Around Jensen's square principle

Young Researchers in Set Theory

Königswinter, Germany
22-March-2011

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Introduction

Ladder systems. A discussion

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Remark

The existence of ladder systems follows from the axiom of choice.

Ladder systems. Famous applications

Partitioning a stationary set

The standard proof of the fact that any stationary subset of ω_1 can be partitioned into uncountably many mutually disjoint stationary sets builds on an analysis of ladder systems over ω_1.

Strong colorings, $\omega_1 \not\to [\omega_1]^2_{\omega_1}$

Todorcevic established the existence of a function $f : [\omega_1]^2 \to \omega_1$ such that $f``[U]^2 = \omega_1$ for every uncountable $U \subseteq \omega_1$. This function f is determined by a ladder system over ω_1.

A particular ladder system

Definition (Jensen, 1960's)

$\square_\lambda$ asserts the existence of a ladder system over λ^+, $\langle C_\alpha \mid \alpha < \lambda^+ \rangle$, such that for all $\alpha < \lambda^+$:

- (Ladders are closed) C_α is a club in α;
- (Ladders are of bounded type) $\mathrm{otp}(C_\alpha) \leq \lambda$;
- (Coherence) if $\sup(C_\alpha \cap \beta) = \beta$, then $C_\alpha \cap \beta = C_\beta$.

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Famous applications

The existence of various sorts of λ^+-trees; The existence of non-reflecting stationary subsets of λ^+; The existence of other incompact objects.

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Today's talk would be centered around the above principle, but let us dedicate some time to discuss abstract ladder systems.

Triviality of ladder systems

Means of triviality

A ladder system $\langle A_\alpha \mid \alpha < \kappa \rangle$ is considered to be trivial, if, in some sense, it is determined by a single κ-sized object.

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If κ is a large cardinal, then we may necessarily face means of triviality.

Fact (Rowbottom, 1970's)

If κ is measurable, then every ladder system $\langle A_\alpha \mid \alpha < \kappa\rangle$, admits a set $A \subseteq \kappa$ such that $A_\alpha = A \cap \alpha$ for stationary many $\alpha < \kappa$.

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"There exists $A \subseteq \kappa$ such that $A_\alpha = A \cap \alpha$ for club many $\alpha < \kappa$."

On the other hand, if κ is non-Mahlo, then for every cofinal $A \subseteq \kappa$, the following set contains a club:

$$\{\alpha < \kappa \mid \mathrm{cf}(\alpha) < \mathrm{otp}(A \cap \alpha)\}.$$

This suggests that non-triviality may be insured here, by setting a global bound on $\mathrm{otp}(A_\alpha)$, e.g., letting $\mathrm{otp}(A_\alpha) = \mathrm{cf}(\alpha)$ for all α.

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It turns out that requiring that $\mathrm{otp}(A_\alpha) = \mathrm{cf}(\alpha)$ for all α does not eliminate all means of triviality. For instance, it may be the case that any sequence of functions defined on the ladders is necessarily induced from a single κ-sized object.

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Fact (Devlin-Shelah, 1978)

MA_{ω_1} implies that any ladder system $\langle A_\alpha \mid \alpha < \omega_1 \rangle$ satisfying $\mathrm{otp}(A_\alpha) = \mathrm{cf}(\alpha)$ for every α, is trivial in the following sense. For every sequence of local functions $\langle f_\alpha : A_\alpha \to 2 \mid \alpha < \omega_1 \rangle$ there exists a global function $f : \omega_1 \to 2$ such that for each α:

$$f_\alpha = f \restriction A_\alpha \text{ (mod finite)}.$$

Nontrivial ladder systems over ω_1

In contrast, the following concept yields a ladder system which is resistant to Devlin and Shelah's notion of triviality.

Definition (Ostaszweski's $\clubsuit$)

$\clubsuit$ asserts the existence of a ladder system $\langle A_\alpha \mid \alpha < \omega_1 \rangle$ such that for every cofinal $A \subseteq \omega_1$, there exists a limit $\alpha < \omega_1$ with $A_\alpha \subseteq A$.

