A tree-based perfectly normal space whose square is not countably metacompact

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This talk is based on the following paper

- Ari Meir Brodsky, A. R., Shira Yadai, **The power of trees**, *Annals of Pure and Applied Logic 177(9), 29pp, 2026*.

Convention

All topological spaces are Hausdorff.

Motivation

A standard question in topology asks what properties are preserved under taking products.

- The product of two regular topological spaces is regular. ✓
- The product of two compact topological spaces is compact. ✓
- The product of two separable topological spaces is separable. ✓
- The Sorgenfrey line $\mathbb{R}_l$ is a regular Lindelöf (hence normal) space whose square is not normal (hence, non-Lindelöf). ✗

Recall that $\mathbb{R}_l$ has underlying set $\mathbb{R}$, but its basic open sets are half-open intervals of the form $[a, b)$, where a and b are real numbers.

The product of normal spaces

Theorem (Dowker, 1951)

For a normal topological space X, the following are equivalent:

- $X \times [0, 1]$ *is normal;*
- X *is countably metacompact (CMC).*

Recall

A space X is:

- *(Countably) compact* iff every (countable) open cover has a finite subcover;
- *(Countably) metacompact* iff every (countable) open cover has a point-finite open refinement.

Normality of products (cont.)

A normal space whose product with $[0,1]$ is not normal is called a Dowker space.

Menachem will tell you all you need to know about Dowker spaces in his upcoming lecture. Let me just mention that Rudin's question asking whether there exists a Dowker space of size $\aleph_1$ is still standing. de Caux (1977) proved that $\clubsuit$ is a sufficient condition. In Rinot–Shelav–Todorčević (2024), it was reduced to $\stick$.

Normality of products (cont.)

A normal space whose product with $[0,1]$ is not normal is called a Dowker space.

A milestone result (with an intricate proof) in this line of study reads as follows:

Theorem (Beslagić, 1992)

Assuming $\diamondsuit$, there is a perfectly normal space X such that X^2 is Dowker.
In particular, X is a CMC space whose square is not CMC.

Recall

A space is *perfectly normal* iff it is:

- perfect (all closed sets are G_δ) — this is stronger than CMC — and
- normal.

An excerpt from Beslagić's construction

We require, at stage $\alpha \in \Lambda \cup \{0\}$, that:

(1) for $i < 3$, $\tau^i_{\alpha+\omega}$ is a Hausdorff topology on $\alpha + \omega$;

(2) for $i < 3$ and every $\beta < \alpha + \omega$, $\{U^i_k(\beta)\colon k < \omega\}$ is a (clopen) basis for the point β in $\langle \alpha + \omega, \tau^i_{\alpha+\omega}\rangle$ consisting of compact sets (in $\langle \alpha + \omega, \tau^i_{\alpha+\omega}\rangle$);

(3) for $i < 3$ and every $\beta < \alpha + \omega$, $\langle U^i_k(\beta)\colon k < \omega\rangle$ is decreasing;

(4) for every $\beta < \alpha + \omega$ and $k < \omega$, $U^0_k(\beta) \cup U^1_k(\beta) \subset U^2_k(\beta)$;

(5) for every $\beta < \alpha + \omega$ and $n < \omega$, if $\beta \in L_n$, then $U^0_0(\beta) \cap U^1_0(\beta) \subset L_n$;

(6) for $i < 3$ and every $\beta < \alpha + \omega$, $\tau^i_\beta = \tau^i_{\alpha+\omega} \cap \mathscr{P}(\beta)$; i.e., $\langle \beta, \tau^i_\beta\rangle$ is an open subspace of $\langle \alpha + \omega, \tau^i_{\alpha+\omega}\rangle$;

(7) for each $i < 3$, $[\alpha, \alpha + \omega) \subset \mathrm{Cl}_{\tau^i_{\alpha+\omega}} B_\alpha$;

(8) if $\alpha \in \Lambda'$ there is an $n < \omega$ and $\beta < \alpha + \omega$ such that
- $C^0_\alpha \cup C^1_\alpha \subset L_n$,
- $\beta \notin L_n$,
- for $i, j, l < 2$, $\langle \beta, \beta\rangle \in \mathrm{Cl}_{\tau^i_{\alpha+\omega} \times \tau^j_{\alpha+\omega}}[(C^l_\alpha \times C^l_\alpha) \cap \Delta]$;

(9) if $\alpha \in \Lambda'$ and $i, j < 2$, then $[\alpha, \alpha + \omega) \times [\alpha, \alpha + \omega) \subset \mathrm{Cl}_{\tau^i_{\alpha+\omega} \times \tau^j_{\alpha\times\omega}} D_\alpha$;

