partition, and it was left open in [CKS20] whether $\omega_1 \not\to_p [\omega_1]^2_\omega$ implies $\omega_1 \not\to_p [\omega_1]^2_{\omega_1}$.

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And if it does, what about higher dimensions? For instance, even in the partition-free context, whether $\mathrm{Pr}_1(\lambda^+, \lambda^+, \lambda, \mathrm{cf}(\lambda))$ implies $\mathrm{Pr}_1(\lambda^+, \lambda^+, \lambda^+, \mathrm{cf}(\lambda))$ for λ singular was open for around 20 years until it was proved by Eisworth in a paper from 2013.

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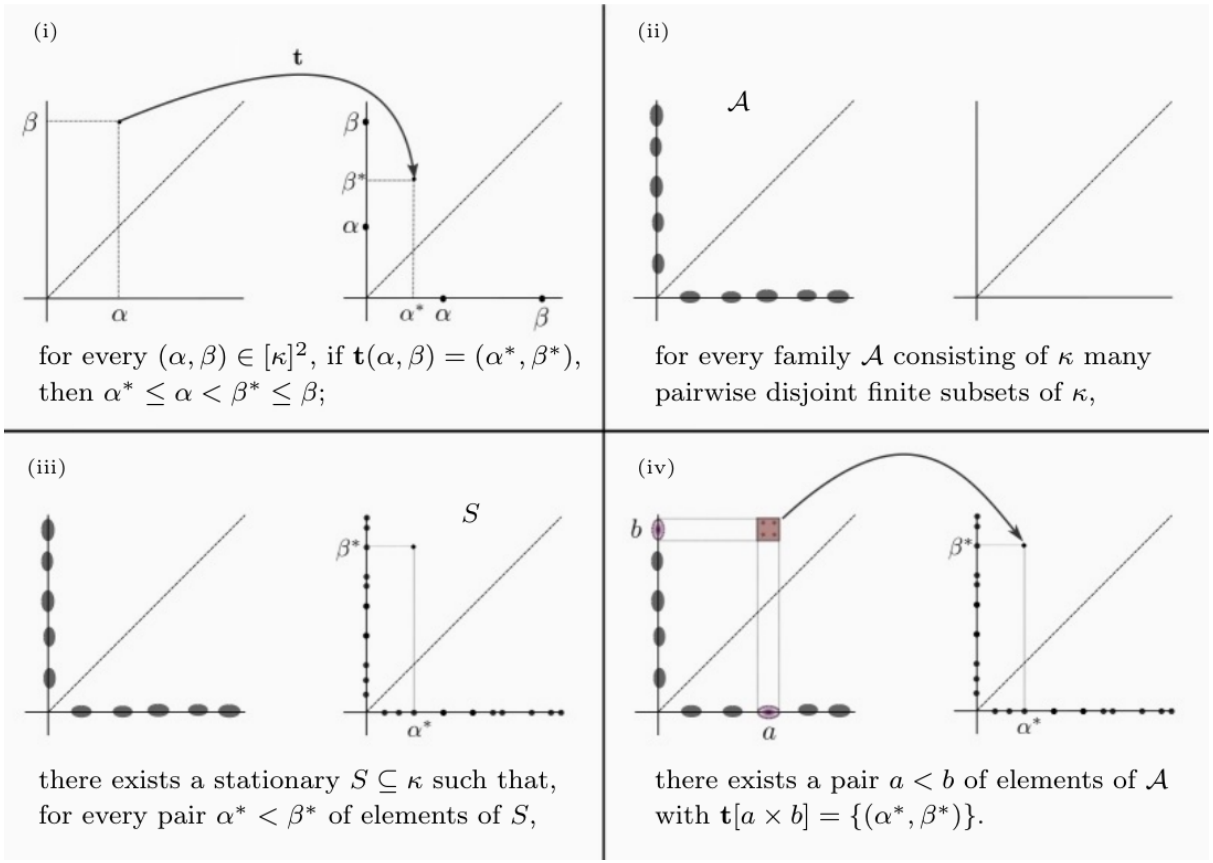
In [13], we slightly pushed Eisworth's work, getting that for any singular λ and any $\theta \leq \lambda^+$, $\mathrm{Pr}_1(\lambda^+, \lambda^+, \theta, \mathrm{cf}(\lambda))$ iff $\lambda^+ \nrightarrow [\lambda^+]^2_\theta$, hence we can just invoke the elementary pump-up fact.