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Indeed, if $\langle A_\alpha \mid \alpha < \omega_1 \rangle$ is a $\clubsuit$-sequence, then for every global $f : \omega_1 \to 2$, there exists a limit $\alpha < \omega_1$ for which $f \restriction A_\alpha$ is constant.
Thus, if $f_\alpha : A_\alpha \to 2$ partitions A_α into two cofinal subsets for all limit α, then no global f trivializes the sequence $\langle f_\alpha \mid \alpha < \omega_1 \rangle$.

Improve your square!

Suppose that $\kappa = \lambda^+$ is a successor cardinal. Thus, we are interested in a ladder system $\langle A_\alpha \mid \alpha < \kappa \rangle$ with ALL of the following features:

1. the set $\{\mathrm{otp}(A_\alpha) \mid \alpha < \kappa\}$ is bounded below κ;
2. the ladders are closed;
3. the ladders cohere;
4. yields a canonical partition of κ into mutually disjoint stationary sets;
5. induces strong colorings;
6. a non-triviality condition à la Devlin-Shelah.

The Ostaszewski square

λ-sequences

We propose a principle which combines $\square_\lambda$ together with $\clubsuit_{\lambda^+}$.

λ-sequences

We propose a principle which combines $\Box_\lambda$ together with $\clubsuit_{\lambda^+}$.
For clarity, let us adopt the next ad-hoc terminology:

Definition

A sequence $\langle A_i \mid i < \lambda \rangle$ is a <u>λ-sequence</u> if the following two holds:

1. each A_i is a cofinal subset of λ^+;
2. if $i < \lambda$ is a limit ordinal, then A_i is moreover closed.

Remark. Clause (2) may be viewed as a continuity condition.

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Definition

$\boxed{\clubsuit}_\lambda$ asserts the existence of a ladder system $\overrightarrow{C} = \langle C_\alpha \mid \alpha < \lambda^+ \rangle$ such that:

- $\mathrm{otp}(C_\alpha) \le \lambda$ for all $\alpha < \lambda^+$;
- C_α is a club in α for all limit $\alpha < \lambda^+$;
- if $\sup(C_\alpha \cap \beta) = \beta$, then $C_\alpha \cap \beta = C_\beta$;

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$\boxed{\clubsuit}_\lambda$ asserts the existence of a ladder system $\overrightarrow{C} = \langle C_\alpha \mid \alpha < \lambda^+ \rangle$ such that:

- $\overrightarrow{C}$ is a $\square_\lambda$-sequence. Let $C_\alpha(i)$ denote the i_{th} element of C_α.
- Suppose that $\langle A_i \mid i < \lambda \rangle$ is a λ-sequence. Then for every cofinal $B \subseteq \lambda^+$, and every limit $\theta < \lambda$, there exists some $\alpha < \lambda^+$ such that:
  1. $\mathrm{otp}(C_\alpha) = \theta$;
  2. for all $i < \theta$, $C_\alpha(i) \in A_i$;
  3. for all $i < \theta$, there exists $\beta_i \in B$ with $C_\alpha(i) < \beta_i < C_\alpha(i+1)$.

The Ostaszewski square (cont.)

$\boxed{\clubsuit}_\lambda$ asserts the existence of a $\square_\lambda$-sequence $\langle C_\alpha \mid \alpha < \lambda^+ \rangle$ such that for every λ-sequence $\langle A_i \mid i < \lambda \rangle$, every cofinal $B \subseteq \lambda^+$, and every limit $\theta < \lambda$, there exists some $\alpha < \lambda^+$ such that:

1. the inverse collapse of C_α is an element of $\prod_{i<\theta} A_i$;

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1. the inverse collapse of C_α is an element of $\prod_{i<\theta} A_i$;
2. if $\gamma < \delta$ belong to C_α, then $B \cap (\gamma, \delta) \neq \emptyset$.

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Feature 1. Club guessing

For every club $D \subseteq \lambda^+$, there exists $\alpha < \lambda^+$ such that $C_\alpha \subseteq D$.