(10) for $i < 2$, $O^i_\alpha \subset \alpha$ is clopen in $\langle \alpha + \omega, \tau^i_{\alpha+\omega}\rangle$, and if $\mathrm{dom}(A_\alpha \cap \Delta)$ is closed in $\langle \alpha + \omega, \tau^i_{\alpha+\omega}\rangle$ then $\mathrm{dom}(A_\alpha \cap \Delta) \subset O^i_\alpha$;

(11) for $i, j < 2$, $O^{i\times j}_\alpha \subset \alpha \times \alpha$ is clopen in $\langle(\alpha + \omega) \times (\alpha + \omega), \tau^i_{\alpha+\omega} \times \tau^j_{\alpha+\omega}\rangle$, and if A_α is closed in $\langle(\alpha + \omega) \times (\alpha + \omega), \tau^i_{\alpha+\omega} \times \tau^i_{\alpha+\omega}\rangle$ then $A_\alpha \subset O^{i\times j}_\alpha$; and

(12) for $i, j < 2$ and each $\beta < \alpha$ with $\beta \in \Lambda \cup \{0\}$, O^i_β, $O^{i\times j}_\beta$ are clopen in $\langle \alpha + \omega, \tau^i_{\alpha+\omega}\rangle$ and $\langle(\alpha + \omega) \times (\alpha + \omega), \tau^i_{\alpha+\omega} \times \tau^j_{\alpha+\omega}\rangle$, respectively.

Amer Beslagić (1959–1997)
Mary E. Rudin (1924–2013)

The interval topology on trees

Definition

For a tree $\mathbf{T} = (T, \prec)$, we write $X_{\mathbf{T}}$ for the topological space with underlying set T equipped with the interval topology, i.e., the one whose base consists of the sets:

(1) $(x, z] := \{y \in T \mid x \prec y \preccurlyeq z\}$ for all pairs $x \prec z$ in T;

(2) $\{x\}$ for all minimal elements $x \in T$.

Note: $x \in T$ is an isolated point of $X_{\mathbf{T}}$ iff its height is either 0 or a successor ordinal.

Problem

Construct a Beslagić-type space, but of the form $X_{\mathbf{T}}$.

Why finding tree-based examples should be interesting

Theorem (Nyikos, 1980)

For every tree $\mathbf{T}$*, if* $X_{\mathbf{T}}$ *is normal, then it is CMC.*

That is, there are no *Dowker trees*!

Non-CMC trees are hard to find

- Any special $\aleph_1$-tree is CMC. Indeed, it is σ-closed-discrete.
- Any Souslin $\aleph_1$-tree is CMC.

The first example of an $\aleph_1$-tree $\mathbf{T}$ that is neither special nor Souslin was presented by Baumgartner (1970), who constructed an $\mathbb{R}$-embeddable nonspecial $\aleph_1$-tree from $\diamondsuit$.

Why finding tree-based examples should be interesting

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Non-CMC trees are hard to find

- Any special $\aleph_1$-tree is CMC. Indeed, it is σ-closed-discrete.
- Any almost-Souslin $\aleph_1$-tree is CMC.

An $\aleph_1$-tree is *almost-Souslin* if the set of levels of any of its antichains is nonstationary.

Why finding tree-based examples should be interesting

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Non-CMC trees are hard to find

- Any special $\aleph_1$-tree is CMC. Indeed, it is σ-closed-discrete.
- Any almost-Souslin $\aleph_1$-tree is CMC.

CMC of $X_{\mathbf{T}}$ is equivalent to a weak uniformization property of $\mathbf{T}$

For a tree $\mathbf{T}$, $X_{\mathbf{T}}$ *is CMC iff for every closed discrete* $A \subseteq X_{\mathbf{T}}$, *for every function* $g : A \to \omega$, *there is a function* $f : T \to \omega$ *such that for every* $y \in A$, $\sup\{x \prec y \mid f(x) \leq g(y)\} \prec y$.

Why finding tree-based examples should be interesting

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Non-CMC trees are hard to find

- Any special $\aleph_1$-tree is CMC. Indeed, it is σ-closed-discrete.
- Any almost-Souslin $\aleph_1$-tree is CMC.

The first example of an $\aleph_1$-tree $\mathbf{T}$ for which $X_{\mathbf{T}}$ is not CMC was constructed by Fleissner (1980) assuming $\diamondsuit^+$, and then by Hanazawa (1983) assuming $\diamondsuit$.