Transformations of the transfinite plane (with Zhang [44])

$\mathsf{P}\ell_1(\kappa)$: $\exists$transformation $\mathbf{t} : [\kappa]^2 \to [\kappa]^2$ satisfying:

1. if $\mathbf{t}(\alpha, \beta) = (\alpha^*, \beta^*)$, then $\alpha^* \leq \alpha < \beta^* \leq \beta$;
2. for every pairwise disjoint family $\mathcal{A} \subseteq [\kappa]^{<\omega}$ with $|\mathcal{A}| = \kappa$, there is a stationary $S \subseteq \kappa$ such that, for every $\alpha^* < \beta^*$ from S, there are $a < b$ from $\mathcal{A}$ such that $\mathbf{t}[a \times b] = \{(\alpha^*, \beta^*)\}$.



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1. if $\mathbf{t}(\alpha, \beta) = (\alpha^*, \beta^*)$, then $\alpha^* \leq \alpha < \beta^* \leq \beta$;
2. for every $\sigma < \chi$, every pairwise disjoint family $\mathcal{A} \subseteq [\kappa]^\sigma$ with $|\mathcal{A}| = \kappa$, there is a stationary $S \subseteq \kappa$ such that for every $\alpha^* < \beta^*$ from S, there are $a < b$ from $\mathcal{A}$ such that $\mathbf{t}[a \times b] = \{(\alpha^*, \beta^*)\}$.

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Note

(i) If c witnesses $\kappa \nrightarrow [\kappa]^2_\theta$ and $\mathbf{t}$ witnesses $\mathrm{P}\ell_1(\kappa, \chi)$, then $c \circ \mathbf{t}$ witnesses $\mathrm{Pr}_1(\kappa, \kappa, \theta, \chi)$.

(ii) It is consistent that $\mathrm{Pr}_1(\kappa, \kappa, \kappa, \omega)$ holds but $\mathrm{P}\ell_1(\kappa, \omega)$ fails.

Pumping-up using transformations?

Theorem ([13])

$\mathsf{P}\ell_1(\lambda^+, \mathrm{cf}(\lambda))$ *holds for every singular cardinal* λ.

Theorem (With Zhang [44],[45])

1. $\mathsf{P}\ell_1(\omega_1, n)$ *holds for every positive* $n < \omega$;
2. $\mathsf{P}\ell_1(\kappa, \chi)$ *holds for* $\chi = \mathrm{cf}(\chi) < \chi^+ < \kappa$, *assuming* $\square(\kappa)$ *or the existence of a nonreflecting stationary subset of* $E^\kappa_{\geq\chi}$;
3. $\mathsf{P}\ell_1(\kappa, \chi)$ *holds for* $\chi < \kappa$ *assuming that* κ *is inaccessible and* $E^\kappa_{\geq\chi}$ *admits a stationary set that does not reflect at regulars;*
4. $\mathsf{P}\ell_1(\kappa, \kappa)$ *holds for* κ *inaccessible, assuming* $\square(\kappa)$ *and* $\diamondsuit(S)$ *for a stationary* $S \subseteq \kappa$ *that does not reflect at regulars.*

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The theory of transformations has been developed to a satisfactory extent, but it does not seem helpful when it comes to partitions.

Pumping-up, after all

Theorem (with Kojman and Steprāns [49])

For every infinite cardinal λ and every coloring $c : [\lambda^+]^2 \to \lambda$ there is a corresponding coloring $c^+ : [\lambda^+]^2 \to \lambda^+$ such that for every partition $p : [\lambda^+]^2 \to \lambda$ and every cardinal $\chi \leq \mathrm{cf}(\lambda)$, if c witnesses $\mathrm{Pr}_1(\lambda^+, \lambda^+, \lambda, \chi)_p$, then c^+ witnesses $\mathrm{Pr}_1(\lambda^+, \lambda^+, \lambda^+, \chi)_p$.

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$$c^+ \text{ witnesses } \mathrm{Pr}_1(\lambda^+, \lambda^+, \lambda^+, \chi)_p.$$

Compared to the classic 1-dimensional stretching formula $c^+(\alpha, \beta) := e_\beta(c(\alpha, \beta))$, here we let $c^+(\alpha, \beta) := e_\gamma(c(\alpha, \beta))$, where γ is computed from the triple $(\alpha, \beta, c(\alpha, \beta))$.

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The exact derivation of γ from $(\alpha, \beta, c(\alpha, \beta))$ depends on whether λ is a regular cardinal, a singular cardinal of countable cofinality, or a singular cardinal of uncountable cofinality (hardest case).

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The proofs demonstrate that the method of walks on ordinals is useful in the context of strong colorings over partitions, contrary to the original impression.