The Ostaszewski square (cont.)

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Feature 2. $\clubsuit_{\lambda^+}$

For every cofinal $A \subseteq \lambda^+$, there exists $\alpha < \lambda^+$ s.t. $\mathrm{nacc}(C_\alpha) \subseteq A$.[a]

[a] $\mathrm{nacc}(C_\alpha) = C_\alpha \setminus \mathrm{acc}(C_\alpha)$, where $\mathrm{acc}(C_\alpha) := \{\beta \in C_\alpha \mid \sup(C_\alpha \cap \beta) = \beta\}$.

The Ostaszewski square (cont.)

$\boxed{\clubsuit}_\lambda$ asserts the existence of a $\square_\lambda$-sequence $\langle C_\alpha \mid \alpha < \lambda^+\rangle$ such that for every λ-sequence $\langle A_i \mid i < \lambda\rangle$, every cofinal $B \subseteq \lambda^+$, and every limit $\theta < \lambda$, there exists some $\alpha < \lambda^+$ such that:

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Feature 3. Canonical partition to stationary sets

Denote $S_\theta := \{\alpha < \lambda^+ \mid \mathrm{otp}(C_\alpha) = \theta\}$.
Then $\langle S_\theta \mid 0 \in \theta \in \mathrm{acc}(\lambda)\rangle$ is a canonical partition of the set of limit ordinals $< \lambda^+$ into λ many mutually disjoint stationary sets.

The Ostaszewski square (cont.)

$\boxed{\clubsuit}_\lambda$ asserts the existence of a $\square_\lambda$-sequence $\langle C_\alpha \mid \alpha < \lambda^+\rangle$ such that for every λ-sequence $\langle A_i \mid i < \lambda\rangle$, every cofinal $B \subseteq \lambda^+$, and every limit $\theta < \lambda$, there exists some $\alpha < \lambda^+$ such that:

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Feature 4. Simultaneous $\clubsuit_{\lambda^+}$ & Club guessing

For every cofinal $A \subseteq \lambda^+$, every club $D \subseteq \lambda^+$, and every $\theta < \lambda$, there exists $\alpha \in S_\theta$ such that $\mathrm{nacc}(C_\alpha) \subseteq A$, and $\mathrm{acc}(C_\alpha) \subseteq D$.

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Further features

We shall now turn to discuss further features.

Simple constructions of higher Souslin trees

λ^+-Souslin trees

Jensen proved that "GCH $+\square_\lambda + \diamondsuit_S$ for all stationary $S \subseteq \lambda^+$" yields the existence of a λ^+-Souslin tree, for every singular λ.
We now suggest a simple construction from a related hypothesis.

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Suppose that λ is an uncountable cardinal.
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Conventions

A κ-tree $\mathbf{T}$ is a tree of height κ, whose underlying set is κ, and levels are of size $< \kappa$.
The α_{th}-level is denoted T_α, and we write $\mathbf{T} \restriction \beta := \bigcup_{\alpha<\beta} T_\alpha$.
$\mathbf{T}$ is κ-Souslin if it is ever-branching and has no κ-sized antichains.

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If $\boxed{\clubsuit}_\lambda + \diamondsuit_{\lambda^+}$ holds, then there exists a λ^+-Souslin tree.

Proof.

Let $\langle C_\alpha \mid \alpha < \lambda^+\rangle$ witness $\boxed{\clubsuit}_\lambda$, and $\langle S_\gamma \mid \gamma < \lambda^+\rangle$ witness $\diamondsuit_{\lambda^+}$.
We build the λ^+-Souslin tree, $\mathbf{T}$, by recursion on the levels.

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Set $T_0 := \{0\}$.