Between normal and perfectly normal

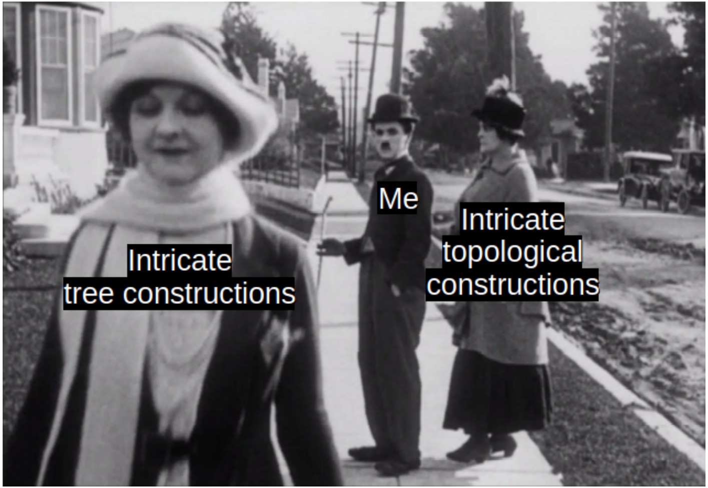
The question of whether $X_{\mathbf{T}}$ for a given $\aleph_1$-tree $\mathbf{T}$ is normal goes back to the work of Jones (1966) on the normal Moore space problem.

$$
\begin{tikzcd}
\mathbf{T} \text{ has property } \gamma \arrow[rr, leftrightarrow, "\text{Hart (1983)}"] \arrow[d] & & \mathbf{X_T} \text{ is hereditarily cwN} \arrow[d] \\
\mathbf{T} \text{ is almost-Souslin} \arrow[rr, leftrightarrow, "\text{Devlin–Shelah (1979)}"] \arrow[dr, "+\mathbb{R}\text{-embeddable}", "\text{Hanazawa (1983)}"'] & & \mathbf{X_T} \text{ is cwH} \arrow[d, "\text{Nyikos (1980)}"] \\
\mathbf{T} \text{ is } \mathbb{R}\text{-embeddable} & \mathbf{X_T} \text{ is perfect} \arrow[l] \arrow[r] & \mathbf{X_T} \text{ is CMC}
\end{tikzcd}
$$

Figure 1: All relevant findings on $\aleph_1$-trees and their corresponding spaces.

Remark: We are unaware of any construction to control the features of $(X_{\mathbf{T}})^2$.

Intermission



Intermission

Together with Ari Brodsky, we devised the *proxy principles* as useful foundations to construct κ-Aronszajn trees for $\kappa > \aleph_1$. Some such foundations are necessary, e.g., by Mitchell (1972), it is consistent that there no $\aleph_2$-Aronszajn trees.
In a series of papers, we developed techniques to construct nontrivial higher trees.

Example 1 (Brodsky–Rinot, 2017)

There consistently exists an $\aleph_6$*-Souslin tree* $\mathbf{T}$ *and a sequence of uniform ultrafilters* $\langle \mathcal{U}_n \mid n < 6\rangle$ *such that* $\mathbf{T}^{\aleph_n}/\mathcal{U}_n$ *is* $\aleph_6$*-Aronszajn iff* $n < 6$ *is not a prime number.*

Example 2 (Rinot–Yadai–You, 2024)

There consistently exists an $\aleph_2$*-Souslin tree such that each of its limit levels omits no more than one potential branch.*

The present work can be seen as a realization at the level of $\aleph_1$ of the experience we gained higher up. The advantage here is that the reader can follow this construction without knowing anything about the proxy principles. This is your opportunity to join!

A tree in need

We are looking for an $\aleph_1$-tree $\mathbf{T} = (T, \prec)$ such that:

(1) $X_{\mathbf{T}}$ is perfectly normal;

(2) $(X_{\mathbf{T}})^2$ is not CMC.

So, we opt to ensure that $\mathbf{T}$ would possess the following features:

(1) $\mathbf{T}$ has property γ (in particular, it is almost-Souslin) and $\mathbb{R}$-embeddable;

(2) $\mathbf{T}^2$ has an anti-uniformization property.

The first example of an almost-Souslin $\mathbb{R}$-embeddable $\aleph_1$-tree $\mathbf{T}$ was constructed by Devlin–Shelah (1979) from $\diamondsuit^*$. The corresponding space $X_{\mathbf{T}}$ is not normal.
Hanazawa (1980) constructed a variation in which $\mathbf{T}$ has property γ.

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The form of anti-uniformization for $\mathbf{T}^2$ to possess

There are an antichain $A \subseteq T^2$, and a function $g : A \to \omega$ such that for every function $f : T^2 \to \omega$, there is a $y \in A$ such that

$$\sup\{x \prec^2 y \mid f(x) = g(y)\} = y.$$

Main result

Theorem (Brodsky–Rinot–Yadai, 2026)

Assuming $\diamondsuit^*$, there exists an $\aleph_1$-tree $\mathbf{T}$ such that:

$X_{\mathbf{T}}$	$(X_{\mathbf{T}})^2$
hereditarily collectionwise normal	*not normal*
perfect, in particular CMC	*not CMC*

Remarks

(1) $\diamondsuit^*$ is not necessary. Forcing over a model of $\diamondsuit$ with any ccc notion of forcing of size $\aleph_1$ that is not ${}^{\omega}\omega$-bounding, we obtain such a tree in the extension.

(2) $\mathfrak{p} > \aleph_1$ implies that $(X_{\mathbf{T}})^2$ is CMC for every $\aleph_1$-tree $\mathbf{T}$ such that $X_{\mathbf{T}}$ is perfect.

Normality of the tree

(From an order-theoretic point of view,) a tree $\mathbf{T}$ is said to be *normal* iff
for every pair $\alpha < \beta$ of ordinals such that both levels T_α and T_β are nonempty,
every node x from T_α admits an extension from T_β.
In our work, we construct elevators as uniform witnesses to normality.
For starters, an elevator is a map $e : T_\alpha \to T_\beta$ such that $e(x)$ is an extension of x.
Our tree $\mathbf{T}$ consists of strictly increasing maps from countable ordinals to $\mathbb{Q}$, so it makes sense to define a *q-elevator* as one satisfying $\sup(\mathrm{Im}(e(x))) < \sup(\mathrm{Im}(x)) + q$, succinctly written as '$e(x) \in U(x, q)$'.

The real power of elevators is in their ability to control the powers of the tree. 😄
In the current construction, we determine a coarsening $\lhd$ of the usual ordering of $\mathbf{T}^2$, and the 1D elevators must satisfy the 2D requirement: $(x_0, x_1) \lhd (e(x_0), e(x_1))$.

~~Normality~~ Coordination of the tree

We impose that for all $\alpha < \beta < \omega_1$, the levels T_α and T_β be coordinated.
This means that for every positive rational number q, the following three hold:

(0) For every finite $W \subseteq T_\beta$, there exists a q-elevator $e : T_\alpha \to T_\beta$ such that $\mathrm{Im}(e) \cap W = \emptyset$.

(1) For every $x \in T_\alpha$, every $y \in U(x,q) \cap T_\beta$ and every finite set $W \subseteq T_\beta \setminus \{y\}$, there exists a q-elevator $e : T_\alpha \to T_\beta$ such that:

- $e(x) = y$, and
- $\mathrm{Im}(e) \cap W = \emptyset$.

(2) For every pair $(x_0, x_1) \in T_\alpha \times T_\alpha$ with $x_0 \neq x_1$, for every pair $(y_0, y_1) \in (U(x_0,q) \cap T_\beta) \times (U(x_1,q) \cap T_\beta)$ such that $(x_0, x_1) \lhd (y_0, y_1)$, and every finite set $W \subseteq T_\beta \setminus \{y_0, y_1\}$, there exists a q-elevator $e : T_\alpha \to T_\beta$ such that:

- $e(x_0) = y_0$,
- $e(x_1) = y_1$, and
- $\mathrm{Im}(e) \cap W = \emptyset$.

An open problem

Note

A space X is CMC iff for every decreasing sequence $\langle D_n \mid n < \omega \rangle$ of closed subsets of X with empty intersection, there is a decreasing sequence $\langle U_n \mid n < \omega \rangle$ of open subsets of X with empty intersection such that $D_n \subseteq U_n$ for all $n < \omega$.

Removing the restriction to closed sets one arrives at the definition of a Δ-space.

Theorem (Leiderman–Szeptycki, 2025)

If $\mathbf{T}$ *is an* $\aleph_1$*-Souslin tree, then* $X_{\mathbf{T}}$ *is not a* Δ*-space.*

There are consistent examples of trees $\mathbf{T}$ such that $X_{\mathbf{T}}$ is a Δ-space, but this is open:

Question (Leiderman)

Is there consistently an $\aleph_1$-tree $\mathbf{T}$ such that $X_{\mathbf{T}}$ is a Δ-space and $(X_{\mathbf{T}})^2$ is not?