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Set $T_0 := \{0\}$. If $\mathbf{T} \restriction \alpha + 1$ is defined, $T_{\alpha+1}$ is obtained by providing each element of T_α with two successors in $T_{\alpha+1}$.
Assume now that α is a limit ordinal; for every $x \in \mathbf{T} \restriction \alpha$, we attach a sequence x_α which is increasing and cofinal in $\mathbf{T} \restriction \alpha$, and then T_α is defined as the limit of all these sequences.
Consequently, the outcome T_α is of size $\leq |\mathbf{T} \restriction \alpha| \leq \lambda$.

λ^+-Souslin trees (cont.)

For every $x \in \mathbf{T} \restriction \alpha$, pick $x_\alpha = \langle x_\alpha(\gamma) \mid \gamma \in C_\alpha \setminus \mathrm{ht}(x) + 1 \rangle$ s.t.:

1. $\mathrm{ht}(x_\alpha(\gamma)) = \gamma$ for all $\gamma \in \mathrm{dom}(x_\alpha)$;
2. $x < x_\alpha(\gamma_1) < x_\alpha(\gamma_2)$ whenever $\gamma_1 < \gamma_2$;
3. If $\gamma \in \mathrm{nacc}(\mathrm{dom}(x_\alpha))$, and S_γ is a maximal antichain in $\mathbf{T} \restriction \gamma$, then $x_\alpha(\gamma)$ happens to be above some element from S_γ.

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If we make sure to choose $x_\alpha(\gamma)$ in a canonical way (e.g., using a well-ordering), then the coherence of the square sequence implies that the branches cohere: $\sup(C_\alpha \cap \delta) = \delta$ implies $x_\delta = x_\alpha \restriction \delta$.
In turn, we get that the whole construction may be carried, ending up with a λ^+-tree.

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Sousliness: towards a contradiction, suppose that $A \subseteq \lambda^+$ is an antichain in $\mathbf{T}$ of size λ^+. By $\diamondsuit_{\lambda^+}$, the following set is stationary

$$A' := \{\gamma < \lambda^+ \mid A \cap \gamma = S_\gamma \text{ is a maximal antichain in } \mathbf{T} \restriction \gamma\}.$$

λ^+-Souslin trees (cont.)

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Let $\langle A_i \mid i < \lambda \rangle$ be a λ-sequence with $A_{i+1} = A'$ for all $i < \lambda$. Pick $\alpha < \lambda^+$ such that $C_\alpha(i) \in A_i$ for all $i < \mathrm{otp}(C_\alpha)$.
Then $\mathrm{nacc}(C_\alpha) \subseteq A'$, and hence clause (3) above applies to the construction of x_α for each and every $x \in \mathbf{T} \restriction \alpha$.

λ^+-Souslin trees (cont.)

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Let $\langle A_i \mid i < \lambda \rangle$ be a λ-sequence with $A_{i+1} = A'$ for all $i < \lambda$. Pick $\alpha < \lambda^+$ such that $C_\alpha(i) \in A_i$ for all $i < \mathrm{otp}(C_\alpha)$.
Then $\mathrm{nacc}(C_\alpha) \subseteq A'$, and hence clause (3) above applies to all x_α.
As every element of T_α is the limit of some x_α, every element of T_α happens to be above some element from $A \cap \alpha$. So, $A \cap \alpha$ is a maximal antichain in $\mathbf{T}$. This is a contradiction. ■

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Suppose we wanted the resulted tree to be, in additional, rigid.
Then fix a $\diamondsuit_{\lambda^+}$ sequence that guesses functions $\langle f_\gamma \mid \gamma < \lambda^+\rangle$.
Given an hypothetical maximal antichain A, and a non-trivial automorphism f, the following sets would be cofinal (in fact, stat.):

$$A_0 := \{\gamma < \lambda^+ \mid A \cap \gamma = S_\gamma \text{ is a maximal antichain in } \mathbf{T} \restriction \gamma\};$$
$$A_1 := \{\gamma < \lambda^+ \mid f \restriction \gamma = f_\gamma \text{ is a n.t. automorphism of } \mathbf{T} \restriction \gamma\}.$$

So, we could find C_α whose odd nacc points are in A_0, and even nacc points are in A_1. Meaning that we could overcome A and f along the way.

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Given an hypothetical maximal antichain A, and a non-trivial automorphism f, the following sets would be cofinal (in fact, stat.):

$$A_0 := \{\gamma < \lambda^+ \mid A \cap \gamma = S_\gamma \text{ is a maximal antichain in } \mathbf{T} \restriction \gamma\};$$
$$A_1 := \{\gamma < \lambda^+ \mid f \restriction \gamma = f_\gamma \text{ is a n.t. automorphism of } \mathbf{T} \restriction \gamma\}.$$

So, we could find C_α whose odd nacc points are in A_0, and even nacc points are in A_1. Meaning that we could overcome A and f along the way. Similarly, we may overcome λ many obstructions in a very elegant way.

λ^+-Souslin trees. The aftermath

Question

What do we gain from the fact that we may guess a λ-sequence if we are only concerned with guessing a single cofinal set?

Answer

We can smoothly construct complicated objects, taking into account λ many independent considerations.

λ^+-Souslin trees. The aftermath

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Question

"smoothly"?

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Question

"smoothly"?

Answer

Jensen's original construction consists of two distinct components; one which is responsible for insuring that the construction may be carried up to height λ^+, and the other responsible for sealing potential large antichains.
This distinction affects the completeness degree of the tree.
In contrast, here, the potential antichains are sealed along the way.

λ^+-Souslin trees. The aftermath

We can smoothly construct complicated objects, having in mind λ many independent considerations.

A complaint

"smoothly"... okay! But Jensen's construction is from

$$\mathsf{GCH} + \square_\lambda + \diamondsuit_S \text{ for all stationary } S \subseteq \lambda^+,$$

while the other construction requires $\boxed{\clubsuit}_\lambda$!!

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Answer

If you are serious about purchasing my $\boxed{\clubsuit}_\lambda$, let me make a price quote.

Ostaszewski square - the price

It should be clear that the usual fine-structural-type of arguments yield that $\boxed{\clubsuit}_\lambda$ holds in L for all λ. But that's an high price to pay.

Ostaszewski square - the price

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Main Theorem

Suppose that $\square_\lambda$ holds for a given cardinal λ.

1. If λ is a limit cardinal, then $\lambda^\lambda = \lambda^+$ entails $\boxed{\clubsuit}_\lambda$.
2. If λ is a successor, then $\lambda^{<\lambda} < \lambda^\lambda = \lambda^+$ entails $\boxed{\clubsuit}_\lambda$.

Corollary

Assume GCH. Then for every uncountable cardinal λ, TFAE:

- $\square_\lambda$;
- $\boxed{\clubsuit}_\lambda$.

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So, for the Jensen setup, you pay no extra!

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Main Theorem

Suppose that $\Box_\lambda$ holds for a given cardinal λ.

1. If λ is a limit cardinal, then $\lambda^\lambda = \lambda^+$ entails $\boxed{\clubsuit}_\lambda$.
2. If λ is a successor, then $\lambda^{<\lambda} < \lambda^\lambda = \lambda^+$ entails $\boxed{\clubsuit}_\lambda$.

Corollary

Assume GCH. Then for every uncountable cardinal λ, TFAE:

- $\Box_\lambda$;
- $\boxed{\clubsuit}_\lambda$.

So, for the Jensen setup, you pay no extra! In fact, you pay less, since $\Box_\lambda$ + GCH implies $\boxed{\clubsuit}_\lambda + \diamondsuit_{\lambda^+}$.

Reflection

Reflection of stationary sets

Definition

We say that a stationary subset $S \subseteq \kappa$ reflects at an ordinal $\alpha < \kappa$, if $S \cap \alpha$ is stationary (as a subset of α).

Fact (Hanf-Scott, 1960's)

If κ is a weakly compact cardinal, then every stationary subset of κ reflects at some $\alpha < \kappa$.

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If κ is a weakly compact cardinal, then every stationary subset of κ reflects at some $\alpha < \kappa$.

Proof.

By Todorcevic, κ is weakly compact iff every ladder system $\langle A_\alpha \mid \alpha < \kappa \rangle$ whose ladders are closed, is trivial in the following sense. There exists a club $C \subseteq \kappa$ such that for all $\beta < \kappa$, there exists $\alpha \geq \beta$ for which $A_\alpha \cap \beta = C \cap \beta$.

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Suppose now that $S \subseteq \kappa$ is stationary and non-reflecting. Then there exists a ladder system as above with $A_\alpha \cap S = \emptyset$ for all limit α. This contradicts the fact that there exists a limit $\beta \in S \cap C$. $\square$

Weak sqaure

A $\square_\lambda$-sequence $\langle C_\alpha \mid \alpha < \lambda^+ \rangle$ is non-trivial in the above sense.

Weak sqaure

A $\Box_\lambda$-sequence $\langle C_\alpha \mid \alpha < \lambda^+ \rangle$ is non-trivial in the above sense. Indeed, since the ladders cohere, $S_\theta = \{\alpha < \lambda^+ \mid \mathrm{otp}(C_\alpha) = \theta\}$ does not reflect for any $\theta < \lambda$, whereas by Fodor's lemma, there must exist some $\theta < \lambda$ for which S_θ is stationary.

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Definition (Jensen, 1960's)

$\square^*_\lambda$ asserts the existence of a ladder system, $\langle C_\alpha \mid \alpha < \lambda^+\rangle$, s.t.:

- $\mathrm{otp}(C_\alpha) \leq \lambda$;
- C_α is closed;
- for all $\beta < \lambda^+$, $\{C_\alpha \cap \beta \mid \alpha < \lambda^+\}$ is of size at most λ.

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$\square^*_\lambda$ follows from $\square_\lambda$, but also from $\lambda^{<\lambda} = \lambda$, hence the main interest in $\square^*_\lambda$ is whenever λ is singular.

Squares and reflection of stationary sets

Theorem (Cummings-Foreman-Magidor, 2001)

It is relatively consistent with the existence of infinitely many supercompact cardinals, that all of the following holds simultaneously:

- GCH;
- $\square^*_{\aleph_\omega}$;
- every stationary subset of $\aleph_{\omega+1}$ reflects.

So, unlike square, weak square does not imply non-reflection.

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Cummings-Foreman-Magidor and Aspero-Krueger-Yoshinobu found that (for a singular λ,) $\square^*_\lambda$ implies sorts of non-reflection, but of generalized stationary sets (in the sense of $\mathcal{P}_\kappa(\lambda), \mathcal{P}_\kappa(\lambda^+)$.)

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- $\square^*_{\aleph_\omega}$;
- every stationary subset of $\aleph_{\omega+1}$ reflects.

We found out that $\square^*_\lambda$ does entail ordinary non-reflection; it is just that the non-reflection takes place in an outer universe...

Weak squares and reflection of stationary sets

Theorem

Suppose that $2^\lambda = \lambda^+$ for a strong limit singular cardinal λ. If $\square^*_\lambda$ holds, then in $V^{Add(\lambda^+,1)}$, there exists a non-reflecting stationary subset of λ^+.

So, this aspect of non-triviality of the weak square system is witnessed in a generic extension.

Weak squares and reflection of stationary sets

Theorem

Suppose that $2^\lambda = \lambda^+$ for a strong limit singular cardinal λ. If $\Box^*_\lambda$ holds, then in $V^{Add(\lambda^+,1)}$, there exists a non-reflecting stationary subset of $\{\alpha < \lambda^+ \mid \mathrm{cf}(\alpha) = \mathrm{cf}(\lambda)\}$.

So, this aspect of non-triviality of the weak square system is witnessed in a generic extension.

Weak squares and reflection of stationary sets

Theorem

Suppose that $2^\lambda = \lambda^+$ for a strong limit singular cardinal λ. If $\square^*_\lambda$ holds, then in $V^{Add(\lambda^+,1)}$, there exists a non-reflecting stationary subset of $\{\alpha < \lambda^+ \mid \mathrm{cf}(\alpha) = \mathrm{cf}(\lambda)\}$.

So, this aspect of non-triviality of the weak square system is witnessed in a generic extension.

Compare with the following.

Example

Suppose that $\lambda > \kappa > \mathrm{cf}(\lambda)$, where λ is a strong limit, and κ is a Laver-indestructible supercompact cardinal.

Then $2^\lambda = \lambda^+$ holds for the strong limit singular cardinal λ, while in $V^{\mathrm{Add}(\lambda^+,1)}$, every stationary subset of $\{\alpha < \lambda^+ \mid \mathrm{cf}(\alpha) = \mathrm{cf}(\lambda)\}$ do reflect.

Strong Colorings

Strong colorings

Suppose that $\overrightarrow{C} = \langle C_\alpha \mid \alpha < \kappa \rangle$ is a ladder system whose ladders are closed. For every $\alpha < \beta < \kappa$, let $\beta = \beta_0 > \cdots > \beta_{k+1} = \alpha$ denote the minimal walk from β down to α along $\overrightarrow{C}$.
Let $[\alpha, \beta]_n$ denote the n_{th} element in the walk from β to α.

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Fact (Todorcevic, Shelah, 1980's)

Suppose that S is a stationary subset of κ such that $S \cap C_\alpha = \emptyset$ for every limit $\alpha < \kappa$. (So, S is non-reflecting).
Then there exists an oscillating function $o : [\kappa]^2 \to \omega$ such that

$$S \setminus \bigcup \left\{ [\alpha, \beta]_{o(\alpha,\beta)} \mid \alpha < \beta \text{ in A} \right\}$$

is non-stationary for every cofinal $A \subseteq \kappa$.

Simply definable strong colorings

Suppose that $\overrightarrow{C} = \langle C_\alpha \mid \alpha < \lambda^+ \rangle$ witnesses $\boxed{\clubsuit}_\lambda$, and let $[\alpha, \beta]_n$ denote the n_{th} element in the $\overrightarrow{C}$-walk from β to α.

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Proposition

For every cofinal $B \subseteq \lambda^+$, there exists an $n < \omega$ such that for every cofinal $A \subseteq \lambda^+$, the set

$$\{[\alpha, \beta]_n \mid \alpha \in A, \beta \in B, \alpha < \beta\}$$

is co-bounded in λ^+.

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Corollary

For every cofinal $B \subseteq \lambda^+$, there exists an $n < \omega$ such that for every cofinal $A \subseteq \lambda^+$, the set

$$\{\mathrm{otp}(C_{[\alpha,\beta]_n}) \mid \alpha \in A, \beta \in B, \alpha < \beta\}$$

contains each and every limit ordinal $< \lambda$.

Simply definable strong colorings

Suppose that $\overrightarrow{C} = \langle C_\alpha \mid \alpha < \lambda^+ \rangle$ witnesses $\clubsuit_\lambda$, and let $[\alpha, \beta]_n$ denote the n_{th} element in the $\overrightarrow{C}$-walk from β to α.

Proposition

For every cofinal $B \subseteq \lambda^+$, there exists an $n < \omega$ such that for every cofinal $A \subseteq \lambda^+$, the set

$$\{[\alpha, \beta]_n \mid \alpha \in A, \beta \in B, \alpha < \beta\}$$

is co-bounded in λ^+.

Remark

The above is optimal in the sense that for every $n < \omega$, there exists a cofinal $B \subseteq \lambda^+$, such that

$$\{[\alpha, \beta]_n \mid \alpha, \beta \in B, \alpha < \beta\}$$

omits any limit ordinal $< \lambda^+$.

Simply definable strong colorings

Suppose that $\overrightarrow{C} = \langle C_\alpha \mid \alpha < \lambda^+ \rangle$ witnesses $♣_\lambda$, and let $[\alpha, \beta]_n$ denote the n_{th} element in the $\overrightarrow{C}$-walk from β to α.

Proposition

For every cofinal $B \subseteq \lambda^+$, there exists an $n < \omega$ such that for every cofinal $A \subseteq \lambda^+$, the set

$$\{[\alpha, \beta]_n \mid \alpha \in A, \beta \in B, \alpha < \beta\}$$

is co-bounded in λ^+.

Conjecture

There exists a one-place function $o : \lambda^+ \to \omega$ such that for every cofinal $A, B \subseteq \lambda^+$, the set

$$\{[\alpha, \beta]_{o(\beta)} \mid \alpha \in A, \beta \in B, \alpha < \beta\}$$

is co-bounded in λ^+.

Squares and small forcings

Squares and small forcing notions

Some people (including the speaker) speculated at some point in time that $\square_\lambda$ cannot be introduced by a forcing notion of size $\ll \lambda$. This indeed sounds plausible, However:

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The idea of the proof is to cook up a model in which $\square_{\aleph_\omega}$ fails, while $\{\alpha < \aleph_{\omega+1} \mid \mathrm{cf}(\alpha) > \omega_1\}$ does carry a so-called partial square. Then, to overcome the lack of coherence over $\{\alpha < \aleph_{\omega+1} \mid \mathrm{cf}(\alpha) = \omega_1\}$, they Levy collapse $\aleph_1$ into countable cardinality.

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Squares and small forcing notions

Theorem (Cummings-Foreman-Magidor, 2001)

It is relatively consistent with the existence of a supercompact cardinal that $\square_{\aleph_\omega}$ is introduced by $\mathrm{coll}(\omega, \omega_1)$.

A rant

Insuring coherence by collapsing cardinals? this is cheating!!
Let me correct my conjecture.

Squares and small forcing notions

Theorem (Cummings-Foreman-Magidor, 2001)

It is relatively consistent with the existence of a supercompact cardinal that $\square_{\aleph_\omega}$ is introduced by coll(ω, ω_1).

Speculation, revised

Square/weak square cannot be introduced by a small forcing that does not collapse cardinals.

Squares and small forcing notions

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It is relatively consistent with the existence of a supercompact cardinal that $\square_{\aleph_\omega}$ is introduced by $\mathrm{coll}(\omega, \omega_1)$.

False speculation

Square/weak square cannot be introduced by a small forcing that does not collapse cardinals.

Theorem

It is relatively consistent with the existence of two supercompact cardinals that $\square^*_{\aleph_{\omega_1}}$ is introduced by a cofinality preserving forcing of size $\aleph_3$.

Squares and small forcing notions

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Theorem

It is relatively consistent with the existence of two supercompact cardinals that $\square^*_{\aleph_{\omega_1}}$ is introduced by a cofinality preserving forcing of size $\aleph_3$.

Conjecture

As $\aleph_1$-sized notion of forcing suffices to introduce $\square_{\aleph_\omega}$, then $\aleph_2$-sized notion of forcing should suffice to introduce (in a cofinality-preserving manner!) $\square^*_{\aleph_{\omega_1}}$.

Open Problems

Two problems

Question

Suppose that $\boxed{\clubsuit}_\lambda + \diamondsuit_{\lambda^+}$ holds for a given singular cardinal λ. Does there exists an homogenous λ^+-Souslin tree?

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Theorem (Dolinar-Džamonja, 2010)

$\square_{\omega_1}$ may be introduced by a forcing notion whose working parts are finite. (that is, the part in the forcing conditions which approximates the generic square sequence is finite.)

Conjecture

$\square^*_{\aleph_{\omega_1}}$ may be introduced by a small, cofinality preserving forcing notion whose working parts are finite.

Epilogue

Summary

- $\boxed{\clubsuit}_\lambda$ is a particular form of $\square_\lambda$ whose intrinsic complexity allows to derive complex objects (such as trees, partitions of stationary sets, and strong colorings) in a canonical way;
- $\boxed{\clubsuit}_\lambda$ and $\square_\lambda$ are equivalent, assuming GCH;
- weak square may be introduced by a small forcing that preserves the cardinal structure;
- weak square implies the existence of a non-reflecting stationary set in a generic extension by Cohen forcing.